Additional exercises

This paper consists of 16 pages including appendices.
Permitted materials: calculator

1 Random variable parameter estimation

A discrete random variable X is defined by

$$X = \begin{cases} -20, & prob. = 1/3 \\ 30, & prob. = 1/2 \\ 10, & prob. = 1/6 \end{cases} \tag{1}$$

(a) find the expected value

(b) find the variance

(c) find the mode

(d) find the coefficient of variation

2 Probability density function

The probability density function of a random variable x is given as:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

(a) Draw a free-hand figure to illustrate the density function and indicate on the figure the probability $prob(x < a)$, $prob(a < x < b)$ and $prob(x > b)$

(b) Draw a free-hand figure to illustrate the cumulative probability distribution function and indicate on the figure the probability $prob(x < a)$, $prob(a < x < b)$ and $prob(x > b)$.

(c) Define or explain the first and second moment of x.

(d) Explain the 68-95-99.7-percent rule

3 Frequency analysis and linear regression

(a) What is the probability to observe at least one 200-years flood or larger within a period of 10 years?

(b) Figure 1A shows a simple linear regression between average runoff and median annual flood. Figure 1B shows the QQ-plot of the residual where the theoretical quantiles were calculated using the normal distribution. Describe which assumption of a simple linear regression is violated in this analysis, and discuss strategies that can be used to improve the analysis.

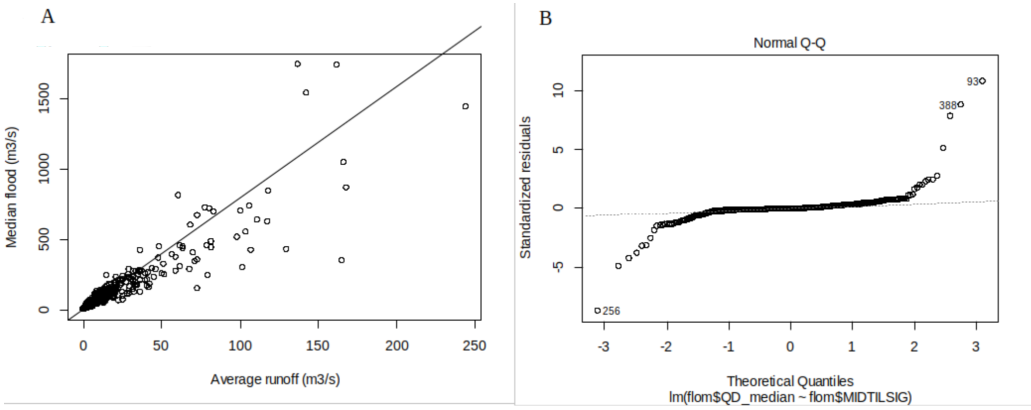


Figure 1: A) linear regression, B) Q-Q plot.

4 Correlation

Consider the following bi-variate sample:

$$X = \{21, 32, 34, 11, 32\}$$

$$Y = \{43, 45, 47, 33, 34\}$$

(a) Calculate the Pearson correlation coefficient of X and Y.

(b) Calculate the Spearman rank correlation of X and Y.

5 Confidence intervals

A sample of 20 random observations produced a mean of 145 and variance of 30.

(a) What is the 95% confidence interval on the mean assuming a normal distribution if

   (i) the true variance is unknown and estimated as 30

   (ii) the true variance is 30

(b) What is the reason for the difference of results in part (i) and part (ii)?

(c) What is the 95% confidence interval on the variance?

6 Hypothesis testing

Below is a 20-year dataset (all values are in m^3/s) with observations of runoff before a human intervention, and a 7-year dataset from after the human intervention in a basin.

Before the intervention:
24, 12, 48, 17, 14, 28, 11, 13, 31, 34, 34, 12, 48, 14, 28, 17, 11, 13, 31, 24 (mean = 23.2, standard deviation = 11.48)

After the intervention:
29, 46, 49, 31, 28, 50, 31 (mean = 37.7, standard deviation = 9.32)

Could it be stated from this small sample that the runoff (both mean and variance) was affected by the intervention? Use a significance level of 5%. What is the probability of making a type II error in your decision?

7 Goodness-of-fit testing

In order to calculate the design flood, you need to know which probability distribution fits your data best. Based on a 35-year record of yearly maximum discharge data from a river station, the following statistics are calculated (with $n = 35$)
Let $Y = \ln Q$ (natural logarithm)
Average: $\bar{Y} = 5.257$
Standard deviation: $S_Y = 0.686$
The 35 years data are classified into 5 classes as given in the table below.

