

# Generalized Least Squares

FW8051 Statistics for Ecologists

Department of Fisheries, Wildlife and Conservation Biology



# Learning Objectives

Learn how to use generalized least squares (GLS) to model data where  $Y_i|X_i$  is normally distributed, but the variance of the residuals is not constant and may depend on one or more predictor variables.

# Generalized Least Squares

Can be used to model data where  $Y_i|X_i$  is normally distributed,  
but we have:

- Non-constant variance (Chapter 5)
- Data that are correlated
  - Multiple measurements on the same sample unit (Chapter 18)
  - Temporal dependence (Chapter 6 of Zuur et al)
  - Spatial dependence (Chapter 7 of Zuur et al)

# Generalized Least Squares

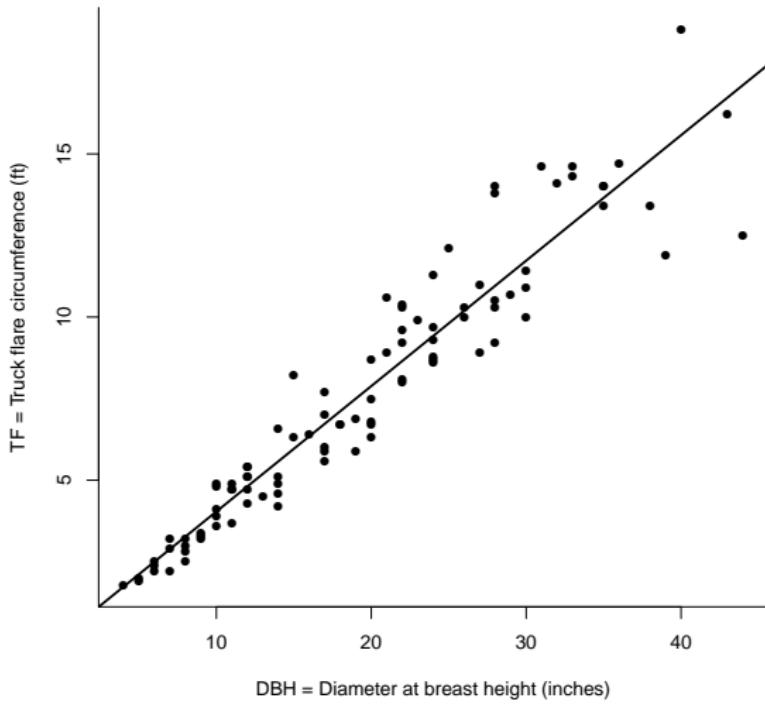
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  - Temporal dependence (Chapter 6 of Zuur et al)
  - Spatial dependence (Chapter 7 of Zuur et al)

For this class, we will focus on non-constant variance and  
multiple measurements on the same sample unit [later in the  
course]

# Trunk Flare Diameter





# Linear Model

$$Y_i = \beta_0 + X_i\beta_1 + \epsilon_i$$

Assume  $\epsilon_i$  are independent, normally distributed, with constant variance.

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$$Y_i \sim N(\mu_i, \sigma^2)$$

$$\mu_i = \beta_0 + X_i\beta_1$$

# Generalized Least Squares: Non-Constant Variance

$$Y_i \sim N(\mu_i, \sigma_i^2)$$

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Model the mean and variance:

- $E[Y_i|X_i] = \beta_0 + X_i\beta_1$
- $Var[Y_i|X_i] = f(X_i; \tau)$ , where  $\tau$  are additional variance parameters.

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`varPower (form=~fitted(.))`

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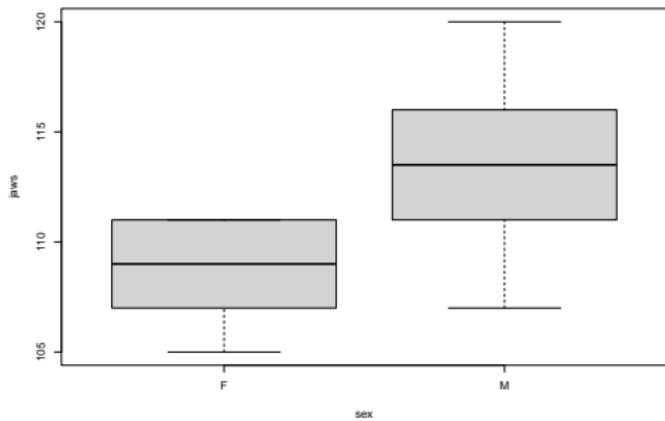
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`varPower(form=~fitted(.))`
- Some combination of the above + other options (see Ch. 4 of Zuur et al.)

