1 Lattices, Haar measures, unipotent elements

Discrete Subgroups of Lie Groups, M. S. Raghunathan [1]
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Abstract

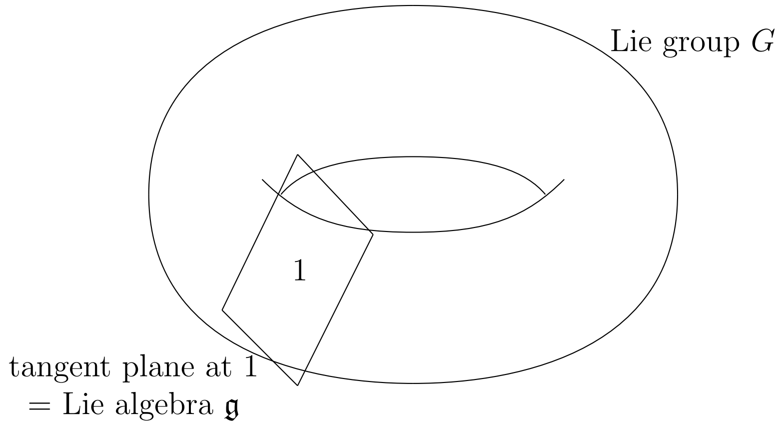
This summary presents several properties of topological groups, with a focus on Lie groups. We discuss measures on locally compact topological groups, building up to the definition of, and important results on, Haar measures. We then define nilpotent Lie groups, their correlation with unipotent elements, and present a theorem which connects unipotent representations to measures on topological groups. Finally, we use properties of the Haar measure to define lattices on topological groups.

1.1 Topological Groups

A topological group is a group G with a Hausdorff topology such that the binary operation and the inverse function on G are continuous functions with respect to the topology. A locally compact group is a topological group G such that every point of G has a compact neighbourhood.

In this summary the objects of interest are studied on topological groups, of which there are many examples. In fact, any group can be made into a topological group if considered with the discrete topology. However, in many cases, our focus will be restricted to a special type of topological groups: Lie groups, which are often discussed in conjunction with Lie algebras.

The information and definitions in the remainder of this subsection are taken from [2], which can be viewed for a more in-depth introduction to Lie groups. A Lie group (over a field $\mathbb{F}$) is a group that is also a finite-dimensional smooth manifold (over $\mathbb{F}$), in which the group operations of multiplication and inversion are smooth maps. When the field $\mathbb{F}$ is not specified we may assume $\mathbb{F} = \mathbb{R}$. Every Lie group G is associated to a Lie algebra $\mathfrak{g}$, which is the tangent space of G at the identity $1 \in G$.



We study Lie algebras because the tangent plane approximates the Lie group, and it is easier to work with.

Definition 1. *A Lie algebra $\mathfrak{g}$ is a vector space with a bilinear function $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ such that*

1. $[x, x] = 0$ *for all* $x \in \mathfrak{g}$,
2. $[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0$ *for all* $x, y, z \in \mathfrak{g}$ *(this is called the Jacobi identity).*

Note that a Lie algebra is not associative, but the Jacobi identity takes the place of associativity. Further, for any $x, y \in \mathfrak{g}$ we have

$$\begin{aligned} 0 &= [x + y, x + y] \\ &= [x, x] + [x, y] + [y, x] + [y, y] \\ &= [x, y] + [y, x]. \end{aligned}$$

Thus this function is antisymmetric: $[x, y] = -[y, x]$.

An example of a Lie group is $GL_n(\mathbb{R})$, the invertible $n \times n$ matrices over $\mathbb{R}$. This has associated Lie algebra $\mathfrak{gl}_n(\mathbb{R})$ of all real $n \times n$ matrices. Here the bilinear function $[\cdot,\cdot]$ is the commutator: For any $x, y \in \mathfrak{gl}_n(\mathbb{R})$, $[x, y] = xy - yx$.

