

## MATH 213 (Important Results)

### Separable ODE

- **Goal:** separate functions and derivatives of  $x$  and  $y$  to either side of the equation

$$\begin{aligned} f(x) &= g(y)y' \\ \implies \int f(x)dx &= \int g(y)dy \quad \text{then integrate both sides as normal} \end{aligned}$$

### Exact ODE

$$\begin{aligned} M(x, y)dx + N(x, y)dy &= 0 \\ &= du \quad \text{where } u \text{ is some function of } x \text{ and } y \end{aligned}$$

- **Goal:** find function  $u(x, y) = C$  (aka. an implicit solution of  $y$ )
- The equation is **exact** if & only if  $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$$\begin{aligned} u &= \int Mdx + k(y) \\ \text{then set } N &= \frac{\partial u}{\partial y} \quad \text{to solve for } k(y) \end{aligned}$$

- Alternatively,

$$\begin{aligned} u &= \int Ndy + k(x) \\ \text{then set } M &= \frac{\partial u}{\partial x} \quad \text{to solve for } k(x) \end{aligned}$$

- If equation is **not exact**, find the **integrating factor**  $\mu$  such that:

$$\begin{aligned} \frac{\partial}{\partial y}\mu M &= \frac{\partial}{\partial x}\mu N \\ \text{then solve as } \mu M(x, y)dx + \mu N(x, y)dy &= 0 \end{aligned}$$

### First-Order Linear ODE (With Variable Coefficients)

- **Homogeneous:**  $y' + p(x)y = 0$

$$y(x) = Ce^{-h}, \quad h = \int p(x)dx$$

- **Nonhomogeneous:**  $y' + p(x)y = q(x)$

$$y(x) = e^{-h} \left( \int e^h q(x)dx + C \right), \quad h = \int p(x)dx$$

## Nth-Order Linear ODE (Homogeneous)

$$y^{(n)} + p_1(x)y^{(n-1)} + \dots + p_{n-1}(x)y' + p_n y = r(x) \\ = L[y] \quad \text{(differential operator)}$$

- **Goal:** find a family of functions which are all solutions of the ODE (**General solution**)

$$y = c_1 y_1 + \dots + c_n y_n$$

where  $y_1 \dots y_n$  are linearly independent particular solutions (a specific function)

## Nth-Order Linear ODE (Homogeneous w/ Constant Coefficients)

- **Characteristic equation** of  $L[y] = 0$  is:

$$\lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_1\lambda + a_0 = 0 \quad \text{with roots } \lambda_1 \dots \lambda_n$$

- **General solution:**

$$y = c_1 e^{\lambda_1 x} + \dots + c_n e^{\lambda_n x}$$

where every  $y = e^{\lambda x}$  is linearly independent if every  $\lambda$  is distinct

- **Repeated roots:** for a root  $\lambda$  of order  $k$ ,

$$e^{\lambda x}, x e^{\lambda x}, x^2 e^{\lambda x} \dots x^{k-1} e^{\lambda x} \quad \text{are linearly independent solutions}$$

- **Complex roots:** for two complex roots  $\lambda_1 = ik, \lambda_2 = -ik$ ,

$$y = c_1 e^{ikx} + c_2 e^{-ikx} = A \cos(kx) + B \sin(kx)$$

## Nth-Order Linear ODE (Nonhomogeneous)

$$L[y] = f_1(x) + \dots + f_k(x)$$

- **General solution:**

$$y = y_h + y_{p1} + \dots + y_{pk}$$

where  $y_h$  is a general solution of  $L[y] = 0$

and  $y_{pi}$  is a particular solution of  $L[y] = f_i(x)$

- **Method of undetermined coefficients:**

- $f_i(x)$   $\rightarrow$   $y_{pi}(x)$  = sum of linearly independent derivatives
- $f_i(x) = k$  (constant term)  $\rightarrow$   $C$
- $f_i(x) = e^{kx}$   $\rightarrow$   $y_{pi}(x) = C e^{kx}$