Class	Observed number
$Q \leq 100$	5
$100 < Q \leq 150$	9
$150 < Q \leq 200$	7
$200 < Q \leq 300$	6
$Q \geq 300$	8

(a) Use Chi square method (with significance level of 5%) to test if the data are log-normally distributed

(b) Could the Kolmogorov-Smirnov test also be used to perform the goodness-of-fit test?

8 Time series analysis

A 12-year time series of annual measurements is given as the sequence:
110, 88, 51, 64, 39, 23, 10, 10, 6, 5, 10, 3
Perform a trend test of your choice to check if the time series contains a trend. Use a significance level of 5%.

9 Machine learning

(a) Why is it common to split the dataset into a training set and a test set when doing machine learning? In your answer, include in a relevant way the terms training error and test error

(b) In many machine learning algorithms you have a parameter that controls the complexity of the model. Why do we want to control this complexity?

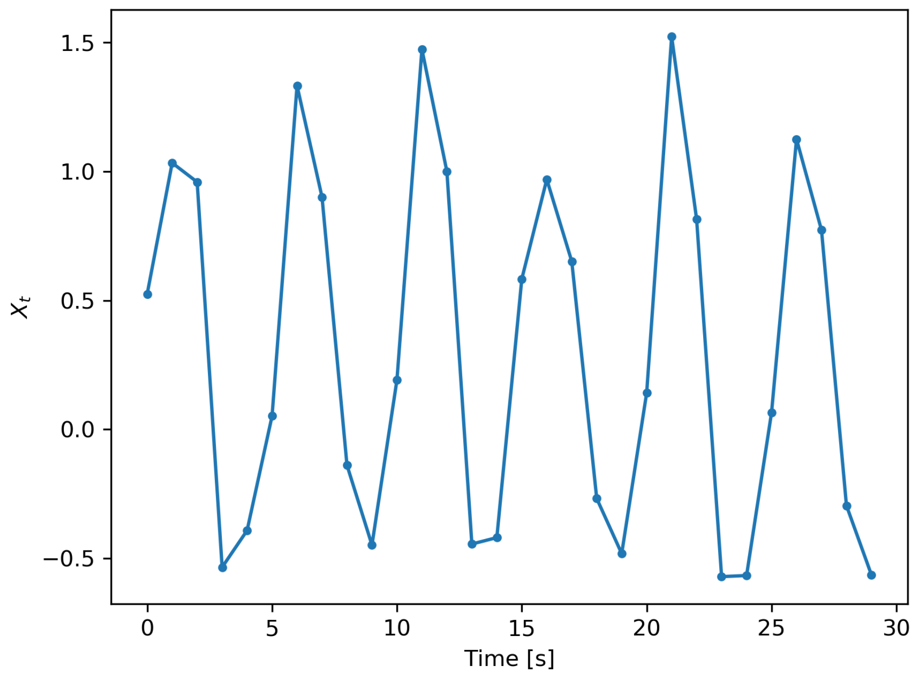
10 Fourier transformation

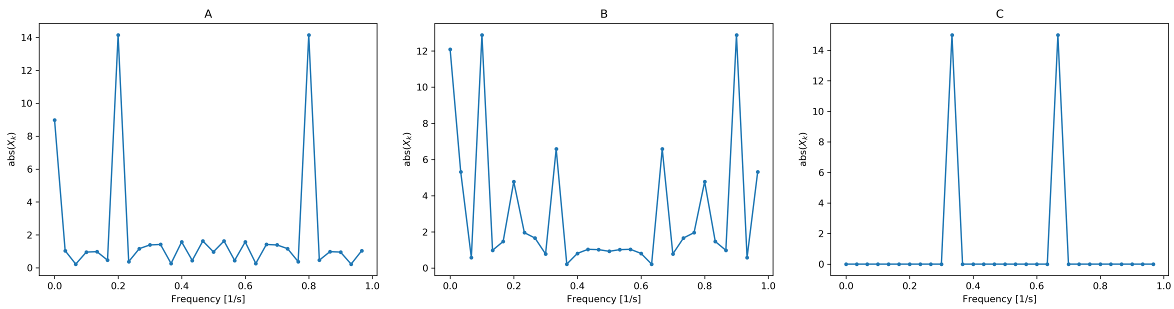
Consider the following time series X_t sampled once per second

(a) The following three graphs show the absolute values for the Fourier coefficients, defined as:

$$X_k = \sum_{n=0}^{N-1} x_n \cdot e^{-i\,2\pi\,k\,n/N}$$

Which one of them (A, B or C) shows the Fourier transform of X_t? Explain your answer.





