Frequentist versus Bayesian statistics

FW8051 Statistics for Ecologists

Department of Fisheries, Wildlife and Conservation Biology

Learning Objectives

- Understand differences in how probability is defined in Frequentist and Bayesian statistics
- Understand how to estimate parameters and their uncertainty using Bayesian methods
- Compare Bayesian and Frequentist inference, starting with a simple problem that we can solve analytically.

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Goal: make 'good' decisions with high probability (across potential repeated experiments)

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Still want to make ‘good’ decisions with high probability (across potential repeated experiments)…*calibrated Bayes*!

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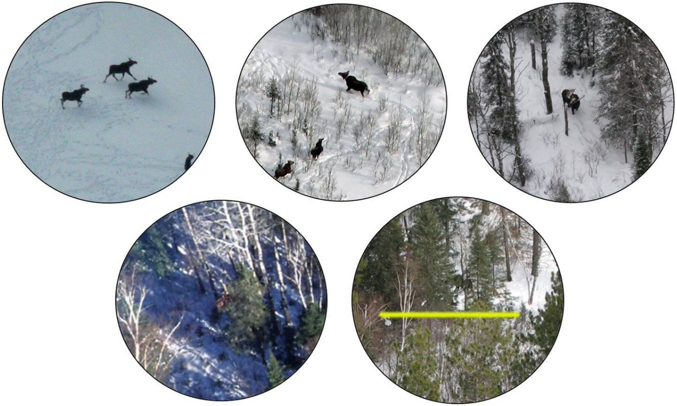
Lets compare inference from the two methods with a simple example

What is the probability, p, that a MNDNR biologist will detect a moose when flying in a helicopter?



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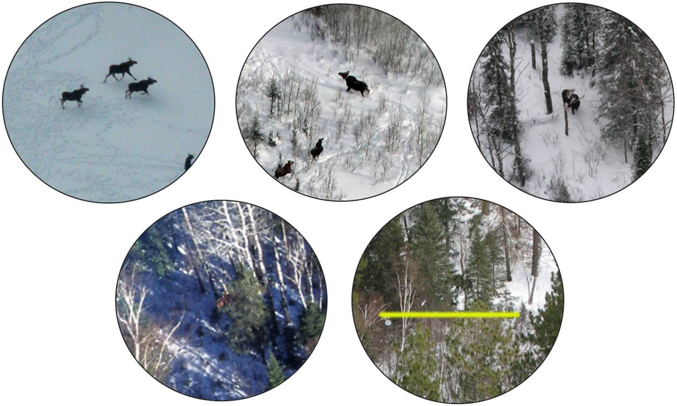

Goal: estimate p and characterize uncertainty wrt p.

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3. Estimate p using Maximum Likelihood

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$var(\hat{p}) = var(y/n) = var(y)/n^2 = p(1-p)/n$

$= I^{-1}(p),$ where $I(p) = E\left(-\frac{\partial^2 logL(p)}{\partial p^2}\right)$

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$$P(\hat{p} + 1.96\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} > p > \hat{p} - 1.96\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}) \approx 0.95$$

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```
# Estimate and SE
(theta.hat<-59/124)
```

```
## [1] 0.4758065
```

```
(se.theta.hat<-sqrt(theta.hat*(1-theta.hat)/124))
```

```
## [1] 0.04484873
```

```
# Confidence Interval
round(rep(theta.hat,2)+ c(-1.96,1.96)*se.theta.hat,2)
```

```
## [1] 0.39 0.56
```

How well do these work?

- Simulate 10,000 binomial random variables using: `x<-rbinom(10,000, size = n, p)` with `n = 15, 30, 100; p=0.1, 0.5, 0.9`
- Estimate 10,000 $\hat{p}$ values (`x/n`) and 10,000 95% CIs
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```
# 10,000 repeated samples of size 124 and theta = theta.hat
ys<-rbinom(10000,size=124,prob=59/124)

# Calculate 10,000 theta^'s, SE(theta^)'s, CI's
theta.hats<-ys/124
se.theta.hats<-sqrt(theta.hats*(1-theta.hats)/124)
up.CIs<-theta.hats+1.96*se.theta.hats
low.CIs<-theta.hats-1.96*se.theta.hats

# Determine coverage
inCI<-I(low.CIs < 59/124 & up.CIs > 59/124)  # true theta is in the in
sum(inCI)/10000
```

```
[1] 0.9414
```

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In reality, we get 1 data set. We ended up with $\hat{p} = 0.48$, with 95% CI = (0.39, 0.56)

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The procedure we used should result in an interval that contains the true parameter 95% of the time.