A Lie subgroup H of a Lie group G is a subgroup which is also a submanifold. We have the following important result on subgroups of Lie groups:

Theorem 2. *Let G be a Lie group.*

1. *Any Lie subgroup is closed in G.*
2. *Any closed subgroup of a Lie group is a Lie group.*

1.2 Haar measure

In this section we will introduce the term Haar measure, which is a measure that assigns an "invariant volume" to subsets of locally compact topological groups and is unique up to multiplication by positive real number.

On a locally compact group G we may define the Borel σ-algebra $\mathcal{B}$ to be the smallest σ-algebra containing all of the open sets of G. The elements of $\mathcal{B}$ are called Borel sets, and a measure defined on $\mathcal{B}$ is called a Borel measure. For the remainder of this section we will let G denote a locally compact group and $\mathcal{B}$ the Borel σ-algebra on G.

Definition 3. *Let μ be a Borel measure on G. It is said that μ is outer regular on $S \in \mathcal{B}$ if $\mu(S) = \inf\{\mu(U) \mid S \subset U : U \text{ open}\}$ and μ is inner regular on $S \in \mathcal{B}$ if $\mu(S) = \sup\{\mu(K) \mid K \subset S : K \text{ compact}\}$.*

A regular Borel measure μ on $\mathcal{B}$ is a measure which is outer regular on all Borel sets, inner regular on all open sets and $\mu(A) < \infty$ for every compact set $A \subseteq G$.

It is standard fact that the Lebesque measure is regular. An example of a measure which is not regular is the measure μ on $\mathbb{R}$ defined as follows:

$$\mu(S) = \begin{cases} 0 & S = \emptyset \\ 0 & S = \{1\} \\ \infty & \text{otherwise} \end{cases}$$

Definition 4. *Let μ be a Borel measure on G. The measure μ is left (respectively, right) translation invariant if for every $S \in \mathcal{B}$, $\mu(gS) = \mu(S)$ for all $g \in G$ (respectively, $\mu(Sg) = \mu(S)$ for all $g \in G$).*

Definition 5. *Let G be a locally compact group. A left (respectively, right) Haar measure on G is a nonzero regular Borel measure μ which is left (respectively, right) translation invariant.*

Such a measure has the property that any nonempty open set has positive measure: if it were not so then there would exist an open set $S \subseteq G$ such that $\mu(S) = 0$. We could then create a covering of every compact subset of G by left translations of the set S, so the measure of an arbitrary compact subset would also be 0. By inner regularity we would get $\mu(G) = 0$ which contradicts μ being nonzero.

Example 6. 1. *The Haar measure on both* $(\mathbb{R}, +)$ *and the circle* S^1 *is the Lebesque measure.*

2. *The Haar measure on any discrete group is the counting measure.*

3. *Let* $G = \left\{ \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \mid x \in \mathbb{R}^\times, y \in \mathbb{R} \right\}$. *The left Haar measure on* G *is* $|x|^{-2}dxdy$ *and the right Haar measure on* G *is* $|x|^{-1}dxdy$. *We will show the left (right) translation invariance: for all* $S \subseteq G$ *and* $g = \begin{pmatrix} r & s \\ 0 & 1 \end{pmatrix}$, *the left translation by* g *is given by*

$$\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} rx & ry+s \\ 0 & 1 \end{pmatrix}.$$

The Jacobian of this transformation is $|r^2|$, *so*

$$\mu(gS) = \int_{gS} |x|^{-2}dx = \int_S |rx|^{-2}|r^2|dx = \int_S |x|^{-2}dx = \mu(S).$$

In the second case, the right translation by g *is given by*

$$\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} rx & xs+y \\ 0 & 1 \end{pmatrix}.$$

and has Jacobian $|r|$, *so*

$$\mu(Sg) = \int_{Sg} |x|^{-1}dx = \int_S |rx|^{-1}|r|dx = \int_S |x|^{-1}dx = \mu(S).$$

4. *The left Haar measure on* $\mathrm{GL}(n, \mathbb{R})$ *is* $\frac{dx}{|\det(x)|^n}$, *where* dx *is the Lebesque measure on* $\mathrm{M}_n(\mathbb{R})$.