# T-test with unequal variances: Jaw data

```
males<-c(120, 107, 110, 116, 114, 111, 113, 117, 114, 112)
females<-c(110, 111, 107, 108, 110, 105, 107, 106, 111, 111)
jawdat <- data.frame(jaws = c(males, females),
                      sex = c(rep("M",10), rep("F", 10)))  
  
boxplot(jaws~sex, data=jawdat)
```



## T-test with unequal variances: Jaw data

$Y_i$  = jaw length for jackal  $i$

$$Y_i \sim N(\mu_i, \sigma_i^2) \quad (1)$$

$$\mu_i = \beta_0 + \beta_1 I(\text{sex}=\text{male})_i \quad (2)$$

$$\sigma_i^2 = \sigma_{\text{sex}}^2 \quad (3)$$

```
gls_ttest <- gls(jaws ~ sex,
                   weights = varIdent(form = ~ 1 | sex),
                   data = jawdat)
```

Estimates of regression parameters are obtained by minimizing:

$$\sum_{i=1}^n \frac{(Y_i - \mu_i)^2}{2\sigma_i^2}$$

```
summary(gls_ttest)
```

```
## Generalized least squares fit by REML
##   Model: jaws ~ sex
##   Data: jawdat
##      AIC      BIC    logLik
## 102.0841 105.6456 -47.04206
##
## Variance function:
##   Structure: Different standard deviations per stratum
##   Formula: ~1 | sex
## Parameter estimates:
##      M      F
## 1.0000000 0.6107279
##
## Coefficients:
##             Value Std.Error t-value p-value
## (Intercept) 108.6 0.7180211 151.24903 0.0000
## sexM         4.8 1.3775993   3.48432  0.0026
##
## Correlation:
##   (Intr)
## sexM -0.521
##
## Standardized residuals:
##      Min      Q1      Med      Q3      Max
## -1.72143457 -0.70466511  0.02689742  0.76657633 1.77522940
##
## Residual standard error: 3.717829
## Degrees of freedom: 20 total; 18 residual
```

## Summary output

Apparent parameterization:

$\sigma_{sex}^2 = \sigma^2 \delta_{sex}^2$ , which is the same as  $\sigma_{sex} = \sigma \delta_{sex}$

with:

- $\hat{\delta}_{males} = 1$
- $\hat{\delta}_{females} = 0.61$
- $\hat{\sigma} = 3.72$  (residual standard error)

## Actual Parameterization

Variance model parameterization actually looks like a linear model on the log scale:

$$\log(\sigma_i) = \gamma_0 + \gamma_1 I(\text{sex} = \text{female})_i$$

Ensures that  $\sigma_i$  is positive when we back-transform using  $\exp()$ .

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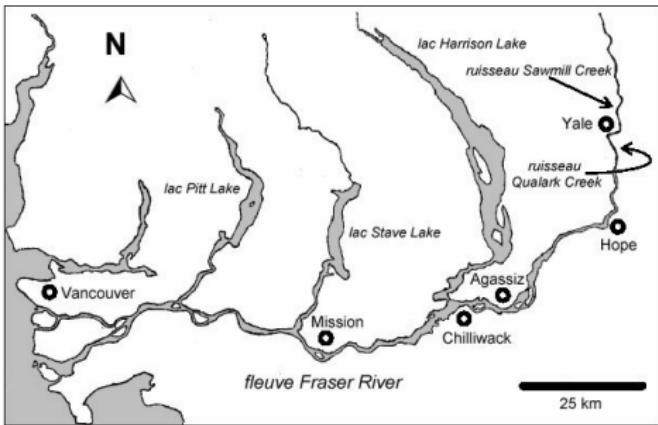
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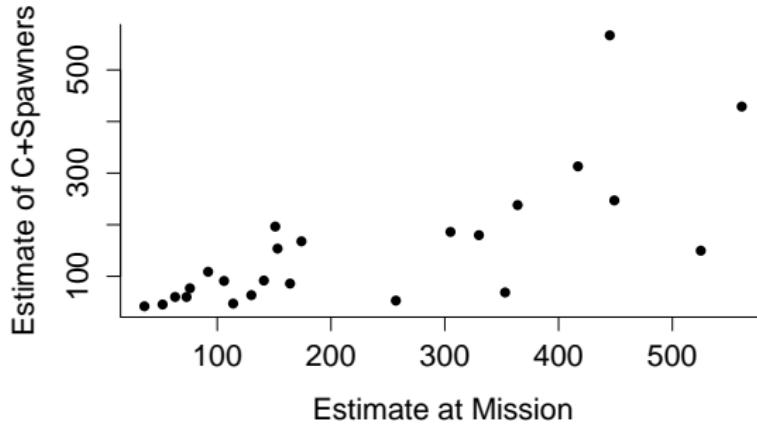
Which is the same as:

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(see book and in-class file for details)

# Fraser River Sockeye





Use historical correlation between the *count at Mission* and  $S_t + C_t$  to manage the fishery

## Variance increasing with $X_i$ or $\mu_i$

1. Fixed variance model:  $\sigma_i^2 = \sigma^2 \text{MisEsc}_i$
2. Power variance model:  $\sigma_i^2 = \sigma^2 |\text{MisEsc}_i|^{2\delta}$
3. Exponential variance model:  $\sigma_i^2 = \sigma^2 e^{2\delta \text{MisEsc}_i}$
4. Constant + power variance model:  $\sigma_i^2 = \sigma^2 (\delta_1 + |\text{MisEsc}_i|^{\delta_2})^2$

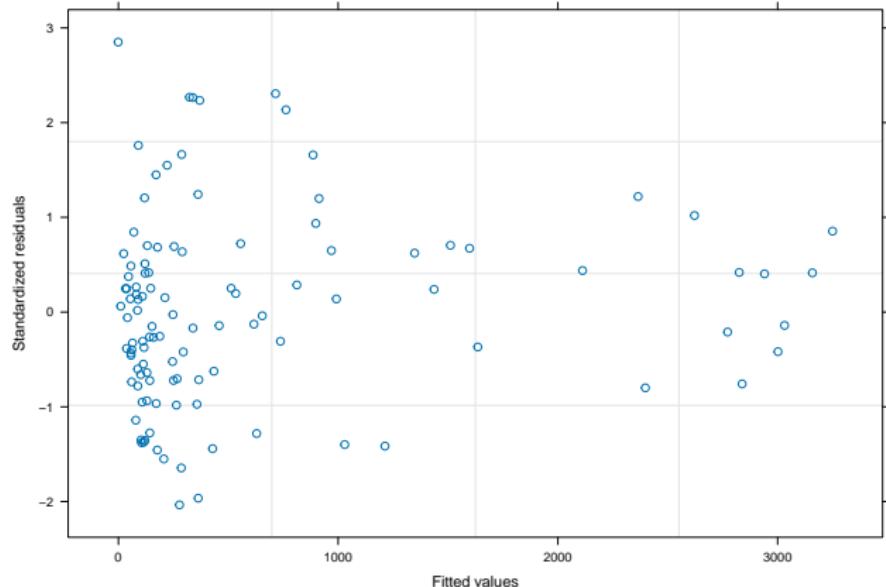
```
varconstp <- gls(SpnEsc ~ MisEsc,  
                      weights = varConstPower(form = ~ MisEsc),  
                      data = sockeye)
```

See the book for syntax associated with the other models.

# Standardized residuals

**Standardized residuals** =  $(Y_i - \hat{Y}_i)/\hat{\sigma}_i$ , should have approximately constant variance:

```
plot(varconstp)
```



## Variance depending on $\mu_i$

$$Y_i \sim N(\mu_i, \sigma_i^2)$$

$$\mu_i = \beta_0 + \beta_1 X_i$$

$$\sigma_i^2 = \mu_i^{2\delta}$$

Fit using `varPower(form = ~ fitted(.))`. See textbook via this link.

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- Choose best set of predictor variables for the mean of  $Y_i$  (using the variance model, chosen above)
- Check diagnostics again and pick best model

... Try to make sense of your results.

# Approximate Confidence and Prediction Intervals

For a confidence interval, we need to consider:

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- Confidence interval = captures uncertainty regarding the average value of  $Y$  (given by the line)
- Prediction interval = captures uncertainty regarding a *particular* value of  $Y$  (need to also consider spread about the line)

# Matrix Multiplication: Expected Value

$$\widehat{E[Y_i|X_i]} = \hat{\beta}_0 + \hat{\beta}_1 X_i$$

$$\widehat{E[Y|X]} = \begin{bmatrix} 1 & X_1 \\ 1 & X_2 \\ \vdots & \vdots \\ 1 & X_n \end{bmatrix} \begin{bmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \end{bmatrix} = \begin{bmatrix} \hat{\beta}_0 + X_1\hat{\beta}_1 \\ \hat{\beta}_0 + X_2\hat{\beta}_1 \\ \vdots \\ \hat{\beta}_0 + X_n\hat{\beta}_1 \end{bmatrix}$$