11 Time series analysis and stochastic modeling

Answer the following statements with True if it is always true, otherwise with False.

(a) Most geoscientific processes (time series) are stochastic processes

(b) Stochastic processes (time series) are stationary

(c) An independent time series is a stationary time series

(d) An independent stationary time series is purely random

(e) A Gaussian time series is an independent series

(f) A Gaussian time series is a stationary time series

(g) A white noise times series is normally distributed

(h) If Xt is a log-normally distributed time series, then Xt is symmetric about its mean

(i) Markov model (i.e. AR(1) model) is used for an independent stationary time series

(j) Markov model (i.e. AR(1) model) is used for a nonstationary time series

Formulary

Statistics

Sample mean:

$$\hat{\mu} = m_X = \frac{1}{N}\sum_{i=1}^{N} x_i$$

Sample variance:

$$\hat{\sigma}^2 = s_X{}^2 = \frac{1}{N-1}\sum_{i=1}^{N}\left(x_i - m_X\right)^2$$

Sample coefficient of variation:

$$C\hat{V}_X = \frac{s_X}{m_X}$$

Sample covariance:

$$\hat{\rho}_{X,Y} = r_{X,Y} = \frac{\sum_{i=1}^{N}\left(x_i - m_X\right)\cdot\left(y_i - m_Y\right)}{\left(N-1\right)\cdot s_X \cdot s_Y}$$

Cumulative probability function:

$$P_X\left(x_i\right) \equiv \Pr\left\{X \le x_i\right\} = \sum_{X \le x_i} p_x\left(x_i\right)$$

Weibull plotting-position:

$$q = \frac{i}{N+1}$$

Probability theory

The *conditional probability* of A to occur, given that B occurs

$$P(A \mid B) = P(A \cap B) / P(B)$$

where:

$P(A|B)$ = the (conditional) probability that event A will occur given that event B has occurred already

$P(A \cap B)$ = the (unconditional) probability that event A and event B occur

The probability that A or B occurs, in case of dependent events:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$P(A \cup B) = P(A)+P(B)- P(A \mid B)\cdot P(B)$	if A and B are dependent
$P(A \cup B) = P(A)+P(B)- P(A) \cdot P(B)$	if A and B are independent
$P(A \cup B) = P(A)+P(B)$	if A and B are mutually exclusive

The probability of both A and B to occur:

$P(A \cap B) = P(A|B)\cdot P(B)$ or $P(A \cap B) = P(B|A) \cdot P(A)$

$P(A \cap B) = P(A) \cdot P(B)$ if A and B are independent

Probability of 'event A does not occur'

$P(A') = 1-P(A)$	probability of A not to occur
$P(A\cap A') = 0$	probability of A occur ***and*** A not occur = 0
$P(A\cup A') = P(A)+P(A')=1$	probability of A occur ***or*** A not occur = 1

Sampling and counting

- In Ordered without replacement case, the first item has *n* ways of selection, and second has *n-1* ways, …. Thus *r* ordered items can be selected from *n* without replacement in:
  $(n)_r = n(n-1)(n-2)\ldots(n-r+1) = n!/(n-r)!$

  (The is called the number of **Permutations** of n items taken r at a time)

- Unordered without replacement case, the number of ways in selecting r items from n items is
  $$\binom{n}{r} = \frac{(n)_r}{r!} = \frac{n!}{(n-r)!r!}$$

(Combinations are arrangements of elements without regard to their order or position).

- In unordered with replacement case, the number of ways in selecting r items from n items is

$$\binom{n+r-1}{r} = \frac{(n+r-1)!}{(n-1)!r!}$$

Probability distribution functions

For a continuous random variable X, if the function f(x) satisfies:

a. f(x)>=0 for all x_i
b. $\int f(x)$ dx = 1 for the whole range of x

then f(x) is a probability density function.

