Transcendental Number Theory in a nutshell — Spring 2013

This information came from Dr. Tubbs' course notes when he taught this as a topics class at CU in spring 2013. For more information, I reccommend the book "Making Transcendence Transparent" by E. Burger and R. Tubbs (2004).

A number is <u>transcendental</u> if it is not algebraic; that is, if it is not the zero of a polynomial with rational coefficients.

Thus transcendental number theory is the branch of mathematics which consists of coming up with ingeneous ways to carry out the difficult task of proving that numbers are transcendental.

My impression is that a lot of proofs in transcendental number theory go like this:

1. Take a number t which you want to show is transcendental.
2. Assume t is algebraic $\Rightarrow$ it is the zero of a polynomial $p(x) \in \mathbb{Z}[x]$.
3. Use some combination of $p(x)$, t and the norm of t (taken from $\mathbb{Q}(t)$ to $\mathbb{Q}$) to get a rational number which is a linear combination of powers of t and algebraic numbers.
4. Clear the denominator of your rational number ~~any~~ and get an integer n:

$$\frac{\text{numenator}}{\text{denomnom}} = n$$

denominators! nom nom nom

5. Show that $0 < n < 1$.
6. Say "there are no rational integers between 0 and 1!! Aagh! contradiction!"

And now you have proven that t is transcendental.

Hilberts 7^{th} Problem (conjectured by Hilbert in 1900, has been proven by someone... possibly in 1930ish?):

- If α and β are algebraic, $\alpha \neq 0$ or 1, and β is irrational $\Rightarrow$ α^{β} is transcendental.

results of this: $2^{\sqrt{2}}$, $i^{\sqrt{2}}$, $e^{\pi} (= (-1)^{-i})$ are all transcendental.

Hilbert's conjecture can also be formulated in two other equivalent statements:

- $l, \beta \in \mathbb{C}$, $l \neq 0$ and β irrational. Then at least one of the following numbers is transcendental: β, e^{l}, $e^{\beta l}$.

- $\alpha, \beta \in \overline{\mathbb{Q}} \setminus \{0\}$. If $\log \alpha / \log \beta$ is irrational then it is transcendental.

* the equivalence of these three statements is relatively easy to show *

Proof that e is transcendental!
(by "proof" I mean ""proof"")

refer to steps on page 1.

1.+2: Assume that e is algebraic. Then there exist integers $r_0, \dots, r_d$, not all zero, such that $P(e) := r_0 + r_1 e + \dots + r_d e^d = 0$

3: $\forall n \in \mathbb{Z}_{>0}$, e^n has series representation $e^n = \sum_{k=0}^{\infty} \frac{n^k}{k!}$.

For $N \in \mathbb{Z}_{>0}$ define $M_N(n) := \sum_{k=0}^{N} \frac{n^k}{k!}$, $T_N(n) = \sum_{k=N+1}^{\infty} \frac{n^k}{k!}$

(i.e. the "main term" and "tail term").

$\Rightarrow e^n = M_N(n) + T_N(n)$ $\rightarrow$ plug this into P.

$$r_0 + r_1 (M_N(1) + T_N(1)) + \cdots + r_d (M_N(d) + T_N(d)) = 0$$

$$\Rightarrow |r_0 + r_1 M_N(1) + \cdots + r_d M_N(d)| = |r_1 T_N(1) + \cdots + r_d T_N(d)|$$

4. Multiplying LHS by $N!$ will clear denominators, yielding an integer:

$$N! \, |r_0 + r_1 M_N(1) + \cdots + r_d M_N(d)| = N! \, | r_1 T_N(1) + \cdots + r_d T_N(d)|$$
$$\text{LHS} \in \mathbb{Z}_{>0} \Rightarrow \text{RHS} \in \mathbb{Z}_{>0}$$

5. For large enough N, $0 < \underbrace{N! \, |r_1 T_N(1) + \cdots + r_d T_N(d)|}_{\in \mathbb{Z}} < 1$

6. This is a contradiction!! e is transcendental.

Louisville numbers: any real number x such that for all $n \in \mathbb{Z}_{>0}$, there exist p, q, $q > 1$, $(p,q) = 1$, such that $\left| x - \frac{p}{q} \right| < \frac{1}{q^n}$.

Theorem (Louisville): Loisville numbers are transcendental.

