Maximum Likelihood

FW8051 Statistics for Ecologists

Department of Fisheries, Wildlife and Conservation Biology

Learning Objectives

Understand how to use Maximum Likelihood to estimate parameters in statistical models

Understand how to create confidence intervals for parameters estimated using Maximum Likelihood

Estimation

We've covered a number of statistical distributions, described by a small set of parameters.

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- How do we determine appropriate values of the parameters?
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Methods of estimation:

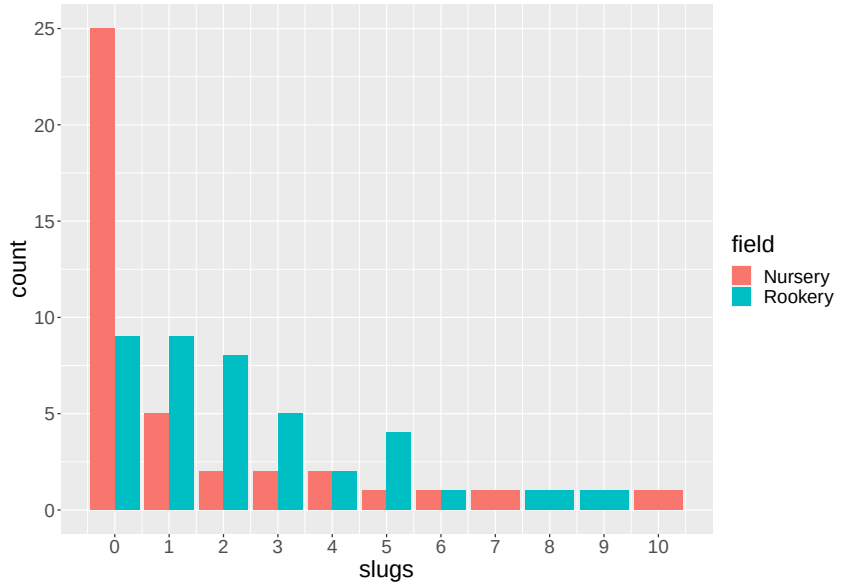
- Least squares
- Maximum likelihood
- Bayesian methods

Example from Crawley 2002. Statistical Computing and also his The R Book (2007).

- Counted slugs in 2 fields (rookery, nursery)
- 40 observations in each

Barplot

```
ggplot(slugs, aes(slugs, fill=field))+
  geom_bar(position=position_dodge())+
  theme(text = element_text(size=20))+
   scale_colour_colorblind()+
   scale_x_continuous(breaks=seq(0,11,1))
```



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What if we want to use a t-test to test $H_0 : \mu_{rookery} = \mu_{nursery}$?
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Also, side note, is this an interesting hypothesis to test?

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Lets start with the simper case of $Y_i \sim Poisson(\lambda)$ (ignoring field type)

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What about the other observations?

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$$= \frac{\exp(-\lambda)(\lambda)^3}{3!} \frac{\exp(-\lambda)(\lambda)^0}{0!} \cdots \frac{\exp(-\lambda)(\lambda)^4}{4!}$$

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For the Poisson distribution:

$$L(\lambda; x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} \frac{\lambda^{x_i} \exp(-\lambda)}{x_i!}$$

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This gives us the Likelihood of the data!

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Calculus (take derivatives with respect to λ and set = 0).

