Correlated Data Overview

FW8051 Statistics for Ecologists

Department of Fisheries, Wildlife and Conservation Biology

Learning Objectives

- Be able to model correlated binary and correlated count data using generalized linear mixed effect models (GLMMs) and generalized estimating equations (GEEs)

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- Be able to interpret parameters in linear and generalized linear mixed effects models
- Be able to describe models and their assumptions using equations and text and match parameters in these equations to estimates in computer output.

Correlated Data Methods in Ecology

For data that are normally distributed:

- Linear mixed effects model
- Generalized Least Squares

For count or binary data:

- Generalized linear mixed effects models (GLMMS)
- Generalized Estimating Equations (GEE)

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Fieberg, J., Rieger, R.H., Zicus, M. C., Schildcrout, J. S. 2009. Regression modelling of correlated data in ecology: subject specific and population averaged response patterns. Journal of Applied Ecology 46:1018-1025.

Mallard Nesting structures

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Mallard Nesting structures

Research Questions: which types of cylinders are best (single or double)? Where should they be placed?

Mallard Data

- 110 nest structures placed in 104 wetlands
- Structure type (single versus double) was chosen randomly at each location
  - 53 single-cylinders, 57 double-cylinders
- Occupancy (0,1) and clutch sizes were recorded in 1997, 1998, 1999

Zicus, M.C., J. Fieberg and D. P. Rave. 2003. Does mallard clutch size vary with landscape composition: a different view. Wilson Bulletin 114:409-413.

Zicus, M. C., D. P. Rave, and J. Fieberg. 2006. Cost effectiveness of single- vs. double-cylinder over-water nest structures. Wildlife Society Bulletin 34:647-655.

Zicus, M. C., Rave, D. P., Das, A., Riggs, M. R., and Buitenwerf, M. L. (2006). Influence of land use on mallard nest-structure occupancy. The Journal of wildlife management, 70(5), 1325-1333.

Example Clutch Size Data

Y_{ij} = clutch size for the i^{th} structure during year j

$\texttt{Init.Date}_{ij}$ = nest initiation date (Julian day) for the i^{th} structure during year j

$\texttt{I(deply=2)}_{i}$ = 0 if i^{th} structure is a single cylinder, 1 if double cylinder

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<u>Model</u>

$$Y_{ij} = (\beta_0 + b_{0i}) + \beta_1 \texttt{Init.Date}_{ij} + \beta_2 \texttt{I(deply=2)}_i + \epsilon_{ij}$$
$$\epsilon_{ij} \sim N(0, \sigma^2)$$
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Assume ϵ_{ij} and b_{0i} are independent.

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Assume ϵ_{ij} and b_{0i} are independent.

Similar to including "structure" as a series of dummy variables with the added assumption that the parameters are drawn from a normal distribution

Generalized Least Squares

For normally distributed response data, we can fit correlated data models without having to resort to random effects:

$$\text{Clutch size}_{ij} = \beta_0 + \beta_1 \texttt{Init.Date}_{ij} + \beta_2 \texttt{I(deply=2)}_i + \epsilon_{ij}$$
$$\epsilon_{ij} \sim N(0, \Omega)$$

Ω = Var/Cov matrix for ϵ. We no longer assume the errors, ϵ_{ij}, are independent!

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Example: compound symmetric covariance matrix for data within each cluster:

$$\Omega = \begin{bmatrix} \Sigma_i & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \Sigma_i \end{bmatrix} \text{ with } \Sigma_i = \begin{bmatrix} \sigma^2 & \rho\sigma^2 & \cdots & \rho\sigma^2 \\ \rho\sigma^2 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \rho\sigma^2 \\ \rho\sigma^2 & \cdots & \rho\sigma^2 & \sigma^2 \end{bmatrix}$$

Correlation between:

- 2 observations from the same cluster = ρ
- 2 observations taken from different clusters = 0

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This model is equivalent to a linear mixed effects model with random intercepts for each structure!

Extensions to Count and Binary Data

Unlike the multivariate normal distribution, Poisson, Negative Binomial, Bernoulli, Binomial distributions

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- Start with generalized linear models (logistic or Poisson regression) and add random coefficients (intercepts, slopes)
- Use generalized estimating equations or cluster-level bootstrap (recognizing clusters serve as independent units)

Option 1: Generalized linear mixed effects models

Option 2: Generalized Estimating equations

Generalized Linear Models

Systematic component: $g(\mu_i) = \eta_i = \beta_0 + \beta_1 x_1 + \dots \beta_p x_p$

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- $f(y_i|x_i)$ is in the exponential family (includes normal, Poisson, binomial, gamma, inverse Gaussian)
- $f(y_i|x_i)$ that describes unmodeled variation about $\mu_i = E[Y_i|X_i]$

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- b_{0i} and b_{1i} allow intercepts and parameters to vary among groups.
- Usually assume: $(b_{0i}, b_{1i}) \sim N(0, D)$

Conditional models

Poisson-normal model:

- $Y_{ij}|b_i \sim \text{Poisson}(\lambda_{ij})$
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Frequentist analysis: the b's are unobserved random variables…we do not estimate them! Rather, we estimate β_0, β_1, D.

