

Logistic regression models for binary data

FW8051 Statistics for Ecologists

Department of Fisheries, Wildlife and Conservation Biology



Learning Objectives

Learning objectives

- Be able to formulate, fit, and interpret logistic regression models appropriate for binary data using R and JAGS
- Be able to compare models and evaluate model fit
- Be able to visualize models using effect plots
- Be able to describe statistical models and their assumptions using equations and text and match parameters in these equations to estimates in computer output

Logistic regression

Model for binary (0/1) data or binomial data (number of 1's out of n trials).

$$Y_i|X_i \sim \text{Bernoulli}(p_i)$$

$$\text{logit}(p_i) = \log\left(\frac{p_i}{1-p_i}\right) = \beta_0 + \beta_1 X_{1,i} + \dots \beta_p X_{p,i}$$

- Random component = Bernoulli or binomial distribution
- Systematic component: $\text{logit}(p_i)$ or $\log(\text{odds})$ = linear combination of predictors

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Remember, for binary data, $E[Y_i|X_i] = p_i$, $\text{Var}[Y_i|X_i] = p_i(1 - p_i)$

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Remember, for binary data, $E[Y_i|X_i] = p_i$, $\text{Var}[Y_i|X_i] = p_i(1 - p_i)$

$$\Rightarrow p_i = \frac{\exp(\beta_0 + \beta_1 X_{1,i} + \dots \beta_p X_{p,i})}{1 + \exp(\beta_0 + \beta_1 X_{1,i} + \dots \beta_p X_{p,i})}$$

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Remember, for binary data, $E[Y_i|X_i] = p_i$, $\text{Var}[Y_i|X_i] = p_i(1 - p_i)$

$$\Rightarrow p_i = \frac{\exp(\beta_0 + \beta_1 X_{1,i} + \dots \beta_p X_{p,i})}{1 + \exp(\beta_0 + \beta_1 X_{1,i} + \dots \beta_p X_{p,i})} \text{ (can use } \text{plogis} \text{ function in R)}$$

Logistic regression

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$$\text{logit}(p_i) = \log\left(\frac{p_i}{1-p_i}\right) = \beta_0 + \beta_1 X_{1,i} + \dots \beta_p X_{p,i}$$

$\frac{p}{1-p}$ is referred to as the **odds**.

The link function, $\log\left(\frac{p}{1-p}\right)$, is referred to as **logit**.

Logistic regression

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Thus, we can describe our model in the following ways:

- We are modeling $\log\left(\frac{p}{1-p}\right)$ as a linear function of X_1, \dots, X_p .
- We are modeling the logit of p as a linear function of X_1, \dots, X_p .
- We are modeling the log odds of p as a linear function of X_1, \dots, X_p .

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If the probability of winning a bet is $= 2/3$, what are the odds of winning?

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Table 6.1. Various probabilities, odds and log odds. The table shows how log odds are calculated from probabilities.

P_i	0.001	0.1	0.3	0.4	0.5	0.6	0.7	0.9	0.999
$1 - P_i$	0.999	0.9	0.7	0.6	0.5	0.4	0.3	0.1	0.001
O_i	0.001	0.11	0.43	0.67	1	1.5	2.33	9	999
$\text{Ln}(O_i)$	-6.91	-2.20	-0.85	-0.41	0	0.41	0.85	2.20	6.91

From Zuur et al. 2007. Analyzing Ecological data

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Odds can vary between 0 and ∞ , so $\log(\text{odds})$ can live on $-\infty$ to ∞ .

Odds Ratios: $\exp(\beta)$

Consider a regression coefficient for a categorical variable:

$$\text{logit}(p_i) = \log\left(\frac{p_i}{1-p_i}\right) = \beta_0 + \beta_1 I(\text{group} = B)_i$$

$I(\text{group} = B)_i = 1$ if observation i is from Group B and 0 if Group A

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- Odds for group A = $\frac{p_A}{1-p_A} = \exp(\beta_0)$

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- Odds for group A = $\frac{p_A}{1-p_A} = \exp(\beta_0)$

Consider the ratio of these odds:

$$\frac{\frac{p_B}{1-p_B}}{\frac{p_A}{1-p_A}} = \frac{\exp(\beta_0 + \beta_1)}{\exp(\beta_0)} = \frac{e^{\beta_0} e^{\beta_1}}{e^{\beta_0}} = \exp(\beta_1)$$

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So, $\exp(\beta_1)$ gives an **odds ratio** (or ratio of odds) for Group B relative to group A.

