A Semantic Proof of Generalised Cut Elimination for Deep Inference

Robert Atkey & Wen Kokke
robert.atkey@strath.ac.uk wen.kokke@pm.me

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From Sequent Calculus to Deep Inference

Multiplicative Linear Logic

$$a^\perp = \overline{a} \qquad \overline{a}^\perp = a \qquad (P \parr Q)^\perp = P^\perp \otimes Q^\perp \qquad (P \otimes Q)^\perp = P^\perp \parr Q^\perp$$

Multiplicative Linear Logic

$$a^{\perp} = \overline{a} \qquad \overline{a}^{\perp} = a \qquad (P \parr Q)^{\perp} = P^{\perp} \otimes Q^{\perp} \qquad (P \otimes Q)^{\perp} = P^{\perp} \parr Q^{\perp}$$

Sequent Calculus

$$\frac{\vdash \Gamma, P, Q}{\vdash \Gamma, P \parr Q}\ \parr \qquad \frac{\vdash \Gamma, P \qquad \vdash \Delta, Q}{\vdash \Gamma, P \otimes Q}\ \otimes \qquad \frac{}{\vdash P, P^{\perp}}\ \text{Ax}$$

$$\frac{\vdash \Gamma, P \qquad \vdash \Delta, P^{\perp}}{\vdash \Gamma, \Delta}\ \text{Cut} \qquad \frac{}{\vdash a, \overline{a}}\ \text{Ax-Atom}$$

Multiplicative Linear Logic

$$a^\perp = \overline{a} \qquad \overline{a}^\perp = a \qquad (P \parr Q)^\perp = P^\perp \otimes Q^\perp \qquad (P \otimes Q)^\perp = P^\perp \parr Q^\perp$$

Sequent Calculus

$$\frac{\vdash \Gamma, P, Q}{\vdash \Gamma, P \parr Q}\ \parr \qquad \frac{\vdash \Gamma, P \qquad \vdash \Delta, Q}{\vdash \Gamma, P \otimes Q}\ \otimes \qquad \frac{}{\vdash P, P^\perp}\ \mathrm{Ax}$$

$$\frac{\vdash \Gamma, P \qquad \vdash \Delta, P^\perp}{\vdash \Gamma, \Delta}\ \mathrm{Cut} \qquad \frac{}{\vdash a, \overline{a}}\ \text{Ax-Atom}$$

Deep Inference

The ideas:

1. Replace sequents with *structures*
2. Use $\otimes$ to combine multiple premises
3. Allow inference rules to be applied to any substructure

Structures

$$P, Q ::= a \mid \bar{a} \mid P \otimes Q \mid P \parr Q \mid I$$

where $(\otimes, I)$ and $(\parr, I)$ are commutative monoids.

$$\frac{\vdash \Gamma, P, Q}{\vdash \Gamma, P \parr Q} \quad \rightsquigarrow \quad \frac{\Gamma \parr P \parr Q}{\Gamma \parr P \parr Q}$$

$$\frac{\vdash \Gamma, P \qquad \vdash \Delta, Q}{\vdash \Gamma, \Delta, P \otimes Q} \quad \rightsquigarrow \quad \frac{(\Gamma \parr P) \otimes (\Delta \parr Q)}{\Gamma \parr \Delta \parr (P \otimes Q)}$$

$$\frac{}{\vdash P, P^{\perp}} \quad \rightsquigarrow \quad \frac{I}{P \parr P^{\perp}}$$

$$\frac{\vdash \Gamma, P \qquad \vdash \Delta, P^{\perp}}{\vdash \Gamma, \Delta} \quad \rightsquigarrow \quad \frac{(\Gamma \parr P) \otimes (\Delta \parr P^{\perp})}{\Gamma \parr \Delta}$$

$$\frac{(\Gamma \parr P) \otimes (\Delta \parr Q)}{\Gamma \parr \Delta \parr (P \otimes Q)} \quad \rightsquigarrow \quad \frac{(P \parr R) \otimes Q}{(P \otimes Q) \parr R}$$

$$\frac{I}{P \parr P^{\perp}}$$

$$\frac{(\Gamma \parr P) \otimes (\Delta \parr P^{\perp})}{\Gamma \parr \Delta} \quad \rightsquigarrow \quad \frac{P \otimes P^{\perp}}{I}$$