Relationship between f(x) and F(x) is

F(x) = ∫f(x)dx
And f(x) = dF(x)/dx

Since F(x) = P(X ≤ x) it follows that

$\int_a^b f(x)dx$ = P(a < X < b) = F (b) - F (a)

Chi-Square distribution:

$$\chi_c^2 = \sum_{j=1}^{n} \frac{(O_j - E_j)^2}{E_j}$$

Hypergeometric distribution: The probability of getting x successes in a sample of size n drawing from population of size N contains k successes is

$$f_x(x; N, n, k) = \binom{k}{x}\binom{N-k}{n-x} \Big/ \binom{N}{n}$$

$$E(x) = \frac{n \cdot k}{N}$$

$$Var(x) = \frac{n \cdot k(N-k)(N-n)}{N^2(N-1)}$$

Binomial distribution:

$$f_x(x; n, p) = \binom{n}{x} p^x q^{n-x}$$

where x- number of success, n – total number of trials, p – probability of one success, q – probability of one failure

$$E(x) = np$$

$$Var(x)= npq$$

Geometric distribution:

$$f_x(x; p) = pq^{x-1}$$

$$E(x)=1/p$$

$$Var(x)=q/p2$$

Flood and extreme value analysis

Return period:

$$TR_X(x) = \frac{1}{1 - F_X(x)}$$

Gumbel distribution:

$$\Pr\{X \leq x\} = F_X(x) = \exp\left[-\exp\left(-\left(\frac{x-\xi}{\alpha}\right)\right)\right]$$

$$\mu_x = \xi + 0.5772\alpha$$

$$\sigma_x^2 = 1.645\alpha^2$$

Usage of the frequency factor, *K*, for the determination of values with return period, *TR*:

for the Gumbel distribution:
K(10) = 1.30
K(100) = 3.14
K(1000) = 5.00

Confidence Interval and Hypothesis Testing

Confidence Interval for the mean of a normal distribution with unknown variance (n small):

$$l = \overline{x} - t_{1-\frac{\alpha}{2},n-1} S_{\overline{x}}$$
$$u = \overline{x} + t_{1-\frac{\alpha}{2},n-1} S_{\overline{x}}$$

Confidence Interval for the mean of a normal distribution with known variance (n large):

$$l = \overline{x} - z_{1-\frac{\alpha}{2}} \sigma_{\overline{x}}$$

$$u = \overline{x} + z_{1-\frac{\alpha}{2}} \sigma_{\overline{x}}$$

Confidence Interval for the variance of a normal distribution:

$$l = \frac{(n-1)S_x^2}{\chi^2_{1-\frac{\alpha}{2},n-1}} \quad , \qquad u = \frac{(n-1)S_x^2}{\chi^2_{\frac{\alpha}{2},n-1}}$$

Hypothesis test statistics

$$t = \frac{\overline{x} - \mu}{S_{\overline{x}}}$$

$$H_o \; is \; rejected \; if \; |t| > t_{1-\frac{\alpha}{2},n-1}$$

$$z = \frac{\overline{x} - \mu}{\sigma_{\overline{x}}}$$

$$H_o \; is \; rejected \; if \; |Z| > z_{1-\frac{\alpha}{2}}$$

$$F_c = S_1^2 / S_2^2$$

$$H_o \; is \; rejected \; if \; F_c > F_{1-\alpha,n1-1,n2-1}$$

Two sample, two-tail:

$$z = \frac{(\overline{x}_{1-} \overline{x}_2 - \delta)}{\sqrt{\frac{\sigma_1^2}{n1} + \frac{\sigma_2^2}{n2}}}$$

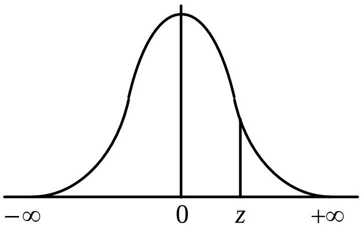
$$H_o \; is \; rejected \; if \; |Z| > z_{1-\frac{\alpha}{2}}$$

$$t = \frac{(\overline{x}_{1-}\ \overline{x}_2 - \delta)}{\sqrt{\frac{s_1^2}{n1} + \ \frac{s_2^2}{n2}}}$$

$$H_o\ is\ rejected\ if\ |t| > \frac{w_1 t_1 + w_2 t_2}{w_1 + w_2}$$

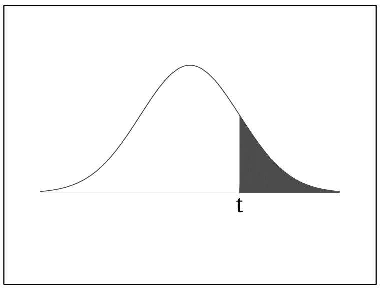
Where, $w_1\ = \ \frac{s_1^2}{n}$, $w_2\ = \ \frac{s_2^2}{n}$, $t_1 = \ t_{1-\frac{\alpha}{2}, n_1 - 1}$, $t_2 = \ t_{1-\frac{\alpha}{2}, n_2 - 1}$