Odds Ratios: $\exp(\beta)$

Consider a continuous predictor, X :

$$\text{logit}(p_i) = \log\left(\frac{p_i}{1-p_i}\right) = \beta_0 + \beta_1 X_i$$

β_1 gives the change in log odds per unit change in X .

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- Odds when $X_i = a$ is given by $\frac{p_i}{1-p_i} = \exp(\beta_0 + \beta_1 a)$
- Odds when $X_j = a + 1$ is given by $\frac{p_j}{1-p_j} = \exp(\beta_0 + \beta_1(a + 1))$

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So, $\exp(\beta_1)$, gives the odds ratio for two observation that differ by 1 unit of X .

Multiple predictors

For multiple predictor models,

$\exp(\beta_i)$ gives the odds ratio for observations where X_i differs by 1 unit, while holding everything else constant!

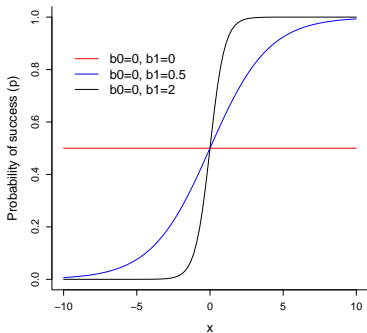
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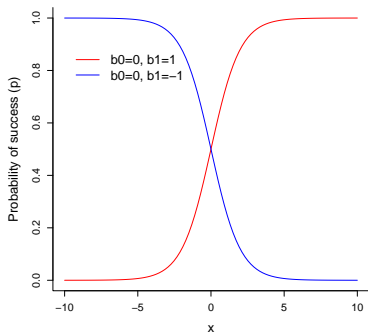
The odds is expected to increase by a factor of $\exp(\beta_i)$ when X_i increases by 1 unit, and everything else is held constant!

$$\text{logit}(p_i) = \log\left(\frac{p_i}{1-p_i}\right) = \beta_0 + \beta_1 X_i$$



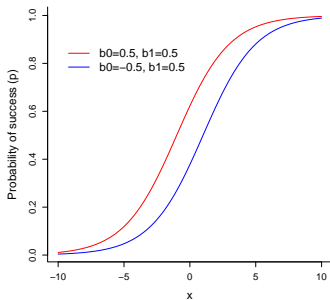
The slope coefficient β_1 controls how quickly we transition from 0 to 1.

$$\text{logit}(p_i) = \log\left(\frac{p_i}{1-p_i}\right) = \beta_0 + \beta_1 x_i$$



The sign of β_1 determines if p increases or decreases as we increase X .

$$\text{logit}(p_i) = \log\left(\frac{p_i}{1-p_i}\right) = \beta_0 + \beta_1 X_i$$

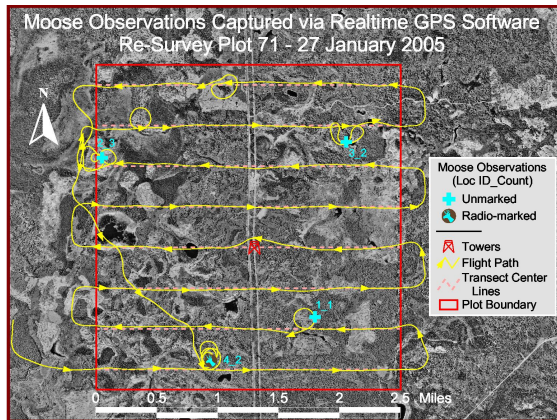


β_0 :

- Controls the height of the curve when $X = 0$.
- Gives the log odds of detection when all predictor variables = 0
- $E[Y_i|X_i = 0] = \frac{\exp(\beta_0)}{1+\exp(\beta_0)}$ (equals 1/2 if $\beta_0 = 0$).

Sightability Surveys: Minnesota Moose

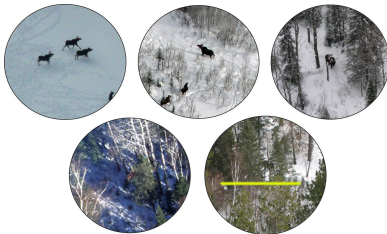
$\frac{124 \text{ 'trials', } 2005-2007}{n_0 = 65 \text{ missed groups}}$
 $n_1 = 59 \text{ observed groups}$



- Binary observations, $Y_i = 0$ (missed) or 1 (seen).
- Covariates thought to influence detection.

