

## Introduction to Generalized Linear Models

FW8051 Statistics for Ecologists

Department of Fisheries, Wildlife and Conservation Biology



- Understand the role of random variables and common statistical distributions in formulating modern statistical regression models
- Be able to fit appropriate models to count data and binary data (yes/no, presence/absence) in both R and JAGS
- Be able to evaluate model goodness-of-fit
- Be able to describe a variety of statistical models and their assumptions using equations and text and match parameters in these equations to estimates in computer output.

## Outline

- Introduction to generalized linear models (today)
- Models for count data (Poisson and Negative Binomial regression)
- Models for Binary data (logistic regression)
- Models for data with lots of zeros

## Linear Regression

Often written in terms of “signal + error”:

$$y_i = \underbrace{\beta_0 + x_i\beta_1}_{\text{Signal}} + \underbrace{\epsilon_i}_{\text{error}}, \text{ with}$$
$$\epsilon_i \sim N(0, \sigma^2)$$

Possible because the Normal distribution has separate parameters that describe:

- mean:  $E[Y_i|X_i] = \mu_i = \beta_0 + x_i\beta_1$
- variance:  $\text{Var}[Y_i|X_i] = \sigma^2$

Remember: for Poisson, Binomial distributions, the variance is a function of the mean.

## Linear Regression

$$Y_i | X_i \sim N(\mu_i, \sigma^2)$$
$$\mu_i = \beta_0 + \beta_1 X_{1,i} + \dots + \beta_p X_{p,i}$$

This description highlights:

1. The distribution of  $Y_i$  depends on a set of predictor variables  $X_i$
2. The distribution of the response variable, conditional on predictor variables is Normal
3. The mean of the Normal distribution depends on predictor variables ( $X_1$  through  $X_p$ ) and regression coefficients (the  $\beta_1$  through  $\beta_p$ )
4. The variance is constant and given by  $\sigma^2$ .

## Linear Regression = General Linear Model

Sometimes referred to as: **General Linear Model**

- t-test (categorical predictor with 2 categories)
- ANOVA (categorical predictor with > 2 categories)
- ANCOVA (continuous and categorical predictor, no interaction so common slope)
- Continuous and categorical variables, with possible interactions

## Generalized Linear Models

**Generalized linear models** further unifies several different regression models:

- **General linear model**
- Logistic regression
- Poisson regression
- ...

Rather elegant general theory developed for exponential family of distributions

## Generalized Linear Models (glm)

**Systematic component:**  $g(\mu_i) = \eta_i = \beta_0 + \beta_1 x_1 + \dots + \beta_p x_p$

Some transformation of the the mean,  $g(\mu_i)$ , results in a linear model.

- $g()$  is called the **link function**
- $\eta_i = \beta_0 + \beta_1 x_1 + \dots + \beta_p x_p$  is called the **linear predictor**.
- $\mu_i = g^{-1}(\eta_i) = g^{-1}(\beta_0 + \beta_1 x_1 + \dots + \beta_p x_p)$

**Random component:**  $Y_i | X_i \sim f(y_i | x_i), i = 1, \dots, n$

- $f(y_i | x_i)$  is in the **exponential family** (includes normal, Poisson, binomial, gamma, inverse Gaussian)
- $f(y_i | x_i)$  describes unmodeled variation about  $\mu_i = E[Y_i | X_i]$

## Linear Regression:

- $f(y_i|x_i) = N(\mu_i, \sigma^2)$
- $E[Y_i|X_i] = \mu_i = \beta_0 + \beta_1 x_1 + \dots + \beta_p x_p$
- $g(\mu_i) = \eta_i = \mu_i$ , the **identity link**
- $\mu_i = g^{-1}(\eta_i) = \eta_i = \beta_0 + \beta_1 x_1 + \dots + \beta_p x_p$

## Poisson regression:

- $f(y_i|x_i) \sim \text{Poisson}(\lambda_i)$
- $E[Y_i|X_i] = \mu_i = \lambda_i$
- $g(\mu_i) = \eta_i = \log(\lambda_i) = \beta_0 + \beta_1 x_1 + \dots + \beta_p x_p$
- $\mu_i = g^{-1}(\eta_i) = \exp(\eta_i) = \exp(\beta_0 + \beta_1 x_1 + \dots + \beta_p x_p)$

