The Measurement of Anisotropic Thermal Conductivity in Snow With Needle Probes

A Thesis Defense

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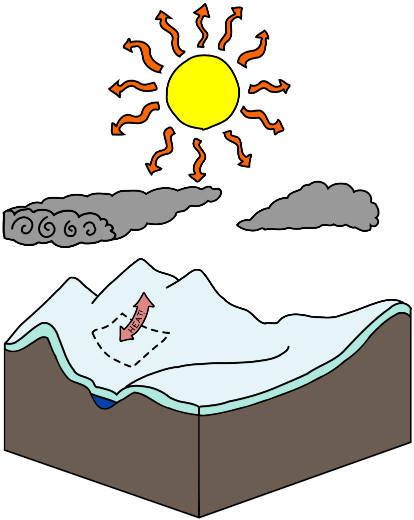
April 4th, 2011

Why Do We Study Snow's Thermal Conductivity?

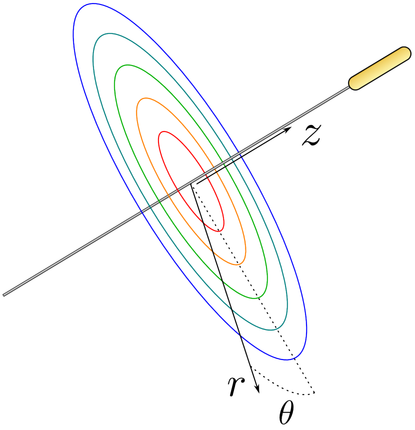


- Climatologists want to model Arctic climates.
- Heat transfer occurs between the earth and the atmosphere.
- Snow gets in the way. It's an *insulating blanket*.

How Do We Measure Thermal Conductivity?

- One way is a guarded hot plate.
- Steady state 1-D conduction: $k = \frac{\dot{q} l}{A \Delta T}$
- Guarded hot plates are good for measuring foam boards.
- Guarded hot plates aren't exactly ideal for snow.

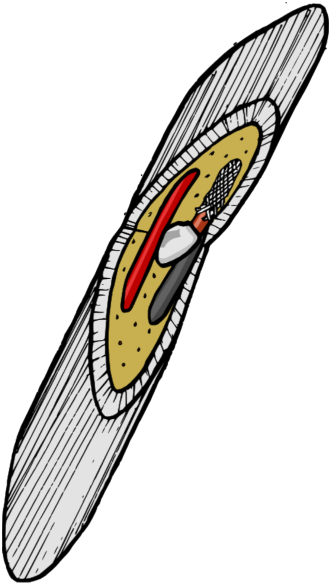
There HAS to be a better way.



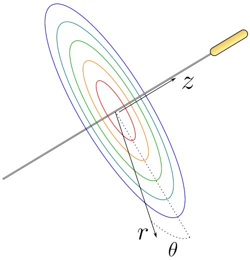
- It's called a needle probe.
- It looks something like this.
- It does *not* use steady-state conduction.
- It depends on a *radial* geometry.
- Okay, I'll admit, "better" is relative here.

What Is A Needle Probe, Exactly?

- Long and thin enough to approximate a line.
- Has a thermocouple in the center.
- Has heat trace running down most of its length.
- This is a drawing of a cross-section of such a needle (The one actually used has a slightly different configuration).



How Do You Measure *Isotropic* Thermal Conductivity With It?



1. Write down The Heat Equation ($-k\nabla^2 T = \rho C \frac{\partial T}{\partial t}$ in 3D Cartesian coordinates—Note the constant k simplification).
2. Pose your problem in cylindrical coordinates, with constant heat flux at $z = 0$
3. Find a book where somebody has solved it for you already (Carslaw & Jaeger) and write down the solution:

$$T(r,t) = -\frac{q}{4\pi k}\mathrm{Ei}\left(-\frac{r^2}{4kt}\right)$$

Ei()? What's *that*?

$$\mathrm{Ei}(x) = -\int_{-x}^{\infty} \frac{e^{-t}}{t} dt$$

- ...in *this* case. Some people define it differently.
- You can't do that integral analytically, unfortunately.
- *However*, there *are* long-time, small-radius approximations. Hooray!

