

Statistics of caustics in the large-scale structure

Job Feldbrugge, Rien van de Weygaert, Johan Hidding

Monday 30th May, 2016, compiled at 15:28

Abstract

test

1 Geometric statistics

Above we described the conditions under which caustics form in Lagrangian fluid dynamics. We here show how the caustics conditions in combination with the fluid dynamics can be used to predict the statistical properties of caustics in Lagrangian fluids with random initial conditions. The study of large-scale structure formation is a good example of a fluid originating from random initial conditions. In this case the initial conditions can accurately be modeled by realizations of Gaussian random fields. We here only give a formal description. For a quantitative analysis we refer to [1].

Considering the expectation value of a statistic $\chi[s_t]$ at time t ,

$$\langle \chi \rangle = \int \chi[s_t] P_E[s_t] \mathcal{D}s_t,$$

which should be interpreted as a functional integral with P_E the probability density function of the displacement map $s_t = s(\cdot, t)$ at time t . Now consider the functional φ_t mapping the initial conditions (we here assume the initial density) ρ_i to the displacement map s_t .

For the Zel'dovich approximation,

$$\varphi_t[\rho_i] = -D_+(t) \nabla_q \Psi(q),$$

with the linearized velocity potential $\Psi(q)$ and growing mode $D_+(t)$. The growing mode can be obtained from linear Eulerian perturbation theory. Up to linear order, the linearized velocity potential is proportional to the linearly extrapolated gravitational potential at the current epoch $\phi_0(q)$, i.e.

$$\Psi(q) = \frac{2}{3\Omega_0 H_0^2} \phi_0(q). \quad (1)$$

If we assume the functional φ_t to be bijective we can express the expectation value in terms of the density function of the initial condition ρ_i as

$$\langle \chi \rangle = \int \chi[\varphi_t[\rho_i]] P_L[\rho_i] \mathcal{D}\rho_i$$

with the change of coordinates

$$P_E[\varphi_t(\rho_i)] = \left| \frac{\delta s_t}{\delta \rho_i} \right| P_L[\rho_i],$$

with P_L the probability density function of the initial condition. For Gaussian initial conditions, the initial density function is given by

$$P_L[\rho_i] \propto e^{-\frac{1}{2} \int \rho_i(q) K(q-q') \rho_i(q') dq dq'},$$

with K the inverse of the two-point correlation function ξ , defined by

$$\int K(q-q') \xi(q'-q'') dq' = \delta(q-q''),$$

with δ the Dirac delta function. We can compute statistics of the caustics in Lagrangian fluids by selecting the statistic χ . Note that for local statistics χ the path integral reduces to an ordinary integral.

For point densities, for example the density of A_3^+ points, we use a generalization of Rice's formula. Consider three independent conditions $c_i[f] = 0$ specifying a set of points, with f a smooth function mapping \mathbb{R}^3 to \mathbb{R} . If the points $\Lambda = \{x \in \mathbb{R}^3 | c_i[f] = 0\}$ are isolated, the number density of points characterized by the statistic

$$\chi[f] = \left| \frac{\partial c}{\partial x} \right| \prod_i \delta(c_i[f]), \quad (2)$$

with $c = (c_1, c_2, c_3)$. The two point correlation function of these points with a second class of points, specified by the conditions $k_i[f] = 0$, is characterized by the statistic

$$\chi[f] = \left| \frac{\partial c}{\partial x} \right| \left| \frac{\partial k}{\partial x} \right| \prod_i \delta(c_i[f]) \delta(k_i[f]). \quad (3)$$

The average line length of curves, also known as the flux, in two-dimensional fluid described by the condition $S[f] = 0$ can be computed using the statistic

$$\chi[f] = \|\nabla S[f]\| \delta(S[f]). \quad (4)$$

For a three-dimensional random field, curves are specified with two independent conditions $S[f] = 0$ and $T[s] = 0$. The average length per volume, or flux, of these lines is characterized by the statistic

$$\chi[f] = \|\nabla S[f] \times \nabla T[f]\| \delta(S[f]) \delta(T[f]). \quad (5)$$

In a similar fashion we can consider other properties, such as the curvature of curves and sheets and average area of sheets in three dimensional simulations.

1.0.1 Area density

The average area of a sheet defined by $S[f] = 0$ per unit volume is given by $\langle \chi(f) \rangle$ with

$$\chi[f] = \sqrt{S_1^2 S_2^2 + S_1^2 S_3^2 + S_2^2 S_3^2} \delta(S[f]) \quad (6)$$

with S_i the derivative of $S[f]$ in the i^{th} direction.

Consider the functional in the x_1 - x_2 -plane. The length of the iso-contour in the x_1 - x_2 plane is the length of the vector $(S_1, S_2, 0)$. In the x_1 - x_3 -plane the length is the length of the vector $(S_1, 0, S_3)$. The area of region spanned by these two vectors is the length of $(S_1, S_2, 0) \times (S_1, 0, S_3)$ which is $\|(S_2 S_3, S_1 S_3, -S_1 S_2)\| = \sqrt{S_1^2 S_2^2 + S_1^2 S_3^2 + S_2^2 S_3^2}$. Note that although in the derivation we singled out the x_1 -direction, the result is symmetric.

1.1 Curvature

The curvature of a line is given by

$$\kappa = \frac{\|u \times \dot{u}\|}{\|u\|^3}, \quad (7)$$

with u a tangent vector along the curve. In two dimensions this gives the statistic

$$\chi[f] = \frac{\|\nabla S[f] \cdot (\nabla \nabla S[f]) \cdot \nabla S[f]\|}{\|\nabla S[f]\|^3} \delta(S[f]) \quad (8)$$

in three dimensions

$$\chi[f] = \frac{\|(\nabla S[f] \times \nabla T[f]) \times [(\nabla S[f] \times T[f]) \cdot \nabla(\nabla S[f] \times \nabla T[f])]\|}{\|\nabla S[f] \times T[f]\|^3} \delta(S[f]) \delta(T[f]). \quad (9)$$