Claim and Proof

Claim: Consider a list of length n containing k true values (T) and $n-k$ false values (F). The list is uniformly shuffled so that each of the $\binom{n}{k}$ ways of placing the k T's among the n positions is equally likely. After shuffling, an algorithm sequentially scans the list from left to right and stops when it encounters the first T. We claim that each individual T is equally likely to be the first encountered T, and thus the probability that *any particular* T is chosen is $\frac{1}{k}$.

Proof

Setup and Notation: Let the set of T's be $\{T_1, T_2, \ldots, T_k\}$ and the set of F's be $\{F_1, F_2, \ldots, F_{n-k}\}$. The total number of ways to arrange these n *distinct* elements is $n!$. However, we are only interested in the relative positions of T's and F's, not their individual identities, so each arrangement of the k T's among the n positions is equally likely. There are $\binom{n}{k}$ such arrangements.

Symmetry in Positioning: Before we know the order, each T is indistinguishable from the others. The uniform shuffle assigns an equal probability

$$\frac{1}{\binom{n}{k}}$$

to each way of distributing the k T's among the n positions. Because of this uniform distribution, each T is equally likely to appear in any specific position of the list. There is no inherent bias toward any particular T.

Event of Interest – "Earliest T": After the shuffle, define the event

$$E_i = \{\text{“}T_i \text{ is the earliest T”}\}.$$

We want to show that

$$P(E_1) = P(E_2) = \cdots = P(E_k).$$

Since one and only one T can be the earliest in any given permutation (ties are impossible in a strict linear order), we have

$$P(E_1) + P(E_2) + \cdots + P(E_k) = 1.$$

Symmetry Argument for Equal Probabilities: Consider any particular T, say T_1. Because all T's are identical before the shuffle and the shuffle is uniform, there is no statistical difference between T_1 and T_2 (or any other T_i). Every distribution of T's where T_1 is earliest has a "mirror" scenario obtained by renaming T's.

More formally, consider a bijection f that permutes the labels of the T's. Given any arrangement where T_i is earliest, applying f yields an arrangement with probability identical under the uniform shuffle but where $f(T_i)$ is earliest. Since all permutations of labels are equally likely, this establishes that

$$P(E_1) = P(E_2) = \cdots = P(E_k).$$

This follows from the principle of symmetry: no T has a privileged position before the shuffle, so no T should have a different probability from any other T of ending up earliest.

Summation to Unity: Since exactly one T is earliest in any given arrangement, and all E_i events are mutually exclusive and collectively exhaustive,

$$P(E_1) + P(E_2) + \cdots + P(E_k) = 1.$$

Given that all these probabilities are equal,

$$k \cdot P(E_1) = 1 \quad \Longrightarrow \quad P(E_1) = \frac{1}{k}.$$

By the symmetry argument, this holds for every i:

$$P(E_i) = \frac{1}{k} \quad \text{for all } i = 1, 2, \ldots, k.$$

Conclusion: Each T is equally likely to be the earliest T encountered by the sequential scan, and thus the probability that any particular T is chosen by the algorithm is $\frac{1}{k}$. This proof relies fundamentally on the symmetry and uniformity of the random permutation, which ensures that no T is favored over another prior to the scanning process.