Useful Approximation Action

Over time, the solution approaches this:

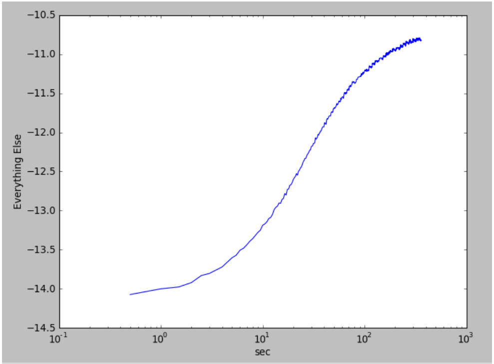
$$T(r,t) = \frac{q}{4\pi k} \ln\left(\frac{4kt}{r^2}\right) - \frac{\gamma q}{4\pi k}$$

But, more usefully:

$$\frac{\partial T}{\partial \ln(t)} = \frac{q}{4\pi k}$$

Finding Conductivities: A Real-Life Example

This is a temperature/time curve for an actual measurement of snow behind my house:

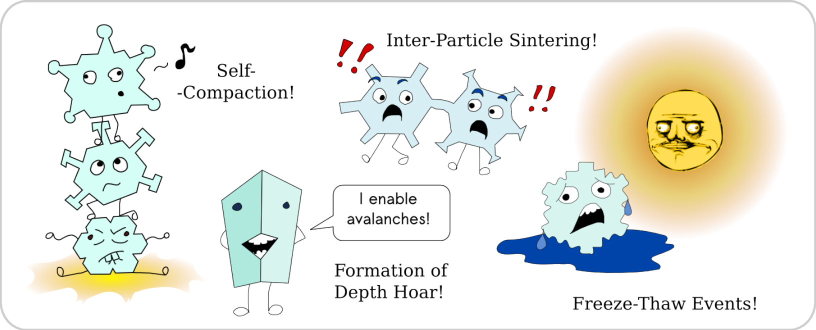


- Early "transient" behavior (The whole situation's transient conduction)
- Mid-range, steady-slope behavior (This is what we're interested in)
- Late-game in-snow convection (Can be mitigated by using less heat)

There's a Cooling Curve, As Well

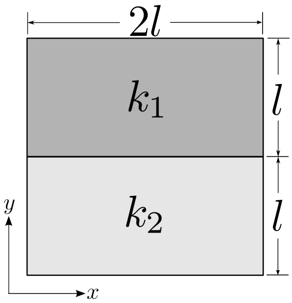
- That was what's called the "heating curve."
- The "cooling curve" is when you watch the needle cool back down.
- By an analogous derivation, we can find a similar long-time solution for cooling curve slope over $\ln(t_{\text{cooling}})$
- Used in measurements, but not modeled. Sorry, guys.

Metamorphic Powder Rangers: How Snow Becomes Anisotropic



These processes cause snow to form regions of varying thermal conductivity—for example, alternating layers of high and low conductivity.

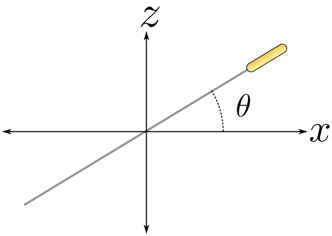
Example Time!



Keep this example in mind for later!

- Vertically: $\frac{1}{2}(k_1 + k_2)$
- Horizontally: $2\left(\frac{1}{k_1} + \frac{1}{k_2}\right)^{-1}$

The Basic Model



- Horizontal and Vertical conductivities only.
- Needle rotated from the horizontal.
- Goal: Find $k_{\mathrm{meas}}(\theta)$ for a given k_{xy} and k_z.

Outline!

- Analytical Approaches
- Numerical Modeling in COMSOL
- Measurements of Snow and Engineered Materials
- Results
- Loose Ends

Analytical Approaches

- We already have a working isotropic theory
- But for the anisotropic case, the conductivity is a 3×3, positive definite matrix!
- Can we adapt the isotropic theory to the anisotropic cases? (Yes.)

$$\nabla \cdot \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{12} & k_{22} & k_{23} \\ k_{13} & k_{23} & k_{33} \end{bmatrix} \nabla T = -\rho C \frac{\partial T}{\partial t}$$

First: `dimension--;`

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In [7]: eig(
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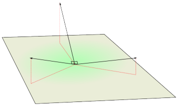

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Out[7]:
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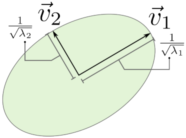


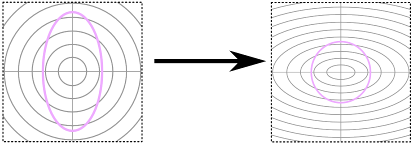
- Remember, there's no temperature gradient in the z direction!
- First, drop the z coordinates, then find the positive eigenvalues.

