MATH 213 (Important Results)

Separable ODE

- **Goal**: separate functions and derivatives of x and y to either side of the equation

$$f(x) = g(y)y'$$
$$\implies \int f(x)dx = \int g(y)dy \qquad \text{then integrate both sides as normal}$$

Exact ODE

$$M(x,y)dx + N(x,y)dy = 0$$
$$= du \quad \text{where } u \text{ is some function of } x \text{ and } y$$

- **Goal**: find function $u(x,y) = C$ (aka. an implicit solution of y)
- The equation is **exact** if & only if $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

$$u = \int M dx + k(y)$$
$$\text{then set} \quad N = \frac{\partial u}{\partial y} \quad \text{to solve for } k(y)$$

- Alternatively,

$$u = \int N dy + k(x)$$
$$\text{then set} \quad M = \frac{\partial u}{\partial x} \quad \text{to solve for } k(x)$$

- If equation is **not exact**, find the **integrating factor** μ such that:

$$\frac{\partial}{\partial y}\mu M = \frac{\partial}{\partial x}\mu N$$

then solve as $\mu M(x,y)dx + \mu N(x,y)dy = 0$

First-Order Linear ODE (With Variable Coefficients)

- **Homogeneous**: $y' + p(x)y = 0$

$$y(x) = Ce^{-h}, \quad h = \int p(x)dx$$

- **Nonhomogeneous**: $y' + p(x)y = q(x)$

$$y(x) = e^{-h}\left(\int e^h q(x)dx + C\right), \quad h = \int p(x)dx$$

Nth-Order Linear ODE (Homogeneous)

$$\begin{aligned} y^{(n)} + p_1(x)y^{(n-1)} + \ldots + p_{n-1}(x)y' + p_n y &= r(x) \\ &= L[y] \quad (\underline{\text{differential operator}}) \end{aligned}$$

- **Goal**: find a <u>family of functions</u> which are all solutions of the ODE (**General solution**)

  $y = c_1 y_1 + \ldots + c_n y_n$
  where $y_1 \ldots y_n$ are linearly independent <u>particular solutions</u> (a specific function)

Nth-Order Linear ODE (Homogeneous w/ Constant Coefficients)

- **Characteristic equation** of $L[y] = 0$ is:

  $\lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + a_0 = 0 \qquad$ with roots $\lambda_1 \ldots \lambda_n$

- **General solution**:

  $y = c_1 e^{\lambda_1 x} + \ldots + c_n e^{\lambda_n x}$
  where every $y = e^{\lambda x}$ is linearly independent if every λ is distinct

- **Repeated roots**: for a root λ of order k,

  $e^{\lambda x}, xe^{\lambda x}, x^2 e^{\lambda x} \ldots x^{k-1}e^{\lambda x} \qquad$ are linearly independent solutions

- **Complex roots**: for two complex roots $\lambda_1 = ik, \lambda_2 = -ik$,

  $y = c_1 e^{ikx} + c_1 e^{-ikx} = A\cos(kx) + B\sin(kx)$

Nth-Order Linear ODE (Nonhomogeneous)

$L[y] = f_1(x) + \ldots + f_k(x)$

- **General solution**:

  $y = y_h + y_{p1} + \ldots + y_{pk}$

  where y_h is a <u>general solution</u> of $L[y] = 0$
  and y_{pi} is a <u>particular solution</u> of $L[y] = f_i(x)$

- **Method of undetermined coefficients**:
  - $f_i(x) \quad \rightarrow \quad y_p i(x) =$ sum of <u>linear independent derivatives</u>
  - $f_i(x) = k$ (constant term) $\quad \rightarrow \quad C$
  - $f_i(x) = e^{kx} \quad \rightarrow \quad y_{pi}(x) = Ce^{kx}$

- $f_i(x) = x^n, n \geq 0 \quad \rightarrow \quad y_{pi}(x) = C_n x^n + \ldots + C_1 x + C_0$
- $f_i(x) = \cos(kx), \sin(kx) \quad \rightarrow \quad y_{pi}(x) = A\cos(kx) + B\sin(kx)$
- $f_i(x) = e^{rx}\cos(kx), e^{rx}\sin(kx) \quad \rightarrow \quad y_{pi}(x) = e^{rx}A\cos(kx) + B\sin(kx)$
- Substitute y_p into LHS and match coefficients with RHS:

$$L[y_p] = f(x) \qquad \text{and solve for constants}$$

Laplace Transform

$$F(s) = L\{f\}(t) = \int_0^\infty f(t)e^{-st}dt$$

- **Transforms of derivatives**:

$$L\{f'\} = sF - f(0)$$

$$L\{f^{(n)}\} = s^n F - s^{n-1}f(0) - \ldots - sf^{(n-2)}(0) - f^{(n-1)}(0)$$

- **Transforms of integrals**:

$$L\{\int_0^t f(r)dr\} = \frac{F}{s}$$

- **Derivatives of transforms** (multiplication):

$$L\{tf(t)\} = -F'(s)$$

- **Integrals of transforms** (division):

$$L\{\frac{f(t)}{t}\} = \int_s^\infty F(r)dr$$

- **S-shifting**:

$$L\{e^{at}f(t)\} = F(s-a)$$

- **T-shifting**:

$$L\{f(t-a)H(t-a)\} = e^{-as}F(s)$$

$$\text{where } H(t) = 0 \qquad \text{if } t < 0$$
$$= 1 \qquad \text{if } t \geq 0$$

