CHAPTER 1

Introduction to Calculus

1.1 Velocity and Distance

The right way to begin a calculus book is with calculus. This chapter will jump directly into the two problems that the subject was invented to solve. You will see what the questions are, and you will see an important part of the answer. There are plenty of good things left for the other chapters, so why not get started?

The book begins with an example that is familiar to everybody who drives a car. It is calculus in action—the driver sees it happening. The example is the relation between the *speedometer* and the *odometer*. One measures the speed (or ***velocity***); the other measures the ***distance traveled***. We will write v for the velocity, and f for how far the car has gone. The two instruments sit together on the dashboard:

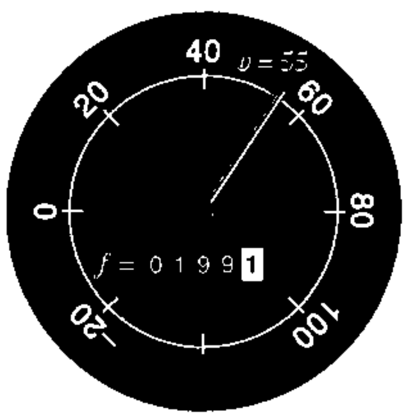


Fig. 1.1 Velocity v and total distance f (at one instant of time).

Notice that the units of measurement are different for v and f. The distance f is measured in kilometers or miles (it is easier to say miles). The velocity v is measured in km/hr or *miles per hour*. A unit of *time* enters the velocity but not the distance. Every formula to compute v from f will have f divided by time.

The central question of calculus is the relation between v and f.

Can you find v if you know f, and vice versa, and how? If we know the velocity over the whole history of the car, we should be able to compute the total distance traveled. In other words, if the speedometer record is complete but the odometer is missing, its information could be recovered. One way to do it (without calculus) is to put in a new odometer and drive the car all over again at the right speeds. That seems like a hard way; calculus may be easier. But the point is that *the information is there.* If we know everything about v, there must be a method to find f.

What happens in the opposite direction, when f is known? If you have a complete record of distance, could you recover the complete velocity? In principle you could drive the car, repeat the history, and read off the speed. Again there must be a better way.

The whole subject of calculus is built on the relation between v and f. The question we are raising here is not some kind of joke, after which the book will get serious and the mathematics will get started. On the contrary, *I am serious now*—and the mathematics has already started. We need to know how to find the velocity from a record of the distance. (That is called ***differentiation***, and it is the central idea of ***differential calculus***.) We also want to compute the distance from a history of the velocity. (That is ***integration***, and it is the goal of ***integral calculus***.)

Differentiation goes from f to v; integration goes from v to f. We look first at examples in which these pairs can be computed and understood.

CONSTANT VELOCITY

Suppose the velocity is fixed at $v = 60$ (miles per hour). Then f increases at this constant rate. After two hours the distance is $f = 120$ (miles). After four hours $f = 240$ and after t hours $f = 60t$. We say that f increases ***linearly*** with time—its graph is a straight line.

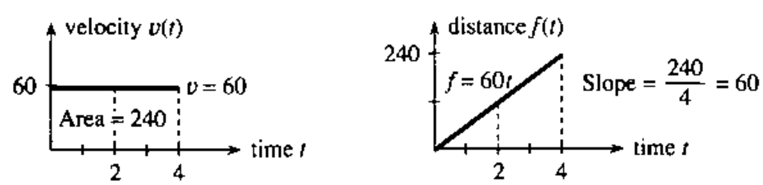


Fig. 1.2 Constant velocity $v = 60$ and linearly increasing distance $f = 60t$.

Notice that this example starts the car at full velocity. No time is spent picking up speed. (The velocity is a "step function.") Notice also that the distance starts at zero; the car is new. Those decisions make the graphs of v and f as neat as possible. One is the horizontal line $v = 60$. The other is the sloping line $f = 60t$. This v, f, t relation needs algebra hut not calculus:

if v is constant and f starts at zero then $f = vt$.