Second: Stretch!

We now have:

$$-\nabla \cdot \left(\begin{bmatrix} k_x & 0 \\ 0 & k_y \end{bmatrix} \nabla T \right) = \rho C \frac{\partial T}{\partial t}$$

- What if we use a different coordinate system (x' and y') to make that matrix a multiple of $[I]$?



Math Alert! Math Alert!

$$x' = a_x x$$
$$y' = a_y y$$
$$a_x = 1 \text{ (WLOG)}$$

$$\frac{dx'}{dx} = 1$$
$$\frac{dy'}{dy} = a_y$$

It's Not Over Yet.

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial x'}\frac{dx'}{dx} = \frac{\partial f}{\partial x'}$$
$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial y'}\frac{dy'}{dy} = a_y\frac{\partial f}{\partial y'}$$

$$\nabla T = \frac{\partial T}{\partial x'}\hat{e}_{x'} + a_y\frac{\partial T}{\partial y'}\hat{e}_{y'}$$
$$[K]\nabla T = k_x\frac{\partial T}{\partial x'}\hat{e}_{x'} + k_y a_y\frac{\partial T}{\partial y'}\hat{e}_{y'}$$
$$\nabla \cdot ([K]\nabla T) = k_x\frac{\partial^2 T}{\partial x'^2} + k_y a_y^2\frac{\partial^2 T}{\partial y'^2}$$

Almost Done!

Suppose the right hand side is equal to the equivalent isotropic expression:

$$k\left(\frac{\partial^2 T}{\partial x'^2} + \frac{\partial^2 T}{\partial y'^2}\right) = k_x \frac{\partial^2 T}{\partial x'^2} + k_y a_y^2 \frac{\partial^2 T}{\partial y'^2}$$

THIS MEANS:

$$k = k_x$$

$$a_y = \sqrt{\frac{k_x}{k_y}}$$

But Then, A Shocking Twist.

$$T(r', t) = \frac{q}{4\pi k_x} \ln\left(\frac{4k_x t}{r'^2}\right) - \frac{\gamma q}{4\pi k_x}$$

- This equation is in terms of r', not r!
- If you try to take your derivative here, it *will not work like you want it to*.
- We need to do something with the r' this time.

What Is r *Anyway?*

- Arguably, our Actual Measurement is the average temperature at some finite r from the center of the needle.
- There are other approaches one may take, such as assuming a non-zero needle thickness in the problem formulation.
- It can be shown (as it is in the thesis) that:

$$||r'||^2 = r_0^2 \left(\cos^2(\theta) + \frac{k_x}{k_y} \sin^2(\theta) \right)$$

- Note, r' is a function of θ.

Elliptical Integral Time

- Taking the averaging approach, our problem can now be described as so:

$$T_{\mathrm{avg}}(t) = \frac{4\pi k_x}{q} \frac{\mathcal{E}(\ln(t), \frac{k_y}{k_x})}{\mathcal{E}(1, \frac{k_y}{k_x})}$$

- What's this $\mathcal{E}$ stuff?

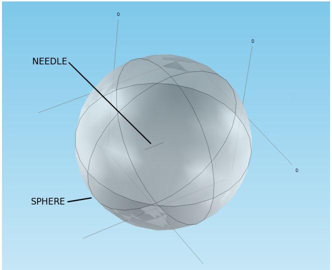
$$\mathcal{E}(f(\phi, \alpha), \alpha) = \int_0^{2\pi} f \sqrt{\cos^2(\phi) + \alpha \sin^2(\phi)} d\phi$$

- Curve fit *this* equation to the $T = A\ln(t) + C$ form, and you now have a k_{meas}!

Implementation Notes

- I used python and numpy/scipy.
- Integrals are evaluated numerically using a quadrature method.
- Source available upon request.

Numerical Modeling!



- 3D Model in COMSOL 3.5a
- Steel needle embedded in sphere **cough**snow**cough**
- "Wait! This medium's not infinite!"
  - It's *pretty big* compared to the needle
  - Zero heat flux boundary conditions on the sphere
  - Monitored temperatures on the boundary showed $O(10^{-14})$ degrees in change, so it's "big enough."