Deep Inference as inference rules

$$\frac{\mathcal{C}[(P \parr R) \otimes Q]}{\mathcal{C}[(P \otimes Q) \parr R]}\ \mathsf{switch} \qquad \frac{\mathcal{C}[I]}{\mathcal{C}[P \parr P^{\perp}]}\ \mathsf{ax} \qquad \frac{\mathcal{C}[P \otimes P^{\perp}]}{\mathcal{C}[I]}\ \mathsf{cut}$$

Deep Inference as a rewrite system

$$
\begin{aligned}
(P \otimes Q) \parr R &\longrightarrow (P \parr R) \otimes Q \\
P \parr P^{\perp} &\longrightarrow I \\
I &\longrightarrow P \otimes P^{\perp}
\end{aligned}
$$

$$
\frac{P \longrightarrow Q}{\mathcal{C}[P] \longrightarrow \mathcal{C}[Q]}
$$

A *derivation* of P from Q: $P \longrightarrow^{*} Q$

A *proof* of P: $P \longrightarrow^{*} I$

Normal proofs

$$\begin{aligned} (P \otimes Q) \parr R &\longrightarrow_n (P \parr R) \otimes Q \\ a \parr \overline{a} &\longrightarrow_n I \end{aligned}$$

$$\frac{P \longrightarrow_n Q}{\mathcal{C}[P] \longrightarrow_n \mathcal{C}[Q]}$$

A *normal proof* of P: $P \longrightarrow_n^* I$

Normal proofs

$$\begin{array}{lll} (P \otimes Q) \parr R & \longrightarrow_n & (P \parr R) \otimes Q \\ a \parr \overline{a} & \longrightarrow_n & I \end{array}$$

$$\frac{P \longrightarrow_n Q}{\mathcal{C}[P] \longrightarrow_n \mathcal{C}[Q]}$$

A *normal proof* of P: $P \longrightarrow_n^* I$

Generalised Cut elimination: if $P \longrightarrow^* I$ then $P \longrightarrow_n^* I$

BV Basic System Virtual (Guglielmi, 2002/2007)

Structures

$$P, Q ::= a \mid \bar{a} \mid P \otimes Q \mid P \parr Q \mid P \lhd Q \mid I$$

where $(\otimes, I)$ and $(\parr, I)$ are commutative monoids, and $(\lhd, I)$ is a monoid.

Duality

$$a^\perp = \bar{a} \qquad \bar{a}^\perp = a \qquad (P \otimes Q)^\perp = P^\perp \parr Q^\perp$$

$$(P \parr Q)^\perp = P^\perp \otimes Q^\perp \qquad (P \lhd Q)^\perp = P^\perp \lhd Q^\perp \qquad I^\perp = I$$

BV *as rewrite rules*

$$
\begin{array}{lcl}
(P \otimes Q) \parr R & \longrightarrow & (P \parr R) \otimes Q \\
(P \lhd Q) \parr (R \lhd S) & \longrightarrow & (P \parr R) \lhd (Q \parr S) \\
(P \otimes Q) \lhd (R \otimes S) & \longrightarrow & (P \lhd R) \otimes (Q \lhd S) \\
P \parr P^{\perp} & \longrightarrow & I \\
I & \longrightarrow & P \otimes P^{\perp}
\end{array}
$$

Example

$$
\begin{array}{rc}
 & (a \lhd b) \multimap (a \parr b) \\
= & ((\overline{a} \lhd \overline{b}) \parr a) \parr b \\
= & ((\overline{a} \lhd \overline{b}) \parr (a \lhd I)) \parr b \\
\longrightarrow & ((\overline{a} \parr a) \lhd (\overline{b} \parr I)) \parr b \\
\longrightarrow & (I \lhd (\overline{b} \parr I)) \parr b \\
= & \overline{b} \parr b \\
\longrightarrow & I
\end{array}
$$

The **need** *for Deep Inference* (Tiu, 2006)

A SYSTEM OF INTERACTION AND STRUCTURE II: THE NEED FOR DEEP INFERENCE

ALWEN TIU

Computer Science Laboratory, Research School in Information Science and Engineering, Australian National University, Canberra ACT 0200, Australia.
e-mail address: Alwen.Tiu@rsise.anu.edu.au

Abstract. This paper studies properties of the logic BV, which is an extension of multiplicative linear logic (MLL) with a self-dual non-commutative operator. BV is presented in the *calculus of structures*, a proof theoretic formalism that supports *deep inference*, in which inference rules can be applied anywhere inside logical expressions. The use of deep inference results in a simple logical system for MLL extended with the self-dual non-commutative operator, which has been to date not known to be expressible in sequent calculus. In this paper, deep inference is shown to be crucial for the logic BV, that is, any restriction on the "depth" of the inference rules of BV would result in a strictly less expressive logical system.

