

The Role of Probability in Regression Models

FW8051 Statistics for Ecologists

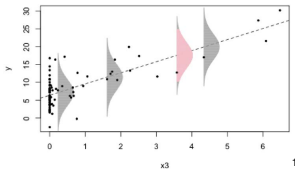
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Understand the role of random variables and common statistical distributions in formulating modern statistical regression models.

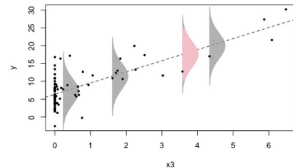
- Will need to know something about other statistical distributions
- Will need to have an understanding of basic probability theory
 - Probability rules and random variables
 - Expected Value
 - Variance
- How to work with probability distributions in R...

$$\text{Linear Regression } y_i = \underbrace{\beta_0 + x_i\beta_1}_{\text{Signal}} + \underbrace{\epsilon_i}_{\text{noise}}$$



- Estimated errors, $\hat{\epsilon}_i$ given by vertical distance between points and the line
- Find the line that minimizes the errors

$$\text{Linear Regression } Y_i \sim N(\beta_0 + \beta_1 X_i, \sigma^2)$$



Instead of errors, think about the normal distribution as a data-generating mechanism:

- The line gives the **expected** (average) value
- Normal curve describes the variability about this expected value.

¹ Code and example from Jack Weiss's Ecol563: <http://www.unc.edu/courses/2010fall/ecol563/001/docs/lectures/lecture4.htm>. Normal distributions, above, extend to 3σ (pink = 2σ , with 1σ in gray)

Generalizing to other probability distributions

Replace the normal distribution as the data-generating mechanism with another probability distribution, but which one?

Leads us to...

- Discrete and continuous random variables
 - Probability mass functions (discrete random variables)
 - Probability density functions (continuous random variables)

See handout for probability rules and distributions!

Sample Space and [Frequentist] Probability

Sample space = the set of all possible outcomes that could occur.

Discrete variables:

- age class = (fawn, adult)
- dice = (1,2,3,4,5,6)

Continuous variables (range of possible values)

- age = (0, ∞)
- (x, y) such that x,y falls within the continental US

The **probability** of event A, $P(A)$, is the long run frequency or proportion of times the event occurs.

Random variables

A **random variable** is a numeric quantity (or numerical event) that changes from trial to trial in a random process.

It is essentially a mapping that takes us from random events to numbers.

- Example: X = number of heads in two coin flips
- Possible events: {HH, TH, HT, TT} (all equally likely)
- Sample space of X = {0, 1, 2}

Discrete Random Variables

A random variable is **discrete** if it can take on a finite (or countably infinite²) set of possible values.

- X = Number of birds seen on a plot
- Y = (0 or 1), representing whether or not a moose calf survives its first year
- G = the species richness value obtained at a beach in the Netherlands {0, 1, 2, ...}

Continuous Random Variables

A random variable is **continuous** if it has values within some interval.

- T = the age at which a randomly selected adult white-tailed deer dies
- W = Mercury level (ppm) in a randomly chosen walleye from Lake Mille Lacs

Probability Mass Function: Discrete Random Variables

A **probability mass function**, $p(x)$ assigns a probability to each value of a discrete random variable, X .

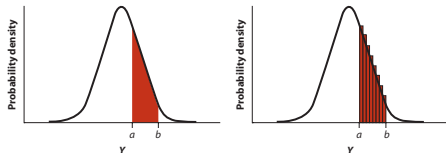
- Example: X = number of heads in two coin flips
- Possible events: {HH, TH, HT, TT} (all equally likely)
- Sample space of $X = \{0, 1, 2\}$

x	0	1	2
p(x)	1/4	1/2	1/4

Note: for any probability mass function $\sum p(x) = 1$

Continuous Distributions, Probability Density Function f(x)

For continuous variables, we define probabilities as areas under a curve, e.g., $P(a \leq X \leq b)$:

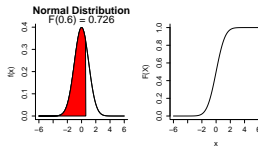


- $P(x < X < x + \Delta x) \approx f(x)\Delta x$
- Probability of any point, $P(X = x) = 0$
- $P(a \leq X \leq b) = P(a < X < b)$

Cumulative Density Function F(x)