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3. Use Bayes rule to determine the posterior distribution of p given the data, $p(p|y)$:

$$p(p|y) = \frac{L(y|p)\pi(p)}{p(y)} = \frac{L(y|p)\pi(p)}{\int L(y|p)\pi(p)dp}$$

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The posterior distribution captures our belief about the parameters after having collected data!

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- This is a continuous version of the *total law of probability* formula we saw previously
- This integral is often difficult to solve, so Markov Chain Monte Carlo (MCMC) is used generate summaries of the posterior distribution, $p(p|y)$

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Keep focus on:

Posterior distribution $\propto$ Likelihood x prior distribution

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[Plot this using `curve(dbeta(x, 1, 1), from=0, to=1)`]

Use $\pi(p)$ and $p(y|p)$ and Bayes Theorem to calculate $p(p|y)$, the posterior distribution.

Bayes Theorem

$$p(p|y) = \frac{p(y|p)\pi(p)}{p(y)} = \frac{p(y|p)\pi(p)}{\int_{-\infty}^{\infty} p(y|p)\pi(p)d(p)}$$

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The posterior distribution gives us the probability distribution of the parameter, given the data and our prior beliefs.

Bayesian Inference

Use `curve` to plot the posterior distribution = Beta(60,66).

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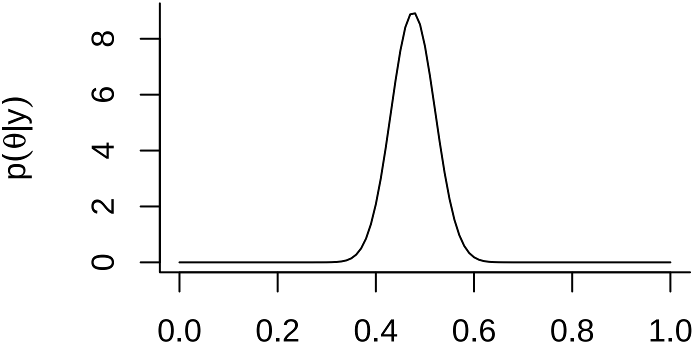
```
par(bty="L", mar=c(2,4.1,1,2.1))
# Plot the Posterior Distribution of theta
plot(curve(dbeta(x, 60, 66), from=0, to=1),
     type="l",xlab=expression(theta), ylab=c(expression(p(group("", theta, "|") *y))))
```



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```
# 95% credible interval
round(qbeta(c(0.025, 0.975), 60,66),2)
```

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[1] 0.39 0.56
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Same endpoints as Frequentist confidence interval, different interpretation!

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Interpretation: p has a 95% chance of being in the interval.

- Represents our *belief* based on *data* and our *prior* assumptions

Frequentist vs. Bayesian

Typical Conclusions

	Frequentist	Bayesian
Estimation	I have 95% confidence that the population mean is between 12.7 and 14.5 mcg/liter.	There is a 95% probability that the population mean is in the interval 136.2 g to 139.6 g.
Hypothesis Testing	If H_0 is true, we would get a result as extreme as the data we saw only 3.2% of the time. Since that is smaller than 5%, we would reject H_0 at the 5% level. These data provide significant evidence for the alternative hypothesis.	The odds in favor of H_0 against H_A are 1 to 3.

{Mary Parker, http://www.austincc.edu/mparker/stat/nov04/}

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- Easy to characterize uncertainty for functions of the parameters
- Intuitive appeal of credibility intervals (vs. confidence intervals)
- Coherent philosophy of statistics
  - All inferences come from the posterior distribution
  - No separate theories for estimation, hypothesis testing, multiple comparisons, etc.

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- Perceived subjectivity
- Computationally demanding when using MCMC

“Ecologists should be aware that Bayesian methods constitute a radically different way of doing science. Bayesian statistics is not just another tool to be added into ecologists’ repertoire of statistical methods. Instead, Bayesians categorically reject various tenets of statistics and the scientific method that are currently widely accepted in ecology and other sciences. The Bayesian approach has split the statistics world into warring factions (ecologists’ “density independence” vs “density dependence” debates of the 1950s pale by comparison), and it is fair to say that the Bayesian approach is growing rapidly in influence” - Brian Dennis (1996, Ecological Applications, p.1095-1103).

Pragmatic Statistician

Many do consider Bayesian methods another tool in the toolbox…

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We will often fit models using both frequentist and Bayesian statistics (often, with similar answers)!

When is Bayesian Inference "Easier" or Prefered?

Dorazio 2016. Population Ecology 58:31-44

- "Hierarchical models of data that link a submodel of sampling processes with a submodel of ecological processes."
- Inference for latent (i.e., unobserved) state variables
- Missing data problems
- Intractable likelihood functions
- Complex models of different sources and types of data (with shared parameters)