Constant Parameters

radius of needle	0.25 mm
length of needle	10 cm
radius of snow	40 cm
density of needle	8000 $\mathrm{kg/m^3}$
C_P of needle	460 $\mathrm{J/kg \cdot K}$
q of needle	0.5 W/m
k of needle	160 W/m·K
density of snow	200 $\mathrm{kg/m^3}$
C_P of snow	2050 $\mathrm{J/kg \cdot K}$

Rotating The Domain (Easier Than Rotating The Needle)

$$K = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix} \begin{bmatrix} k_{xy} & 0 & 0 \\ 0 & k_{xy} & 0 \\ 0 & 0 & k_z \end{bmatrix} \begin{bmatrix} \cos(\theta) & 0 & -\sin(\theta) \\ 0 & 1 & 0 \\ \sin(\theta) & 0 & \cos(\theta) \end{bmatrix}$$

- Instead of rotating the needle, we rotate the planes of anisotropy.
- Both approaches work, but COMSOL handles arbitrary K better than changes in domain geometry.
- That means it crashes less.

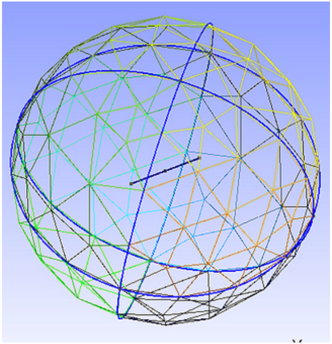
“Parameter Studies,” COMSOL and MATLAB

- COMSOL 3.5 doesn’t have built-in facilities for complex parameter studies (COMSOL 4.1 is better!)
- However, there *are* MATLAB bindings for scriptage
- Good to go!

Program Structure

- mesher.m does the meshing
- solver.m does the solving
- fitter.m does the curve-fitting
- worker.m does the bossing around
- Master m-file and a bash command tell the worker what to do

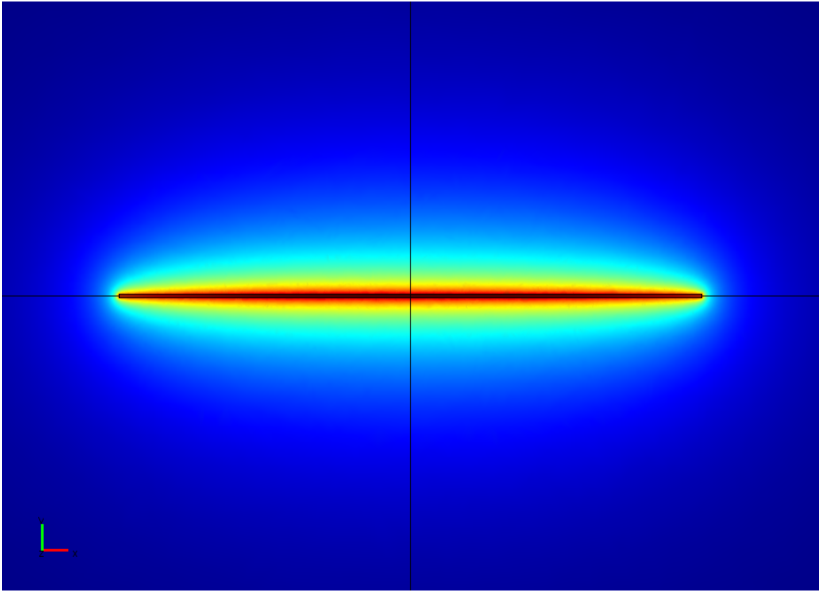
mesher.m



- Mostly auto-generated code, wrapped in a function call
- Assigns geometry parameters (lengths, widths, etc.)
- Generates a mesh to do the solving on
- Returns “fem” structure

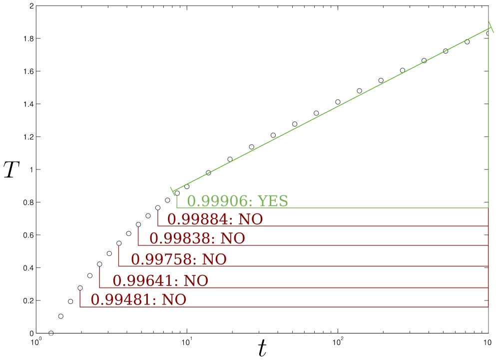
solver.m

- Also mostly auto-generated and "function-ized"
- Takes fem object from mesher as an argument
- Assigns material properties (conductivity, heat capacity, etc.)
- Solves the problem
- Returns (T, t) data for center of probe and averaged final temperature on cardinal surface points
- A variation: Saves fem structure to .mph file for exploration in COMSOL GUI



fitter.m

- Find straight portion by finding maximum number of consecutive “samples” from the tail-end that are sufficiently straight
- Used correlation coefficient as a measure of “straightness”
- This only works in idealized simulations, not in real life. Simulations don’t mess up.