Probability density function, $f(X)$

Cumulative distribution function, $F(X) = P(X \leq x) = \int_{-\infty}^x f(x)dx$



- Unlike probabilities $f(x)$ can be greater than 1
- $\int f(x)dx = 1$ (area under the curve is one)
- $F(x)$ goes from 0 to 1

Mean of a Discrete Random Variable

The **mean** for a discrete random variable with probability function, $p(x)$, is given by:

$$E[x] = \sum x p(x)$$

Example: Calculate $E[x]$, where X = sum of two dice

```
x<-2:12
px<-c(1:6,5:1)/36
sum(x*px)
```

```
[1] 7
```

Total on dice	Pairs of dice	Probe
2	1+1	1/36
3	1+2, 2+1	2/36
4	1+3, 2+2, 3+1	3/36
5	1+4, 2+3, 3+2, 4+1	4/36
6	1+5, 2+4, 3+3, 4+2, 5+1	5/36
7	1+6, 2+5, 3+4, 4+3, 5+2, 6+1	6/36

Variance and Standard Deviation

The **variance** for a discrete random variable with probability function, $p(x)$, and mean $E[x]$ is given by:

$$\text{var}(x) = E(X - E(X))^2 = \sum (x - E[x])^2 p(x) = E[x^2] - (E[x])^2$$

The **standard deviation** is $\sigma = \sqrt{\text{var}(x)}$

For continuous random variables

Normal Distribution $X \sim N(\mu, \sigma^2)$

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

Parameters:

- $\mu = E[X]$
- $\sigma^2 = \text{Var}[x]$

Characteristics:

- Mean and variance are independent (knowing one tells us nothing about the other)... this is unique!
- X can take on any value (i.e., the range goes from $-\infty$ to ∞)
- R normal functions: **dnorm**, **pnorm**, **qnorm**, **rnorm**.
- JAGS: **dnorm**

Mean: $E[x] = \mu = \int_{-\infty}^{\infty} x f(x) dx$

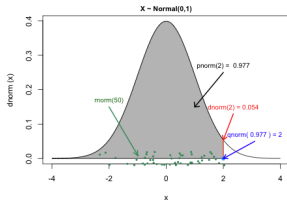
Variance: $\int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx$

Distributions in R

For each probability distribution in R, there are 4 basic probability functions, starting with either - **d**, **p**, **q**, or **r**:

- **d** is for “density” and returns the value of $f(x)$ - **probability density function** (continuous distributions) - **probability mass function** (discrete distributions).
- **p** is for “probability”; returns a value of $F(x)$, **cumulative distribution function**.
- **q** is for “quantile”; returns a value from the inverse of $F(X)$; also known as the quantile function.
- **r** is for “random”; generates a random value from the given distribution.

Functions in R



Use this graph, and R help functions if necessary, to complete Exercise 9.1 in the companion book.

Why is the Normal Distribution so Popular

- Central limit theorem (as n gets large, \bar{x} , $\sum x$) become normally distributed
- Model for measurements that are influenced by a large number of factors that act in an additive way

Other notes:

- In JAGS, WinBugs, specified in terms of precision $\tau = 1/\sigma^2$
- In R, specified in terms of σ not σ^2 .
- Often used for priors (Bayesian analysis) to express ignorance (e.g., $N(0,100)$ for regression parameters).

log-normal Distribution: $X \sim \text{Lognormal}(\mu, \sigma)$

- X has a log-normal distribution if $\log(X) \sim N(\mu, \sigma^2)$
- μ and σ are the mean and variance of $\log(X)$ not X
- Range: > 0
- R: **dlnorm**, **plnorm**, **qlnorm**, **rlnorm** with parameters **meanlog** and **sdlog**
- $E[X] = \exp(\mu + 1/2\sigma^2)$
- $\text{Var}(X) = \exp(2\mu + \sigma^2)(\exp(\sigma^2) - 1)$
- $\text{Var}(X) = kE[X]^2$

Motivation for log-normal

CLT: if we sum a lot of independent things, then we get a normal distribution.

If we multiply a lot of independent things, we get a log-normal distribution, since:

$$\log(X_1 X_2 \cdots X_n) = \log(X_1) + \log(X_2) + \dots \log(X_n)$$

Possible examples in biology? population dynamic models...