Non-linear models: Modeling Occupancy Probability

Logistic-normal model:

$$Y_{ij}|b_i \sim \text{Bernoulli}(p_{ij})$$
$$\log[p_{ij}/(1-p_{ij})|b_i] = \beta_0 + b_{0i} + \beta_1 VOM_{ij} + \beta_2 \texttt{I(deply=2)}_i$$
$$b_{0i} \sim N(0, \tau^2)$$

Structures have different “propensities” of being occupied:

- Depending on visual obstruction (VOM), structure type, and other unmeasured characteristics (b_{0i}) associated with the structure and the landscape in which it is placed.

Parameter Estimation

To estimate parameters using Maximum Likelihood, we need to determine:

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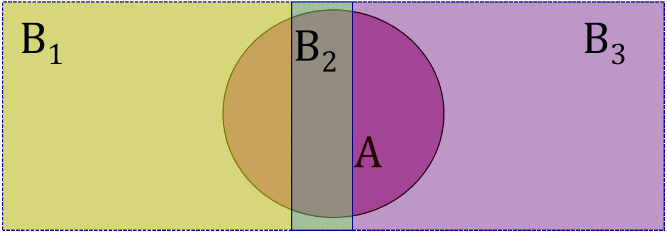
Total law of Probability If events $B_1, B_2, \ldots, B_k$ are mutually exclusive and together make up all possibilities, then:

$P(A) = \sum_i P(A|B_i)P(B)$



Unconditional Model and Likelihood

To determine the distribution of Y_{ij}, we integrate over the random effects:

$$L(Y_{ij}|\beta, D) = \int f(Y_{ij}|b_i) f(b_i) db_i$$

- β = fixed effects parameters in $f(Y_{ij}|b_i)$
- D are variance parameters of the random effects distribution, $f(b_i)$

Normally Distributed Data: Linear Mixed Effects Models

$$
\begin{gathered}
Y_{ij} | b \sim N(\mu_{ij}, \sigma^2) \\
\mu_{ij} = X_{ij}\beta + Z_i b \\
b \sim N(0, D)
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If we average over (or integrate out) the random effects (b), we get the marginal Distribution of Y:

$$Y \sim MVN(X\beta, \Omega)$$
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$$I = \underset{(n \times n)}{\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}}$$

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Random Intercept Model:

$$\Omega = \begin{bmatrix} \Sigma_i & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \Sigma_i \end{bmatrix} \text{ with } \Sigma_i = \begin{bmatrix} \tau^2 + \sigma^2 & \tau^2 & \cdots & \tau^2 \\ \tau^2 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \tau^2 \\ \tau^2 & \cdots & \tau^2 & \tau^2 + \sigma^2 \end{bmatrix}$$

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GLS Model with compound-symmetric correlation structure:

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Logistic-normal random intercept model

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- $f(Y_{ij}|b_i) = p_{ij}^{Y_{ij}}(1 - p_{ij})^{1-Y_{ij}}$
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$$L(Y_{ij}|\beta, D) = \int f(Y_{ij}|b_i) f(b_i) db_i$$

$$\int \left[\frac{exp(\beta_0+b_{0i}+\beta_1 x_{ij})}{1+exp(\beta_0+b_{0i}+\beta_1 x_{ij})}\right]^{Y_{ij}} \left[\frac{1}{1+exp(\beta_0+b_{0i}+\beta_1 x_{ij})}\right]^{1-Y_{ij}} \frac{1}{\sqrt{2\pi}\sigma_b} e^{\frac{(b_{0i}-0)^2}{2\sigma_b^2}} db_{0i}$$

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No closed-form solution!

Unconditional Model and Likelihood

How do we use maximum likelihood to estimate parameters?

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- Use numerical integration (can be slow, difficult, particularly with multiple random effects)
- Add priors, and use Bayesian techniques

Numerical Integration: glmer

`nAGQ`

- specifies number of points per axis for evaluating the adaptive Gauss-Hermite approximation to the log-likelihood.
- default = 1 (Laplace approximation)
- values > 1 produce greater accuracy in the evaluation of the log-likelihood at the expense of speed.
- a value of zero uses a faster but less exact form of parameter estimation for GLMMs (penalized iteratively reweighted least squares step).

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See also `mixed_model` in the `GLMMadaptive` package.

Linear versus Generalized Linear Mixed Effects Models

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Parameters in generalized linear mixed effects models have a "subject-specific", but not "population-average" interpretation.

Parameter Interpretation: Linear mixed effects models

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Assume ϵ_{ij} and b_{0i} are independent.

How does clutch size vary with nest initiation date and structure type for a "typical" structure (i.e., one with $b_{0i} = 0$)?