On the next page, I give a summary of some numbers which are known to be transcendental - there are more, but I think this covers the most popular results!

transcendental number	remarks
e	
e^{α}, $\alpha \neq 0$, $\alpha \in \overline{\mathbb{Q}}$	this is the result of the Hermite-Lindemann theorem
e^{π}	Gelfond, 1929
$\alpha^{\sqrt{-r}}$	$\alpha \neq 0, 1$ algebraic, $r \in \mathbb{Q}_{>0}$
$\sum_{k=1}^{\infty} 10^{-k!}$	the Loisville constant, which is a Louisville number (~1845)
$2^{\sqrt{2}}$	Kuzmin, 1930

The Six Exponentials Theorem:
$\{x_1, x_2\}$ and $\{y_1, y_2, y_3\}$ $\mathbb{Q}$-linearly independent sets of complex numbers. Then at least one of the six numbers $e^{x_i y_j}$, $i \in \{1,2\}$, $j \in \{1,2,3\}$ is transcendental.

<u>example</u>: I like examples which "collapse" the number of exponentials to something < 6. For example, if $\{x_1, x_2\} = \{1, e\}$ and $\{y_1, y_2, y_3\} = \{e, e^2, e^3\}$, then at least one of the following is transcendental:

$$e^{e},\ e^{e^2},\ e^{e^3},\ e^{e^4}$$

Another theme in transcendental number theory is the use of "algebraic independence" or "algebraic dependence" of sets of numbers to show that one (or more) of the numbers in the set is transcendental.

So what is algebraic dependence?
If K, L are fields, $K \subseteq L$, then two elements $\alpha_1, \alpha_2 \in L$ are <u>algebraically dependent</u> over K if there exists $p(x_1, x_2) \neq 0$ with coefficients in K such that $p(\alpha_1, \alpha_2) = 0$.
Otherwise, α_1 and α_2 are <u>algebraically independent</u> over K.

For example, $\Gamma(1/4)^2/\sqrt{\pi}$ is transcendental.
This is shown by relating a Weierstrass $\wp$ function to an improper integral I and showing that I evaluates to a finite complex number and is transcendental. We then use properties of the gamma function to show that $\Gamma(1/4)^2/\sqrt{\pi}$ and I are algebraically dependent over $\mathbb{Q}(\sqrt{2})$. Since I is transcendental, this implies that $\Gamma(1/4)^2/\sqrt{\pi}$ must also be transcendental.
Obviously, I left out a <u>lot</u> of details here, but this at least shows how this concept of algebraic dependence can be useful... and some of the main theorems in this subject draw on this idea!

The Lindemann-Weierstrass Theorem:
Let $\alpha_1, \ldots, \alpha_n$ be algebraic numbers which are linearly independent over $\mathbb{Q}$. Then $e^{\alpha_1}, \ldots, e^{\alpha_n}$ are algebraically independent.

ex. π is transcendental
Assume π is algebraic $\Rightarrow \pi i$ is algebraic $\Rightarrow \alpha_1 = 0, \alpha_2 = \pi i$ algebraic but linearly independent over $\mathbb{Q}$ $\Rightarrow e^0, e^{\pi i}$ must

be algebraically independent (over $\mathbb{Q}$). But $e^0 = 1$, $e^{\pi i} = -1$ so these numbers are clearly not algebraically independent over $\mathbb{Q}$. This gives a contradiction $\Rightarrow$ π must be transcendental.

The Schneider-Lang Theorem

def. We say an entire function $f(z)$ has order of growth κ if for all $\varepsilon > 0$ $|f(z)| < e^{|z|^{\kappa+\varepsilon}}$ for $|z|$ sufficiently large.

Thm. $f_1(z), f_2(z)$ two algebraically independent meromorphic functions with orders of growth $\rho_1, \rho_2 < \infty$.
Further suppose that there exists a collection of functions $f_3, \ldots, f_n$ such that $\frac{d}{dz}$ maps a ring $F[f_1, f_2, f_3, \ldots, f_n]$ onto itself. Then for any number field $E \supseteq F$,

$$\#\{z \in \mathbb{C} \mid f_1(z), \ldots, f_n(z) \in E\} \leq (\rho_1 + \rho_2)[E : \mathbb{Q}].$$

The way one would generally apply this theorem is by contradiction: if t is a number which you wish to show is transcendental, you should start by assuming it is algebraic. Then take E to be a finite degree extension of $\mathbb{Q}(t)$, so that $[E : \mathbb{Q}] < \infty$.
If we choose f_1 and f_2 well, we can find functions where there are infinitely many points z such that $f_1(z), f_2(z) \in E$ (this is because E is "bigger" than we are assuming $\rightarrow$ if t is transcendental, E won't be a finite degree extension of $\mathbb{Q}$). This of course gives a contradiction, telling us that t is actually transcendental.

The End!