Lognormal Distribution

Explore briefly in R:

```
curve(dlnorm(x, meanlog=0, sdlog=2), from=0, to=1000)
eps<-rlnorm(10000, meanlog=0, sdlog=2)
mean(eps)
var(eps)
```

Compare to the expressions for the mean and variance as a function of (μ, σ) :

- $E[X] = \exp(\mu + 1/2\sigma^2)$
- $Var(X) = \exp(2\mu + \sigma^2)(\exp(\sigma^2) - 1)$

Bernouli Distribution: $X \sim \text{Bernouli}(p)$

$$f(x) = P(X = x) = p^x (1 - p)^{1-x}$$

Discrete random variable with two possible outcomes

x	0	1
p(x)	1-p	p

- One parameter, p , the probability of 'success' = $P(X = 1)$
 - $0 \leq p \leq 1$
- $E[X] = \sum xp(x) = 0(1 - p) + 1p = p$
- $Var[x] = \sum (x - E[x])^2 p(x)$

$$= (0 - p)^2(1 - p) + (1 - p)^2 p = p(1 - p)$$
- JAGS and WinBugs: **dbern**
- R has only Binomial distribution (next)

Binomial random variable: $X \sim \text{Binomial}(n, p)$

A **binomial random variable** counts the the number of "successes" (any outcome of interest) in a sequence of trials where

- The number of trials, n , is fixed in advance
- The probability of success, p , is the same on each trial
- Successive trials are independent of each other

Formally, a binomial random variable arises from a sum of *independent* Bernoulli random variables, each with parameter, p :

$$Y = X_1 + X_2 + \dots X_n$$

Binomial: $X \sim \text{Binomial}(n, p)$

- $E[X] = np$
- $\text{Var}(x) = np(1 - p)$
- In R: `dbinom`, `pbinom`, `qbinom`, `rbinom`
- `size = n` and `prob = p` when using these functions.

Examples:

- X = Number of heads in 2 coin flips ($n = 2$, $p = 0.5$)
- Y = number of males in a clutch, class, herd
- Z = number of animals detected among N present

Calculating Binomial Probabilities

YAHTZEE! Count the number of sixes in five dice rolls

On each roll:

- Success (S) = a "6"
- Fail (F) = any other number

X = number of S's in 5 trials:

- $P(s) = p = 1/6$
- $n = 5$

$$P(X = 5) = P(SSSSS) = P(S)P(S)P(S)P(S)P(S) = \left(\frac{1}{6}\right)^5 = 0.00013$$

$$P(X = 0) = P(F)^5 = \left(\frac{5}{6}\right)^5 = 0.4019$$

Calculating Binomial Probabilities

X = number of S's in 5 trials:

- $p = 1/6$
- $n = 5$

$$P(X = 1)$$

$$= P(SFFFF) + P(FSFFF) + P(FFSFF) + P(FFFSF) + P(FFFSS)$$

$$= 5 \frac{1}{6} \left(\frac{5}{6}\right)^4 = 0.419$$

- 5 = number of arrangements with one S and four F
- Probability of each arrangement = $\frac{1}{6} \left(\frac{5}{6}\right)^4$

Binomial Probability Function

For a binomial random variable with n trials and probability of success p on each trial, the probability of exactly k successes in the n trials is:

$$P(x = k) = \binom{n}{k} p^k (1 - p)^{n-k}$$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \text{ with } n! = n(n-1)(n-2) \cdots (2)1$$

Calculate $P(X = 3)$ in the YAHTZEE example ($n = 5$, $p = 1/6$)

$$= \binom{5}{3} \frac{1}{6}^3 \frac{5}{6}^2 = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{(3 \cdot 2 \cdot 1)(2 \cdot 1)} \frac{1}{216} \frac{25}{36} = 0.0322$$

Free Throws

Raymond Felton's free throw percentage during the 2004-2005 season at North Carolina was 70%. If we assume successive attempts are independent, what is the probability that he would hit **at least 4** out of 6 free throws in 2005 Championship Game (he hit 5)?

