Understanding and Dealing with Collinearity

FW8051 Statistics for Ecologists

Department of Fisheries, Wildlife and Conservation Biology

Learning Objectives: Collinearity

- What is collinearity/multicollinearity?
- How does one assess collinearity?
- What are the different types of collinearity?
- What are the effects of collinearity on
    - Parameter estimates
    - Standard errors

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Will draw from:

- A lecture by Todd Steury, Auburn University

What is Collinearity?

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Multicollinearity implies one of the explanatory variables can be predicted by the others with a high degree of accuracy.

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- Compositional variables: have to sum to 1 (the last category is completely determined by the others)
  - e.g., percent cover of different habitat types.
- Structural collinearity: occurs when variables are measured on very different scales (e.g., X, X^2)
- Incidental collinearity: variables may have separate effects but happen to covary in the data set possibly due to small sample sizes

Examples of Collinearity

- Habitat attributes: riparian areas also tend to have thick understory cover
- Urban areas have lots of impervious surface, minimal forest cover, high density of humans
- Areas farther north tend to be colder, get more snow, less sunlight in winter.

[Think-pair-share] Do you have similar examples from your study systems?

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- Expected change in Y as we change X by 1 unit, while holding all other predictor variables constant.

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Interpretation of β in a multiple regression:

- Expected change in Y as we change X by 1 unit, while holding all other predictor variables constant.

- When predictors are collinear, changes in X tend to occur with changes in other variables too (making β hard to estimate)

Variance Inflation Factors

Multicollinearity can be measured using a variance inflation factor (VIF)

$$VIF(\hat{\beta}_j) = \frac{1}{1-R^2_{x_j|x_1,\ldots,x_{j-1},x_{j+1},x_p}}, \text{ where}$$

$R^2_{x_j|x_1,\ldots,x_{j-1},x_{j+1},x_p}$ = multiple R^2 from:

$$\texttt{lm(}x_j \sim x_1 + \ldots + x_{j-1} + x_{j+1} + x_p\texttt{)}$$

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Calculate in R: `vif` in the `car` package

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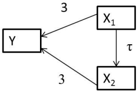
Calculate in R: `vif` in the `car` package

Rules of Thumb in Published Literature:

- Many suggest VIFs ≥ 10 are problematic
- Graham (2003): VIFs as small as 2 can have significant impacts

Simulation study: Confounding Variables



Truth: $Y_i = 10 + 3X_{1,i} + 3X_{2,i} + \epsilon_i$ with $\epsilon_i \sim N(0, 2)$

- $X_{1,i} \sim U(0, 10)$
- $X_{2,i} = \tau X_{1,i} + \gamma_i$ with $\gamma_i \sim N(0, 4)$
- Varied τ from 0 to 9 by 3 (`tau<-seq(0,9,3)`)

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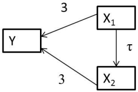

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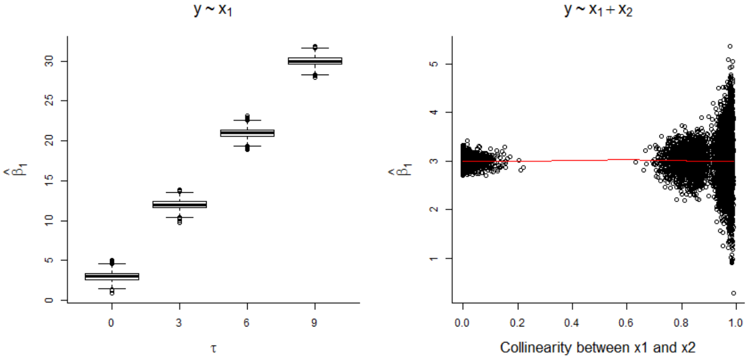
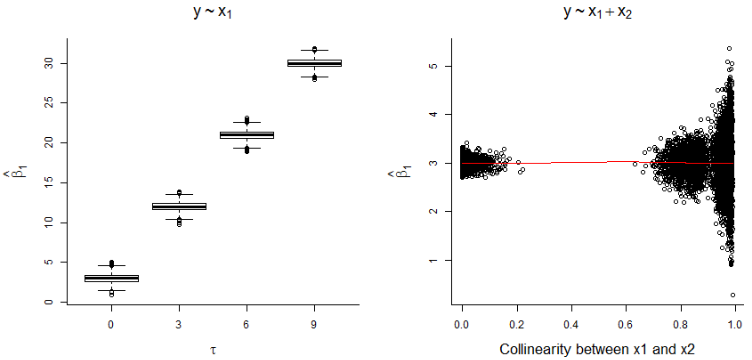
Simulated 2000 data sets and to each fit:

- `lm(Y` $\sim X_1$`)`
- `lm(Y` $\sim X_1 + X_2$`)`



- coefficient for X_1 is biased when X_2 is not included (unless $\tau = 0$)
- magnitude of the bias increases with the correlation between X_1 and X_2 (i.e., with τ)



- coefficient for X_1 is biased when X_2 is not included (unless $\tau = 0$)
- magnitude of the bias increases with the correlation between X_1 and X_2 (i.e., with τ)
- coefficient for X_1 is unbiased when X_2 is included, but SE increases when X_1 and X_2 are highly correlated

Mathematically…

$$Y_i = 10 + 3X_{1,i} + 3X_{2,i} + \epsilon_i \text{ and } X_{2,i} = \tau X_{1,i} + \gamma_i$$

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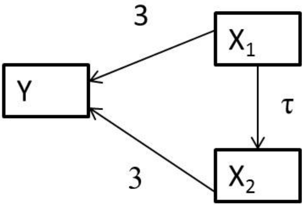
$$Y_i = 10 + 3X_{1,i} + 3(\tau X_{1,i} + \gamma_i) + \epsilon_i$$

$$Y_i = 10 + (3 + 3\tau)X_{1,i} + (3\gamma_i + \epsilon_i)$$

Causal Networks



X_1 captures the effect of both X_1 and X_2 when X_2 is left out of the model!

When we leave X_2 out of the model, the coefficient for X_1 captures the direct of effect of X_1 on Y and also the indirect effect of X_1 on Y (mediated by X_2)

Trade-offs

Models with collinear variables

- Large standard errors

Models in which confounding variables are left out

- Misleading estimates of effect due to omission of important variables

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Models in which confounding variables are left out

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When possible, try to eliminate confounding variables via study design (e.g., experiments, matching)

Strategies for dealing with multicollinearity

If the only goal is prediction, may choose to ignore multicollinearity

- Often has minimal impact on prediction error (versus SEs for individual coefficients)
- Can be problematic if making out-of-sample predictions where the extent and nature of collinearity changes

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For estimation, there are methods that introduce some bias to improve precision

- Ridge regression, LASSO (ch 8)

Other methods

Graham (2003) and the textbook also briefly consider:

- Residual and sequential regression
- Principal component regression
- Structural equation models (related to causal inference methods in Ch 7)

Example from Graham (2003):

- OD = wave orbital displacement (in meters)
- BD = wave breaking depth (in meters)
- LTD = average tidal height (in meters)
- W = wind velocity (in meters/s).

```r
library(car)
vif(lm(Response~OD+BD+LTD+W, data=Kelp))
```

```
      OD       BD      LTD        W
2.574934 2.355055 1.175270 2.094319
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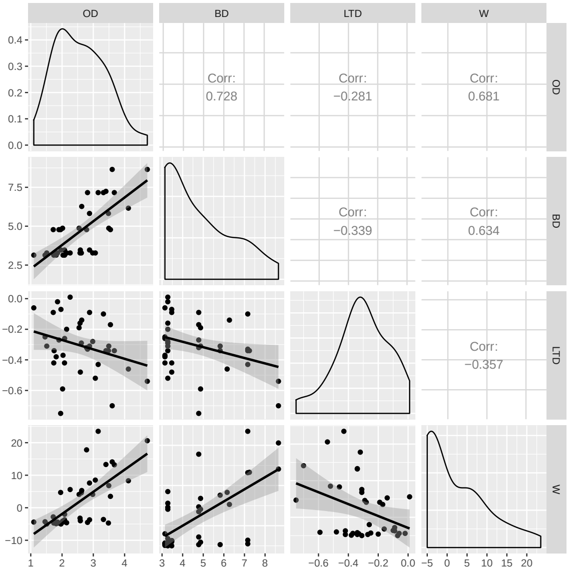
Always look at the relationship among your predictors (without the response variables) as a first step to assessing collinearity!