worker.m

- Stores solved data for every angle on-disk
- Stores list of unsolved angles on-disk
- Uses mapping functions over lists of k_{xy} and k_z meshgrids to apply solver and fitter to varying situations
- Combined with “while :; do” loop in bash, restarts after crashes
- Only recovers by-angle, can not recover half-solved k-value combination sets.

Resulting Data Structure

- 2-D Cell Array (more like arrays in other languages than MATLAB's matrices/vectors) with indexes corresponding to k_{xy} and k_z MATLAB vector indices
- Each "slot" in this cell array is *another* cell array.

1 :	k_{meas}
2 :	$[\text{time}, \text{temperature}]^T$
3 :	Avg. Surf. Temp.

Convergence Study

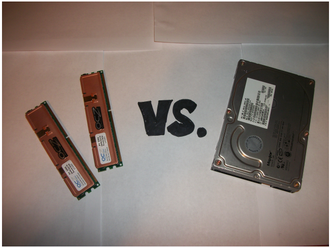
- Goal: Show that the solution as a function of mesh size converges on a particular solution.
- Why: Makes sure solution "has meaning" & that mesh size is reasonable
- A Problem: Increasing mesh size makes the problem *harder*.

Convergence Study Runtimes

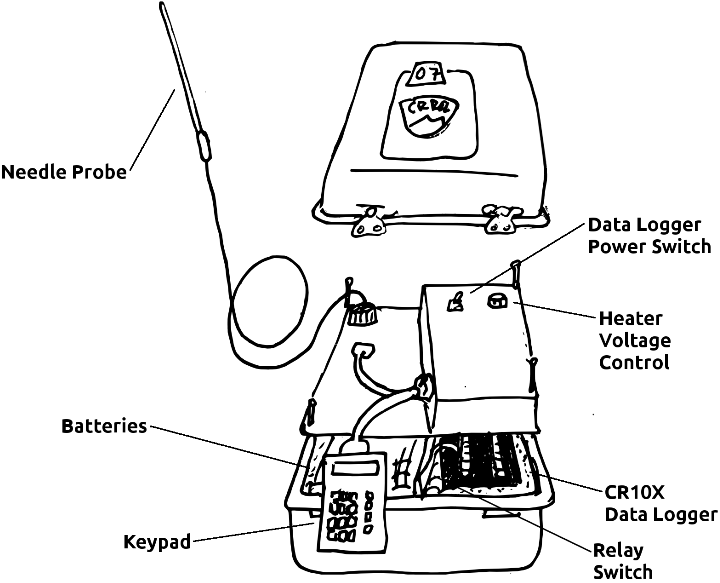
	Number of Elements	Time to Complete
"Classic"	35892 elements	$\approx$ 10 minutes
"Intermediate"	159641 elements	$\approx$ 2 hours
"Nightmare"	528945 elements	??? ($>$ 3 weeks)

Q: What the heck happened in "Nightmare" mode?

A:



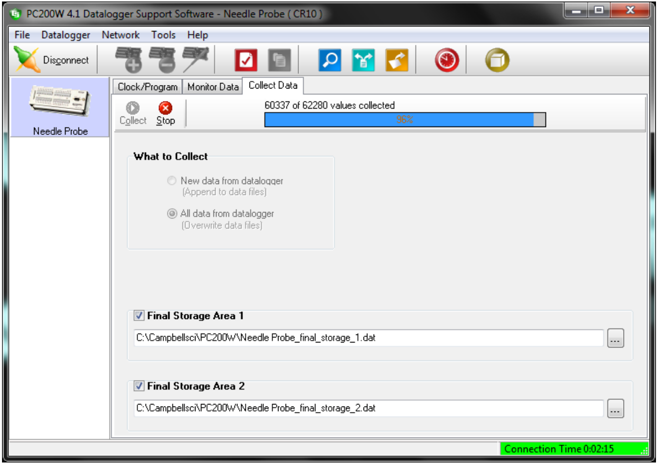
Needle Probe Fundamentals



How To Use:

1. Turn on the device.
2. Insert the needle into medium being measured.
3. Turn on the first register by pressing "* 6 A D 1". This makes the apparatus measure temperature.
4. Turn on the second register by pressing "* 6 A D 2". This turns on the heating element, effectively starting the test.
5. Wait 20 minutes for test to complete (5 min. heating, 10 min. cooling).