NORMAL DISTRIBUTION TABLE



	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

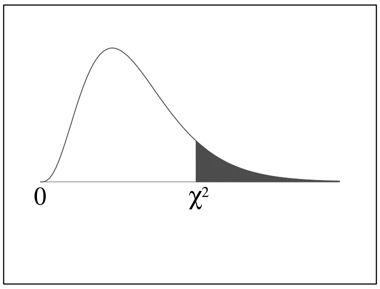
t-Distribution Table



The shaded area is equal to α for $t = t_\alpha$.

df	$t_{.100}$	$t_{.050}$	$t_{.025}$	$t_{.010}$	$t_{.005}$
1	3.078	6.314	12.706	31.821	63.657
2	1.886	2.920	4.303	6.965	9.925
3	1.638	2.353	3.182	4.541	5.841
4	1.533	2.132	2.776	3.747	4.604
5	1.476	2.015	2.571	3.365	4.032
6	1.440	1.943	2.447	3.143	3.707
7	1.415	1.895	2.365	2.998	3.499
8	1.397	1.860	2.306	2.896	3.355
9	1.383	1.833	2.262	2.821	3.250
10	1.372	1.812	2.228	2.764	3.169
11	1.363	1.796	2.201	2.718	3.106
12	1.356	1.782	2.179	2.681	3.055
13	1.350	1.771	2.160	2.650	3.012
14	1.345	1.761	2.145	2.624	2.977
15	1.341	1.753	2.131	2.602	2.947
16	1.337	1.746	2.120	2.583	2.921
17	1.333	1.740	2.110	2.567	2.898
18	1.330	1.734	2.101	2.552	2.878
19	1.328	1.729	2.093	2.539	2.861
20	1.325	1.725	2.086	2.528	2.845
21	1.323	1.721	2.080	2.518	2.831
22	1.321	1.717	2.074	2.508	2.819
23	1.319	1.714	2.069	2.500	2.807
24	1.318	1.711	2.064	2.492	2.797
25	1.316	1.708	2.060	2.485	2.787
26	1.315	1.706	2.056	2.479	2.779
27	1.314	1.703	2.052	2.473	2.771
28	1.313	1.701	2.048	2.467	2.763
29	1.311	1.699	2.045	2.462	2.756
30	1.310	1.697	2.042	2.457	2.750
32	1.309	1.694	2.037	2.449	2.738
34	1.307	1.691	2.032	2.441	2.728
36	1.306	1.688	2.028	2.434	2.719
38	1.304	1.686	2.024	2.429	2.712
∞	1.282	1.645	1.960	2.326	2.576

Chi-Square Distribution Table



The shaded area is equal to α for $\chi^2 = \chi^2_\alpha$.

