

# Delta Method

## FW8051 Statistics for Ecologists

Department of Fisheries, Wildlife and Conservation Biology



# Learning Objectives

Understand how we can use the **delta method** to calculate SEs for functions of parameters

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Understand how we can use the **delta method** to calculate SEs for functions of parameters

See also:

## Approximating Variance of Demographic Parameters Using the Delta Method: A Reference for Avian Biologists

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*The Condor*, Volume 109, Issue 4, 1 November 2007, Pages 949–954,

<https://doi.org/10.1093/condor/109.4.949>

In the GLS section, we learned how to calculate  $\text{var}(\hat{\beta}_0 + X_i\hat{\beta}_1)$  using matrix multiplication

And, more generally:  $\text{var}(X\beta)$  for design matrix  $X$  :

$$\begin{bmatrix} 1 & X_1 \\ 1 & X_2 \\ \vdots & \vdots \\ 1 & X_n \end{bmatrix} \begin{bmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \end{bmatrix} \quad \Sigma = \begin{bmatrix} \sigma_{\hat{\beta}_0}^2 & \sigma_{\hat{\beta}_0, \hat{\beta}_1}^2 \\ \sigma_{\hat{\beta}_0, \hat{\beta}_1}^2 & \sigma_{\hat{\beta}_1}^2 \end{bmatrix}$$

- $\sigma_X^2, \sigma_Y^2 = \text{variance of } \hat{\beta}_0, \hat{\beta}_1$
- $\sigma_{\hat{\beta}_0, \hat{\beta}_1}^2 = \text{covariance of } \hat{\beta}_0 \text{ and } \hat{\beta}_1$

Recall:  $\text{var}(X\beta) = X\Sigma X^T$

What if we are interested in non-linear functions of parameters?

$$Length_i = L_{\infty}(1 - \exp(-kAge_i))$$

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$$Length_i = L_{\infty}(1 - \exp(-kAge_i))$$

Options:

- Bootstrap
- Delta method
- Bayesian inference

$$Length_i = L_{\infty}(1 - \exp(-kAge_i))$$

Want to calculate a confidence interval for the length at a particular age,  $Age_i$ :

$$var(\hat{L}_{\infty}(1 - \exp(-\hat{k}Age_i))) = f(\hat{L}_{\infty}, \hat{k})$$

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If we estimate  $\theta = (L_{\infty}, k)$  using Maximum likelihood, and our sample size is large, we know:

$$\hat{\theta} \sim MVN(\theta, I^{-1}(\theta)) \text{ with:}$$

- $I(\theta) = \left[ \frac{\partial^2 \log L(\theta)}{\partial \theta^2} \right]$  is the Hessian matrix



# Delta Method

Let:

- $f(L_\infty, k) = L_\infty(1 - \exp(-kAge_i))$
- $f'(L_\infty, k) = (\frac{\partial f}{\partial L_\infty}, \frac{\partial f}{\partial k})$
- $\Sigma$  be the asymptotic variance/covariance matrix of  $(L_\infty, k)$  given by the inverse of the Hessian matrix

**Delta Method** (derived using a Taylor's series approximation of  $f$ ):

$$var(\hat{L}_\infty(1 - \exp(-\hat{k}Age_i))) \approx f'(\hat{L}_\infty, \hat{k})\Sigma f'(\hat{L}_\infty, \hat{k})^T|_{L_\infty=\hat{L}_\infty, k=\hat{k}}$$

# Delta Method

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**Delta Method** (derived using a Taylor's series approximation of  $f$ ):

$$\text{var}(\hat{L}_\infty(1 - \exp(-\hat{k} \text{Age}_i))) \approx f'(\hat{L}_\infty, \hat{k}) \Sigma f'(\hat{L}_\infty, \hat{k})^T |_{L_\infty = \hat{L}_\infty, k = \hat{k}}$$

More generally:

$$\text{var}(f(\theta)) \approx f'(\theta) \Sigma f'(\theta)^T |_{\theta = \hat{\theta}}$$

# Implementation

In R:

- use the `detavar` function in the `emdbook` package to calculate the derivatives and variance (see `FemalesvonB.R`)
- or, calculate the derivatives yourself (or using <https://www.symbolab.com/solver/derivative-calculator>), then roll your own with `%*%` for matrix multiplication.

$$f(\theta) = L_{\infty}(1 - \exp(-kAge_i))$$

$$f'(\theta) = (1 - \exp(-kAge_i), L_{\infty}Age_i \exp(-kAge_i))$$

# Derivatives

## Derivative Calculator

Differentiate functions step-by-step

### Derivatives

- First Derivative
- Second Derivative
- Third Derivative
- Higher Order Derivatives
- Derivative at a point
- Partial Derivative
- Implicit Derivative
- Second Implicit Derivative (new)
- Derivative using Definition (new)

### Derivative Applications

### Limits

### Integrals

### Integral Applications (new)

### Series

### ODE

### Laplace Transform

### Taylor/Maclaurin Series

### Fourier Series (new)

full pad »

$x^2$	$x^{\square}$	$\log_{\square}$	$\sqrt{\square}$	$\sqrt[n]{\square}$	$\leq$	$\geq$	$\ln$	$\cdot$	$\div$	$x^{\circ}$	$\pi$
$(\square)^{\circ}$	$\frac{d}{dx}$	$\frac{\partial}{\partial x}$	$\int$	$\int_{\square}^{\infty}$	$\lim$	$\sum$	$\infty$	$\theta$	$(f \circ g)$	$H_2O$	$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Most Used Actions

simplify

solve for

inverse

tangent

line

See All ▾

$$\frac{d}{dL} \left( L \cdot (1 - e^{-k \cdot A}) \right)$$

Go

Graph » Examples »



Solution

Keep Practicing >

Show Steps ▾

$$\frac{d}{dL} \left( L(1 - e^{-kA}) \right) = 1 - e^{-kA}$$

### Steps

$$\frac{d}{dL} \left( L(1 - e^{-kA}) \right)$$

Take the constant out:  $(a \cdot f)' = a \cdot f'$

$$= (1 - e^{-kA}) \frac{d}{dL} (L)$$

Apply the common derivative:  $\frac{d}{dL} (L) = 1$

$$= (1 - e^{-kA}) \cdot 1$$

Simplify

$$= 1 - e^{-kA}$$

[click here to practice derivatives »](#)