PC200W



You use this software to get data off the data logger and eventually convert it to .csv format

CSV Headers

1. An instrument ID (constant in this case).
2. Ordinal day, out of 366. For example, March 17th is day 76.
3. hh:mm portion of timestamp. For example, 6:30pm is represented as 1830.
4. Seconds portion of timestamp.
5. Needle temperature, in Celcius.
6. Reference temperature, in Celcius.
7. Voltage across needle probe, in millivolts.
8. Experiment timer, in seconds.

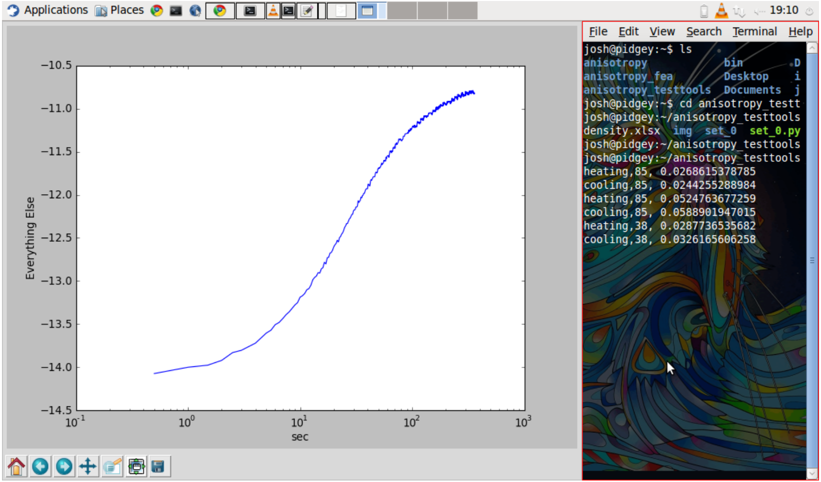
Metadata

- Each measurement has associated metadata (ie, angle) that is not kept track of by the needle probe.
- One must be careful to keep the right metadata associated with the right time series!
- I did this with folder structure, comments in calculation scripts, and associated .csv files.

How do you measure angle, anyway?

- A protractor and a plumb bob works *okay*.
- Simple, but with non-negligible error, especially in the case of snow.
- Unimplemented alternative: 3-axis accelerometer (bonus points: compass bearing).

PLEASE hand-inspect real-world measurements!



Measuring Snow

"It's non-trivial."

—Dr. Jerome Johnson

1. Cold wires can be difficult to work with.
2. Cold electronics can be difficult to work with.
3. Snow is fragile.
4. Snow does not behave as nicely as COMSOL models.

This Is A Picture of Snowpack:

Also-Measured

- Height above soil (tape measure)
- Density of snow in area being measured (cardboard cylinder & scale)
- This could have been done better.
- Density interesting within a larger context (density/conductivity relations)

Control Volume	736.76 mL
Mass	161.14 g
Density	0.219 g/mL
	219 kg/m^3

Engineered Materials

This is the apparatus:

It's based on a "standard" involving glycerine.

Raw Materials

After a long search, I finally came across two reasonable materials:

- Table salt
- Table sugar

Raw Salt & Sugar Measurements

Material	#	Heating	Cooling	Average	Standard Deviation
Pure salt	1	0.222	0.220	0.225	0.015
	2	0.218	0.256		
	3	0.216	0.219		
Pure sugar	1	0.108	0.113	0.106	0.008
	2	0.098	0.109		
	3	0.094	0.113		