- $f_i(x) = x^n, n \geq 0 \quad \rightarrow \quad y_{pi}(x) = C_n x^n + \dots + C_1 x + C_0$
- $f_i(x) = \cos(kx), \sin(kx) \quad \rightarrow \quad y_{pi}(x) = A \cos(kx) + B \sin(kx)$
- $f_i(x) = e^{rx} \cos(kx), e^{rx} \sin(kx) \quad \rightarrow \quad y_{pi}(x) = e^{rx} (A \cos(kx) + B \sin(kx))$
- Substitute  $y_p$  into LHS and match coefficients with RHS:

$$L[y_p] = f(x) \quad \text{and solve for constants}$$

## Laplace Transform

$$F(s) = L\{f\}(t) = \int_0^\infty f(t)e^{-st}dt$$

- **Transforms of derivatives:**

$$L\{f'\} = sF - f(0)$$

$$L\{f^{(n)}\} = s^n F - s^{n-1}f(0) - \dots - sf^{(n-2)}(0) - f^{(n-1)}(0)$$

- **Transforms of integrals:**

$$L\left\{\int_0^t f(r)dr\right\} = \frac{F}{s}$$

- **Derivatives of transforms (multiplication):**

$$L\{tf(t)\} = -F'(s)$$

- **Integrals of transforms (division):**

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(r)dr$$

- **S-shifting:**

$$L\{e^{at}f(t)\} = F(s - a)$$

- **T-shifting:**

$$L\{f(t - a)H(t - a)\} = e^{-as}F(s)$$

$$\begin{aligned} \text{where } H(t) &= 0 & \text{if } t < 0 \\ &= 1 & \text{if } t \geq 0 \end{aligned}$$

- **Dirac's delta function:**

$$L\{\delta(t - a)f(t)\} = e^{-as}f(a)$$

$$\begin{aligned} \text{where } \delta(t) &= \begin{cases} \infty & \text{if } t = 0 \\ 0 & \text{elsewhere} \end{cases} \end{aligned}$$

$$\text{and } \int_0^\infty \delta(t) dt = 1$$

- **Periodic functions:** if  $f(t + T) = f(t)$  for all  $t$  in domain,

$$L\{f(t)\} = \frac{1}{1 - e^{-sT}} \int_0^T f(t) e^{-st} dt$$

- **Convolution:**

$$F(s)G(s) = L\{f(t) * g(t)\} = L\left\{\int_0^t f(r)g(t-r)dr\right\}$$

- **Initial value theorem:** if  $f$  is continuous,  $f'$  is piecewise continuous, and  $f, f'$  are of exponential order,

$$\lim_{s \rightarrow \infty} sF(s) = f(0)$$

- **Using Laplace in ODEs:**

- Given with initial conditions  $y(0), y'(0)$ :

$$y'' + ay' + by = f(t)$$

$$L\{y'' + ay' + by\} = F(s)$$

$$(s^2Y - sy(0) - y'(0)) + a(sY - y(0)) + bY = F(s)$$

- Collect like terms, solve for  $Y$
- Solve for  $y(t) = L^{-1}\{Y\}$

## Vector Spaces

### Fourier Series

- For  $f(x)$  with period  $2L$ , defined over  $-L < x < L$

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left( a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right)$$

$$a_0 = \frac{1}{2L} \int_{-L}^L f(x) dx$$

$$a_n = \frac{1}{L} \int_{-L}^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx$$

$$b_n = \frac{1}{L} \int_{-L}^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

- If  $f(x)$  is even, the series only has  $a_0$  and  $a_n$  terms (**Fourier cosine series**) where

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

$$a_0 = \frac{1}{L} \int_0^L f(x) dx$$

$$a_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx$$

- If  $f(x)$  is odd, the series only has  $b_n$  terms (**Fourier sine series**) where

