

This example is set up in Papyrus.

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\usepackage[no-math]{fontspec}  
\setmainfont[Mapping=tex-text]{Papyrus}  
\usepackage[defaultmathsizes]{mathastext}
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Typeset with mathastext 1.15d (2012/10/13).  
(compiled with Xe<sup>Λ</sup>T<sub>E</sub>X)

Theorem 1. Let there be given indeterminates  $u_i, v_i, k_i, x_i, y_i, l_i$ , for  $1 \leq i \leq n$ . We define the following  $n \times n$  matrices

$$U_n = \begin{pmatrix} u_1 & u_2 & \dots & u_n \\ k_1 v_1 & k_2 v_2 & \dots & k_n v_n \\ k_1^2 u_1 & k_2^2 u_2 & \dots & k_n^2 u_n \\ \vdots & \dots & \dots & \vdots \end{pmatrix} \quad V_n = \begin{pmatrix} v_1 & v_2 & \dots & v_n \\ k_1 u_1 & k_2 u_2 & \dots & k_n u_n \\ k_1^2 v_1 & k_2^2 v_2 & \dots & k_n^2 v_n \\ \vdots & \dots & \dots & \vdots \end{pmatrix} \quad (1)$$

where the rows contain alternatively  $u$ 's and  $v$ 's. Similarly:

$$X_n = \begin{pmatrix} x_1 & x_2 & \dots & x_n \\ l_1 y_1 & l_2 y_2 & \dots & l_n y_n \\ l_1^2 x_1 & l_2^2 x_2 & \dots & l_n^2 x_n \\ \vdots & \dots & \dots & \vdots \end{pmatrix} \quad Y_n = \begin{pmatrix} y_1 & y_2 & \dots & y_n \\ l_1 x_1 & l_2 x_2 & \dots & l_n x_n \\ l_1^2 y_1 & l_2^2 y_2 & \dots & l_n^2 y_n \\ \vdots & \dots & \dots & \vdots \end{pmatrix} \quad (2)$$

There holds

$$\det \left( \frac{u_i y_j - v_i x_j}{l_j - k_i} \right)_{1 \leq i, j \leq n} = \frac{1}{\prod_{i,j} (l_j - k_i)} \begin{vmatrix} U_n & X_n \\ V_n & Y_n \end{vmatrix}_{2n \times 2n} \quad (3)$$

Proof. Let  $A, B, C, D$  be  $n \times n$  matrices, with  $A$  and  $C$  invertible. Using  $\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix} \begin{pmatrix} I & A^{-1}B \\ 0 & C^{-1}D \end{pmatrix}$  we obtain

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = |A||C| |C^{-1}D - A^{-1}B| \quad (4)$$

where vertical bars denote determinants. Let  $d(u) = \text{diag}(u_1, \dots, u_n)$  and  $p_u = \prod_{1 \leq i \leq n} u_i$ . We define similarly  $d(v), d(x), d(y)$  and  $p_v, p_x, p_y$ . From the previous identity we get

$$\begin{vmatrix} Ad(u) & Bd(x) \\ Cd(v) & Dd(y) \end{vmatrix} = |A||C| p_u p_v \left| d(v)^{-1} C^{-1} D d(y) - d(u)^{-1} A^{-1} B d(x) \right| \\ = |A||C| \left| d(u) C^{-1} D d(y) - d(v) A^{-1} B d(x) \right| \quad (5)$$

The special case  $A = C, B = D$ , gives

$$\begin{vmatrix} Ad(u) & Bd(x) \\ Ad(v) & Bd(y) \end{vmatrix}_{2n \times 2n} = \det(A)^2 \det \left( (u_i y_j - v_i x_j) (A^{-1} B)_{ij} \right) \quad (6)$$

Let  $W(k)$  be the Vandermonde matrix with rows  $(1 \dots 1)$ ,  $(k_1 \dots k_n)$ ,  $(k_1^2 \dots k_n^2)$ , ..., and  $\Delta(k) = \det W(k)$  its determinant. Let

$$K(t) = \prod_{1 \leq m \leq n} (t - k_m) \quad (7)$$

and let  $C$  be the  $n \times n$  matrix  $(c_{im})_{1 \leq i, m \leq n}$ , where the  $c_{im}$ 's are defined by the partial fraction expansions:

$$1 \leq i \leq n \quad \frac{t^{i-1}}{K(t)} = \sum_{1 \leq m \leq n} \frac{c_{im}}{t - k_m} \quad (8)$$

We have the two matrix equations:

$$C = W(k) \operatorname{diag}(K'(k_1)^{-1}, \dots, K'(k_n)^{-1}) \quad (9a)$$

$$C \cdot \left( \frac{1}{l_j - k_m} \right)_{1 \leq m, j \leq n} = W(l) \operatorname{diag}(K(l_1)^{-1}, \dots, K(l_n)^{-1}) \quad (9b)$$

This gives the (well-known) identity:

$$\left( \frac{1}{l_j - k_m} \right)_{1 \leq m, j \leq n} = \operatorname{diag}(K'(k_1), \dots, K'(k_n)) W(k)^{-1} W(l) \operatorname{diag}(K(l_1)^{-1}, \dots, K(l_n)^{-1}) \quad (10)$$

We can thus rewrite the determinant we want to compute as:

$$\left| \frac{u_i y_j - v_i x_j}{l_j - k_i} \right|_{1 \leq i, j \leq n} = \prod_m K'(k_m) \prod_j K(l_j)^{-1} \left| (u_i y_j - v_i x_j) (W(k)^{-1} W(l))_{ij} \right|_{n \times n} \quad (11)$$

We shall now make use of (6) with  $A = W(k)$  and  $B = W(l)$ .

$$\begin{aligned} \left| \frac{u_i y_j - v_i x_j}{l_j - k_i} \right|_{1 \leq i, j \leq n} &= \Delta(k)^{-2} \prod_m K'(k_m) \prod_j K(l_j)^{-1} \left| \begin{array}{cc} W(k) d(u) & W(l) d(x) \\ W(k) d(v) & W(l) d(y) \end{array} \right| \\ &= \frac{(-1)^{\frac{n(n-1)}{2}}}{\prod_{i,j} (l_j - k_i)} \left| \begin{array}{cc} W(k) d(u) & W(l) d(x) \\ W(k) d(v) & W(l) d(y) \end{array} \right|_{2n \times 2n} \end{aligned} \quad (12)$$

The sign  $(-1)^{n(n-1)/2} = (-1)^{\lfloor \frac{n}{2} \rfloor}$  is the signature of the permutation which exchanges rows  $i$  and  $n + i$  for  $i = 2, 4, \dots, 2\lfloor \frac{n}{2} \rfloor$  and transforms the

determinant on the right-hand side into  $\begin{vmatrix} U_n & X_n \\ V_n & Y_n \end{vmatrix}$ . This concludes the proof.  $\square$