# MATH 239 Summary Sheet

## Graphs

- $V(G)$ = set of **vertices** in the graph $G$
- $E(G)$ = set of **edges** in the graph $G$
- "$k$-regular graph" = every vertex has degree $k$
- "Complete graph" $(K_n)$ = all vertices are adjacent; i.e. $(n-1)$-regular
- "Complete bipartite graph" $(K_{m,n})$ = all vertices in one partition are adjacent to all vertices in other partition
- **Handshaking Lemma:**

$$\sum_{v \in V(G)} deg(v) = 2|E(G)|$$

## Paths and Cycles

- **Theorem:** $\exists$ a <u>walk</u> between $u, v$ in $G \implies \exists$ a <u>path</u> between $u, v$ in $G$
  - **Corollary:** $\exists$ a path between $x, y$ AND $\exists$ a path between $y, z \implies \exists$ a path between $x, z$
- **Theorem:** every vertex in $G$ has degree $\geq 2 \implies G$ contains a cycle

## Connectedness

- Fix vertex $v$ in $G$;
  - $\forall$ vertex $w$ in $G$, $\exists$ path between $v, w \implies G$ is **connected**
- Let $X \subset V(G)$;
  - "Cut induced by $X$" = set of edges with exactly one vertex $\in X$
  - **Theorem:** $G$ is <u>not connected</u> $\iff \exists\, X$ such that cut induced by $X$ is empty
- "Eulerian circuit" = closed walk that contains every edge exactly once
- **Theorem:** $G$ has Eulerian circuit $\iff G$ is connected AND every vertex has even degree
- **Lemma:** $G$ is connected AND $e$ is a bridge $\implies G - e$ has exactly 2 components
- **Theorem:** $e$ is a bridge $\iff e$ is not contained in any cycle
  - **Corollary:** $\exists$ 2 distinct paths between $u, v$ in $G \implies G$ contains a cycle

## Trees

- "Tree" = <u>connected graph</u> with <u>no cycles</u>
- "Leaf" = vertex in a tree with degree 1

- Let $T$ be a tree;
  - **Lemma:** $\exists$ a unique path between every $u, v$ in $T$
  - **Lemma:** every edge in $T$ is a bridge
  - **Theorem:** $T$ has $\geq 2$ vertices $\implies$ $T$ has $\geq 2$ leaves
  - **Theorem:** $|E(T)| = |V(T)| - 1$
- **Theorem:** $G$ is connected $\iff$ $G$ has a spanning tree
  - **Corollary:** $G$ is connected AND $G$ has $p$ vertices and $q = p - 1$ edges $\implies$ $G$ is a tree
- **Theorem:** $T$ is a spanning tree of $G$ AND $e$ is an edge $\notin T$ $\implies$ $T + e$ contains exactly 1 cycle $C$
  - Also: $e'$ is an edge $\in C$ $\implies$ $T + e - e'$ is also a spanning tree of $G$

### Bipartites

- **Theorem:** all trees are bipartite
- **Theorem:** $G$ is bipartite $\iff$ $G$ contains no <u>odd cycles</u>

### Minimum Spanning Tree

- **Prim's Algorithm:**
  - Begin with a vertex in $G$ and add it to $T$
  - At each step, find the <u>lowest-weight edge</u> that joins a vertex in $T$ with a vertex not in $T$
  - Follow this edge and add the vertex to $T$; repeat