Left Haar measure usually does not coincide with right Haar measure. But if it does, we call the group unimodular (later we will see another case when group is called unimodular and the conditions are equivalent). Examples of unimodular groups are abelian, compact, topologically simple, discrete, connected semisimple Lie or connected nilpotent Lie groups.

We will now state the theorem about the existence and uniqueness of Haar measure. The proof however will not be included, since it is rather long and uses advanced techniques. If case of interest, the proof can be found for example in [5].

Theorem 7. *Let G be a locally compact group. Then, there exists a left Haar measure on G and it is (up to a multiple by positive real number) unique.*

Lemma 8. *A locally compact group G has a finite Haar measure (i.e. $\mu(G) < \infty$) if and only if G is compact.*

Proof. Let U be a compact neighbourhood of 1 and V open such that $VV^{-1} \subseteq U$. Since $\mu(V) > 0$ and $\mu(G) < \infty$, we can find maximal number of n such that the sets $g_iV \subseteq G$, $i = 1, \ldots, n$ are pairwise disjoint. Take $g \in G$ arbitrary. Then gV must intersect one of the g_iV, so $g \in g_iVV^{-1} \subseteq g_iU$. Therefore $G = \bigcup_{i=1}^{n} g_iU$ which is a finite union of compact sets, so G is compact. $\square$

In the case of G being a compact group, we can determine the left or right Haar measure uniquely requiring $\int_G d\mu = 1$.

1.3 Unipotent Elements

Let R be a ring with unity. An element $r \in R$ is called *unipotent* if $(r-1)^n = 0$ for some positive integer n. Unipotent elements are particularly interesting to study in the representation ring of Lie groups, and we will focus on this case in the remainder of this subsection.

Definition 9. *A nilpotent Lie algebra is a Lie algebra $\mathfrak{g}$ with lower central series which eventually vanishes. That is, setting $\mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}]$ and inductively letting $\mathfrak{g}^k = [\mathfrak{g}, \mathfrak{g}^{k-1}]$ we have that $\mathfrak{g}^\ell = 0$ for some $\ell \in \mathbb{Z}_{>0}$.*

Definition 10. *A nilpotent Lie group G is a Lie group whose associated Lie algebra $\mathfrak{g}$ is nilpotent.*

Remark 11. *Any real nilpotent Lie group is diffeomorphic to Euclidean space $\mathbb{R}^n$.*

Example 12. *Let G be the Lie group of invertible 3×3 upper triangular matrices over $\mathbb{R}$ with 1's on the diagonal. Any element of G is of the form*

$$\begin{pmatrix} 1 & a_{12} & a_{13} \\ 0 & 1 & a_{23} \\ 0 & 0 & 1 \end{pmatrix}$$

for $a_{ij} \in \mathbb{R}$.

The associated Lie algebra $\mathfrak{g}$ is all 3×3 upper triangular matrices over $\mathbb{R}$ with 1's on the diagonal, where its bilinear function is the commutator $[x, y] = xy - yx$. The group G is nilpotent because the lower central series of $\mathfrak{g}$ eventually vanishes:

$$\mathfrak{g}^1 = \begin{pmatrix} 0 & b_{12} & b_{13} \\ 0 & 0 & b_{23} \\ 0 & 0 & 0 \end{pmatrix}$$

$$\mathfrak{g}^2 = \begin{pmatrix} 0 & 0 & c_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\mathfrak{g}^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

If G is any Lie group, a representation $\tau : G \to GL_n(\mathbb{R})$ of a Lie group is *unipotent* if $\tau(x)$ is unipotent for all $x \in G$. It is standard fact that if G is nilpotent and simply connected then G admits a faithful unipotent representation $\rho : G \to GL_n(\mathbb{R})$.

In example 12 we can just set the representation τ to be the identity map. Then since any $m - I \in G$ will eventually become the zero matrix upon exponentiation, we clearly have that τ is unipotent.