```r
library(GGally)
ggpairs(Kelp[,c("OD", "BD", "LTD", "W")],
        lower = list(continuous = "smooth"))
```



Residual and sequential regression

Prioritize different variables to include sequentially:

- Include x_1 (unique and shared contributions)
- Then, residuals of `lm(`$x_2 \sim x_1$`)` (part of x_2 not shared with x_1)
- Then, residuals of `lm(`$x_3 \sim x_1 + x_2$`)` (part of x_3 not shared with x_1 or x_2)
- ...

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How to Prioritize?

- Instincts and intuition
- Previously collected data

Graham considered (newly formed) predictors in this order:

- OD = captures unique effect of OD + shared effect with other variables
- W | OD = captures effect of W not shared with OD
- LTD | OD, W = captures effect of LTD that is not shared with OD or W
- BD | OD, W, LTD = captures effect of BD not shared with OD, W, LTD

```
Kelp$W.g.OD<-lm(W~OD, data=Kelp)$resid
Kelp$LTD.g.W.OD<-lm(LTD~W+OD, data=Kelp)$resid
Kelp$BD.g.W.OD.LTD<-lm(BD~W+OD+LTD, data=Kelp)$resid
```

```r
seq.lm<-lm(Response~OD+W.g.OD+LTD.g.W.OD+BD.g.W.OD.LTD, data=Kelp)
summary(seq.lm)
```

```
Call:
lm(formula = Response ~ OD + W.g.OD + LTD.g.W.OD + BD.g.W.OD.LTD,
    data = Kelp)

Residuals:
      Min        1Q    Median        3Q       Max
-0.284911 -0.098861 -0.002388  0.099031  0.301931

Coefficients:
               Estimate Std. Error t value Pr(>|t|)
(Intercept)    2.747588   0.078192  35.139  < 2e-16 ***
OD             0.194243   0.028877   6.726 1.16e-07 ***
W.g.OD         0.008082   0.003953   2.045   0.0489 *
LTD.g.W.OD    -0.055333   0.141350  -0.391   0.6980
BD.g.W.OD.LTD -0.004295   0.021137  -0.203   0.8402
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.1431 on 33 degrees of freedom
Multiple R-squared:  0.6006,    Adjusted R-squared:  0.5522
F-statistic: 12.41 on 4 and 33 DF,  p-value: 2.893e-06
```

```r
seq.lm2<-lm(Response~OD+W.g.OD, data=Kelp)
summary(seq.lm2)$coef
```

```
               Estimate  Std. Error   t value     Pr(>|t|)
(Intercept) 2.747587774 0.076148241 36.082091 2.796476e-29
OD          0.194243475 0.028122800  6.906975 5.038589e-08
W.g.OD      0.008082141 0.003849614  2.099468 4.305538e-02
```

Regression parameter estimates did not change.

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Disadvantages:

- Requires prioritization (which may not be reflect functional importance of the variables)

Principal Components Regression

Form new predictors as linear combinations of the correlated variables:

$pca_1 = \lambda_{1,1}X_1 + \lambda_{1,2}X_2 + \ldots \lambda_{1,p}x_p$

$pca_2 = \lambda_{2,1}X_1 + \lambda_{2,2}X_2 + \ldots \lambda_{2,p}x_p$

$\ldots$

$pca_p = \lambda_{p,1}X_1 + \lambda_{p,2}X_2 + \ldots \lambda_{p,p}x_p$, where

- The pca_i's are all orthogonal (statistically independent)
- pca_1 accounts for the greatest variation in $(x_1, x_2, ..., x_p)$
- pca_2 accounts for greatest amount of remaining variation in $(x_1, x_2, ..., x_p)$, not accounted for by pca_1
- $\ldots$

