CS 341

These notes are meant to be supplementary to lecture slides & the textbook, and so may not contain all covered materials. Here I've chosen content which might not be easy to remember and/or is helpful to look at when doing assignments.

Asymptotic Analysis

- **Order Notations**:
  - $f(n) \in O(g(n))$ if $\exists\ c > 0$ and $n_0 > 0$ such that $0 \leq f(n) \leq cg(n)\ \forall\ n \geq n_0$
    - f "grows no faster than" g
  - $f(n) \in \Omega(g(n))$ if $\exists\ c > 0$ and $n_0 > 0$ such that $0 \leq cg(n) \leq f(n)\ \forall\ n \geq n_0$
    - f "grows no slower than" g
  - $f(n) \in \Theta(g(n))$ if $\exists\ c_1, c_2 > 0$ and $n_0 > 0$ such that $0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n)\ \forall\ n \geq n_0$
    - f ang g have the same complexity
  - $f(n) \in o(g(n))$ if $\forall\ c > 0, \exists\ n_0 > 0$ such that $0 \leq f(n) < cg(n)\ \forall\ n \geq n_0$
    - f has lower complexity than g
  - $f(n) \in \omega(g(n))$ if $\forall\ c > 0, \exists\ n_0 > 0$ such that $0 \leq cg(n) < f(n)\ \forall\ n \geq n_0$
    - f has higher complexity than g
  - $f \in O(g)$ and $f \in \Omega(g) \iff f \in \Theta(g)$
- **Limit method**: suppose $L = \lim_{n\to\infty} \dfrac{f(n)}{g(n)}$
  - $f \in o(g)$ if $L = 0$
  - $f \in \Theta(g)$ if $0 < L < \infty$
  - $f \in \omega(g)$ if $L = \infty$
- Useful facts for first-principles proofs:
  - $\log n \geq 1\ \forall\ n \geq 2$; i.e. $\log n$ grows faster than 1
  - $\log n \geq n\ \forall\ n \geq 0$; i.e. $\log n$ grows faster than n
- Some math rules:
  - Summing a polynomial: $\sum_{i=1}^{n} i^k \in \Theta(n^{k+1})$
  - Summing an exponential (special case of geometric series):

$$\sum_{i=1}^{n} c^i \in \begin{cases} \Theta(c^{n+1}) & \text{if } c > 1 \\ \Theta(n) & \text{if } c = 1 \\ \Theta(1) & \text{if } c < 1 \end{cases}$$

  - $a^{\log_b n} = n^{\log_b a}$ (Useful for recursion trees)

- Geometric series:
$$\sum_{i=0}^{n-1} ar^i = \begin{cases} a\frac{r^n-1}{r-1} \in \Theta(r^n) & \text{if } r > 1 \\ na \in \Theta(n) & \text{if } r = 1 \\ a\frac{1-r^n}{1-r} \in \Theta(1) & \text{if } r < 1 \end{cases}$$
- $\lim_{x\to c} \dfrac{f(x)}{g(x)} = \lim_{x\to c} \dfrac{f'(x)}{g'(x)}$ if it exists (L'Hopital's Rule)
- $\log(n!) \in \Theta(n \log n)$ (Stirling's Approximation)
- $\sum_{i=1}^{n} \dfrac{1}{i} \in \Theta(\log n)$ (Harmonic series)

Divide and Conquer

- **Recursion-tree method**:
  - E.g. given the recurrence $T(n) = aT(n/b) + f(n), T(1) = c$:
    - a is the # of recursive calls made in the function
    - b is the # by which the input size n is divided in each recursive call
    - $f(n)$ is the runtime of the "work done outside of the recursive calls"
    - c is the constant-time work done in each recursive call in the base case
  - Each node of the recursion tree represents the cost of the work done other than making recursive calls
  - Each row represents the total cost of work done in all recursive calls at that recursion "level"
  - The height of the tree depends on the factor that the input size is divided by; i.e. $\log_b n$
  - E.g.: Picture this as a tree where each node (except for leaves) has a children;

Level 0:	$f(n)$			total $= f(n)$
Level 1:	$f(n/b)$	$\dots$	$f(n/b)$	total $= af(n/b)$
Level 2:	$f(n/b^2)$	$\dots$	$f(n/b^2)$	total $= a^2 f(n/b^2)$
$\dots$				
Level k:	$f(n/b^k)$	$\dots$	$f(n/b^k)$	total $= a^k f(n/b^k)$
$\dots$				
Level $\log_b n$:	c	$\dots$	c	total $= ca^{\log_b n} = cn^{\log_b a}$

    - Total runtime of recursion tree is (summing every row total):

$$T(n) \in \Theta\left(\sum_{i=0}^{\log_b n - 1} a^i f(n/b^i)\right) + \Theta(n^{\log_b a})$$