$$P(X \geq 4) = P(X = 4) + P(X = 5) + P(X = 6)$$

$$= \binom{6}{4} 0.7^4 0.3^2 + \binom{6}{5} 0.7^5 0.3^1 + 0.7^6$$

```
choose(6,4) * (0.7)^4 * (0.3)^2 + choose(6,5) * (0.7)^5 * (0.3) + 0.7^6
```

```
[1] 0.74431
```

```
sum(dbinom(4:6, size=6, p=0.7))
```

```
[1] 0.74431
```

```
pbinom(3, size=6, p=0.7, lower.tail=FALSE)
```

```
[1] 0.74431
```



Multinomial Distribution

$$X \sim \text{Multinomial}(n, p_1, p_2, \dots, p_k)$$

- Records the number of events falling into each of k different categories out of n trials.
- Parameters: p_1, p_2, \dots, p_k (associated with each category)
- $p_k = 1 - \sum_{j=1}^{k-1} p_j$
- Generalizes the binomial to more than 2 (unordered) categories
- R: **dmultinom**, **pmultinom**, **qmultinom**, **rmultinom**.
- JAGS: **dmulti**

Multinomial distribution

$X = (x_1, x_2, \dots, x_k)$ a multivariate random variable recording the number of events in each category

If (n_1, n_2, \dots, n_k) is the observed number of events in each category, then:

$$P((x_1, x_2, \dots, x_k) = (n_1, n_2, \dots, n_k)) = \frac{n!}{n_1! n_2! \dots n_k!} p_1^{n_1} p_2^{n_2} \dots p_k^{n_k}$$

Poisson Distribution: $N_t \sim \text{Poisson}(\lambda)$

Let N_t = number of events occurring in a time interval of length t . What is the probability of observing k events in this interval?

$$P(N_t = k) = \frac{\exp(-\lambda t)(\lambda t)^k}{k!}$$

Events in 2-D space, if events occur at a constant rate, the probability of observing k events in an area of size A :

$$P(N_A = k) = \frac{\exp(-\lambda A)(\lambda A)^k}{k!}$$

If A or t is constant:

$$P(N = k) = \frac{\exp(-\lambda)(\lambda)^k}{k!}$$

Poisson distribution

- Single parameter, $\lambda = \text{lambda}$.
- $E[X] = \text{Var}(x) = \lambda$
- R: **dpois**, **ppois**, **qpois**, and **rpois**.
- JAGS: **dpois**

Examples:

- Spatial statistics (null model of “complete spatial randomness”)
- Can be motivated by random event processes with constant rates of occurrence in space or time
- $\text{Binomial}(n, p) \rightarrow \text{Poisson}(\lambda = np)$ as $n \rightarrow \infty$ if $p \rightarrow 0$ (such that $np \rightarrow \text{a constant}$)

Poisson distribution

Suppose a certain region of California experiences about 5 earthquakes a year. Assume occurrences follow a Poisson distribution. What is the probability of 3 earthquakes in a given year?

```
dpois(3, lambda=5)
```

```
## [1] 0.1403739
```

```
5^3*exp(-5) / (3*2)
```

```
## [1] 0.1403739
```

Geometric Distribution

Number of failures until you get your first success.

$$f(x) = P(X = x) = (1 - p)^x p$$

- Parameter = p (probability of success)
- Range: $\{0, 1, 2, \dots\}$
- $E[x] = \frac{1}{p} - 1$
- $\text{Var}[x] = \frac{(1-p)}{p^2}$
- ***geom**

Negative Binomial: Classic Parameterization

X_r = Number of failures, x , before you get r successes; $X_r \sim \text{NegBinom}(p)$

- Total of $n = x + r$ trials
- Last trial is a success (p)
- The preceding $x + r - 1$ trials had x failures (equiv. to a binomial experiment)

$$P(X = x) = \binom{x+r-1}{x} p^{r-1} (1-p)^x p$$

or

$$P(X = x) = \binom{x+r-1}{x} p^r (1-p)^x$$

- $E[x] = \frac{r(1-p)}{p}$
- $\text{Var}[x] = \frac{r(1-p)}{p^2}$

Ecological Parameterization

Express p in terms of mean, μ and r :

$$\mu = \frac{r(1-p)}{p} \Rightarrow p = \frac{r}{\mu+r} \text{ and}$$
$$1-p = \frac{\mu}{\mu+r}$$

Plugging these values in to $f(x)$ and changing r to θ , we get:

$$P(X=x) = \binom{x+\theta-1}{x} \left(\frac{\theta}{\mu+\theta}\right)^\theta \left(\frac{\mu}{\mu+\theta}\right)^x$$