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

$$b_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

- **Complex exponential form:**

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L}$$

$$c_n = \frac{1}{2L} \int_{-L}^L f(x) e^{-in\pi x/L} dx$$

- Where

$$\sum_{n=-\infty}^{\infty} a_n \equiv \lim_{N \rightarrow \infty} \sum_{-N}^N a_n$$

- **Half/quarter-range expansions:**

- Half-range cosine expansion

- Given  $f(x)$  on  $0 < x < L$ , extend as an even periodic function on  $-\infty < x < \infty$
    - Identical to Fourier cosine series (with restriction on  $0 < x < L$ )

- Half-range sine expansion

- Given  $f(x)$  on  $0 < x < L$ , extend as an odd periodic function on  $-\infty < x < \infty$
    - Identical to Fourier sine series (with restriction on  $0 < x < L$ )

- Quarter-range cosine expansion

$$f(x) = \sum_{n=1,3,\dots}^{\infty} a_n \cos\left(\frac{n\pi x}{2L}\right) \quad (0 < x < L)$$

$$a_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{2L}\right)$$

- Quarter-range sine expansion

$$f(x) = \sum_{n=1,3,\dots}^{\infty} b_n \sin\left(\frac{n\pi x}{2L}\right) \quad (0 < x < L)$$

$$b_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{2L}\right)$$

- Useful integrals:

$$\int x \cos(nx) dx = \frac{x \sin(nx)}{n} + \frac{\cos(nx)}{n^2}$$

$$\int x \sin(nx) dx = -\frac{x \cos(nx)}{n} + \frac{\sin(nx)}{n^2}$$

$$\int_0^{2\pi} \sin(ax) dx = \int_0^{2\pi} \cos(ax) dx = \int_0^{2\pi} \sin(ax) \cos(bx) dx = 0 \quad \text{for any } a, b$$

- Useful identities:

$$\sin(x) \cos(y) = \frac{1}{2}(\sin(x+y) + \sin(x-y))$$

$$\cos(ax) = \frac{1}{2}(e^{iax} + e^{-iax})$$

$$\sin(ax) = \frac{1}{2i}(e^{iax} - e^{-iax})$$

$$\cos(n\pi) = (-1)^n$$

## Fourier Transform

- **Fourier integral:**

$$f(x) = \int_0^{\infty} (A(\omega) \cos \omega x + B(\omega) \sin \omega x) d\omega$$

$$A(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \cos \omega x dx$$

$$B(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \sin \omega x dx$$

- **Fourier transform:**

$$F\{f(x)\} = \hat{f}(\omega) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx$$

$$F^{-1}\{\hat{f}(\omega)\} = f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x} d\omega$$

- **Transforms of derivatives:**

$$F\{f'\} = i\omega \hat{f}$$

$$F\{f^{(n)}\} = (i\omega)^n \hat{f}$$

- **Transforms of integrals:**

$$F\left\{\int_{-\infty}^x f(r) dr\right\} = \frac{1}{i\omega} \hat{f}$$

- **Derivatives of transforms (multiplication):**

$$F\{x^n f\} = i^n \hat{f}^{(n)}$$

- **x-shifting:**

$$F\{f(x-a)\} = e^{-ia\omega} \hat{f}$$

- **w-shifting:**

$$F\{e^{iax} f\} = \hat{f}(\omega - a)$$

- **Horizontal scaling:**

$$F\{f(ax)\} = \frac{1}{|a|} \hat{f}\left(\frac{\omega}{a}\right)$$

- **Fourier convolution:**

$$\hat{f} \cdot \hat{g} = F\{f * g\} = F\left\{\int_{-\infty}^{\infty} f(x-r)g(r)dr\right\}$$

- **Fourier cosine & sine transforms:**

$$F_C\{f'\} = \omega \hat{f}_S - f(0)$$

$$F_S\{f'\} = -\omega \hat{f}_C$$

$$F_C\{f''\} = -\omega^2 \hat{f}_C - f'(0)$$

$$F_S\{f''\} = -\omega^2 \hat{f}_S + \omega f(0)$$