Models for Data with Zero Inflation

FW8051 Statistics for Ecologists

Department of Fisheries, Wildlife and Conservation Biology

Learning Objectives

- Be able to fit models to response data with lots of zeros (hurdle and zero-inflated models)

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- Be able to describe these models and their assumptions using equations and text and match parameters in these equations to estimates in computer output.

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Also relevant to:

- Binary data (occupancy models, Kery Ch 20)
- Continuous data (e.g., Friederichs et al. 2011. Oikos 120:756-765)

Abundance Data and Zero Inflation

{From of Matt Russell, UMN}

Top 4 reasons why you might get a 0 when counting critters?

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Top 4 reasons why you might get a 0 when counting critters?

- Sites are not suitable for the species
- Density effects: a site is suitable, but unoccupied
- Design errors: sampling for too short of a time period, or during the wrong times
- Observer error: some species are difficult to identify/detect

Sampling and modeling macroinvertebrates

- Mayflies sampled using stratified random sampling along the Upper Mississippi River
- Characterized by a low-flow environment
- Samples collected with a 23 cm x 23cm sampler
- 43% of sample locations yielded zero mayflies

Gray 2005. *Ecol Model* 185:1-12

Univ. of Michigan

Center for Coastal Resources Management

Some examples:
ingrowth of trees in a forest inventory

US Forest Service

- We don't measure all trees when sampling
- Typically establish a minimum diameter to sample (say 5.0 inches DBH)

PLOTID	ForestType	Year1	Year2	Number of ingrowth trees ha^{-1}
1	Aspen	2010	2015	0
2	Red pine	2010	2015	0
3	Aspen	2010	2015	20
4	Red Pine	2010	2015	0
5	Red Pine	2010	2015	15

Zeros and common statistical distributions

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For continuous data:

- We do not expect a "piling" up of zeros
- We can apply "mixture models" (similar to the models you will here see for count data)
- For an example, see: Friederichs et al. 2011. Oikos 120:756-765. (on Moodle)

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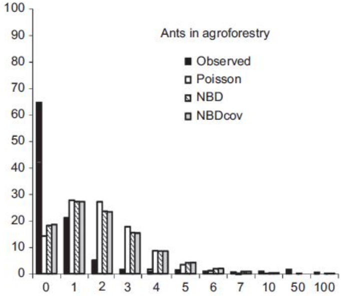

Sileshi 2008 (on Moodle)

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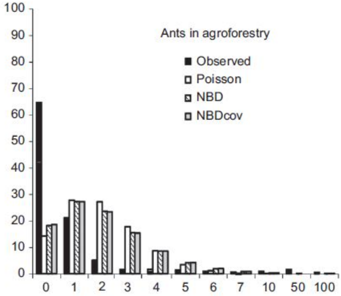

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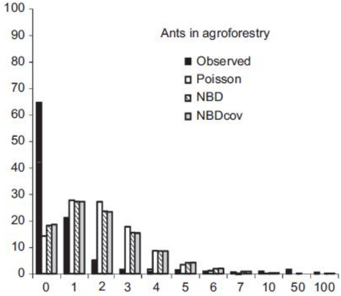

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- Can also test for overdispersion (variation > mean?)

Modeling Zero-Inflated Data

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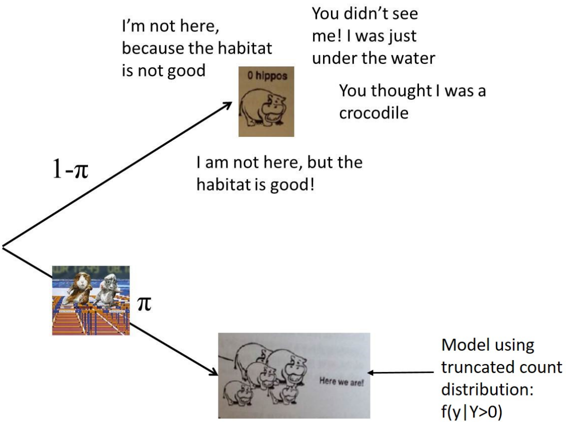
- Mixture models: allow for multiple ways to get a 0

For the in-class exercise, we will focus on the latter approach.

