Delta Method

FW8051 Statistics for Ecologists

Department of Fisheries, Wildlife and Conservation Biology

Learning Objectives

Understand how we can use the delta method to calculate SEs for functions of parameters

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See also:

Approximating Variance of Demographic Parameters Using the Delta Method: A Reference for Avian Biologists FREE

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In the GLS section, we learned how to calculate $var(\hat{\beta}_0 + X_i\hat{\beta}_1)$ using matrix multiplication

And, more generally: $var(X\beta)$ for design matrix X :

$$\begin{bmatrix} 1 & X_1 \\ 1 & X_2 \\ \vdots & \vdots \\ 1 & X_n \end{bmatrix} \begin{bmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \end{bmatrix} \qquad \Sigma = \begin{bmatrix} \sigma^2_{\hat{\beta}_0} & \sigma^2_{\hat{\beta}_0,\hat{\beta}_1} \\ \sigma^2_{\hat{\beta}_0,\hat{\beta}_1} & \sigma^2_{\hat{\beta}_1} \end{bmatrix}$$

- σ^2_X, σ^2_Y = variance of $\hat{\beta}_0$, $\hat{\beta}_1$
- $\sigma^2_{\hat{\beta}_0,\hat{\beta}_1}$ = covariance of $\hat{\beta}_0$ and $\hat{\beta}_1$

Recall: $\text{var}(X\beta) = X\Sigma X^T$

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Options:

- Bootstrap
- Delta method
- Baysian inference

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Want to calculate a confidence interval for the length at a particular age, Age_i:

$$var(\hat{L}_\infty(1 - \exp(-\hat{k}Age_i))) = f(\hat{L}_\infty, \hat{k})$$

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If we estimate $\theta = (L_\infty, k)$ using Maximum likelihood, and our sample size is large, we know:

$$\hat{\theta} \sim MVN(\theta, I^{-1}(\theta)) \text{ with:}$$

- I(θ) $= \left[\frac{\partial^2 logL(\theta)}{\partial\theta^2}\right]$ is the Hessian matrix

Delta Method

Let:

- $f(L_\infty, k) = L_\infty(1 - \exp(-kAge_i))$
- $f'(L_\infty, k) = (\frac{\partial f}{\partial L_\infty}, \frac{\partial f}{\partial k})$
- Σ be the asympototic variance/covariance matrix of (L_∞, k) given by the inverse of the Hessian matrix

Delta Method (derived using a Taylor's series approximation of f):

$$var(\hat{L}_\infty(1 - \exp(-\hat{k}Age_i))) \approx f'(\hat{L}_\infty, \hat{k})\Sigma f'(\hat{L}_\infty, \hat{k})^T|_{L_\infty=\hat{L}_\infty, k=\hat{k}}$$

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More generally:

$$var(f(\theta)) \approx f'(\theta)\Sigma f'(\theta)^T|_{\theta=\hat{\theta}}$$

Implementation

In R:

- use the `detavar` function in the `emdbook` package to calculate the derivatives and variance (see `FemalesvonB.R`)
- or, calculate the derivatives yourself (or using https://www.symbolab.com/solver/derivative-calculator), then roll your own with `%*%` for matrix multiplication.

$$f(\theta) = L_\infty(1 - \exp(-kAge_i))$$

$$f'(\theta) = (1 - \exp(-kAge_i), L_\infty Age_i \exp(-kAge_i))$$

Derivatives