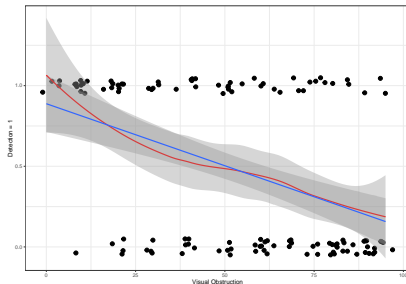
Covariates

- Visual obstruction
- Survey year (may be due to different observers)



Visual Obstruction

```
ggplot(exp.m, aes(voc, observed)) + theme_bw() +  
  geom_point(position = position_jitter(w = 2, h = 0.05), size = 3) +  
  geom_smooth(colour = "red") + geom_smooth(method = "lm") +  
  xlab("Visual Obstruction") +  
  ylab("Detection = 1")
```



- lm would eventually predict $p_i \geq 1$ and $p_i \leq 0$
- lm assumes constant variance rather than $\text{var}(p_i) = p_i(1 - p_i)$

$$Y_i|X_i \sim \text{Bernoulli}(p_i)$$

$$\text{logit}(p_i) = \log\left(\frac{p_i}{1-p_i}\right) = \beta_0 + \beta_1 \text{voc}_i$$

Assumptions:

- observations are independent
- log odds is a linear function of voc
- mean and variance depend on voc

$E[Y_i|X_i] = p_i; \text{Var}[Y_i|X_i] = p_i(1-p_i)$ with:

$$p_i = \frac{\exp(\beta_0 + \beta_1 \text{voc}_i)}{1 + \exp(\beta_0 + \beta_1 \text{voc}_i)}$$

```
mod1<-glm(observed~voc, data=exp.m, family=binomial())
summary(mod1)
```

Call:

```
glm(formula = observed ~ voc, family = binomial(), data = exp.m)
```

Deviance Residuals:

Min	1Q	Median	3Q	Max
-1.8056	-0.9071	-0.6218	0.9745	1.8647

Coefficients:

	Estimate	Std. Error	z value	Pr(> z)	
(Intercept)	1.759933	0.460137	3.825	0.000131	***
voc	-0.034792	0.007753	-4.487	7.21e-06	***

Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

(Dispersion parameter for binomial family taken to be 1)

Null deviance: 171.61 on 123 degrees of freedom
Residual deviance: 147.38 on 122 degrees of freedom
AIC: 151.38

Number of Fisher Scoring iterations: 4

```
mod1$coef
```

```
(Intercept)          voc  
1.75993309 -0.03479153
```

Regression coefficient for voc (visual obstruction) = -0.039.

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- The log odds of being detected decreases by 0.039 per unit increase in visual obstruction


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Regression coefficient for voc (visual obstruction) = -0.039.

- The log odds of being detected decreases by 0.039 per unit increase in visual obstruction
- The odds of being detected decreases by a factor of $\exp(0.039) = 0.96$ per unit increase in visual obstruction

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Intercept = 2.12 =

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mod1$coef
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- The odds of being detected decreases by a factor of $\exp(0.039) = 0.96$ per unit increase in visual obstruction

Intercept = 2.12 = $\log(\text{odds})$ of detection when VOC = 0.

```
mod1$coef
```

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```

Regression coefficient for voc (visual obstruction) = -0.039.

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- The odds of being detected decreases by a factor of $\exp(0.039) = 0.96$ per unit increase in visual obstruction

Intercept = 2.12 = $\log(\text{odds})$ of detection when VOC = 0.

```
#  $p(Y=1 | \text{voc}=0) = \exp(\text{coef(mod1)[1]}) / (1 + \exp(\text{coef(mod1)[1]}))$   
plogis(coef(mod1)[1])
```

```
(Intercept)  
0.8532013
```

```
mod1$coef
```

```
(Intercept)          voc  
1.75993309 -0.03479153
```

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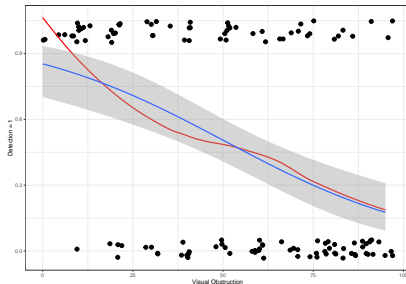
Intercept = 2.12 = $\log(\text{odds})$ of detection when VOC = 0.

```
# p(Y=1|voc=0) = exp(coef(mod1)[1]) / (1+exp(coef(mod1)[1]))  
plogis(coef(mod1)[1])
```

```
(Intercept)  
0.8532013
```