Then, let θ = dispersion parameter take on any positive number (not just integers as in the original parameterization)

Negative Binomial: $X \sim \text{NegBin}(\mu, \theta)$

$$P(X=x) = \binom{x+\theta-1}{x} \left(\frac{\theta}{\mu+\theta}\right)^\theta \left(\frac{\mu}{\mu+\theta}\right)^x$$

- $E[x] = \mu$
- $\text{Var}(x) = \mu + \frac{\mu^2}{\theta}$
- In R: ***nbinom**, with parameters (prob = p, size = n) or (mu = μ , size = θ)
- JAGS: **dnegbin** with parameters (p, θ)

Overdispersed relative to Poisson ($\text{Var}(x)/E[x] = 1 + \frac{\mu}{\theta}$) versus 1 for Poisson

Poisson is a limiting case (when $\theta \rightarrow \infty$)

Negative Binomial

Its appeal for use as a probability generating mechanism in ecology includes the following.

- Allows for non-constant variance typical of count data.
- It often fits zero-inflated data well (and much better than a Poisson distribution).
- It respects the discreteness of the data (no need to transform).
- It can be motivated biologically - e.g.:

If: $X_i \sim \text{Poisson}(\lambda_i)$, with $\lambda_i \sim \text{Gamma}(\alpha, \beta)$, then X_i has a negative binomial distribution.

Continuous Uniform

If observations are equally likely within an interval (A,B):

$$f(x) = \frac{1}{b-a}$$

- Two parameters (a, b)
- Model of ignorance for prior distributions
- $E[x] = (a+b)/2$
- $\text{Var}(x) = \frac{(b-a)^2}{12}$
- ***unif**
- JAGS: **dunif(lower, upper)**

Gamma Distribution: $X \sim \text{Gamma}(\alpha, \beta)$

$$f(x) = \frac{1}{\Gamma(\alpha)} x^{\alpha-1} \beta^\alpha \exp(-\beta x)$$

- Range 0 to ∞
- $\Gamma(\alpha)$ is a generalization of the factorial function (!) that we've seen earlier
- α and β are parameters > 0 .
- $E[x] = \frac{\alpha}{\beta}$
- $\text{Var}[x] = \frac{\alpha}{\beta^2}$
- R: ***gamma**

Beta Distribution: $X \sim \text{Beta}(\alpha, \beta)$

$$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1} (1-x)^{\beta-1}$$

- ranges from 0 to 1.
- α and β are parameters > 0 .
- $E[x] = \frac{\alpha}{\alpha+\beta}$
- $\text{Var}[x] = \frac{\alpha}{(\alpha+\beta)^2(\alpha+\beta+1)}$
- R: ***beta**

Exponential: $X \sim \text{Exp}(\lambda)$

$$f(x) = \lambda \exp(-\lambda x)$$

- Range 0 to ∞
- $\lambda > 0$
- $E[x] = \frac{1}{\lambda}$
- $\text{Var}[x] = \frac{1}{\lambda^2}$
- R: ***exp**

Distributions

How do we choose an appropriate distribution for our data? (Zuur et al. ch 8.7.1):

- Presence-absence (0,1) at M sites \rightarrow Bernoulli distribution
- Counts, fixed number of sites/trials/etc \rightarrow Binomial distribution
- Counts, in a fixed unit of time, area
 - Normal distribution if the counts are large
 - Poisson distribution: if $E[Y|X] \approx \text{Var}(Y|X)$
 - If $\text{Var}(Y|X) > E(Y|X)$: Negative Binomial, Quasipoisson, Poisson-normal model
- Continuous response variable: normal distribution (usual default)
 - gamma (if Y must be > 0)
 - lognormal (if skewed data)
- Time to event: exponential, Weibull

Other useful information

For a diagram showing links between distributions, see:

• [Diagram of distribution relationships](#)

- [{http://www.johndcook.com/distribution/_chart.html}](http://www.johndcook.com/distribution/_chart.html)

See handout with distributions (note that some can be written in multiple ways):

For example, gamma: $f(x) = \frac{1}{\Gamma(\alpha)} x^{\alpha-1} \beta^\alpha \exp(-\beta x)$

$$f(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} \exp(-x/\beta)$$