No “shallow” sequent calculus.

MAV Multiplicative Additive System Virtual (Horne, 2015)

Structures

$$P, Q ::= a \mid \overline{a} \mid P \otimes Q \mid P \parr Q \mid P \lhd Q \mid P \mathbin{\&} Q \mid P \oplus Q \mid I$$

where $(\otimes, I)$ and $(\parr, I)$ are commutative monoids, and $(\lhd, I)$ is a monoid.

Duality

$$a^\perp = \overline{a} \qquad \overline{a}^\perp = a \qquad (P \otimes Q)^\perp = P^\perp \parr Q^\perp$$

$$(P \parr Q)^\perp = P^\perp \otimes Q^\perp \qquad (P \lhd Q)^\perp = P^\perp \lhd Q^\perp$$

$$(P \mathbin{\&} Q)^\perp = P^\perp \oplus Q^\perp \qquad (P \oplus Q)^\perp = P^\perp \mathbin{\&} Q^\perp \qquad I^\perp = I$$

MAV *as rewrite rules* (normal rules only)

$$
\begin{array}{lcl}
(P \otimes Q) \parr R & \longrightarrow & (P \parr R) \otimes Q \\
(P \lhd Q) \parr (R \lhd S) & \longrightarrow & (P \parr R) \lhd (Q \parr S) \\
P \parr P^{\perp} & \longrightarrow & I \\
I \,\&\, I & \longrightarrow & I \\
P \oplus Q & \longrightarrow & P \\
P \oplus Q & \longrightarrow & Q \\
(P \,\&\, Q) \parr R & \longrightarrow & (P \parr R) \,\&\, (Q \parr R) \\
(P \lhd Q) \,\&\, (R \lhd S) & \longrightarrow & (P \,\&\, R) \lhd (Q \,\&\, S)
\end{array}
$$

Proving Cut-elimination

Syntactic proof with key *splitting* lemma (Guglielmi, 2007)

If $\mathcal{C}[P \parr Q] \longrightarrow^* I$, then exist S_1, S_2 such that for all R:

1. $\mathcal{C}[R] \longrightarrow^* R \otimes (S_1 \parr S_2)$
2. $P \parr S_1 \longrightarrow^* I$
3. $Q \parr S_2 \longrightarrow^* I$

Similarly for $P \otimes Q$.

Long syntactic proof. Subsequently extended by Horne for MAV and Guglielmi and Straßburger for BV+exponentials (NEL).

Semantic Cut-elimination / Normalisation by Evaluation

1. Make a poset A from cut-free proofs
   $$P \sqsubseteq Q \text{ iff } P \longrightarrow_n^* Q$$
2. Complete A to $\hat{A}$, a model of the whole system with an order embedding $\eta : A \to \hat{A}$
3. such that $[\![P]\!] \sqsubseteq \neg\eta(P)$

Then for a proof $P \longrightarrow^* I$:

1. Interpret as $I \sqsubseteq [\![P]\!]$ in $\hat{A}$ (soundness)
2. So $\neg\eta(I) \sqsubseteq \neg\eta(P)$ (properties of η)
3. So $\eta(P) \sqsubseteq \eta(I)$ (contravariance of $\neg$)
4. So $P \longrightarrow_n^* I$ (order embedding)

Okada's Semantic Cut-elimination Proof (Okada, 1999)

Okada's construction: use the phase semantics.

1. $(M, \cdot, \epsilon)$ a commutative monoid, $\perp \subseteq M$ is the "pole"
2. $\alpha \subseteq M$ are *pre-facts*
3. Define $M^{\perp} = \{x \mid \forall x \in M. x \cdot y \in \perp\}$.
4. Facts are pre-facts M s.t. $M^{\perp\perp} = M$
5. Facts ordered by inclusion form a model of MALL.

Okada: let M be the monoid of cut-free provable sequents...deduce cut-elimination property.

Why not adapt Okada's proof?

To handle $P \triangleleft Q$ we could try:

1. Let $(M, \cdot, \epsilon)$ be a partially ordered monoid
2. Assume another monoid structure $(\triangleright, \epsilon)$ with the right relationship with $(\cdot, \epsilon)$ (*duoidal*).
3. Take the lattice of facts again.

Why not adapt Okada's proof?