We see roughly 85% of moose if there is no visual obstruction.

```
ggplot(exp.m, aes(voc,observed))+ theme_bw() +
  geom_point(position = position_jitter(w = 2, h = 0.05), size=3) +
  xlab("Visual Obstruction") + geom_smooth(se=F, colour="red") +
  stat_smooth(method="glm", method.args = list(family = "binomial"))
ylab("Detection = 1")
```



```
exp.m$year<-as.factor(exp.m$year)
mod2<-glm(observed~voc+year, data=exp.m, family=binomial())
summary(mod2)
```

Call:

```
glm(formula = observed ~ voc + year, family = binomial(), data = exp.m)
```

Deviance Residuals:

	Min	1Q	Median	3Q	Max
	-1.9351	-0.8411	-0.4561	0.9493	1.8680

Coefficients:

	Estimate	Std. Error	z value	Pr(> z)
(Intercept)	2.453203	0.622248	3.942	8.06e-05 ***
voc	-0.037391	0.008199	-4.560	5.11e-06 ***
year2006	-0.453862	0.516567	-0.879	0.3796
year2007	-1.111884	0.508269	-2.188	0.0287 *

Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

(Dispersion parameter for binomial family taken to be 1)

Null deviance: 171.61 on 123 degrees of freedom
 Residual deviance: 142.23 on 120 degrees of freedom
 AIC: 150.23

Number of Fisher Scoring iterations: 4

```
coef(mod2)
```

```
(Intercept)          voc    year2006    year2007  
2.45320264 -0.03739118 -0.45386154 -1.11188432
```



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(Intercept)          voc    year2006    year2007  
2.45320264 -0.03739118 -0.45386154 -1.11188432
```

Year 2005: $\log(p_i/(1 - p_i)) = 2.45 - 0.037VOC$

Year 2006: $\log(p_i/(1 - p_i)) = 2.45 - 0.037VOC - 0.45$

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```
(Intercept)      voc    year2006    year2007  
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Year 2005: $\log(p_i/(1 - p_i)) = 2.45 - 0.037VOC$

Year 2006: $\log(p_i/(1 - p_i)) = 2.45 - 0.037VOC - 0.45$

So, -0.45 gives the difference in log odds between years 2005 and 2004 (if we hold VOC constant).

```
coef(mod2)
```

```
(Intercept)      voc    year2006    year2007  
2.45320264 -0.03739118 -0.45386154 -1.11188432
```

Year 2005: $\log(p_i/(1 - p_i)) = 2.45 - 0.037VOC$

Year 2006: $\log(p_i/(1 - p_i)) = 2.45 - 0.037VOC - 0.45$

So, -0.45 gives the difference in log odds between years 2005 and 2004 (if we hold VOC constant).

$\exp(-0.45) = 0.63 = \text{odds ratio (year 2006 to year 2005)}$

$\text{odds ratio} = \frac{p_{2006}/(1-p_{2006})}{p_{2005}/(1-p_{2005})} = 0.63$

Supporting Theory

The estimates of β are maximum likelihood estimates, found by maximizing:

$$L(\beta; y, x) = \prod_{i=1}^n p_i^{y_i} (1 - p_i)^{1-y_i}, \text{ with}$$

$$p_i = \frac{e^{\beta_0 + \beta_1 x_1 + \dots + \beta_k x_k}}{1 + e^{\beta_0 + \beta_1 x_1 + \dots + \beta_k x_k}}$$

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Remember, for large samples, $\hat{\beta} \sim N(\beta, \Sigma)$.

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We can use this theory to conduct tests (z-statistics and p-values in output by the `summary` function) and to get confidence intervals.

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We can use this theory to conduct tests (z-statistics and p-values in output by the `summary` function) and to get confidence intervals.

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mod2$coef
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  2.45320264 -0.03739118 -0.45386154 -1.11188432
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(Intercept)          voc    year2006    year2007  
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exp(rep(0.006879, 2)+c(-1.96, 1.96)*0.53664) # exp(beta +/-1.96SE)
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95% Confidence interval for odds ratio = (0.35, 2.88) includes 1
(not statistically significant)

Confin

```
(ci.prof<-confint(mod2))
```

Waiting for profiling to be done...

	2.5 %	97.5 %
(Intercept)	1.30341777	3.7586692
voc	-0.05448153	-0.0221268
year2006	-1.48479529	0.5516852
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```
exp(ci.prof[3,])
```

	2.5 %	97.5 %
	0.2265487	1.7361764

Profile-likelihood based intervals should have better statistical properties with small data sets (better **coverage** rates).