- $E[Y|X, b_{0i} = 0] = \beta_0 + \beta_1 \texttt{Init.Date} + \beta_2 \texttt{I(deply=2)}$

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Fixed effects parameters have both population-averaged and subject-specific interpretations!

Parameter Interpretation: Generalized Linear Mixed Effects Models

$$Y_i|b_i \sim \text{Binomial}(1, p_i)$$
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The fixed effects parameters in logistic regression models **only** have a **subject-specific** interpretation when we transform back to scales of interest!

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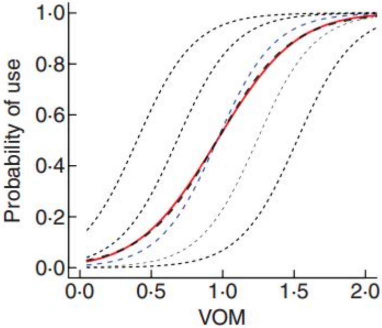
is β_2 meaningful? What if the explanatory variable was "sex" in a model where animals were repeatedly sampled?

Interpretation of Parameters: GLMMS



$E[Y_{ij}|X]$ (red curve) is no longer the same as $E[Y_{ij}|X, b_{0i} = 0]$ (blue curve)!

GLMMs and Population-Average Response Patterns

$$Y_{ij}|b_{0i}, b_{1i} \sim f(y_{ij}|b_{0i}, b_{1i})$$
$$(b_{0i}, b_{1i}) \sim N(0, D), \text{ with}$$

$f(y_i|b_{0i}, b_{1i})$ given by Poisson, binomial, negative binomial.

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- `mixed_model` + `marginal_coefs` in `GLMMadaptive` package to estimate equivalent "marginal coefficients" (based on Hedeker et al. 2018).

Hedeker, D., du Toit, S. H., Demirtas, H. and Gibbons, R. D. (2018), A note on marginalization of regression parameters from mixed models of binary outcomes. Biometrics 74, 354-361.

GLMM Resources on Canvas

Readings:

- Bolker et al. 2008. Generalized linear mixed models: a practical guide for ecology and evolution
- Bolker et al. 2013: Strategies for fitting nonlinear ecological models in R, AD Model Builder, and BUGS

Useful Links

- GLMM wiki
- GLMMS worked examples

Option 1: Generalized linear mixed effects models

Option 2: Generalized Estimating equations

Generalized Estimating Equations

- motivation (least squares, maximum likelihood...)
- assumptions and implementation of GEE approach

Least Squares and Maximum Likelihood

For Normally distributed data:

$$L(\mu, \sigma^2; y_1, y_2, \ldots, y_n) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(y_i-\mu)^2}{2\sigma^2}\right)$$

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&= \frac{1}{(\sqrt{2\pi}\sigma)^n} \exp\left(-\sum_{i=1}^{n} \frac{(y_i-\beta_0+x_i\beta_1)^2}{2\sigma^2}\right)
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$$\Rightarrow \text{maximizing logL} \Rightarrow \text{minimizing} \sum_{i=1}^{n} \frac{(y_i-\beta_0+x_i\beta_1)^2}{2\sigma^2}$$

$$\text{or, equivalently} \sum_{i=1}^{n}(y_i - \beta_0 - x_i\beta_1)^2$$

Least squares

$$\sum_{i=1}^{n}(y_i - E[Y_i|X_i])^2 = \sum_{i=1}^{n}(y_i - (\beta_0 + x_{1i}\beta_1 + \ldots))^2 \qquad (1)$$

Least squares leads to the following set of equations for estimating parameters (take the derivative and set = 0):

$$2\sum_{i=1}^{n}\frac{\partial E[Y_i|X_i]}{\partial \beta}(Y_i - E[Y_i|X_i]) = 0 \qquad (2)$$

Or, equivalently…

$$\sum_{i=1}^{n} X_i(Y_i - E[Y_i|X_i]) = 0 \qquad (3)$$

Generalized Linear Models

Maximum Likelihood estimators are found by solving:

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Poisson Regression:

- $E[Y_i|X_i] = exp(X_i\beta)$
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Generalized Estimating Equations (GEE)

GEE: $\hat{\beta}$ solves: $\sum_{i=1}^{n} \frac{\partial \mu_i}{\partial \beta} V_i(\alpha)^{-1}(Y_i - E[Y_i|X_i]) = 0.$

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R_i = working correlation model that describes within subject correlation.

- Examples include exchangeable (equal correlation among all observations), Ar(1) (time series), unstructured

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- Fit using `geeglm` in `geepack` library

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Estimates will be asymptotically unbiased (think "large number of clusters")

- Even if the assumed variance & covariance models are wrong

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Uses “robust” (or “sandwich”) standard errors, treating clusters as independent observational units

- $\hat{\beta} \pm 1.96SE$ gives valid CIs (for large numbers of clusters)