df	$\chi^2_{.995}$	$\chi^2_{.990}$	$\chi^2_{.975}$	$\chi^2_{.950}$	$\chi^2_{.900}$	$\chi^2_{.100}$	$\chi^2_{.050}$	$\chi^2_{.025}$	$\chi^2_{.010}$	$\chi^2_{.005}$
1	0.000	0.000	0.001	0.004	0.016	2.706	3.841	5.024	6.635	7.879
2	0.010	0.020	0.051	0.103	0.211	4.605	5.991	7.378	9.210	10.597
3	0.072	0.115	0.216	0.352	0.584	6.251	7.815	9.348	11.345	12.838
4	0.207	0.297	0.484	0.711	1.064	7.779	9.488	11.143	13.277	14.860
5	0.412	0.554	0.831	1.145	1.610	9.236	11.070	12.833	15.086	16.750
6	0.676	0.872	1.237	1.635	2.204	10.645	12.592	14.449	16.812	18.548
7	0.989	1.239	1.690	2.167	2.833	12.017	14.067	16.013	18.475	20.278
8	1.344	1.646	2.180	2.733	3.490	13.362	15.507	17.535	20.090	21.955
9	1.735	2.088	2.700	3.325	4.168	14.684	16.919	19.023	21.666	23.589
10	2.156	2.558	3.247	3.940	4.865	15.987	18.307	20.483	23.209	25.188
11	2.603	3.053	3.816	4.575	5.578	17.275	19.675	21.920	24.725	26.757
12	3.074	3.571	4.404	5.226	6.304	18.549	21.026	23.337	26.217	28.300
13	3.565	4.107	5.009	5.892	7.042	19.812	22.362	24.736	27.688	29.819
14	4.075	4.660	5.629	6.571	7.790	21.064	23.685	26.119	29.141	31.319
15	4.601	5.229	6.262	7.261	8.547	22.307	24.996	27.488	30.578	32.801
16	5.142	5.812	6.908	7.962	9.312	23.542	26.296	28.845	32.000	34.267
17	5.697	6.408	7.564	8.672	10.085	24.769	27.587	30.191	33.409	35.718
18	6.265	7.015	8.231	9.390	10.865	25.989	28.869	31.526	34.805	37.156
19	6.844	7.633	8.907	10.117	11.651	27.204	30.144	32.852	36.191	38.582
20	7.434	8.260	9.591	10.851	12.443	28.412	31.410	34.170	37.566	39.997
21	8.034	8.897	10.283	11.591	13.240	29.615	32.671	35.479	38.932	41.401
22	8.643	9.542	10.982	12.338	14.041	30.813	33.924	36.781	40.289	42.796
23	9.260	10.196	11.689	13.091	14.848	32.007	35.172	38.076	41.638	44.181
24	9.886	10.856	12.401	13.848	15.659	33.196	36.415	39.364	42.980	45.559
25	10.520	11.524	13.120	14.611	16.473	34.382	37.652	40.646	44.314	46.928
26	11.160	12.198	13.844	15.379	17.292	35.563	38.885	41.923	45.642	48.290
27	11.808	12.879	14.573	16.151	18.114	36.741	40.113	43.195	46.963	49.645
28	12.461	13.565	15.308	16.928	18.939	37.916	41.337	44.461	48.278	50.993
29	13.121	14.256	16.047	17.708	19.768	39.087	42.557	45.722	49.588	52.336
30	13.787	14.953	16.791	18.493	20.599	40.256	43.773	46.979	50.892	53.672
40	20.707	22.164	24.433	26.509	29.051	51.805	55.758	59.342	63.691	66.766
50	27.991	29.707	32.357	34.764	37.689	63.167	67.505	71.420	76.154	79.490
60	35.534	37.485	40.482	43.188	46.459	74.397	79.082	83.298	88.379	91.952
70	43.275	45.442	48.758	51.739	55.329	85.527	90.531	95.023	100.425	104.215
80	51.172	53.540	57.153	60.391	64.278	96.578	101.879	106.629	112.329	116.321
90	59.196	61.754	65.647	69.126	73.291	107.565	113.145	118.136	124.116	128.299
100	67.328	70.065	74.222	77.929	82.358	118.498	124.342	129.561	135.807	140.169

F distribution critical value landmarks

Table entries are critical values for *F** with probably *p* in right tail of the distribution.

Figure of *F* distribution (like in Moore, 2004, p. 656) here.

