CS 341

These notes are meant to be supplementary to lecture slides & the textbook, and so may not contain all covered materials. Here I've chosen content which might not be easy to remember and/or is helpful to look at when doing assignments.

Asymptotic Analysis

- **Order Notations**:
  - $f(n) \in O(g(n))$ if $\exists\ c > 0$ and $n_0 > 0$ such that $0 \leq f(n) \leq cg(n)\ \forall\ n \geq n_0$
    - f "grows no faster than" g
  - $f(n) \in \Omega(g(n))$ if $\exists\ c > 0$ and $n_0 > 0$ such that $0 \leq cg(n) \leq f(n)\ \forall\ n \geq n_0$
    - f "grows no slower than" g
  - $f(n) \in \Theta(g(n))$ if $\exists\ c_1, c_2 > 0$ and $n_0 > 0$ such that $0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n)\ \forall\ n \geq n_0$
    - f ang g have the same complexity
  - $f(n) \in o(g(n))$ if $\forall\ c > 0, \exists\ n_0 > 0$ such that $0 \leq f(n) < cg(n)\ \forall\ n \geq n_0$
    - f has lower complexity than g
  - $f(n) \in \omega(g(n))$ if $\forall\ c > 0, \exists\ n_0 > 0$ such that $0 \leq cg(n) < f(n)\ \forall\ n \geq n_0$
    - f has higher complexity than g
  - $f \in O(g)$ and $f \in \Omega(g) \iff f \in \Theta(g)$
- **Limit method**: suppose $L = \lim_{n\to\infty} \dfrac{f(n)}{g(n)}$
  - $f \in o(g)$ if $L = 0$
  - $f \in \Theta(g)$ if $0 < L < \infty$
  - $f \in \omega(g)$ if $L = \infty$
- Useful facts for first-principles proofs:
  - $\log n \geq 1\ \forall\ n \geq 2$; i.e. $\log n$ grows faster than 1
  - $\log n \leq n\ \forall\ n \geq 0$; i.e. $\log n$ grows slower than n
- Useful limit laws:
  - $\lim_{n\to\infty} \dfrac{f(n)}{g(n)} = \lim_{n\to\infty} \dfrac{\log(f(n))}{\log(g(n))}$
  - $\lim_{n\to\infty} f(n)^k = (\lim_{n\to\infty} f(n))^k$
- Some math rules:
  - Summing a polynomial: $\sum_{i=1}^{n} i^k \in \Theta(n^{k+1})$

- Summing an exponential (special case of geometric series):

$$\sum_{i=1}^{n} c^i \in \begin{cases} \Theta(c^{n+1}) & \text{if } c > 1 \\ \Theta(n) & \text{if } c = 1 \\ \Theta(1) & \text{if } c < 1 \end{cases}$$

- $a^{\log_b n} = n^{\log_b a}$ (Useful for recursion trees)
- Geometric series:

$$\sum_{i=0}^{n-1} ar^i = \begin{cases} a\frac{r^n-1}{r-1} \in \Theta(r^n) & \text{if } r > 1 \\ na \in \Theta(n) & \text{if } r = 1 \\ a\frac{1-r^n}{1-r} \in \Theta(1) & \text{if } r < 1 \end{cases}$$

- $a \geq b + c$ if $a \geq 2 \cdot \max(b, c)$ (useful for asymtotic proofs)
- $\lim_{x \to c} \dfrac{f(x)}{g(x)} = \lim_{x \to c} \dfrac{f'(x)}{g'(x)}$ if it exists (L'Hopital's Rule)
- $\log(n!) \in \Theta(n \log n)$ (Stirling's Approximation)
- $\sum_{i=1}^{n} \dfrac{1}{i} \in \Theta(\log n)$ (Harmonic series)