To handle $P \lhd Q$ we could try:

1. Let $(M, \cdot, \epsilon)$ be a partially ordered monoid
2. Assume another monoid structure $(\rhd, \epsilon)$ with the right relationship with $(\cdot, \epsilon)$ (*duoidal*).
3. Take the lattice of facts again.

But: we don't get a self-dual $\rhd$ on facts. We get two distinct but dual operators. Not a model of BV or MAV.

Semantics of MAV

$\ast$-autonomous posets

A *$\ast$-autonomous partial order* is a structure $(A, \leq, \otimes, I, \neg)$ where:

1. $(\otimes, I)$ is a pomonoid on $(A, \leq)$
2. $\neg : A^{\mathrm{op}} \longrightarrow A$ is anti-monotone and involutive
3. $x \otimes y \leq \neg z$ iff $x \leq \neg(y \otimes z)$

$\ast$-autonomous partial order satisfies *mix* if $\neg I = I$

Duoidal monoids

A pomonoid $(\bullet, i)$ is *duoidal over* another pomonoid $(\triangleleft, j)$ on a partial order $(A, \leq)$ if the following inequalities hold:

1. $(w \triangleleft x) \bullet (y \triangleleft z) \leq (w \bullet y) \triangleleft (x \bullet z)$
2. $j \bullet j \leq j$
3. $i \leq i \triangleleft i$
4. $i \leq j$

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2. $j \bullet j \leq j$
3. $i \leq i \triangleleft i$
4. $i \leq j$

- If $i = j$, then last three are automatic
- If $\bullet$ is a join or $\triangleleft$ is a meet, then all are automatic

(Aguiar and Mahajan, 2010)

Algebraic Models of MAV

An *MAV-algebra* is a structure $(A, \leq, \otimes, \lhd, I, \neg)$ s.t.:

1. $(A, \leq, \otimes, I, \neg)$ is $*$-autonomous and satisfies *mix*.
2. $(A, \leq, \lhd, I)$ is a pomonoid.
3. $\lhd$ is self dual: $\neg(x \lhd y) = (\neg x) \lhd (\neg y)$.
4. $(\otimes, I)$ is duoidal over $(\lhd, I)$.
5. $(A, \leq)$ has binary meets, which we write as $x \mathbin{\&} y$.

Let $(A, \leq, \otimes, \lhd, I, \neg)$ be a MAV-algebra.

1. There is another commutative pomonoid structure $(⅋, I)$ on $(A, \leq)$, defined as $x ⅋ y = \neg(\neg x \otimes \neg y)$.
2. $(\otimes, I)$ and $(⅋, I)$ are linearly distributive: $x \otimes (y ⅋ z) \leq (x \otimes y) ⅋ z$
3. $(A, \leq)$ has binary joins, given by $x \oplus y = \neg(\neg x \,\&\, \neg y)$
4. $\oplus$ distributes over $\otimes$: $x \otimes (y \oplus z) = (x \otimes y) \oplus (x \otimes z)$
5. $\&$ distributes over $⅋$: $(x ⅋ z) \,\&\, (y ⅋ z) = (x \,\&\, y) ⅋ z$
6. $\lhd$ duoidal over $⅋$: $(w ⅋ x) \lhd (y ⅋ z) \leq (w \lhd y) ⅋ (x \lhd z)$
7. $\lhd$ duoidal over $\&$: $(w \,\&\, x) \lhd (y \,\&\, z) \leq (w \lhd y) \,\&\, (x \lhd z)$
8. $\oplus$ duoidal over $\lhd$: $(w \lhd x) \oplus (y \lhd z) \leq (x \oplus y) \lhd (x \oplus z)$

Interpretation *and* **Soundness**

Let $(A, \leq, \otimes, \lhd, I, \neg)$ be a MAV-algebra.

Assume $V(a) \in A$ for each atom a.

Interpret $[\![P \otimes Q]\!]$ as $[\![P]\!] \otimes [\![Q]\!]$ and so on.

Lemma *(Duality)*: $[\![P^\perp]\!] = \neg[\![P]\!]$

Thm *(Soundness)*: $P \longrightarrow^* I$ implies $I \leq [\![P]\!]$.

MAV frames

An *MAV-frame* is a structure $(F, \leq, \parr, \lhd, i, +)$ where:

1. $(F, \leq)$ is a partial order
2. $(F, \leq, \parr, i)$ is a commutative pomonoid
3. $(F, \leq, \lhd, i)$ is a pomonoid
4. $+$ is a binary monotone function

Satisfying:

1. $(w \lhd x) \parr (y \lhd z) \leq (w \parr y) \lhd (x \parr z)$
2. $(x + y) \parr z \leq (x \parr z) + (y \parr z)$
3. $(w \lhd x) + (y \lhd z) \leq (w + y) \lhd (x + z)$
4. $i + i \leq i$

Two duoidal relationships and a distributivity law.

