MATH 239 Summary Sheet

Graphs

- $V(G)$ = set of **vertices** in the graph G
- $E(G)$ = set of **edges** in the graph G
- "k-regular graph" = every vertex has degree k
- "Complete graph" (K_n) = all vertices are adjacent; i.e. $(n-1)$-regular
- "Complete bipartite graph" $(K_{m,n})$ = all vertices in one partition are adjacent to all vertices in other partition
- **Handshaking Lemma:**

$$\sum_{v \in V(G)} deg(v) = 2|E(G)|$$

Paths and Cycles

- **Theorem:** $\exists$ a <u>walk</u> between u, v in $G \implies \exists$ a <u>path</u> between u, v in G
  - **Corollary:** $\exists$ a path between x, y AND $\exists$ a path between $y, z \implies \exists$ a path between x, z
- **Theorem:** every vertex in G has degree $\geq 2 \implies G$ contains a cycle

Connectedness

- Fix vertex v in G;
  - $\forall$ vertex w in G, $\exists$ path between $v, w \implies G$ is **connected**
- Let $X \subset V(G)$;
  - "Cut induced by X" = set of edges with exactly one vertex $\in X$
  - **Theorem:** G is <u>not connected</u> $\iff \exists\ X$ such that cut induced by X is empty
- "Eulerian circuit" = closed walk that contains every edge exactly once
- **Theorem:** G has Eulerian circuit $\iff G$ is connected AND every vertex has even degree
- **Lemma:** G is connected AND e is a bridge $\implies G - e$ has exactly 2 components
- **Theorem:** e is a bridge $\iff e$ is not contained in any cycle
  - **Corollary:** $\exists$ 2 distinct paths between u, v in $G \implies G$ contains a cycle

Trees

- "Tree" = <u>connected graph</u> with <u>no cycles</u>
- "Leaf" = vertex in a tree with degree 1

- Let T be a tree;
  - **Lemma:** $\exists$ a unique path between every u, v in T
  - **Lemma:** every edge in T is a bridge
  - **Theorem:** T has ≥ 2 vertices $\implies$ T has ≥ 2 leaves
  - **Theorem:** $|E(T)| = |V(T)| - 1$
- **Theorem:** G is connected $\iff$ G has a spanning tree
  - **Corollary:** G is connected AND G has p vertices and $q = p - 1$ edges $\implies$ G is a tree
- **Theorem:** T is a spanning tree of G AND e is an edge $\notin T$ $\implies$ $T + e$ contains exactly 1 cycle C
  - Also: e' is an edge $\in C$ $\implies$ $T + e - e'$ is also a spanning tree of G

Bipartites

- **Theorem:** all trees are bipartite
- **Theorem:** G is bipartite $\iff$ G contains no <u>odd cycles</u>

Minimum Spanning Tree

- **Prim's Algorithm:**
  - Begin with a vertex in G and add it to T
  - At each step, find the <u>lowest-weight edge</u> that joins a vertex in T with a vertex not in T
  - Follow this edge and add the vertex to T; repeat

Planarity

- **Handshaking Lemma for Faces (Faceshaking Lemma):**

$$\sum_{f \in faces} deg(f) = 2|E(G)|$$

- **Euler's Formula:** let G be a connected graph with p vertices and q edges; if G has a planar embedding with f faces, then

$$p - q + f = 2$$

- **Theorem:** a graph is planar $\iff$ it can be drawn on the surface of a sphere
- "Platonic graph" = graph whose planar embedding has vertices all with degree $d \geq 3$ and faces all with degree $d^* \geq 3$
- **Theorem:** there are exactly 5 platonic graphs
- **Lemma:** (d, d^*) pairs are: $(3,3), (3,4), (4,3), (3,5), (5,3)$

- **Lemma:** G is a platonic graph with p vertices of degree d, q edges, and f faces of degree d^*, then:

$$q = \frac{2dd^*}{2d + 2d^* - dd*} \qquad\qquad p = \frac{2q}{d} \qquad\qquad f = \frac{2q}{d^*}$$

- **Theorem:** if a graph is connected and planar with $p \geq 3$ vertices and q edges, then $q \leq 3p-6$
    - **Corollary:** a planar graph has a vertex of degree < 6
- Note: K_5 and $K_{3,3}$ are not planar
- **Kuratowski's Theorem:** G is <u>not planar</u> $\iff$ G contains a edge subdivision of K_5 or $K_{3,3}$
- **Theorem:** G is 2-colourable $\iff$ G is bipartite
- **Theorem:** K_n is n-colourable
- **Theorem:** every vertex of G has degree $\leq d \implies G$ is $(d+1)$-colourable
- **Four Colour Theorem:** every planar graph is 4-colourable

Matchings and Covers

- **Lemma:** M has an augmenting path $\implies$ M is not a maximum matching
- **Lemma:** M is a matching of G AND C is a cover of $G \implies |M| \leq |C|$
- **Lemma:** $|M| = |C| \implies M$ is a maximum matching and C is a minimum cover
- **Konig's Theorem:** in a bipartite graph, the max size of a matching = min size of a cover