This example is set up in Romande ADF. It uses:

```
\usepackage[T1]{fontenc}
\usepackage{fourier}
\usepackage{romande}
\usepackage[italic,defaultmathsizes,noasterisk]{mathastext}
\renewcommand{\itshape}{\swashstyle}
```

Typeset with mathastext 1.13 (2011/03/11).

To illustrate some Hilbert Space properties of the co-Poisson summation, we will assume $K=\mathbf{Q}$. The components (a_v) of an adele a are written a_p at finite places and a_r at the real place. We have an embedding of the Schwartz space of test-functions on $\mathbf{R}$ into the Bruhat-Schwartz space on $\mathbf{A}$ which sends $\psi(x)$ to $\varphi(a)=\prod_p \mathbf{1}_{|a_p|_p\leq 1}(a_p)\cdot\psi(a_r)$, and we write $E'_{\mathbf{R}}(g)$ for the distribution on $\mathbf{R}$ thus obtained from $E'(g)$ on $\mathbf{A}$.

Theorem 1. *Let g be a compact Bruhat-Schwartz function on the ideles of $\mathbf{Q}$. The co-Poisson summation $E'_{\mathbf{R}}(g)$ is a square-integrable function (with respect to the Lebesgue measure). The $L^2(\mathbf{R})$ function $E'_{\mathbf{R}}(g)$ is equal to the constant $-\int_{\mathbf{A}^\times} g(v)|v|^{-1/2}d^*v$ in a neighborhood of the origin.*

Proof. We may first, without changing anything to $E'_{\mathbf{R}}(g)$, replace g with its average under the action of the finite unit ideles, so that it may be assumed invariant. Any such compact invariant g is a finite linear combination of suitable multiplicative translates of functions of the type $g(v)=\prod_p \mathbf{1}_{|v_p|_p=1}(v_p)\cdot f(v_r)$ with $f(t)$ a smooth compactly supported function on $\mathbf{R}^\times$, so that we may assume that g has this form. We claim that:

$$\int_{\mathbf{A}^\times}|\varphi(v)|\sum_{q\in\mathbf{Q}^\times}|g(qv)|\sqrt{|v|}\,d^*v<\infty$$

Indeed $\sum_{q\in\mathbf{Q}^\times}|g(qv)|=|f(|v|)|+|f(-|v|)|$ is bounded above by a multiple of $|v|$. And $\int_{\mathbf{A}^\times}|\varphi(v)||v|^{3/2}\,d^*v<\infty$ for each Bruhat-Schwartz function on the adeles (basically, from $\prod_p(1-p^{-3/2})^{-1}<\infty$). So

$$E'(g)(\varphi)=\sum_{q\in\mathbf{Q}^\times}\int_{\mathbf{A}^\times}\varphi(v)g(qv)\sqrt{|v|}\,d^*v-\int_{\mathbf{A}^\times}\frac{g(v)}{\sqrt{|v|}}d^*v\int_{\mathbf{A}}\varphi(x)\,dx$$

$$E'(g)(\varphi)=\sum_{q\in\mathbf{Q}^\times}\int_{\mathbf{A}^\times}\varphi(v/q)g(v)\sqrt{|v|}\,d^*v-\int_{\mathbf{A}^\times}\frac{g(v)}{\sqrt{|v|}}d^*v\int_{\mathbf{A}}\varphi(x)\,dx$$

Let us now specialize to $\varphi(a)=\prod_p \mathbf{1}_{|a_p|_p\leq 1}(a_p)\cdot\psi(a_r)$. Each integral can be evaluated as an infinite product. The finite places contribute 0 or 1 according to whether $q\in\mathbf{Q}^\times$ satisfies $|q|_p<1$ or not. So only the inverse integers $q=1/n$, $n\in\mathbf{Z}$, contribute:

$$E'_{\mathbf{R}}(g)(\psi)=\sum_{n\in\mathbf{Z}^\times}\int_{\mathbf{R}^\times}\psi(nt)f(t)\sqrt{|t|}\frac{dt}{2|t|}-\int_{\mathbf{R}^\times}\frac{f(t)}{\sqrt{|t|}}\frac{dt}{2|t|}\int_{\mathbf{R}}\psi(x)\,dx$$

