Delta Method

FW8051 Statistics for Ecologists

Department of Fisheries, Wildlife and Conservation Biology

Learning Objectives

Understand how we can use the delta method to calculate SEs for functions of parameters

See also:

Approximating Variance of Demographic Parameters Using the Delta Method: A Reference for Avian Biologists FREE

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In the GLS section, we learned how to calculate $var(\hat{\beta}_0 + X_i\hat{\beta}_1)$ using matrix multiplication

And, more generally: $var(X\beta)$ for design matrix X :

$$\begin{bmatrix} 1 & X_1 \\ 1 & X_2 \\ \vdots & \vdots \\ 1 & X_n \end{bmatrix} \begin{bmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \end{bmatrix} \quad \Sigma = \begin{bmatrix} \sigma^2_{\hat{\beta}_0} & \sigma^2_{\hat{\beta}_0,\hat{\beta}_1} \\ \sigma^2_{\hat{\beta}_0,\hat{\beta}_1} & \sigma^2_{\hat{\beta}_1} \end{bmatrix}$$

- σ^2_X, σ^2_Y = variance of $\hat{\beta}_0$, $\hat{\beta}_1$
- $\sigma^2_{\hat{\beta}_0,\hat{\beta}_1}$ = covariance of $\hat{\beta}_0$ and $\hat{\beta}_1$

Recall: var$(X\beta) = X\Sigma X^T$

What if we are interested in non-linear functions of parameters?

$$Length_i = L_\infty(1 - \exp(-kAge_i))$$

Options:

- Bootstrap
- Delta method
- Baysian inference

$$Length_i = L_\infty(1 - \exp(-kAge_i))$$

Want to calculate a confidence interval for the length at a particular age, Age_i:

$var(\hat{L}_\infty(1 - \exp(-\hat{k}Age_i))) = f(\hat{L}_\infty, \hat{k})$

If we estimate $\theta = (L_\infty, k)$ using Maximum likelihood, and our sample size is large, we know:

$$\hat{\theta} \sim MVN(\theta, I^{-1}(\theta)) \text{ with:}$$

- $\mathrm{I}(\theta) = \left[\frac{\partial^2 logL(\theta)}{\partial\theta^2}\right]$ is the Hessian matrix

Delta Method

Let:

- $f(L_\infty, k) = L_\infty(1 - \exp(-kAge_i))$
- $f'(L_\infty, k) = (\frac{\partial f}{\partial L_\infty}, \frac{\partial f}{\partial k})$
- Σ be the asympototic variance/covariance matrix of (L_∞, k) given by the inverse of the Hessian matrix

Delta Method (derived using a Taylor's series approximation of f):

$var(\hat{L}_\infty(1 - \exp(-\hat{k}Age_i))) \approx f'(\hat{L}_\infty, \hat{k})\Sigma f'(\hat{L}_\infty, \hat{k})^T|_{L_\infty = \hat{L}_\infty, k = \hat{k}}$

More generally:

$var(f(\theta)) \approx f'(\theta)\Sigma f'(\theta)^T|_{\theta=\hat{\theta}}$

Implementation

In R:

- use the `detavar` function in the `emdbook` package to calculate the derivatives and variance (see `FemalesvonB.R`)
- or, calculate the derivatives yourself (or using https://www.symbolab.com/solver/derivative-calculator), then roll your own with `%*%` for matrix multiplication.

$f(\theta) = L_\infty(1 - \exp(-kAge_i))$

$f'(\theta) = (1 - \exp(-kAge_i), L_\infty Age_i \exp(-kAge_i))$

Derivatives