Hurdle Models

Group all 0′s into a single category:



Hurdle: positive counts arise if you exceed some threshold (with probability π)

Hurdle Models

1. Presence-absence subcomponent:

$$Z_i = \left\{ \begin{array}{ll} 0 \text{ when } y = 0 & \text{occurs with probability } (1-\pi) \\ 1 \text{ when } y > 0 & \text{occurs with probability } \pi \end{array} \right\}$$

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2. Count model subcomponent:

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- Poisson or negative binomial, modified to exclude the possibility of a 0

Can do this in two steps or use a single modeling framework (see Hurdle models Ch 11.5 in Zuur et al).

The non-zeros

Truncated distributions for non-zero count data:

$P(Y = y|Y > 0) = \frac{P(Y=y)}{P(Y>0)} = \frac{f(y)}{(1-f(0))}$

remember, $P(A|B)$=P(A and B)/P(B)

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- Need to be careful when plotting fitted model or constructing Bayesian p-values
- See Zuur et al. p. 288 for expressions for $E[Y|X]$ and $Var[Y|X]$

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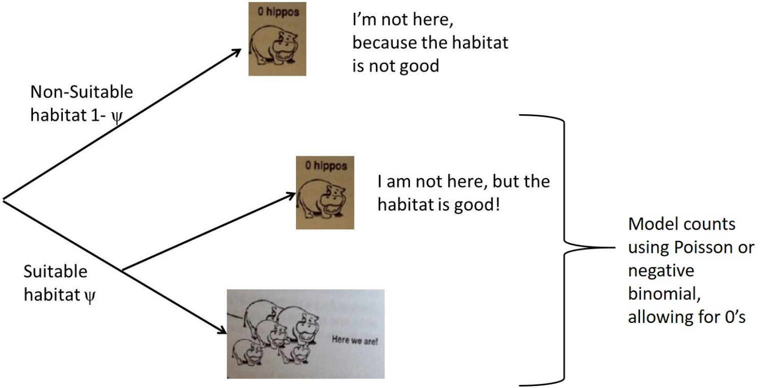
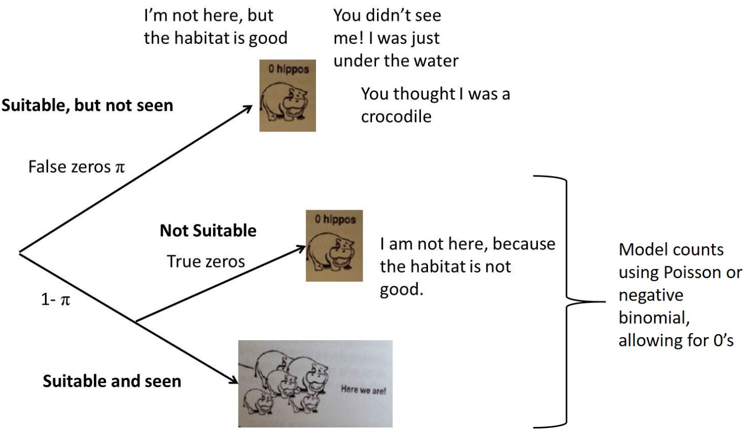
Which function in R is used to determine $F(Y)$? `pnorm`!

Mixture Model: Suitable and Non-Suitable Habitat (Kery)

Two ways to get a 0:



Mixture Models: true and false zeros (Zuur et al)



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Zero-inflation:

- Kery suggests we think of the extra zeros as arising from non-suitable habitat
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See comments on this blog:
https://statisticalhorizons.com/zero-inflated-models

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Probability Mass Function: $f(y) = \frac{e^{-\lambda}\lambda^{y}}{y!}$

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Zuur and `zeroinfl` function in `pscl` R package:

- Parameterizes in terms of π = the probability of a zero-inflated response

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ZINB model: Zero-inflated Negative Binomial

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Fitting Models in R

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Can also code models in JAGS (see Kery Ch 14) and fit using other packages (e.g. `glmmTMB`)

zeroinfl versus Kery

Remember:

- `zeroinf`: models probability of a zero-inflated response (i.e., "false" zero) = π_i
- Kery: models the probability of a NON zero-inflated response (i.e., probability of a "true" zero or a count > 0) = ψ_i

As a result, the sign of the coefficients will differ between the two approaches.

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Can compare Poisson, Negative Binomial, Zero-inflation models

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Also, zero-inflated negative binomial models can sometimes be difficult to fit (past homework problem)