A process algebra reading

Change $\parr$ to $\|$, $\lhd$ to $;$, and $\leq$ to $\longrightarrow$:

1. $(w;x) \| (y;z) \longrightarrow (w \| y);(x \| z)$
2. $(x+y) \| z \longrightarrow (x \| z) + (y \| z)$
3. $(w;x) + (y;z) \longrightarrow (w+y);(x+z)$
4. $i + i \longrightarrow i$

A *bit* like a CCS-style process algebra with sequencing or Concurrent Kleene Algebra, Hoare et al. 2011

Relations as an MAV frame

Let $(A, \cdot, I)$ be a commutative monoid (of “states”).

Take relations $R, S : A \times A \to \Omega$, with:

1. $(R \lhd S)(a, c) = \exists b. R(a, b) \wedge S(b, c)$
2. $(R \parr S)(a, b) = \exists a_1, a_2, b_1, b_2.$
$$(a_1 \cdot a_2 = a) \wedge (b_1 \cdot b_2 = b) \wedge$$
$$R(a_1, b_1) \wedge S(a_2, b_2)$$
3. $(R + S)(a, b) = R(a, b) \cup S(a, b)$

Claim: this is an MAV frame.

Normal derivations as an MAV frame

Normal proofs

$$P \longrightarrow_n^* Q$$

form an MAV frame with structures as the elements, ordered by $\longrightarrow_n^*$. Use $P \& Q$ for $P + Q$.

Ignores the $\otimes$, $\oplus$ part of the structure.

From MAV frames to MAV algebras

Lower Sets

Let $\hat{A}$ be lower subsets of A:

$$F \in \hat{A} \Leftrightarrow \forall x, y. x \in F \wedge y \leq x \Rightarrow y \in F$$

Ordered by subset inclusion.

Embedding: $\eta : A \to \hat{A}$; $\eta(x) = \{y \mid y \leq x\}$.

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Embedding: $\eta : A \rightarrow \hat{A}$; $\eta(x) = \{y \mid y \leq x\}$.

1. $\hat{A}$ has meets and joins
2. For any monoid $(\bullet, i)$, define (Day, 1970)
$$F \hat{\bullet} G = \{z \mid z \leq x \bullet y, x \in F, y \in G\} \qquad \hat{i} = \eta(i)$$
residuated and $\eta(x \bullet y) = \eta(x) \hat{\bullet} \eta(y)$.
3. If $(\bullet, i)$ is duoidal over $(\lhd, j)$ in A,
then $(\hat{\bullet}, \hat{i})$ is duoidal over $(\hat{\lhd}, \hat{j})$ in $\hat{A}$

$+$-closed Lower Sets

A lower set F is $+$-closed if

$$\forall x, y. x \in F \wedge y \in F \Rightarrow x + y \in F$$

$+$-closed lower sets $\hat{A}^{+}$, ordered by inclusion.

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$+$-closed lower sets $\hat{A}^+$, ordered by inclusion.

There are functions:

- $U : \hat{A}^+ \to \hat{A}$ forgetful
- $\alpha : \hat{A} \to \hat{A}^+$ close

such that $\alpha(UF) = F$ and $F \subseteq \alpha(UF)$.

Embedding $\eta^+ = \alpha \circ \eta : A \to \hat{A}^+$.

$+$-closed Lower Sets

1. Meets $F \wedge G = F \cap G$ and joins $F \vee G = \alpha(UF \cup UG)$.
   with $\eta^+(x+y) \subseteq \eta^+(x) \vee \eta^+(y)$.

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   with $\eta^+(x+y) \subseteq \eta^+(x) \vee \eta^+(y)$.
2. For a monoid $(\bullet, i)$ that distributes over $+$, then:

$$F \,\hat{\bullet}^+\, G = \alpha(UF \,\hat{\bullet}\, UG) \qquad\qquad \hat{i}^+ = \alpha\hat{i}$$

   is a monoid, s.t. $\eta^+(x \bullet y) = \eta^+(x) \,\hat{\bullet}^+\, \eta^+(y)$
   and is residuated.