Divide and Conquer

- **Recursion-tree method**:
  - Given the recurrence $T(n) = aT(n/b) + f(n), T(1) = c$:
    - a is the # of recursive calls made (# of subproblems)
    - b is the # by which the input size n is divided in each recursive call
    - $f(n)$ is the runtime of the "work done outside of the recursive calls"
    - c is the constant-time work done in each recursive call in the base case
  - Each node of the recursion tree represents the cost of the work done other than making recursive calls
  - Each row represents the total cost of work done in all recursive calls at that recursion "level"
  - The height of the tree depends on the factor that the input size is divided by; i.e. $\log_b n$
  - E.g.: Picture this as a tree where each node (except for leaves) has a children;

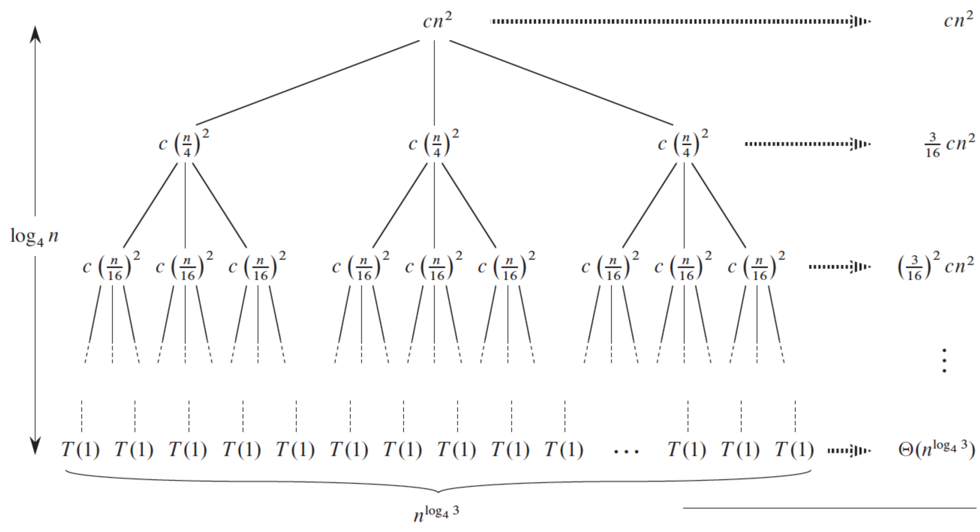
Level 0:	$f(n)$			total $= f(n)$
Level 1:	$f(n/b)$	$\ldots$	$f(n/b)$	total $= af(n/b)$
Level 2:	$f(n/b^2)$	$\ldots$	$f(n/b^2)$	total $= a^2 f(n/b^2)$
$\ldots$				
Level k:	$f(n/b^k)$	$\ldots$	$f(n/b^k)$	total $= a^k f(n/b^k)$
$\ldots$				
Level $\log_b n$:	c	$\ldots$	c	total $= ca^{\log_b n} = cn^{\log_b a}$

    - Total runtime of recursion tree is (summing every row total):

    $$T(n) \in \Theta\left(\sum_{i=0}^{\log_b n - 1} a^i f(n/b^i)\right) + \Theta(n^{\log_b a})$$

    - Use geometric series formula to find a simplified value

- Example: $T(n) = 3T(n/4) + n^2$



(d)

$$T(n) \in \Theta\left(\sum_{k=0}^{\log_4 n-1}\left(\frac{3}{16}\right)^k n^2 + n^{\log_4 3}\right) \in \Theta(n^2)$$

- **Master method**:
  - Given the recurrence $T(n) = aT(n/b) + f(n)$ where $f(n) \in \Theta(n^d)$:

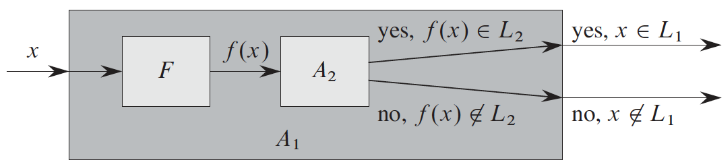
$$T(n) \in \begin{cases} \Theta(n^{\log_b a}) & \text{if } a > b^d \\ \Theta(n^d \log n) & \text{if } a = b^d \\ \Theta(n^d) & \text{if } a < b^d \end{cases}$$