We can now revert the steps, but this time on $\mathbf{R}^\times$ and we get:

$$E'_{\mathbf{R}}(g)(\psi) = \int_{\mathbf{R}^\times} \psi(t) \sum_{n\in\mathbf{Z}^\times} \frac{f(t/n)}{\sqrt{|n|}} \frac{dt}{2\sqrt{|t|}} - \int_{\mathbf{R}^\times} \frac{f(t)}{\sqrt{|t|}} \frac{dt}{2|t|} \int_{\mathbf{R}} \psi(x)\, dx$$

Let us express this in terms of $\alpha(y) = (f(y)+f(-y))/2\sqrt{|y|}$:

$$E'_{\mathbf{R}}(g)(\psi) = \int_{\mathbf{R}} \psi(y) \sum_{n\geq 1} \frac{\alpha(y/n)}{n} dy - \int_0^\infty \frac{\alpha(y)}{y} dy \int_{\mathbf{R}} \psi(x)\, dx$$

So the distribution $E'_{\mathbf{R}}(g)$ is in fact the even smooth function

$$E'_{\mathbf{R}}(g)(y) = \sum_{n\geq 1} \frac{\alpha(y/n)}{n} - \int_0^\infty \frac{\alpha(y)}{y} dy$$

As $\alpha(y)$ has compact support in $\mathbf{R}\setminus\{0\}$, the summation over $n \geq 1$ contains only vanishing terms for $|y|$ small enough. So $E'_{\mathbf{R}}(g)$ is equal to the constant $-\int_0^\infty \frac{\alpha(y)}{y} dy = -\int_{\mathbf{R}^\times} \frac{f(y)}{\sqrt{|y|}} \frac{dy}{2|y|} = -\int_{\mathbf{A}^\times} g(t)/\sqrt{|t|}\, d^*t$ in a neighborhood of 0. To prove that it is L^2, let $\beta(y)$ be the smooth compactly supported function $\alpha(1/y)/2|y|$ of $y \in \mathbf{R}$ ($\beta(0) = 0$). Then ($y \neq 0$):

$$E'_{\mathbf{R}}(g)(y) = \sum_{n\in\mathbf{Z}} \frac{1}{|y|} \beta(\frac{n}{y}) - \int_{\mathbf{R}} \beta(y)\, dy$$

From the usual Poisson summation formula, this is also:

$$\sum_{n\in\mathbf{Z}} \gamma(ny) - \int_{\mathbf{R}} \beta(y)\, dy = \sum_{n\neq 0} \gamma(ny)$$

where $\gamma(y) = \int_{\mathbf{R}} \exp(i2\pi yw)\beta(w)\, dw$ is a Schwartz rapidly decreasing function. From this formula we deduce easily that $E'_{\mathbf{R}}(g)(y)$ is itself in the Schwartz class of rapidly decreasing functions, and in particular it is is square-integrable. □

It is useful to recapitulate some of the results arising in this proof:

Theorem 2. *Let g be a compact Bruhat-Schwartz function on the ideles of $\mathbf{Q}$. The co-Poisson summation $E'_{\mathbf{R}}(g)$ is an even function on $\mathbf{R}$ in the Schwartz class of rapidly decreasing functions. It is constant,* *as well as its Fourier Transform, in a neighborhood of the origin. It may be written as*

$$E'_{\mathbf{R}}(g)(y) = \sum_{n\geq 1} \frac{\alpha(y/n)}{n} - \int_0^\infty \frac{\alpha(y)}{y} dy$$

with a function $\alpha(y)$ *smooth with compact support away from the origin, and conversely each such formula corresponds to the co-Poisson summation* $E'_{\mathbf{R}}(g)$ *of a compact Bruhat-Schwartz function on the ideles of* $\mathbf{Q}$*. The Fourier transform* $\int_{\mathbf{R}} E'_{\mathbf{R}}(g)(y) \exp(i2\pi wy)\, dy$ *corresponds in the formula above to the replacement* $\alpha(y) \mapsto \alpha(1/y)/|y|$.

Everything has been obtained previously.