Goodness-of-fit

Can adapt our general approach for testing goodness-of-fit using Pearson residuals (r_i)

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- $Var[Y_i|X_i] = p_i(1 - p_i)$

See textbook for an implementation of this test...

Hosmer-Lemeshow test (similar test)

Group Observations by deciles of their predicted values to form groups, then calculate the expected and observed number of successes and failures for each group:

Model Results				
	$G_1 = [0, \hat{\pi}_1]$	$G_2 = (\hat{\pi}_1, \hat{\pi}_2]$...	$G_{10} = (\hat{\pi}_9, 1]$
Successes	$\sum_{x_i \in G_1} \hat{\pi}_i$	$\sum_{x_i \in G_2} \hat{\pi}_i$...	$\sum_{x_i \in G_{10}} \hat{\pi}_i$
Failures	$n_1 - \sum_{x_i \in G_1} \hat{\pi}_i$	$n_2 - \sum_{x_i \in G_2} \hat{\pi}_i$...	$n_{10} - \sum_{x_i \in G_{10}} \hat{\pi}_i$

Observed Results				
	$G_1 = [0, \hat{\pi}_1]$	$G_2 = (\hat{\pi}_1, \hat{\pi}_2]$...	$G_{10} = (\hat{\pi}_9, 1]$
Successes	$\sum_{x_i \in G_1} y_i$	$\sum_{x_i \in G_2} y_i$...	$\sum_{x_i \in G_{10}} y_i$
Failures	$n_1 - \sum_{x_i \in G_1} y_i$	$n_2 - \sum_{x_i \in G_2} y_i$...	$n_{10} - \sum_{x_i \in G_{10}} y_i$

See: **Goodness of fit with binary data** here:

<http://www.unc.edu/courses/2010fall/ecol/563/001/docs/lectures/lecture21.htm>

Hosmer-Lemeshow Test

$$\chi^2 = \sum_{i=1}^{n_g} \frac{(O_i - E_i)^2}{E_i} \sim \chi_{g-2}^2$$

where g = number of groups.

Hosmer-Lemeshow test

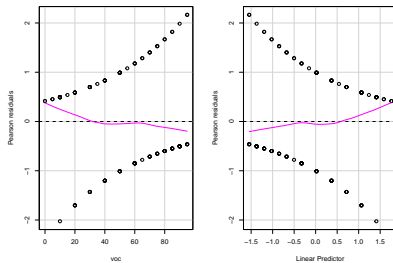
```
library(ResourceSelection)
hoslem.test(exp.m$observed, fitted(mod1), g=8)
```

Hosmer and Lemeshow goodness of fit (GOF) test

```
data: exp.m$observed, fitted(mod1)
X-squared = 3.2505, df = 6, p-value = 0.7768
```

Residual plots

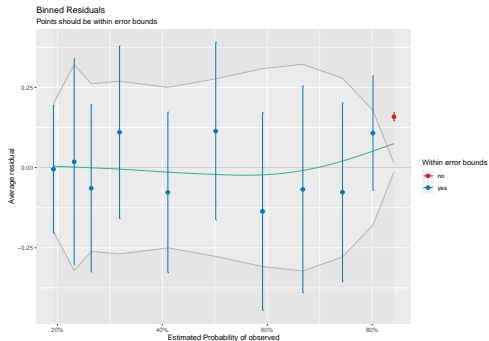
```
car::residualPlots(mod1)
```



```
Test stat Pr(>|Test stat|)
voc      0.6873      0.4071
```


Binned residual plot

```
binplot<-performance::binned_residuals(mod1)
plot(binplot)
```



Likelihood ratio tests

We can again use difference in deviances (equivalent to likelihood ratio tests) to compare full and reduced models.

```
drop1(mod2, test="Chisq")
```

Single term deletions

Model:

observed ~ voc + year

	Df	Deviance	AIC	LRT	Pr(>Chi)
<none>		142.23	150.23		
voc	1	168.20	174.20	25.9720	3.464e-07 ***
year	2	147.38	151.38	5.1558	0.07593 .

Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

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voc is an important predictor, the importance of year is less clear.

ANOVA function (car package)

Or use Anova in car package

```
library(car)
```

```
Anova(mod2)
```

```
Analysis of Deviance Table (Type II tests)
```

```
Response: observed
```

```
      LR Chisq Df Pr(>Chisq)  
voc    25.9720  1  3.464e-07 ***  
year    5.1558  2   0.07593 .  
---
```

```
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

AIC

We can compare nested or non-nested models using the AIC function

```
AIC(mod1, mod2)
```

	df	AIC
mod1	2	151.3824
mod2	4	150.2266

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We can use `predict(model, newdata=, type="link", se=TRUE)` to get predictions on logit scale.

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We can use `predict(model, newdata=, type="link", se=TRUE)` to get predictions on logit scale.

Then use `plogis(p.hat$fit +/- 1.95*p.hat$se.fit)` to transform the limits back to the probability scale.

A note on model visualization

Model 2: $\text{observed} \sim \text{voc} + \text{year}$ is additive on the logit scale

- Differences in $\text{logit}(p)$ among years will not depend on voc
- Differences in p , will however, depend on voc!

See: Section 16.6.3 in the book

A note on model visualization

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- Differences in p , will however, depend on voc!

See: Section 16.6.3 in the book

- Can always create your own “effect” plots by calculating predicted values for different combinations of your predictors
- Can use the `effects` package or `ggeffects` to do something similar

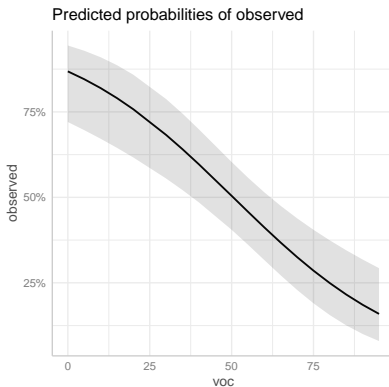
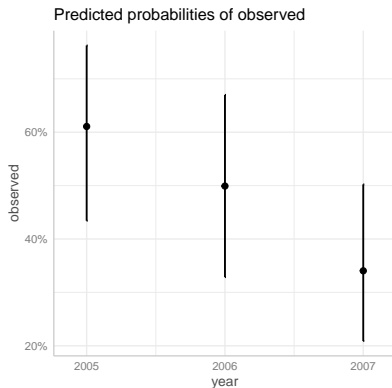
Effect plots on probability scale

Use `effects` or `ggeffects` package:

- Fixes all continuous covariates (other than the one of interest) at their mean values
- Categorical predictors: averages predictions on link scale, weighted by proportion of data in each category, then back transforms to probability scale
- These are referred to as marginal predictions by `ggeffects`

Effect plots

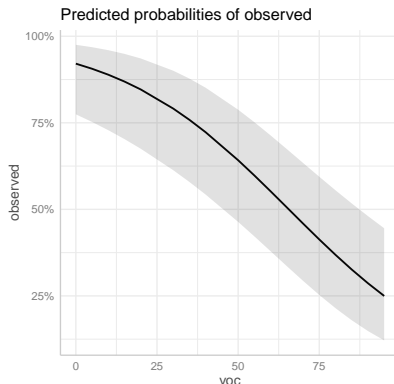
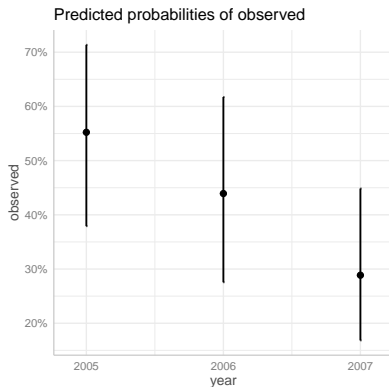
```
library(ggeffects); library(patchwork)
p1 <- plot(ggeffect(mod2, "year"))
p2 <- plot(ggeffect(mod2, "voc"))
p1 + p2
```



Adjusted plots

Instead of averaging predictions across years, we could set year to a specific value. This leads to adjusted plots.

```
library(ggeffects); library(patchwork)
p1 <- plot(ggpredict(mod2, "year"))
p2 <- plot(ggpredict(mod2, "voc"))
p1 + p2
```



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In class exercise: adapt the JAGS code for fitting mod1 (voc only) to allow fitting of mod2 (voc + year).