We include one final theorem, which summarizes connections between nilpotent Lie groups and some of the concepts discussed in the previous subsection:

Theorem 13. *Let G be a simply connected nilpotent Lie group and $H \subset G$ a closed subgroup. The following conditions on G are equivalent:*

1. *For some faithful unipotent representation $\rho : G \to GL(n, \mathbb{R})$, $\rho(G)$ and $\rho(H)$ have the same Zariski closure in $GL(n, \mathbb{C})$.*

2. *G/H is compact.*

3. *G/H carries a finite invariant measure.*

4. *There are no proper connected closed subgroups of G containing H.*

5. *For any unipotent representation $\rho : G \to GL(n, \mathbb{R})$, $\rho(H)$ and $\rho(G)$ have the same Zariski closure in $GL(n, \mathbb{C})$*

1.4 Lattices

Throughout this section G is a locally compact group and H a closed subgroup of G. We fix left Haar measures μ_G, μ_H on G and H, respectively.

Recall that a function $f : G \to \mathbb{C}$ has compact support if the set $\overline{\{x \in G \mid f(x) \neq 0\}}$ is compact. Let $C_c(G)$ be the set of all continuous complex valued (Borel measurable) functions with compact support. For $f \in C_c(G)$, we will denote the integral of f over G with respect to the left Haar measure by $\int_G f(g)dg$.

We can easily define a topology on the quotient group G/H as follows: the set $S \subseteq G$ is open if the preimage $\pi^{-1}(S)$ of the canonical projection is open.

Definition 14. *Let μ be a Borel measure on G/H and $\chi : G \to \mathbb{R}^+$ a continuous homomorphism. Then μ is said to be semi-G-invariant with character χ if for every $g \in G$ and measurable set $E \subset G/H$, we have $\mu(gE) = \chi(g) \cdot \mu(E)$*

Remark 15. *Not all spaces admit a semi-invariant measure.*

We will now denote $\Delta_G : G \to \mathbb{C}$ a modular function on G which is defined as follows: for $x \in G$ and a continuous complex valued function $f : G \to \mathbb{C}$ we have

$$\Delta_G(x) \int_G f(g)dg = \int_G f(gx)dg.$$

The existence of Δ_G is due to the function $\mu_x : 2^G \to \mathbb{R}_0^+$, $\mu_x(S) = \mu_G(Sx)$ being also a left Haar measure for all $x \in G$ and so (by 7) there exists a positive real number c, such that $\mu_x = c \cdot \mu_G$ and we simply put $\Delta_G(x) = c$.

Example 16. *Let $G = \left\{ \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \mid x \in \mathbb{R}^\times, y \in \mathbb{R} \right\}$. Then the modular function $\Delta_G\left(\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\right) = |a|^{-1}$. Indeed for any continuous complex valued function $f : G \to \mathbb{C}$ with compact support*

$$|a|^{-1} \int_G f(g)dg = \int_G |a|^{-1} f(gx)|a|dg = \int_G f(gx)dg.$$

Note that if we have a subgroup H of a group G it does not mean that $\Delta_H = \Delta_G|_H$.

Example 17. *Let* $G = \mathrm{SL}(2,\mathbb{R})$ *with subgroup* $H = \left\{ \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}) \right\}$. *The modular function of* G *is* $\Delta_G = 1$ *and the modular function of* H *is*

$$\Delta_H \left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \right) = |ac^2|^{-1}.$$

The modular function Δ_G *is given by the following condition: for any continuous function* $f : G \to \mathbb{C}$ *with compact support and for all* $x \in G$ *we have*

$$\Delta_G(x) \int_G f(g)dg = \int_G f(gx)dg = \int_G f(g)|\det(x)|^{-2}dg = \int_G f(g)dg.$$

Therefore it is easy to see that $\Delta_G(x) = 1$. *On the other hand, for any continuous function* $f : H \to \mathbb{C}$ *with compact support and for all* $x = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in H$

$$\Delta_H(x) \int_H f(h)dh = \int_H f(hx)dh = \int_H f(h)|ac^2|^{-1}dh,$$

so $\Delta_H = |ac^2|^{-1} \neq 1$.

We will now state a necessary and sufficient condition for the existence of a semi-invariant measure:

Lemma 18. *The homogeneous space* G/H *admits a semi-invariant measure if and only if the homomorphism* $\Delta_G \Delta_H^{-1} : H \to \mathbb{R}^+$ *can be extended to a continuous homomorphism on all of* G.