Log-likelihood

For practical and theoretical reasons, we usually work with the log-likelihood (maximizing the log-likelihood is equivalent to maximizing the likelihood)

$$logL(\lambda; x_1, x_2, \ldots, x_n) = log(L(\lambda; x_1, x_2, \ldots, x_n))$$

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For the Poisson model:

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To maximize, take derivatives and set the expression = 0, giving:

$$-n + \frac{\sum_{i=1}^{n} X_i}{\lambda} = 0$$

$$\Rightarrow \hat{\lambda} = \textstyle\sum_{i=1}^{n} \frac{X_i}{n}$$

Some notes

To verify that $\hat{\lambda}$ maximizes (rather than minimizes) $logL(\lambda|x)$:

- Verify that the $\frac{\partial^2 logL(\lambda;x)}{\partial \lambda^2}$ evaluated at $\lambda = \hat{\lambda} = \bar{x} < 0$

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Some constants, e.g., $\sum_{i=1}^{n} \log(x_i!)$ do not matter when maximizing the likelihood

- Statistical software may drop/ignore these
- Can matter when comparing models for different probability distributions using AIC

Finding the "best" value of λ

What if we do not remember calculus? How can we find the value of λ that maximizes:

$$L(\lambda; x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} \frac{\lambda^{x_i} \exp(-\lambda)}{x_i!}$$

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Graph this expression for different values of λ

[Excel in-class exercise]

Finding the "best" values for λ_1 and λ_2

What if we had a function of more than 1 parameter? How could we numerically find the value of λ that maximizes:

$$L(\lambda_1, \lambda_2; x_1, x_2, \ldots, x_n) =$$
$$\prod_{i=1}^{n_{field}} \frac{\lambda_1^{x_i} \exp(-\lambda_1)}{x_i!} \prod_{j=1}^{n_{rookery}} \frac{\lambda_2^{x_j} \exp(-\lambda_2)}{x_j!}$$

Use *solver* in Excel or `optim` (or `glm`) in R

[In-class exercise R]

Optim? When would you use something like this?

Bolker, B.M. 2008. Ecological Models and Data in R. Princeton University Press, Oxford, UK.

Tadpole predation: Example 6.3.1.1 starting on p. 182

$$p = \frac{a}{1+ahN}$$

$$k \sim \text{Binomial}(p, N)$$

- N = number of tadpoles in a tank
- k = number eaten by predators

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We will come back to this or a similar example (in this section & later after introducing Bayesian methods).

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$\hat{\theta}$ = maximum likelihood estimate of θ.

For large n (asymptotically):

- Maximum likelihood estimators are unbiased (not always true for small n):
  - $\sigma^2_{MLE} = \sum(x_i - \mu)^2/n$ (biased by a factor of $n/(n-1)$)

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$I(\theta)$ is called the Information matrix

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The matrix of second derivatives of logL with respect to θ is called the Hessian:

$\text{Hessian}(\theta) = \left[\frac{\partial^2 logL(\theta)}{\partial \theta^2}\right]$

Observed Information Matrix

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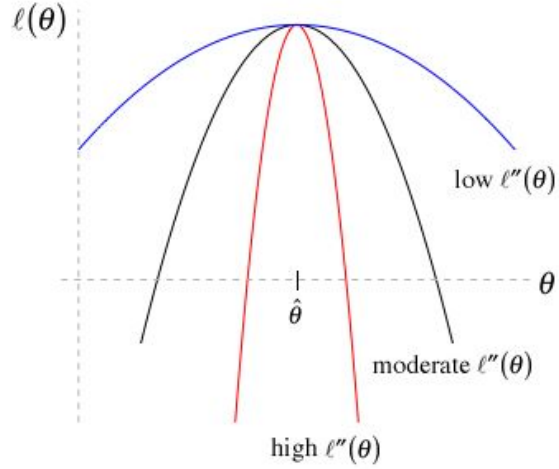
- Inverse of observed information matrix is usually what is reported as $var(\hat{\theta})$ by statistical software

Hessian

The Hessian(θ) = $\left[\frac{\partial^2 logL(\theta)}{\partial \theta^2}\right]$ describes curvature in the log-likelihood curve (surface)