Material k-ratio to Anisotropic k-ratio

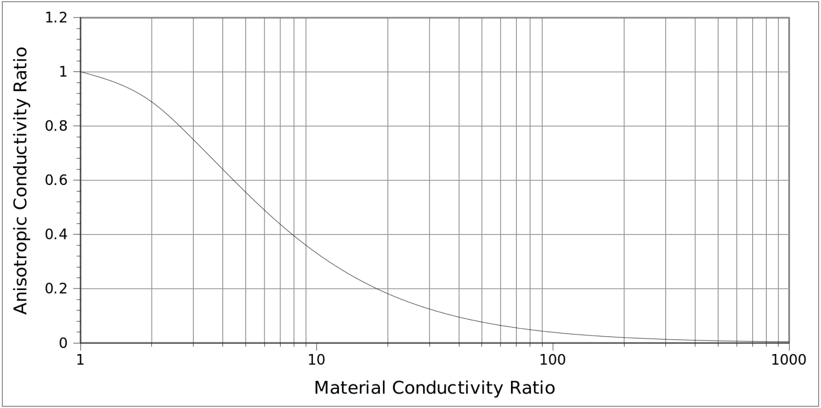
(Remember how I said to remember that example? It's the same geometry.)

$$k_{xy} = \frac{1}{2}\left(k_{\text{salt}} + k_{\text{sugar}}\right) = \boxed{0.166}$$

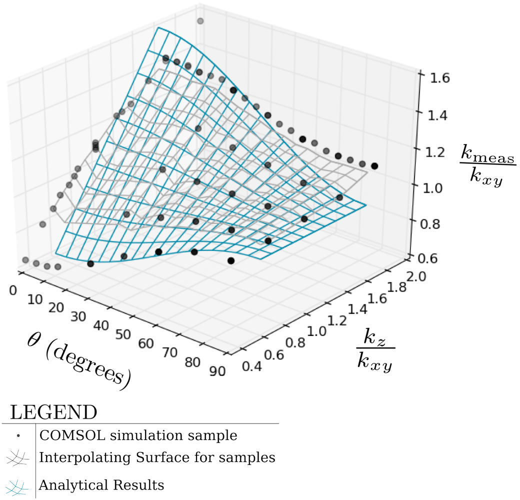
$$k_z = 2\left(\frac{1}{k_{\text{salt}}} + \frac{1}{k_{\text{sugar}}}\right)^{-1} = \boxed{0.144}$$

$$\frac{k_z}{k_{xy}} = \boxed{0.870}$$

What? Only 0.87?!



Model Results



Comments On The Chart:

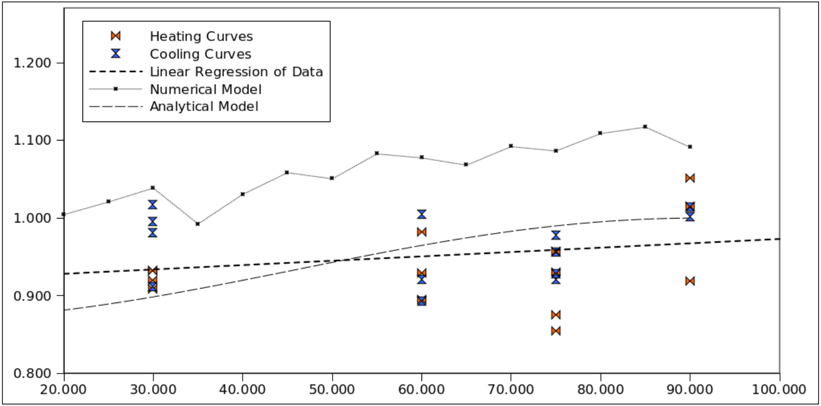
- General agreement. Encouraging!
- Potential Explanation 1: Relative lack of mesh refinement
- Potential Explanation 2: Edge Effects

Benchtop Results:

Angle (degrees)	#	Heating	Cooling
90	1	0.223	0.243
	2	0.247	0.246
	3	0.256	~~0.318~~
75	1	0.213	0.224
	2	0.233	0.232
	3	0.208	0.226
	4	0.226	0.238
60	2	0.239	0.244
	3	0.226	0.224
	4	0.218	0.217
30	1	0.227	0.238
	2	0.226	0.247
	3	0.221	0.242
	4	0.223	0.221

(The strikeout is a statistical outlier.)

Compared to the Models:



What Can We Conclude?