Example 19. *In example* 17 *we showed that the group*

$$H = \left\{ \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}) \right\}$$

is not unimodular. Take its subgroup $H_0 = \left\{ \begin{pmatrix} x & y \\ 0 & x^{-1} \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}) \right\}$. *Then this group is not a subgroup of a kernel of* Δ_H, *and is itself unimodular. Then* $\Delta_H \Delta_{H_0}^{-1} = \Delta_H|_{H_0}$, *which can be clearly extended on* H, *so* H/H_0 *admits semi-invariant measure.*

Lemma 20. *Let G be a locally compact group and H a closed subgroup of G. Given any homomorphism $u : G \to \mathbb{R}^+$ such that $u|_H = \Delta_G \Delta_H^{-1}$, G/H admits a semi-invariant measure with character u. This measure is unique up to a scalar multiple.*

Note that this also means, that the character of the semi-invariant measure from Lemma 18 is the extension of the homomorphism $\Delta_G \Delta_H^{-1}$.

Definition 21. *A semi-invariant measure on G/H is invariant if the associated character is trivial; i.e. for every $g \in G$ and measurable set $E \subset G/H$, we have $\mu(gE) = \mu(E)$.*

Lemma 22. *Let G be a locally compact group and H a closed subgroup of G. Then G/H admits an invariant measure if and only if $\Delta_G|_H = \Delta_H$.*

Proof. If $\Delta_G|_H = \Delta_H$, then by Lemma 20 $\Delta_G \Delta_H^{-1} = 1$, which can be extended on G and there is a semi-invariant measure on G/H with trivial character. Contrary if G/H admits an invariant measure then $\Delta_G \Delta_H^{-1}$ can be extended on G and the extension is the character belonging to the semi-invariant measure. Since the measure has trivial character, $\Delta_G \Delta_H^{-1} = 1$. □

Example 23. *The group G from example 19 does not admit invariant measure, since $H \not\subseteq \ker(\Delta_G)$, there exists $h \in H$ such that $\Delta_G(h) \neq 1 = \Delta_H(h)$.*

Lemma 24. *Let G be a locally compact group and H a closed subgroup of G. If a semi-invariant measure μ on G/H is finite, it is invariant.*

Proof. Since μ is semi-invariant, $\mu(gG/H) = \chi(g) \cdot \mu(G/H)$ for all $g \in G$. If $\mu(G/H) < \infty$ it follows, that $\chi(g) = 1$, so the character is trivial. □

Lemma 25. *Let H_1, H_2 be closed subgroups of G such that $H_2 \subseteq H_1$. Then G/H_2 carries a G-invariant finite measure if and only if G/H_1 and H_1/H_2 also carry invariant finite measure.*

Definition 26. *A discrete subgroup H of a group G is a lattice if G/H carries a finite invariant measure.*

Remark 27. *A lattice is sometimes defined to be a discrete subgroup Λ of G of co-finite volume. A closed subgroup H of G having a co-finite volume is defined as G/H admitting a finite G-invariant measure, so the definitions are equivalent.*

Some may encounter the term lattice L in $\mathbb{R}^n$ as being a discrete subgroup of $\mathbb{R}^n$. But in this case the invariant measure is just the product Lebesque measure, which is finite because $\lambda(\mathbb{R}^n/L)$ is the measure of the fundamental domain.

Remark 28. *If G admits a lattice then G is unimodular, i.e. the modular function $\Delta_G = 1$.*

Definition 29. *A subgroup H of a group G is called uniform if G/H is compact.*

Remark 30. *In some texts, the term co-compact is used instead of uniform. Both terms have the same meaning.*

Example 31. *Every discrete uniform subgroup H of a group G is a lattice. We can show that since G/H is compact it is unimodular: this is because $\Delta_G(G)$ must be a compact subgroup of $(0, \infty)$, which is only the trivial subgroup. Therefore G/H admits an invariant measure which is the left Haar measure. Since G/H is compact, the left Haar measure is finite by 8.*

A lattice Λ in G is called uniform (co-compact) if Λ is a uniform subgroup of G.

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