- **Dirac's delta function**:

$$L\{\delta(t-a)f(t)\} = e^{-as}f(a)$$

where $\delta(t) = \text{“}\infty\text{”} \quad \text{if } t = 0$

$\qquad\quad = 0 \quad \text{elsewhere}$

and $\int_0^\infty \delta(t)dt = 1$

- **Periodic functions**: if $f(t+T) = f(t)$ for all t in domain,

  $$L\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T f(t)e^{-st}dt$$

- **Convolution**:

  $$F(s)G(s) = L\{f(t) * g(t)\} = L\{\int_0^t f(r)g(t-r)dr\}$$

- **Initial value theorem**: if f is continuous, f' is piecewise continuous, and f, f' are of exponential order,

  $$\lim_{s\to\infty} sF(s) = f(0)$$

- **Using Laplace in ODEs**:
  - Given with initial conditions $y(0), y'(0)$:

    $$\begin{aligned} y'' + ay' + by &= f(t) \\ L\{y'' + ay' + by\} &= F(s) \\ (s^2Y - sy(0) - y'(0)) + a(sY - y(0)) + bY &= F(s) \end{aligned}$$

  - Collect like terms, solve for Y
  - Solve for $y(t) = L^{-1}\{Y\}$

Vector Spaces

Fourier Series

- For $f(x)$ with period $2L$, defined over $-L < x < L$

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right)$$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

- If $f(x)$ is even, the series only has a_0 and a_n terms (**Fourier cosine series**) where

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

$$a_0 = \frac{1}{L}\int_{0}^{L} f(x)dx$$

$$a_n = \frac{2}{L}\int_{0}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

- If $f(x)$ is odd, the series only has b_n terms (**Fourier sine series**) where

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

$$b_n = \frac{2}{L}\int_{0}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

- **Complex exponential form**:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L}$$

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-in\pi x/L}dx$$

  - Where

$$\sum_{n=-\infty}^{\infty} a_n \equiv \lim_{N\to\infty}\sum_{-N}^{N} a_n$$

- **Half/quarter-range expansions**:
  - Half-range cosine expansion
    - Given $f(x)$ on $0 < x < L$, extend as an even periodic function on $-\infty < x < \infty$
    - Identical to Fourier cosine series (with restriction on $0 < x < L$)
  - Half-range sine expansion
    - Given $f(x)$ on $0 < x < L$, extend as an odd periodic function on $-\infty < x < \infty$
    - Identical to Fourier sine series (with restriction on $0 < x < L$)
  - Quarter-range cosine expansion

$$f(x) = \sum_{n=1,3,\ldots}^{\infty} a_n \cos\left(\frac{n\pi x}{2L}\right) \qquad (0 < x < L)$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{2L}\right)$$

  - Quarter-range sine expansion

$$f(x) = \sum_{n=1,3,\ldots}^{\infty} b_n \sin\left(\frac{n\pi x}{2L}\right) \qquad (0 < x < L)$$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{2L}\right)$$

- Useful integrals:

$$\int x\cos(nx)dx = \frac{x\sin(nx)}{n} + \frac{\cos(nx)}{n^2}$$

$$\int x\sin(nx)dx = -\frac{x\cos(nx)}{n} + \frac{\sin(nx)}{n^2}$$

$$\int_0^{2\pi} \sin(ax)dx = \int_0^{2\pi} \cos(ax)dx = \int_0^{2\pi} \sin(ax)\cos(bx)dx = 0 \qquad \text{for any } a, b$$

- Useful identities:

$$\sin(x)\cos(y) = \frac{1}{2}(\sin(x+y) + \sin(x-y))$$

$$\cos(ax) = \frac{1}{2}(e^{iax} + e^{-iax})$$

$$\sin(ax) = \frac{1}{2i}(e^{iax} - e^{-iax})$$

$$\cos(n\pi) = (-1)^n$$

Fourier Transform

- **Fourier integral**:

$$f(x) = \int_0^\infty (A(\omega)\cos\omega x + B(\omega)\sin\omega x)d\omega$$

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos\omega x dx$$

$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin\omega x dx$$

- **Fourier transform**:

$$F\{f(x)\} = \hat{f}(\omega) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx$$

$$F^{-1}\{\hat{f}(\omega)\} = f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega x}d\omega$$

  - **Transforms of derivatives**:

$$F\{f'\} = i\omega\hat{f}$$

$$F\{f^{(n)}\} = (i\omega)^n\hat{f}$$

  - **Transforms of integrals**:

$$F\{\int_{-\infty}^{x} f(r)dr\} = \frac{1}{i\omega}\hat{f}$$

  - **Derivatives of transforms** (multiplication):

$$F\{x^n f\} = i^n\hat{f}^{(n)}$$

  - **x-shifting**:

$$F\{f(x-a)\} = e^{-ia\omega}\hat{f}$$

  - **w-shifting**:

$$F\{e^{iax}f\} = \hat{f}(\omega - a)$$

  - **Horizontal scaling**:

$$F\{f(ax)\} = \frac{1}{|a|}\hat{f}\left(\frac{\omega}{a}\right)$$

  - **Fourier convolution**:

$$\hat{f}\cdot\hat{g} = F\{f * g\} = F\{\int_{-\infty}^{\infty} f(x-r)g(r)dr\}$$

- **Fourier cosine & sine transforms**:

$$F_C\{f'\} = \omega \hat{f}_S - f(0)$$
$$F_S\{f'\} = -\omega \hat{f}_C$$
$$F_C\{f''\} = -\omega^2 \hat{f}_C - f'(0)$$
$$F_S\{f''\} = -\omega^2 \hat{f}_S + \omega f(0)$$