Degrees of freedom in denominator (df2)		Degrees of freedom in numerator (df1)										
	p	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**12**	**24**	**1000**
1	0.100	39.86	49.50	53.59	55.83	57.24	58.20	58.91	59.44	60.71	62.00	63.30
	0.050	161.4	199.5	215.7	224.6	230.2	234.0	236.8	238.9	243.9	249.1	254.2
	0.025	647.8	799.5	864.2	899.6	921.8	937.1	948.2	956.6	976.7	997.3	1017.8
	0.010	4052	4999	5404	5624	5764	5859	5928	5981	6107	6234	6363
	0.001	405312	499725	540257	562668	576496	586033	593185	597954	610352	623703	636101
2	0.100	8.53	9.00	9.16	9.24	9.29	9.33	9.35	9.37	9.41	9.45	9.49
	0.050	18.51	19.00	19.16	19.25	19.30	19.33	19.35	19.37	19.41	19.45	19.49
	0.025	38.51	39.00	39.17	39.25	39.30	39.33	39.36	39.37	39.41	39.46	39.50
	0.010	98.50	99.00	99.16	99.25	99.30	99.33	99.36	99.38	99.42	99.46	99.50
	0.001	998.38	998.84	999.31	999.31	999.31	999.31	999.31	999.31	999.31	999.31	999.31
3	0.100	5.54	5.46	5.39	5.34	5.31	5.28	5.27	5.25	5.22	5.18	5.13
	0.050	10.13	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.74	8.64	8.53
	0.025	17.44	16.04	15.44	15.10	14.88	14.73	14.62	14.54	14.34	14.12	13.91
	0.010	34.12	30.82	29.46	28.71	28.24	27.91	27.67	27.49	27.05	26.60	26.14
	0.001	167.06	148.49	141.10	137.08	134.58	132.83	131.61	130.62	128.32	125.93	123.52
4	0.100	4.54	4.32	4.19	4.11	4.05	4.01	3.98	3.95	3.90	3.83	3.76
	0.050	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	5.91	5.77	5.63
	0.025	12.22	10.65	9.98	9.60	9.36	9.20	9.07	8.98	8.75	8.51	8.26
	0.010	21.20	18.00	16.69	15.98	15.52	15.21	14.98	14.80	14.37	13.93	13.47
	0.001	74.13	61.25	56.17	53.43	51.72	50.52	49.65	49.00	47.41	45.77	44.09
5	0.100	4.06	3.78	3.62	3.52	3.45	3.40	3.37	3.34	3.27	3.19	3.11
	0.050	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.68	4.53	4.37
	0.025	10.01	8.43	7.76	7.39	7.15	6.98	6.85	6.76	6.52	6.28	6.02
	0.010	16.26	13.27	12.06	11.39	10.97	10.67	10.46	10.29	9.89	9.47	9.03
	0.001	47.18	37.12	33.20	31.08	29.75	28.83	28.17	27.65	26.42	25.13	23.82
6	0.100	3.78	3.46	3.29	3.18	3.11	3.05	3.01	2.98	2.90	2.82	2.72
	0.050	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.00	3.84	3.67
	0.025	8.81	7.26	6.60	6.23	5.99	5.82	5.70	5.60	5.37	5.12	4.86
	0.010	13.75	10.92	9.78	9.15	8.75	8.47	8.26	8.10	7.72	7.31	6.89
	0.001	35.51	27.00	23.71	21.92	20.80	20.03	19.46	19.03	17.99	16.90	15.77
7	0.100	3.59	3.26	3.07	2.96	2.88	2.83	2.78	2.75	2.67	2.58	2.47
	0.050	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.57	3.41	3.23
	0.025	8.07	6.54	5.89	5.52	5.29	5.12	4.99	4.90	4.67	4.41	4.15
	0.010	12.25	9.55	8.45	7.85	7.46	7.19	6.99	6.84	6.47	6.07	5.66
	0.001	29.25	21.69	18.77	17.20	16.21	15.52	15.02	14.63	13.71	12.73	11.72
8	0.100	3.46	3.11	2.92	2.81	2.73	2.67	2.62	2.59	2.50	2.40	2.30
	0.050	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.28	3.12	2.93
	0.025	7.57	6.06	5.42	5.05	4.82	4.65	4.53	4.43	4.20	3.95	3.68
	0.010	11.26	8.65	7.59	7.01	6.63	6.37	6.18	6.03	5.67	5.28	4.87
	0.001	25.41	18.49	15.83	14.39	13.48	12.86	12.40	12.05	11.19	10.30	9.36
9	0.100	3.36	3.01	2.81	2.69	2.61	2.55	2.51	2.47	2.38	2.28	2.16
	0.050	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.07	2.90	2.71
	0.025	7.21	5.71	5.08	4.72	4.48	4.32	4.20	4.10	3.87	3.61	3.34
	0.010	10.56	8.02	6.99	6.42	6.06	5.80	5.61	5.47	5.11	4.73	4.32
	0.001	22.86	16.39	13.90	12.56	11.71	11.13	10.70	10.37	9.57	8.72	7.84