## Logistic regression:

- $f(y_i|x_i) \sim \text{Bernoulli}(p_i)$
- $E[Y_i|X_i] = p_i$
- $g(\mu_i) = \eta_i = \text{logit}(p_i) = \log\left(\frac{p_i}{1-p_i}\right) = \beta_0 + \beta_1 x_1 + \dots + \beta_p x_p$
- $\mu_i = g^{-1}(\eta_i) = \frac{\exp^{\eta_i}}{1+\exp^{\eta_i}} = \frac{\exp^{\beta_0 + \beta_1 x_1 + \dots + \beta_p x_p}}{1+\exp^{\beta_0 + \beta_1 x_1 + \dots + \beta_p x_p}}$

## Link functions and sample space

Link functions allow the “structural component”  $(\beta_0 + \beta_1 x_1 + \dots + \beta_p x_p)$  to live on  $(-\infty, \infty)$  while keeping the  $\mu_i$  consistent with the range of the response variable.

Poisson (counts) =  $0, 1, 2, \dots, \infty$

- $g(\mu_i) = \eta_i = \log(\lambda_i) = \beta_0 + \beta_1 x_1 + \dots + \beta_p x_p$ , range =  $(-\infty, \infty)$
- $\mu_i = \exp(\beta_0 + \beta_1 x_1 + \dots + \beta_p x_p)$ , range =  $[0, \infty]$

## Logistic regression:

- $g(\mu_i) = \eta_i = \log\left(\frac{p_i}{1-p_i}\right) = \beta_0 + \beta_1 x_1 + \dots + \beta_p x_p$ , range =  $(-\infty, \infty)$
- $\mu_i = g^{-1}(\eta_i) = \frac{\exp^{\eta_i}}{1+\exp^{\eta_i}} = \frac{\exp^{\beta_0 + \beta_1 x_1 + \dots + \beta_p x_p}}{1+\exp^{\beta_0 + \beta_1 x_1 + \dots + \beta_p x_p}}$ , range =  $(0, 1)$



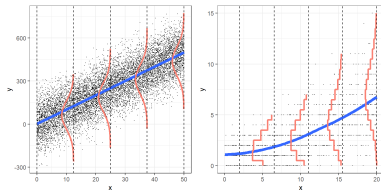
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- $X_i$  = amount of grassland cover
- $Y_i|X_i \sim \text{Poisson}(\lambda_i)$
- $\log(\lambda_i) = \beta_0 + \beta_1 X_i$

Because the mean of the Poisson distribution is  $\lambda$ :

- $E[Y_i|X_i] = \lambda_i = \exp(\beta_0 + \beta_1 X_i) = \exp(\beta_0) \exp(\beta_1 X_i)$
- The mean number of pheasants increases by a factor of  $\exp(\beta_1)$  as we increase  $X_i$  by 1 unit.

1. **Poisson Response:** The response variable is a count per unit of time or space, described by a Poisson distribution.
2. **Independence:** The observations must be independent of one another.
3. **Mean=Variance:** By definition, the mean of a Poisson random variable must be equal to its variance.
4. **Linearity:** The log of the mean rate,  $\log(\lambda)$ , must be a linear function of  $x$ .



From Broadening Your Statistical Horizons: <https://bookdown.org/toback/bookdown-bysh/>

## Next Steps

- Understand how Maximum Likelihood is used to fit modern statistical regression models (*glm*)
- Be able to fit regression models appropriate for count data in R and JAGS
  - Poisson regression models
  - Quasi-Poisson (R only)
  - Negative Binomial regression
- Interpret estimated coefficients and describe their uncertainty using confidence and credible intervals
- Use simple tools to assess model fit
  - Residuals (deviance and Pearson)
  - Goodness-of-fit tests
- Use deviances and AIC to compare models.
- Use an offset to model rates and densities, accounting for variable survey effort
- Be able to describe statistical models and their assumptions using equations and text and match parameters in these equations to estimates in computer output.