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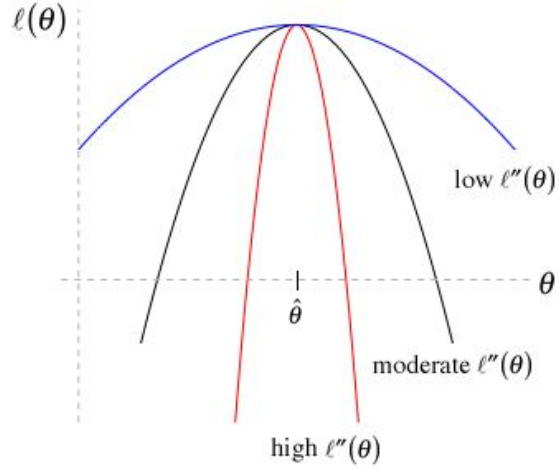
- The likelihood surface is flat
- LogL is similar across a range of parameter values



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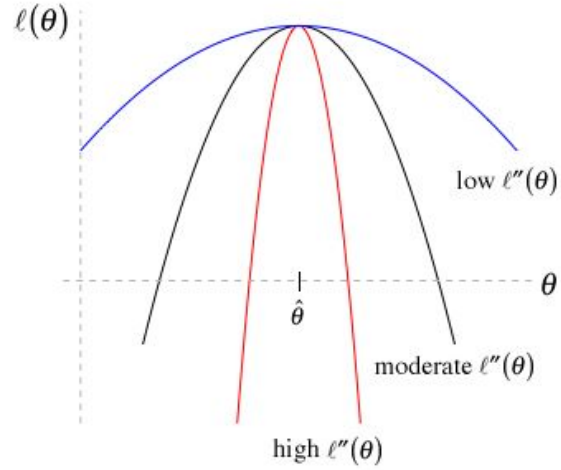

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Leads to larger confidence intervals since $\widehat{var}(\hat{\theta}) = I^{-1}(\theta) =$ Hessian$^{-1}(\theta)$

Curvature	Information	$\mathrm{Var}(\hat{\theta})$	Confidence interval for θ
high	high	low	narrow
low	low	high	wide

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Slug data example:

Full model:

- $Y_i|$ Nursery $\sim Poisson(\lambda_1)$
- $Y_i|$ Rookery $\sim Poisson(\lambda_2)$

Reduced model:

- $Y_i \sim Poisson(\lambda)$ (i.e., $\lambda_2 = \lambda_1$)

Likelihood Ratio Test

Test statistic:

$$LR = 2log\left[\frac{L(\lambda_1,\lambda_2|Y)}{L(\lambda|Y)}\right] = 2[logL(\lambda_1,\lambda_2|Y) - logL(\lambda|Y)]$$

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Null distribution (appropriate when n is large):

$$LR \sim \chi^2_1$$

. . . and more generally χ^2_p, where p is the difference in the number of parameters in the two models.

[See Section 10.9 in book]

Profile Likelihood Confidence Intervals

Can "invert" the LR test to get profile likelihood-based confidence intervals. Consider generating a CI for λ under the common λ model.

We could use the likelihood ratio test to evaluate $H_0 : \lambda = \lambda_0$ vs. $H_A : \lambda \neq \lambda_0$:

$$LR = 2log\left[\frac{L(\hat{\lambda}|Y)}{L(\lambda_0|Y)}\right] \sim \chi^2_1$$

where $\hat{\lambda}$ is the MLE of λ.