```r
pcas<-prcomp(~OD+BD+LTD+W, data=Kelp, scale=TRUE)
pcas$rotation
```

```
           PC1        PC2         PC3         PC4
OD   0.5479919 -0.2901058 -0.15915149 -0.76825404
BD   0.5453470 -0.1793692 -0.58088137  0.57706165
LTD -0.3384653 -0.9335391  0.06706729  0.09720099
W    0.5364166 -0.1103180  0.79545560  0.25949479
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W    0.5364166 -0.1103180  0.79545560  0.25949479
```
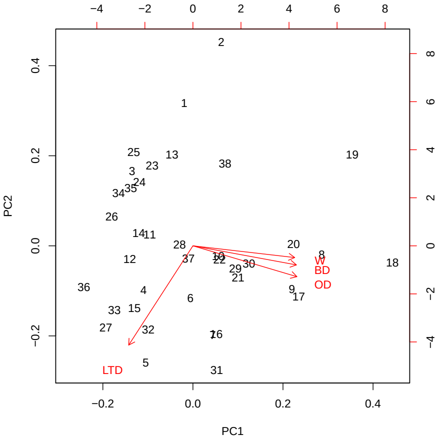

```r
head(cbind(Kelp[,2:5], pcas$x))
```

```
      OD   BD   LTD    W         PC1        PC2         PC3
1 2.0176 4.87 -0.59 -4.1 -0.19127827  1.7527358 -0.66278941
2 1.9553 4.78 -0.75  4.7  0.62234092  2.5023873  0.18091063
3 1.8131 3.14 -0.38 -4.9 -1.33268779  0.9190480 -0.03361542
4 2.5751 3.28 -0.16 -3.2 -1.08056344 -0.5416139  0.01891911
5 2.2589 3.28  0.01  5.6 -1.03524778 -1.4381622  1.00570204
6 2.5448 4.87 -0.19  4.1 -0.05452203 -0.6398905  0.18695547
          PC4
1  0.24694830
2  0.46900655
3 -0.05590063
4 -0.55322453
5  0.11858908
6  0.22937274
```

Biplot

```r
biplot(pcas)
```



Principal Components Regression

```r
summary(pcas)
```

```
Importance of components:
                          PC1    PC2     PC3     PC4
Standard deviation     1.6017 0.8975 0.60895 0.50822
Proportion of Variance 0.6413 0.2014 0.09271 0.06457
Cumulative Proportion  0.6413 0.8427 0.93543 1.00000
```

The first principal component explains 64% of the variation in (OD, BD, LTD, W)

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The first principal component explains 64% of the variation in (OD, BD, LTD, W)

Choose one or more pca_i to include as new regressors (Graham 2003 suggests including all of them).

- pca_1 explains the greatest variation in $(x_1, x_2, ..., x_p)$ (not necessarily the greatest variation in Y)
- Since the pca_i's are orthogonal, the coefficients will not change as other pca_i's are added or dropped.

Principal Components Regression

```r
Kelp<-cbind(Kelp, pcas$x)
lm.pca<-lm(Response~ PC1+PC2+PC3+PC4, data=Kelp)
summary(lm.pca)
```

```
Call:
lm(formula = Response ~ PC1 + PC2 + PC3 + PC4, data = Kelp)

Residuals:
      Min        1Q    Median        3Q       Max
-0.284911 -0.098861 -0.002388  0.099031  0.301931

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  3.24984    0.02321 140.035  < 2e-16 ***
PC1          0.09806    0.01468   6.678 1.33e-07 ***
PC2         -0.02971    0.02620  -1.134    0.265
PC3          0.03612    0.03862   0.935    0.356
PC4         -0.07826    0.04628  -1.691    0.100
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.1431 on 33 degrees of freedom
Multiple R-squared:  0.6006,    Adjusted R-squared:  0.5522
F-statistic: 12.41 on 4 and 33 DF,  p-value: 2.893e-06
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Principal Components Regression

The main disadvantage is the principal components can be difficult to interpret.

Options:

- Can apply separately to groups of like variables (“weather”, “vegetation”, etc)
- Consider other “rotations” (that ensure that some $\lambda_{i,j} = 0$)
- Other variable clustering methods that group variables (Harrell 2001. Regression Modeling Strategies).

Structural Equation Modeling

- Chapter on Causal Models (on Moodle)
- Allows for direct and indirect effects
- Can account for unique and shared contributions (the latter through latent variables)
- Focuses on *a priori* modeling and testing of hypothesized relationships

Conclusions from Graham (2003)

"The suite of techniques described herein compliment each other and offer ecologists useful alternatives to standard multiple regression for identifying ecologically relevant patterns in collinear data. Each comes with its own set of benefits and limitations, yet together they allow ecologists to directly address the nature of shared variance contributions in ecological data."