Greedy Algorithms

- Two methods of proving greedy algorithms:
- **Greedy stays ahead (induction)**
  - Show that S_g is better than S at every step
  - <u>Base case</u>: show $S_g[1] > S[1]$
  - <u>Inductive hypothesis</u>: assume $S_g[k-1] > S[k-1]$; show that $S_g[k] > S[k]$
    - Often by contradiction; i.e. suppose $\exists\, S^*$ such that $S_g[k] < S^*[k]$, and derive some contradiction
- **Exchange argument (swapping)**
  - Let S_g = greedy solution, S = some arbitrary solution
  - Show that (any) S can be transformed into S_g step-by-step <u>without getting worse at any point</u>
    - i.e. compare the cost of before & after swapping 2 elements in S, show that it doesn't change or improves

Intractability

- P: solvable in polynomial time
- NP: verifiable in polynomial time
    - A verification algorithm takes 2 inputs:
    - Problem instance – a particular set of parameters of the problem
        - e.g. a graph, with vertices and edges
    - Certificate – a given "solution" of the problem that shows the instance returns *yes*
        - e.g. a ordered list of vertices, used to verify that a Hamiltonian cycle indeed exists
- $P \subseteq NP$
- **Reducibility**:
    - $L_1 \leq_P L_2$ – "L_1 is polynomial-time reducible to L_2"
    - There exists a P-time computable function f that maps instances of L_1 to L_2
    - e.g. If problem instance I returns *yes* for L_1, then $f(I)$ also returns *yes* for L_2



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    - If $L_1 \leq_P L_2$, then $L_2 \in P \implies L_1 \in P$
        - i.e. if L_2 is solvable in P-time, then L_1 is as well
        - i.e. L_1 is no harder than L_2
    - Contrapositive:
        - i.e. if L_1 is known to be *not* solvable in P-time, then L_2 can't be either
        - i.e. L_2 is as hard as L_1
    - Mainly used for reducing decision problems as part of NPC proofs
- **Turing reduction**:
    - A reduction $L_1 \leq^T L_2$ used to solve L_1, where L_2 is known, e.g. available as a subroutine
    - $L_1 \leq_P^T L_2$ – polynomial-time Turing reduction
    - L_1, L_2 don't have to be decision problems
    - Mainly used for reducing optimization problem $\rightarrow$ decision problem
- **NP-Completeness**:
    - L is NPC if:
        - $L \in NP$, and
        - $\forall\, L' \in NP, L' \leq_P L$ (this is definition of NP-hard)
    - i.e. NPC = set of problems in NP such that all other NP problems can be can be

    reduced to X
  - NPC = NP $\cap$ NP-hard
  - Because of these properties, to prove a problem L is NPC:
    - Show that $L \in$ NP
    - Show that $L' \leq_P L$ for some $L' \in$ NPC
- Reduction relationships between NPC problems:

$$
\begin{tikzcd}
 & \textbf{CNF-SAT} \arrow[d] & \\
 & \textbf{3-CNF-SAT} \arrow[d] & \\
 & \textbf{Clique} \arrow[d] & \\
 & \textbf{Vertex Cover} \arrow[dl] \arrow[dr] & \\
\textbf{Subset Sum} \arrow[d] & & \textbf{Hamiltonian Cycle} \arrow[d] \\
\textbf{0-1 Knapsack} & & \textbf{TSP-Decision}
\end{tikzcd}
$$

- **Undecidability**:
  - No algorithm can solve the problem
  - Show that the existence of an algorithm leads to a contradiction
  - Example: the Halting Problem
  - Reduce an undecidable problem to another problem to show that it's also undecidable