

This example is set up in Libertine. It uses:

```
\usepackage[T1]{fontenc}  
\usepackage{libertine}  
\usepackage[greek=n]{libgreek}  
\usepackage[noasterisk,defaultmathsizes]{mathastext}
```

Typeset with mathastext 1.13 (2011/03/11).

To illustrate some Hilbert Space properties of the co-Poisson summation, we will assume  $K = \mathbf{Q}$ . The components  $(a_v)$  of an adele  $a$  are written  $a_p$  at finite places and  $a_r$  at the real place. We have an embedding of the Schwartz space of test-functions on  $\mathbf{R}$  into the Bruhat-Schwartz space on  $\mathbf{A}$  which sends  $\psi(x)$  to  $\phi(a) = \prod_p \mathbf{1}_{|a_p|_p \leq 1}(a_p) \cdot \psi(a_r)$ , and we write  $E'_R(g)$  for the distribution on  $\mathbf{R}$  thus obtained from  $E'(g)$  on  $\mathbf{A}$ .

**Theorem 1.** *Let  $g$  be a compact Bruhat-Schwartz function on the ideles of  $\mathbf{Q}$ . The co-Poisson summation  $E'_R(g)$  is a square-integrable function (with respect to the Lebesgue measure). The  $L^2(\mathbf{R})$  function  $E'_R(g)$  is equal to the constant  $-\int_{\mathbf{A}^\times} g(v)|v|^{-1/2} d^*v$  in a neighborhood of the origin.*

*Proof.* We may first, without changing anything to  $E'_R(g)$ , replace  $g$  with its average under the action of the finite unit ideles, so that it may be assumed invariant. Any such compact invariant  $g$  is a finite linear combination of suitable multiplicative translates of functions of the type  $g(v) = \prod_p \mathbf{1}_{|v_p|_p=1}(v_p) \cdot f(v_r)$  with  $f(t)$  a smooth compactly supported function on  $\mathbf{R}^\times$ , so that we may assume that  $g$  has this form. We claim that:

$$\int_{\mathbf{A}^\times} |\phi(v)| \sum_{q \in \mathbf{Q}^\times} |g(qv)| \sqrt{|v|} d^*v < \infty$$

Indeed  $\sum_{q \in \mathbf{Q}^\times} |g(qv)| = |f(|v|)| + |f(-|v|)|$  is bounded above by a multiple of  $|v|$ . And  $\int_{\mathbf{A}^\times} |\phi(v)| |v|^{3/2} d^*v < \infty$  for each Bruhat-Schwartz function on the adeles (basically, from  $\prod_p (1 - p^{-3/2})^{-1} < \infty$ ). So

$$\begin{aligned} E'(g)(\phi) &= \sum_{q \in \mathbf{Q}^\times} \int_{\mathbf{A}^\times} \phi(v) g(qv) \sqrt{|v|} d^*v - \int_{\mathbf{A}^\times} \frac{g(v)}{\sqrt{|v|}} d^*v \int_{\mathbf{A}} \phi(x) dx \\ E'(g)(\phi) &= \sum_{q \in \mathbf{Q}^\times} \int_{\mathbf{A}^\times} \phi(v/q) g(v) \sqrt{|v|} d^*v - \int_{\mathbf{A}^\times} \frac{g(v)}{\sqrt{|v|}} d^*v \int_{\mathbf{A}} \phi(x) dx \end{aligned}$$

Let us now specialize to  $\phi(a) = \prod_p \mathbf{1}_{|a_p|_p \leq 1}(a_p) \cdot \psi(a_r)$ . Each integral can be evaluated as an infinite product. The finite places contribute 0 or 1 according to whether  $q \in \mathbf{Q}^\times$  satisfies  $|q|_p < 1$  or not. So only the inverse integers  $q = 1/n$ ,  $n \in \mathbf{Z}$ , contribute:

$$E'_R(g)(\psi) = \sum_{n \in \mathbf{Z}^\times} \int_{\mathbf{R}^\times} \psi(nt) f(t) \sqrt{|t|} \frac{dt}{2|t|} - \int_{\mathbf{R}^\times} \frac{f(t)}{\sqrt{|t|}} \frac{dt}{2|t|} \int_{\mathbf{R}} \psi(x) dx$$