Critical values computed with Excel 9.0

Degrees of freedom in denominator (df2)		Degrees of freedom in numerator (df1)										
	p	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**12**	**24**	**1000**
10	0.100	3.29	2.92	2.73	2.61	2.52	2.46	2.41	2.38	2.28	2.18	2.06
	0.050	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	2.91	2.74	2.54
	0.025	6.94	5.46	4.83	4.47	4.24	4.07	3.95	3.85	3.62	3.37	3.09
	0.010	10.04	7.56	6.55	5.99	5.64	5.39	5.20	5.06	4.71	4.33	3.92
	0.001	21.04	14.90	12.55	11.28	10.48	9.93	9.52	9.20	8.45	7.64	6.78
12	0.100	3.18	2.81	2.61	2.48	2.39	2.33	2.28	2.24	2.15	2.04	1.91
	0.050	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.69	2.51	2.30
	0.025	6.55	5.10	4.47	4.12	3.89	3.73	3.61	3.51	3.28	3.02	2.73
	0.010	9.33	6.93	5.95	5.41	5.06	4.82	4.64	4.50	4.16	3.78	3.37
	0.001	18.64	12.97	10.80	9.63	8.89	8.38	8.00	7.71	7.00	6.25	5.44
14	0.100	3.10	2.73	2.52	2.39	2.31	2.24	2.19	2.15	2.05	1.94	1.80
	0.050	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.53	2.35	2.14
	0.025	6.30	4.86	4.24	3.89	3.66	3.50	3.38	3.29	3.05	2.79	2.50
	0.010	8.86	6.51	5.56	5.04	4.69	4.46	4.28	4.14	3.80	3.43	3.02
	0.001	17.14	11.78	9.73	8.62	7.92	7.44	7.08	6.80	6.13	5.41	4.62
16	0.100	3.05	2.67	2.46	2.33	2.24	2.18	2.13	2.09	1.99	1.87	1.72
	0.050	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.42	2.24	2.02
	0.025	6.12	4.69	4.08	3.73	3.50	3.34	3.22	3.12	2.89	2.63	2.32
	0.010	8.53	6.23	5.29	4.77	4.44	4.20	4.03	3.89	3.55	3.18	2.76
	0.001	16.12	10.97	9.01	7.94	7.27	6.80	6.46	6.20	5.55	4.85	4.08
18	0.100	3.01	2.62	2.42	2.29	2.20	2.13	2.08	2.04	1.93	1.81	1.66
	0.050	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.34	2.15	1.92
	0.025	5.98	4.56	3.95	3.61	3.38	3.22	3.10	3.01	2.77	2.50	2.20
	0.010	8.29	6.01	5.09	4.58	4.25	4.01	3.84	3.71	3.37	3.00	2.58
	0.001	15.38	10.39	8.49	7.46	6.81	6.35	6.02	5.76	5.13	4.45	3.69
20	0.100	2.97	2.59	2.38	2.25	2.16	2.09	2.04	2.00	1.89	1.77	1.61
	0.050	4.35	3.49	3.10	2.87	2.71	2.60	2.51	2.45	2.28	2.08	1.85
	0.025	5.87	4.46	3.86	3.51	3.29	3.13	3.01	2.91	2.68	2.41	2.09
	0.010	8.10	5.85	4.94	4.43	4.10	3.87	3.70	3.56	3.23	2.86	2.43
	0.001	14.82	9.95	8.10	7.10	6.46	6.02	5.69	5.44	4.82	4.15	3.40
30	0.100	2.88	2.49	2.28	2.14	2.05	1.98	1.93	1.88	1.77	1.64	1.46
	0.050	4.17	3.32	2.92	2.69	2.53	2.42	2.33	2.27	2.09	1.89	1.63
	0.025	5.57	4.18	3.59	3.25	3.03	2.87	2.75	2.65	2.41	2.14	1.80
	0.010	7.56	5.39	4.51	4.02	3.70	3.47	3.30	3.17	2.84	2.47	2.02
	0.001	13.29	8.77	7.05	6.12	5.53	5.12	4.82	4.58	4.00	3.36	2.61
50	0.100	2.81	2.41	2.20	2.06	1.97	1.90	1.84	1.80	1.68	1.54	1.33
	0.050	4.03	3.18	2.79	2.56	2.40	2.29	2.20	2.13	1.95	1.74	1.45
	0.025	5.34	3.97	3.39	3.05	2.83	2.67	2.55	2.46	2.22	1.93	1.56
	0.010	7.17	5.06	4.20	3.72	3.41	3.19	3.02	2.89	2.56	2.18	1.70
	0.001	12.22	7.96	6.34	5.46	4.90	4.51	4.22	4.00	3.44	2.82	2.05
100	0.100	2.76	2.36	2.14	2.00	1.91	1.83	1.78	1.73	1.61	1.46	1.22
	0.050	3.94	3.09	2.70	2.46	2.31	2.19	2.10	2.03	1.85	1.63	1.30
	0.025	5.18	3.83	3.25	2.92	2.70	2.54	2.42	2.32	2.08	1.78	1.36
	0.010	6.90	4.82	3.98	3.51	3.21	2.99	2.82	2.69	2.37	1.98	1.45
	0.001	11.50	7.41	5.86	5.02	4.48	4.11	3.83	3.61	3.07	2.46	1.64
1000	0.100	2.71	2.31	2.09	1.95	1.85	1.78	1.72	1.68	1.55	1.39	1.08
	0.050	3.85	3.00	2.61	2.38	2.22	2.11	2.02	1.95	1.76	1.53	1.11
	0.025	5.04	3.70	3.13	2.80	2.58	2.42	2.30	2.20	1.96	1.65	1.13
	0.010	6.66	4.63	3.80	3.34	3.04	2.82	2.66	2.53	2.20	1.81	1.16
	0.001	10.89	6.96	5.46	4.65	4.14	3.78	3.51	3.30	2.77	2.16	1.22

Use StaTable, WinPepi > WhatIs, or other reliable software to determine specific *p* values