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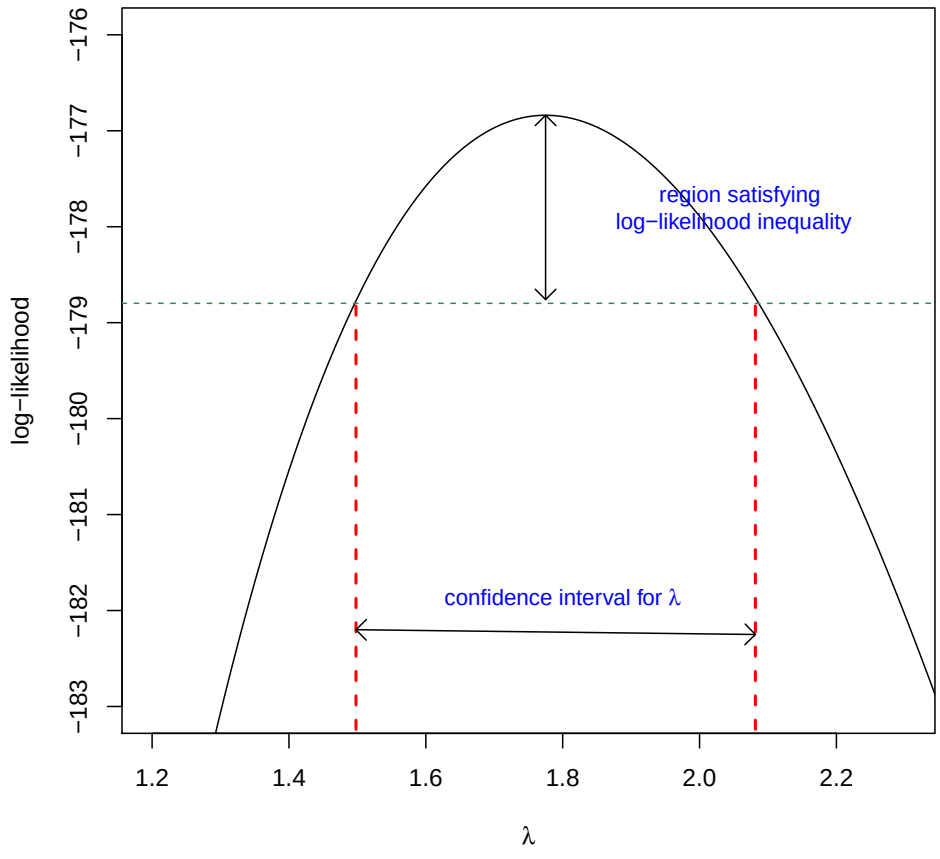
- Reject $\lambda = \lambda_0$ at $\alpha = 0.05$ if LR $> \chi^2_1(0.95)$, where $\chi^2_1(0.95)$ is the 95% of the χ^2_1 distribution.
- Fail to reject if LR $< \chi^2_1(0.95)$ (these values are plausible, given the data)

CI for λ: include all values for which we do not reject the null hypothesis

Profile Likelihood Intervals

So, include in our CI all values of λ that lie within $\chi^2_1(0.95) =$ `qchisq(alpha, df=1)/2` = 1.92 units of the maximum.



Profile Likelihood Intervals

- Can extend to multi-parameter models
- Typically more accurate than normal-based CIs (Wald intervals) when n is small.

See Chapter 6 in Bolker's book (listed in **Readings** section on Canvas).

Least Squares and Maximum Likelihood

For Normally distributed data:

$$L(\mu, \sigma^2; y_1, y_2, \ldots, y_n) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(y_i-\mu)^2}{2\sigma^2}\right)$$

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With linear regression, we assume $Y_i \sim N(\beta_0 + x_i\beta_1, \sigma^2)$, so...

$$L(\beta_0, \beta_1, \sigma; x) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(y_i-\beta_0+x_i\beta_1)^2}{2\sigma^2}\right)$$

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$$\Rightarrow logL = -nlog(\sigma) - \frac{n}{2}log(2\pi) - \sum_{i=1}^{n} \frac{(y_i-\beta_0+x_i\beta_1)^2}{2\sigma^2}$$

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$$\Rightarrow \text{maximizing logL} \Rightarrow \text{minimizing } \sum_{i=1}^{n} \frac{(y_i-\beta_0+x_i\beta_1)^2}{2\sigma^2}$$

$$\text{or, equivalently } \sum_{i=1}^{n}(y_i - \beta_0 - x_i\beta_1)^2$$