We can now revert the steps, but this time on  $\mathbf{R}^\times$  and we get:

$$E'_R(g)(\psi) = \int_{\mathbf{R}^\times} \psi(t) \sum_{n \in \mathbf{Z}^\times} \frac{f(t/n)}{\sqrt{|n|}} \frac{dt}{2\sqrt{|t|}} - \int_{\mathbf{R}^\times} \frac{f(t)}{\sqrt{|t|}} \frac{dt}{2|t|} \int_{\mathbf{R}} \psi(x) dx$$

Let us express this in terms of  $\alpha(y) = (f(y) + f(-y))/2\sqrt{|y|}$ :

$$E'_R(g)(\psi) = \int_R \psi(y) \sum_{n \geq 1} \frac{\alpha(y/n)}{n} dy - \int_0^\infty \frac{\alpha(y)}{y} dy \int_R \psi(x) dx$$

So the distribution  $E'_R(g)$  is in fact the even smooth function

$$E'_R(g)(y) = \sum_{n \geq 1} \frac{\alpha(y/n)}{n} - \int_0^\infty \frac{\alpha(y)}{y} dy$$

As  $\alpha(y)$  has compact support in  $R \setminus \{0\}$ , the summation over  $n \geq 1$  contains only vanishing terms for  $|y|$  small enough. So  $E'_R(g)$  is equal to the constant  $-\int_0^\infty \frac{\alpha(y)}{y} dy = -\int_{R^\times} \frac{f(y)}{\sqrt{|y|}} \frac{dy}{2|y|} = -\int_{A^\times} g(t)/\sqrt{|t|} d^*t$  in a neighborhood of 0.

To prove that it is  $L^2$ , let  $\beta(y)$  be the smooth compactly supported function  $\alpha(1/y)/2|y|$  of  $y \in R$  ( $\beta(0) = 0$ ). Then ( $y \neq 0$ ):

$$E'_R(g)(y) = \sum_{n \in Z} \frac{1}{|y|} \beta\left(\frac{n}{y}\right) - \int_R \beta(y) dy$$

From the usual Poisson summation formula, this is also:

$$\sum_{n \in Z} \gamma(ny) - \int_R \beta(y) dy = \sum_{n \neq 0} \gamma(ny)$$

where  $\gamma(y) = \int_R \exp(i2\pi yw) \beta(w) dw$  is a Schwartz rapidly decreasing function. From this formula we deduce easily that  $E'_R(g)(y)$  is itself in the Schwartz class of rapidly decreasing functions, and in particular it is square-integrable.  $\square$

It is useful to recapitulate some of the results arising in this proof:

**Theorem 2.** *Let  $g$  be a compact Bruhat-Schwartz function on the ideles of  $\mathbf{Q}$ . The co-Poisson summation  $E'_R(g)$  is an even function on  $R$  in the Schwartz class of rapidly decreasing functions. It is constant, as well as its Fourier Transform, in a neighborhood of the origin. It may be written as*

$$E'_R(g)(y) = \sum_{n \geq 1} \frac{\alpha(y/n)}{n} - \int_0^\infty \frac{\alpha(y)}{y} dy$$

*with a function  $\alpha(y)$  smooth with compact support away from the origin, and conversely each such formula corresponds to the co-Poisson summation  $E'_R(g)$  of a compact Bruhat-Schwartz function on the ideles of  $\mathbf{Q}$ . The Fourier transform  $\int_R E'_R(g)(y) \exp(i2\pi wy) dy$  corresponds in the formula above to the replacement  $\alpha(y) \mapsto \alpha(1/y)/|y|$ .*

Everything has been obtained previously.