

Delta Method

FW8051 Statistics for Ecologists

Department of Fisheries, Wildlife and Conservation Biology



Learning Objectives

Understand how we can use the **delta method** to calculate SEs for functions of parameters

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Understand how we can use the **delta method** to calculate SEs for functions of parameters

See also:

Approximating Variance of Demographic Parameters Using the Delta Method: A Reference for Avian Biologists

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In the GLS section, we learned how to calculate $\text{var}(\hat{\beta}_0 + X_i\hat{\beta}_1)$ using matrix multiplication

And, more generally: $\text{var}(X\beta)$ for design matrix X :

$$\begin{bmatrix} 1 & X_1 \\ 1 & X_2 \\ \vdots & \vdots \\ 1 & X_n \end{bmatrix} \begin{bmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \end{bmatrix} \quad \Sigma = \begin{bmatrix} \sigma_{\hat{\beta}_0}^2 & \sigma_{\hat{\beta}_0, \hat{\beta}_1}^2 \\ \sigma_{\hat{\beta}_0, \hat{\beta}_1}^2 & \sigma_{\hat{\beta}_1}^2 \end{bmatrix}$$

- $\sigma_X^2, \sigma_Y^2 = \text{variance of } \hat{\beta}_0, \hat{\beta}_1$
- $\sigma_{\hat{\beta}_0, \hat{\beta}_1}^2 = \text{covariance of } \hat{\beta}_0 \text{ and } \hat{\beta}_1$

Recall: $\text{var}(X\beta) = X\Sigma X^T$

What if we are interested in non-linear functions of parameters?

$$Length_i = L_{\infty}(1 - \exp(-kAge_i))$$

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$$Length_i = L_{\infty}(1 - \exp(-kAge_i))$$

Options:

- Bootstrap
- Delta method
- Bayesian inference

$$Length_i = L_{\infty}(1 - \exp(-kAge_i))$$

Want to calculate a confidence interval for the length at a particular age, Age_i :

$$var(\hat{L}_{\infty}(1 - \exp(-\hat{k}Age_i))) = f(\hat{L}_{\infty}, \hat{k})$$

$$Length_i = L_\infty(1 - \exp(-kAge_i))$$

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If we estimate $\theta = (L_\infty, k)$ using Maximum likelihood, and our sample size is large, we know:

$$\hat{\theta} \sim MVN(\theta, I^{-1}(\theta)) \text{ with:}$$

- $I(\theta) = \left[\frac{\partial^2 \log L(\theta)}{\partial \theta^2} \right]$ is the Hessian matrix

Delta Method

Let:

- $f(L_\infty, k) = L_\infty(1 - \exp(-kAge_i))$
- $f'(L_\infty, k) = (\frac{\partial f}{\partial L_\infty}, \frac{\partial f}{\partial k})$
- Σ be the asymptotic variance/covariance matrix of (L_∞, k) given by the inverse of the Hessian matrix

Delta Method (derived using a Taylor's series approximation of f):

$$var(\hat{L}_\infty(1 - \exp(-\hat{k}Age_i))) \approx f'(\hat{L}_\infty, \hat{k})\Sigma f'(\hat{L}_\infty, \hat{k})^T|_{L_\infty=\hat{L}_\infty, k=\hat{k}}$$

Delta Method

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Delta Method (derived using a Taylor's series approximation of f):

$$\text{var}(\hat{L}_\infty(1 - \exp(-\hat{k} \text{Age}_i))) \approx f'(\hat{L}_\infty, \hat{k}) \Sigma f'(\hat{L}_\infty, \hat{k})^T |_{L_\infty = \hat{L}_\infty, k = \hat{k}}$$

More generally:

$$\text{var}(f(\theta)) \approx f'(\theta) \Sigma f'(\theta)^T |_{\theta = \hat{\theta}}$$

Implementation

In R:

- use the `detavar` function in the `emdbook` package to calculate the derivatives and variance (see `FemalesvonB.R`)
- or, calculate the derivatives yourself (or using <https://www.symbolab.com/solver/derivative-calculator>), then roll your own with `%*%` for matrix multiplication.

$$f(\theta) = L_{\infty}(1 - \exp(-kAge_i))$$

$$f'(\theta) = (1 - \exp(-kAge_i), L_{\infty}Age_i \exp(-kAge_i))$$

Derivatives

Derivative Calculator

Differentiate functions step-by-step

Derivatives

- First Derivative
- Second Derivative
- Third Derivative
- Higher Order Derivatives
- Derivative at a point
- Partial Derivative
- Implicit Derivative
- Second Implicit Derivative (new)
- Derivative using Definition (new)

Derivative Applications

Limits

Integrals

Integral Applications (new)

Series

ODE

Laplace Transform

Taylor/Maclaurin Series

Fourier Series (new)

full pad »

x^2	x^{\square}	\log_{\square}	$\sqrt{\square}$	$\sqrt[n]{\square}$	\leq	\geq	\ln	\cdot	\div	x°	π
$(\square)^{\circ}$	$\frac{d}{dx}$	$\frac{\partial}{\partial x}$	\int	\int_{\square}^{∞}	\lim	Σ	∞	θ	$(f \circ g)$	H_2O	$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Most Used Actions

simplify

solve for

inverse

tangent

line

See All ▾

$$\frac{d}{dL} \left(L \cdot (1 - e^{-k \cdot A}) \right)$$

Go

Graph » Examples »



Solution

Keep Practicing >

Show Steps ▾

$$\frac{d}{dL} \left(L(1 - e^{-kA}) \right) = 1 - e^{-kA}$$

Steps

$$\frac{d}{dL} \left(L(1 - e^{-kA}) \right)$$

Take the constant out: $(a \cdot f)' = a \cdot f'$

$$= (1 - e^{-kA}) \frac{d}{dL} (L)$$

Apply the common derivative: $\frac{d}{dL} (L) = 1$

$$= (1 - e^{-kA}) \cdot 1$$

Simplify

$$= 1 - e^{-kA}$$

[click here to practice derivatives »](#)