

Delta Method

FW8051 Statistics for Ecologists

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In the GLS section, we learned how to calculate $\text{var}(\hat{\beta}_0 + X_i \hat{\beta}_1)$ using matrix multiplication

And, more generally: $\text{var}(X\beta)$ for design matrix X :

$$\begin{bmatrix} 1 & X_1 \\ 1 & X_2 \\ \vdots & \vdots \\ 1 & X_n \end{bmatrix} \begin{bmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \end{bmatrix} \quad \Sigma = \begin{bmatrix} \sigma_{\hat{\beta}_0}^2 & \sigma_{\hat{\beta}_0, \hat{\beta}_1}^2 \\ \sigma_{\hat{\beta}_0, \hat{\beta}_1}^2 & \sigma_{\hat{\beta}_1}^2 \end{bmatrix}$$

- σ_X^2, σ_Y^2 = variance of $\hat{\beta}_0, \hat{\beta}_1$
- $\sigma_{\hat{\beta}_0, \hat{\beta}_1}^2$ = covariance of $\hat{\beta}_0$ and $\hat{\beta}_1$

Recall: $\text{var}(X\beta) = X\Sigma X^T$

Understand how we can use the **delta method** to calculate SEs for functions of parameters

See also:

Approximating Variance of Demographic Parameters Using the Delta Method: A Reference for Avian Biologists

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The Condor, Volume 109, Issue 4, 1 November 2007, Pages 949–954,
<https://doi.org/10.1093/condor/109.4.949>

What if we are interested in non-linear functions of parameters?

$$\text{Length}_i = L_\infty(1 - \exp(-k\text{Age}_i))$$

Options:

- Bootstrap
- Delta method
- Bayesian inference

Delta Method

$$Length_i = L_{\infty}(1 - \exp(-kAge_i))$$

Want to calculate a confidence interval for the length at a particular age, Age_i :

$$var(\hat{L}_{\infty}(1 - \exp(-\hat{k}Age_i))) = f(\hat{L}_{\infty}, \hat{k})$$

If we estimate $\theta = (L_{\infty}, k)$ using Maximum likelihood, and our sample size is large, we know:

$$\hat{\theta} \sim MVN(\theta, I^{-1}(\theta)) \text{ with:}$$

- $I(\theta) = \left[\frac{\partial^2 \log L(\theta)}{\partial \theta^2} \right]$ is the Hessian matrix

Let:

- $f(L_{\infty}, k) = L_{\infty}(1 - \exp(-kAge_i))$
- $f'(L_{\infty}, k) = \left(\frac{\partial f}{\partial L_{\infty}}, \frac{\partial f}{\partial k} \right)$
- Σ be the asymptotic variance/covariance matrix of (L_{∞}, k) given by the inverse of the Hessian matrix

Delta Method (derived using a Taylor's series approximation of f):

$$var(\hat{L}_{\infty}(1 - \exp(-\hat{k}Age_i))) \approx f'(\hat{L}_{\infty}, \hat{k}) \Sigma f'(\hat{L}_{\infty}, \hat{k})^T \big|_{L_{\infty}=\hat{L}_{\infty}, k=\hat{k}}$$

More generally:

$$var(f(\theta)) \approx f'(\theta) \Sigma f'(\theta)^T \big|_{\theta=\hat{\theta}}$$

Implementation

In R:

- use the `detavar` function in the `emdbook` package to calculate the derivatives and variance (see `FemalesvonB.R`)
- or, calculate the derivatives yourself (or using <https://www.symbolab.com/solver/derivative-calculator>), then roll your own with `%*%` for matrix multiplication.