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   is a monoid, s.t. $\eta^+(x \bullet y) = \eta^+(x) \mathbin{\hat{\bullet}^+} \eta^+(y)$
   and is residuated.
3. For a monoid $(\lhd, j)$ that is duoidal over $+$, then:
   - $F \mathbin{\hat{\bullet}} G$ is $+$-closed when F and G are; and
   - $\hat{j}$ is $+$-closed

$+$-closed Lower Sets

1. Meets $F \wedge G = F \cap G$ and joins $F \vee G = \alpha(UF \cup UG)$.
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   and is residuated.
3. For a monoid $(\lhd, j)$ that is duoidal over $+$, then:
   - $F \,\hat{\bullet}\, G$ is $+$-closed when F and G are; and
   - $\hat{j}$ is $+$-closed
4. If $(\bullet, i)$ is duoidal over $(\lhd, j)$ in A,
   then $(\hat{\bullet}^+, \hat{i}^+)$ is duoidal over $(\hat{\lhd}^+, \hat{j}^+)$ in $\hat{A}^+$.

Chu construction (Barr, Chu, 1979)

Let $(A, \bullet, \to, \wedge)$ be a residuated $\wedge$-pomonoid, $k \in A$

Define $\mathrm{Chu}(A, k)$ as:

- Elements $(a^+, a^-) \in A \times A$ such that $a^+ \bullet a^- \leq k$.
- $(a^+, a^-) \sqsubseteq (b^+, b^-)$ when $a^+ \leq b^+$ and $b^- \leq a^-$.

$\mathrm{Chu}(A, k)$ is then $*$-autonomous, with $\neg(a^+, a^-) = (a^-, a^+)$.

If A has joins, then $\mathrm{Chu}(A, k)$ has meets and joins:

$$(a^+, a^-) \sqcup (b^+, b^-) = (a^+ \wedge b^+, a^- \vee b^-)$$

Self-dual operators on $\mathrm{Chu}(A, k)$

If we have $(\lhd, j)$ on A such that:

1. $(\bullet, i)$ is duoidal over $(\lhd, j)$;
2. $k \lhd k \leq k$;
3. $j \leq k$

then

$$(a^+, a^-) \lhd (b^+, b^-) = (a^+ \lhd b^+, a^- \lhd b^-) \qquad J = (j, j)$$

is a self dual monoid on $\mathrm{Chu}(A, k)$.

Moreover, $(\otimes, I)$ is duoidal over $(\lhd, J)$.

Putting it all together

If $(F, \leq, \parr, \lhd, \iota, +)$ is an MAV-frame,
then $\mathrm{Chu}(\hat{F}^+, \hat{\iota}^+)$ is an MAV-algebra,
with an order embedding $\eta : F \to \mathrm{Chu}(\hat{F}^+, \hat{\iota}^+)$.

Putting it all together

If $(F, \leq, \parr, \lhd, i, +)$ is an MAV-frame,
then $\mathrm{Chu}(\hat{F}^+, \hat{i}^+)$ is an MAV-algebra,
with an order embedding $\eta : F \to \mathrm{Chu}(\hat{F}^+, \hat{i}^+)$.

In particular, if F is the MAV frame of normal proofs,
then for all structures P,

$$[\![P]\!] \sqsubseteq \neg\eta(P)$$

Putting it all together

If $(F, \leq, \parr, \lhd, i, +)$ is an MAV-frame,
then $\mathsf{Chu}(\hat{F}^+, \hat{i}^+)$ is an MAV-algebra,
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In particular, if F is the MAV frame of normal proofs,
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$$[\![P]\!] \sqsubseteq \neg\eta(P)$$

So we can apply the recipe to deduce that all MAV proofs can be normalised.

Frame semantics of MAV

As a corollary, MAV is sound and complete for a semantics in MAV frames:

$$P \longrightarrow^* I$$

iff

$$\text{for all MAV frames } A.\, I \sqsubseteq [\![P]\!] \text{ in } \mathrm{Chu}(\hat{A}^+, I)$$

Extensions

Technique is adaptable:

1. Scales down to *BV*
2. *MAUV*: MAV with additive units $\top$ and $\mathbf{0}$.
3. *NEL* (Guglielmi and Straßburger, 2011) : BV with exponentials.

Summary

1. Semantics proof of Cut-elimination for MAV
2. ... and BV, MAUV, and NEL
3. constructed from modular well-known components.
4. Entire development has been formalised in Agda
   and is executable so can actually normalise proofs

Summary

1. Semantics proof of Cut-elimination for MAV
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Future work

1. MAUVE, BI, Modal Logics
2. Fixpoints, incl. Kleene Star
3. Proof-relevant semantics, categorify everything