MATH 213 (Important Results)

Separable ODE

- **Goal**: separate functions and derivatives of x and y to either side of the equation

$$f(x) = g(y)y'$$
$$\implies \int f(x)dx = \int g(y)dy \qquad \text{then integrate both sides as normal}$$

Exact ODE

$$M(x,y)dx + N(x,y)dy = 0$$
$$= du \quad \text{where } u \text{ is some function of } x \text{ and } y$$

- **Goal**: find function $u(x,y) = C$ (aka. an implicit solution of y)
- The equation is **exact** if & only if $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

$$u = \int M dx + k(y)$$
$$\text{then set} \quad N = \frac{\partial u}{\partial y} \quad \text{to solve for } k(y)$$

- Alternatively,

$$u = \int N dy + k(x)$$
$$\text{then set} \quad M = \frac{\partial u}{\partial x} \quad \text{to solve for } k(x)$$

- If equation is **not exact**, find the **integrating factor** μ such that:

$$\frac{\partial}{\partial y}\mu M = \frac{\partial}{\partial x}\mu N$$

then solve as $\mu M(x,y)dx + \mu N(x,y)dy = 0$

First-Order Linear ODE (With Variable Coefficients)

- **Homogeneous**: $y' + p(x)y = 0$

$$y(x) = Ce^{-h}, \quad h = \int p(x)dx$$

- **Nonhomogeneous**: $y' + p(x)y = q(x)$

$$y(x) = e^{-h}(\int e^h q(x)dx + C), \quad h = \int p(x)dx$$

Nth-Order Linear ODE (Homogeneous)

$$\begin{aligned} y^{(n)} + p_1(x)y^{(n-1)} + \ldots + p_{n-1}(x)y' + p_n y &= r(x) \\ &= L[y] \quad (\underline{\text{differential operator}}) \end{aligned}$$

- **General solution**:

  $y = c_1 y_1 + \ldots + c_n y_n$
  where $y_1 \ldots y_n$ are linearly independent <u>particular solutions</u>

Nth-Order Linear ODE (Homogeneous w/ Constant Coefficients)

- **Characteristic equation** of $L[y] = 0$ is:

  $\lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + a_0 = 0$ with roots $\lambda_1 \ldots \lambda_n$

- **General solution**:

  $y = c_1 e^{\lambda_1 x} + \ldots + c_n e^{\lambda_n x}$
  where every $y = e^{\lambda x}$ is linearly independent if every λ is distinct

- **Repeated roots**: for a root λ of order k,

  $e^{\lambda x}, xe^{\lambda x}, x^2 e^{\lambda x} \ldots x^{k-1} e^{\lambda x}$ are linearly independent solutions

- **Complex roots**: for two complex roots $\lambda_1 = ik, \lambda_2 = -ik$,

  $y = c_1 e^{ikx} + c_1 e^{-ikx} = A\cos(kx) + B\sin(kx)$

Nth-Order Linear ODE (Nonhomogeneous)

$L[y] = f_1(x) + \ldots + f_k(x)$

- **General solution**:

  $y = y_h + y_{p1} + \ldots + y_{pk}$

  where y_h is a <u>general solution</u> of $L[y] = 0$
  and y_{pi} is a <u>particular solution</u> of $L[y] = f_i(x)$

- **Method of undetermined coefficients**:
  - $f_i(x) \quad \rightarrow \quad y_p i(x) =$ sum of <u>linear independent derivatives</u>
  - $f_i(x) = e^{kx} \quad \rightarrow \quad y_p i(x) = Ce^{kx}$
  - $f_i(x) = x^n, n \geq 0 \quad \rightarrow \quad y_p i(x) = C_n x^n + \ldots + C_1 x + C_0$

- $f_i(x) = \cos(kx), \sin(kx) \quad \to \quad y_pi(x) = A\cos(kx) + B\sin(kx)$
- $f_i(x) = e^{rx}\cos(kx), e^{rx}\sin(kx) \quad \to \quad y_pi(x) = e^{rx}A\cos(kx) + B\sin(kx)$
- Substitute y_p into LHS and match coefficients with RHS:

$$L[y_p] = f(x) \qquad \text{and solve for constants}$$

Laplace Transform

$$F(s) = L\{f\}(t) = \int_0^\infty f(t)e^{-st}dt$$

- **Transforms of derivatives**:

$$L\{f'\} = sF - f(0)$$

$$L\{f^{(n)}\} = s^nF - s^{n-1}f(0) - \ldots - sf^{(n-2)}(0) - f^{(n-1)}(0)$$

- **Transforms of integrals**:

$$L\{\int_0^t f(r)dr\} = \frac{F}{s}$$

- **Derivatives of transforms** (multiplication):

$$L\{tf(t)\} = -F'(s)$$

- **Integrals of transforms** (division):

$$L\{\frac{f(t)}{t}\} = \int_s^\infty F(r)dr$$

- **S-shifting**:

$$L\{e^{at}f(t)\} = F(s-a)$$

- **T-shifting**:

$$L\{f(t-a)H(t-a)\} = e^{-as}F(s)$$

- **Dirac's delta function**:

$$L\{\delta(t-a)f(t)\} = e^{-as}f(a)$$

- **Periodic functions**: if $f(t+T) = f(t)$ for all t in domain,

$$L\{f(t)\} = \frac{1}{1-e^{-sT}}\int_0^T f(t)e^{-st}dt$$

- **Convolution**:

$$F(s)G(s) = L\{f(t) * g(t)\} = L\{\int_0^t f(r)g(t-r)dr\}$$

- **Initial value theorem**: if f is continuous, f' is piecewise continuous, and f, f' are of exponential order,

$$\lim_{s \to \infty} sF(s) = f(0)$$