## Matrix multiplication: Variances

$$\begin{aligned} \text{var}(\widehat{E[Y|X]}) &= \text{Var}(\hat{\beta}_0 + X\hat{\beta}_1) = \\ \text{Var}(\hat{\beta}_0) + X^2\text{Var}(\hat{\beta}_1) + 2X\text{Cov}(\hat{\beta}_0, \hat{\beta}_1) \end{aligned}$$

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Define:

$$\hat{\Sigma} = \begin{bmatrix} \sigma_{\hat{\beta}_0}^2 & \sigma_{\hat{\beta}_0, \hat{\beta}_1}^2 \\ \sigma_{\hat{\beta}_0, \hat{\beta}_1}^2 & \sigma_{\hat{\beta}_1}^2 \end{bmatrix}$$

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$$\text{Let } \mathbf{X} = \begin{bmatrix} 1 & X_1 \\ 1 & X_2 \\ \vdots & \vdots \\ 1 & X_n \end{bmatrix} \text{ and } \mathbf{X}' \text{ be its transpose}$$

$$\widehat{var}(E[Y|X]) = Var(\hat{\beta}_0 + \hat{\beta}_1 X) = \mathbf{X}\hat{\Sigma}\mathbf{X}'$$

# Calculations in R

We can use matrix multiplication to efficiently calculate intervals for multiple observations.

In R, we do this by using `%*%`

$$\text{Let } X = \text{design matrix} = \begin{bmatrix} 1 & X_1 \\ 1 & X_2 \\ \vdots & \vdots \\ 1 & X_n \end{bmatrix}$$

$\widehat{E[Y|X]} = X \%*\% \text{coef}(\text{modelname})$  gives us predicted values for all rows of the  $X$  matrix.

# Calculations in R

$$\widehat{var}(E[\widehat{Y}|X]) = X \%*\% \text{vcov}(\text{modelname}) \%*\% t(X)$$

- `vcov(model)` =  $\hat{\Sigma}$  = estimated variance-covariance matrix of  $\hat{\beta}$  (works for `lm`, `gls`, maybe others)
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We end up with a matrix that looks something like:

$$\begin{bmatrix} var(\hat{Y}_1) & cov(\hat{Y}_1, \hat{Y}_2) & \cdots & cov(\hat{Y}_1, \hat{Y}_n) \\ cov(\hat{Y}_2, \hat{Y}_1) & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & cov(\hat{Y}_{n-1}, \hat{Y}_n) \\ cov(\hat{Y}_n, \hat{Y}_1) & \cdots & cov(\hat{Y}_n, \hat{Y}_{n-1}) & var(\hat{Y}_n) \end{bmatrix}$$

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Pull off the diagonal elements (the variances) - see the textbook for code.

# Prediction Intervals

$$var(\hat{Y}_i|X_i) = var(\hat{\beta}_0 + X\hat{\beta}_1 + \epsilon_i)$$

- $var(\epsilon_i) = var(Y_i|X_i) = \sigma_i^2$  and estimated by  $\hat{\sigma}_i^2$ .
- In many cases,  $\hat{\sigma}_i^2$  is independent of  $\begin{bmatrix} \hat{\beta}_0 & \hat{\beta}_1 \end{bmatrix}$
- This implies  $cov(\hat{\sigma}_i^2, \hat{\beta}_0) = cov(\hat{\sigma}_i^2, \hat{\beta}_1) = 0$

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- This implies  $\text{cov}(\hat{\sigma}_i^2, \hat{\beta}_0) = \text{cov}(\hat{\sigma}_i^2, \hat{\beta}_1) = 0$

So, to construct a prediction interval, we approximate  $\text{var}(\hat{Y}_i|X_i)$  with:

$$\text{var}(\hat{Y}_i|X_i) \approx \widehat{\text{var}}(\hat{\beta}_0 + X_i\hat{\beta}_1) + \hat{\sigma}_i^2 = X_i \hat{\Sigma} X_i' + \hat{\sigma}_i^2.$$

## Additional Notes

Note, these estimates are approximate in that:

- They rely on asymptotic normality (central limit theorem)  
[think difference between  $t$  and  $z$ ]
- They ignore uncertainty in the variance parameters

# Temporal or Spatial Correlation

$$\begin{aligned}y_i &= \beta_0 + \beta_1 x_i + \epsilon_i \\ \epsilon_i &\sim N(0, \Omega)\end{aligned}$$

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If these interest you, I highly recommend taking Brian Aukema's class.