$$f(\theta) = L_{\infty}(1 - \exp(-kAge_i))$$

$$f'(\theta) = (1 - \exp(-kAge_i), L_{\infty}Age_i \exp(-kAge_i))$$

Derivatives

Derivative Calculator

Differentiate functions step-by-step

Full post >

x^3 x^2 $\ln(x)$ $\log_e(x)$ \sqrt{x} $\frac{1}{x}$ $\frac{1}{x^2}$ $\frac{1}{x^3}$ $\frac{1}{x^4}$ $\frac{1}{x^5}$ $\frac{1}{x^6}$ $\frac{1}{x^7}$ $\frac{1}{x^8}$ $\frac{1}{x^9}$ $\frac{1}{x^{10}}$ $\frac{1}{x^{11}}$ $\frac{1}{x^{12}}$ $\frac{1}{x^{13}}$ $\frac{1}{x^{14}}$ $\frac{1}{x^{15}}$ $\frac{1}{x^{16}}$ $\frac{1}{x^{17}}$ $\frac{1}{x^{18}}$ $\frac{1}{x^{19}}$ $\frac{1}{x^{20}}$ $\frac{1}{x^{21}}$ $\frac{1}{x^{22}}$ $\frac{1}{x^{23}}$ $\frac{1}{x^{24}}$ $\frac{1}{x^{25}}$ $\frac{1}{x^{26}}$ $\frac{1}{x^{27}}$ $\frac{1}{x^{28}}$ $\frac{1}{x^{29}}$ $\frac{1}{x^{30}}$ $\frac{1}{x^{31}}$ $\frac{1}{x^{32}}$ $\frac{1}{x^{33}}$ $\frac{1}{x^{34}}$ $\frac{1}{x^{35}}$ $\frac{1}{x^{36}}$ $\frac{1}{x^{37}}$ $\frac{1}{x^{38}}$ $\frac{1}{x^{39}}$ $\frac{1}{x^{40}}$ $\frac{1}{x^{41}}$ $\frac{1}{x^{42}}$ $\frac{1}{x^{43}}$ $\frac{1}{x^{44}}$ $\frac{1}{x^{45}}$ $\frac{1}{x^{46}}$ $\frac{1}{x^{47}}$ $\frac{1}{x^{48}}$ $\frac{1}{x^{49}}$ $\frac{1}{x^{50}}$ $\frac{1}{x^{51}}$ $\frac{1}{x^{52}}$ $\frac{1}{x^{53}}$ $\frac{1}{x^{54}}$ $\frac{1}{x^{55}}$ $\frac{1}{x^{56}}$ $\frac{1}{x^{57}}$ $\frac{1}{x^{58}}$ $\frac{1}{x^{59}}$ $\frac{1}{x^{60}}$ $\frac{1}{x^{61}}$ $\frac{1}{x^{62}}$ $\frac{1}{x^{63}}$ $\frac{1}{x^{64}}$ $\frac{1}{x^{65}}$ $\frac{1}{x^{66}}$ $\frac{1}{x^{67}}$ $\frac{1}{x^{68}}$ $\frac{1}{x^{69}}$ $\frac{1}{x^{70}}$ $\frac{1}{x^{71}}$ $\frac{1}{x^{72}}$ $\frac{1}{x^{73}}$ $\frac{1}{x^{74}}$ $\frac{1}{x^{75}}$ $\frac{1}{x^{76}}$ $\frac{1}{x^{77}}$ $\frac{1}{x^{78}}$ $\frac{1}{x^{79}}$ $\frac{1}{x^{80}}$ $\frac{1}{x^{81}}$ $\frac{1}{x^{82}}$ $\frac{1}{x^{83}}$ $\frac{1}{x^{84}}$ $\frac{1}{x^{85}}$ $\frac{1}{x^{86}}$ $\frac{1}{x^{87}}$ $\frac{1}{x^{88}}$ $\frac{1}{x^{89}}$ $\frac{1}{x^{90}}$ $\frac{1}{x^{91}}$ $\frac{1}{x^{92}}$ $\frac{1}{x^{93}}$ $\frac{1}{x^{94}}$ $\frac{1}{x^{95}}$ $\frac{1}{x^{96}}$ $\frac{1}{x^{97}}$ $\frac{1}{x^{98}}$ $\frac{1}{x^{99}}$ $\frac{1}{x^{100}}$

Most Used Actions

$\frac{d}{dx} (L_{\infty}(1 - e^{-kAge_i}))$

Graph > Examples >

Solution

$\frac{d}{dx} (L_{\infty}(1 - e^{-kx})) = L_{\infty} e^{-kx}$

Steps

$\frac{d}{dx} (L_{\infty}(1 - e^{-kx}))$

Take the constant out: $(L_{\infty} e^{-kx})' = L_{\infty} (e^{-kx})'$

$= L_{\infty} (1 - e^{-kx})' \frac{d}{dx} (L_{\infty})$

Apply the common derivative: $\frac{d}{dx} (L_{\infty}) = 1$

$= L_{\infty} (1 - e^{-kx}) \cdot 1$

Simplify

$= L_{\infty} e^{-kx}$

[click here to practice derivatives >](#)