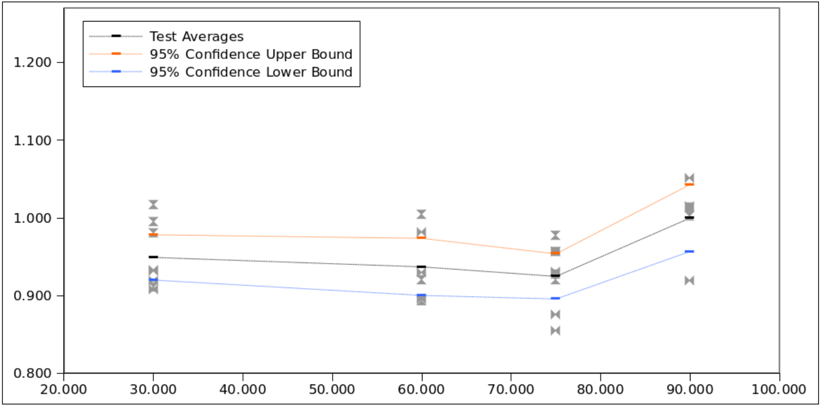
- The 30 degree limit is painful.
- Needs *way* more data.
- There *is* a slope, but is it statistically significant?
- (Note, that outlier makes the slope more convenient for us.)
- Keep in mind, our layers are thick enough that the model *might just not be that good*.

Let's Do Some Basic Stats:

Ignoring the outlier at 90 degrees:

	$k_{\mathrm{meas}}/\bar{k_{xy}}$		
Angle	Mean	Standard Deviation	95% Confidence
90	1.000	0.0491	0.0431
75	0.923	0.0419	0.0291
60	0.937	0.0459	0.0367
30	0.949	0.0420	0.0291

Statistical Conclusions



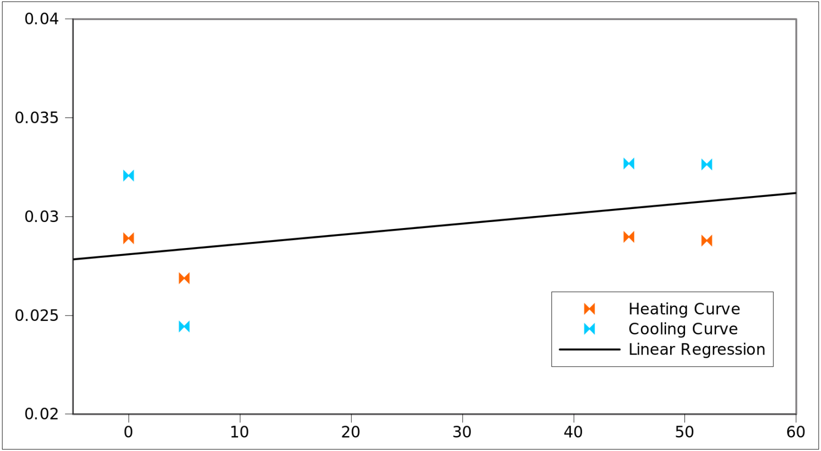
- Doesn't look good!
- Consulted statisticians used other appropriate tests to reach similar conclusions.

What About Snow?

Angle	Heating	Cooling
0	0.0289	0.0321
5	0.0269	0.0244
45	0.0290	0.0327
52	0.0288	0.0326

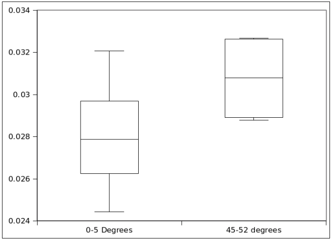
- Even worse with respect to # measurements.
- I took three times this many measurements, but these are the ones that *worked out*.

Snow, Graphically:



Snow, Statistically:

- Still way too few measurements!
- On the other hand, the trend looks very promising. Exciting!



Some Future Work (There's A Lot!)

- Port model to COMSOL 4, examine more configurations
- Better convergence study
- Edge effects in analytical model and/or finite thickness needle model
- Cooling curve analysis
- More manageable benchtop apparatus
- **Comprehensive benchtop measurement study**, backed with stats
- More thorough, careful analysis of snow

Quick Recap:

- Snow is a very complex material, but understanding it is very important for climate studies
- The isotropic model for needle probes has been applied to anisotropic situations
- The anisotropic situation has been modelled in COMSOL 3.5
- A method for testing predictions was devised and used
- Actual snow measurements, while limited, show promise!
- Lots more to do!

A Few Acknowledgements:

- CIFAR for grant money
- ARSC for computer access
- Rorik Peterson for being a great advisor
- Dr. Johnson and Dr. Sturm for their help and advice
- Dr. Bueler for helping with the mathematical rigour
- Everybody that came today (Hi Mom!)