The opposite is also true. When f increases linearly, v is constant. *The division by time gives the slope.* The distance is $f_1 = 120$ miles when the time is $t_1 = 2$ hours. Later $f_2 = 240$ at $t_2 = 4$. At both points, the ratio f/t is 60 miles/hour. Geometrically, ***the velocity is the slope of the distance graph***:

$$\text{slope} = \frac{\text{change in distance}}{\text{change in time}} = \frac{vt}{t} = v.$$

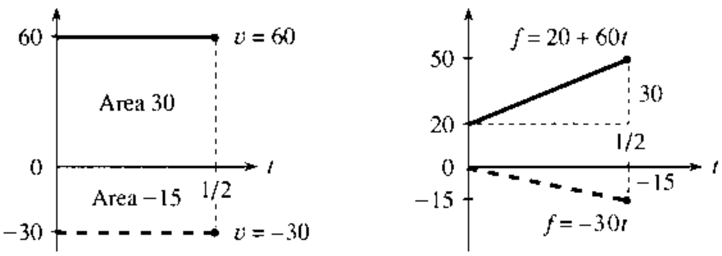


Fig. 1.3 Straight lines $f = 20 + 60t$ (slope 60) and $f = -30t$ (slope -30).

The slope of the f-graph gives the v-graph. Figure 1.3 shows two more possibilities:

1. The distance starts at 20 instead of 0. The distance formula changes from $60t$ to $20 + 60t$. The number 20 cancels when we compute *change* in distance—so the slope is still 60.
2. When v is *negative*, the graph of f goes *downward*. The car goes backward and the slope of $f = -30t$ is $v = -30$.

I don't think speedometers go below zero. But driving backwards, it's not that safe to watch. If you go fast enough, Toyota says they measure "absolute values"—the speedometer reads $+30$ when the velocity is -30. For the odometer, as far as I know it just stops. It should go backward.†

VELOCITY *vs.* DISTANCE: SLOPE *vs.* AREA

How do you compute f from v? The point of the question is to see *$f = vt$ on the graphs.* We want to start with the graph of v and discover the graph of f. Amazingly, the opposite of slope is *area*.

The distance f is the area under the v-graph. When v is constant, the region under the graph is a rectangle. Its height is v, its width is t, and its area is v times t. This is *integration*, to go from v to f by computing the area. We are glimpsing two of the central facts of calculus.

> **1A** *The slope of the f-graph gives the velocity v. The area under the v-graph gives the distance f.*

That is certainly not obvious, and I hesitated a long time before I wrote it down in this first section. The best way to understand it is to look first at more examples. The whole point of calculus is to deal with velocities that are *not* constant, and from now on v has several values.

EXAMPLE ***(Forward and back)*** There is a motion that you will understand right away. The car goes forward with velocity V, and comes back at the same speed. To say it more correctly, the *velocity in the second part is* $-V$. If the forward part lasts until $t = 3$, and the backward part continues to $t = 6$, ***the car will come back where it started.*** The total distance after both parts will be $f = 0$.

†This actually happened in *Ferris Bueller's Day Off*, when the hero borrowed his father's sports car and ran up the mileage. At home he raised the car and drove in reverse. I forget if it worked.

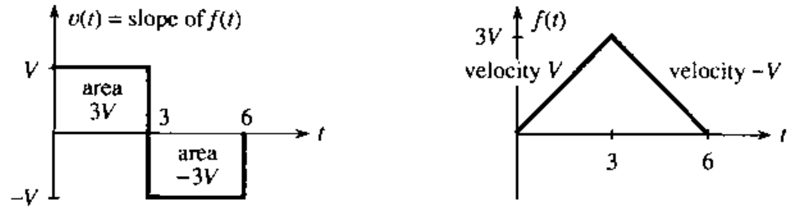


Fig. 1.4 Velocities $+V$ and $-V$ give motion forward and back, ending at $f(6)=0$.

The v-graph shows velocities $+V$ and $-V$. The distance starts up with slope $+V$ and reaches $f=3V$. Then the car starts backward. The distance goes down with slope $-V$ and returns to $f=0$ at $t=6$.

Notice what that means. The total area "under" the v-graph is zero! A negative velocity makes the distance graph go *downward* (*negative slope*). The car is moving backward. ***Area below the axis in the v-graph is counted as negative.***

FUNCTIONS

This forward-back example gives practice with a crucially important idea—the concept of a "***function***." We seize this golden opportunity to explain functions:

The number $v(t)$ is the value of the function v at the time t.

The time t is the ***input*** to the function. The velocity $v(t)$ at that time is the ***output***. Most people say "v *of* t" when they read $v(t)$. The number "v *of* 2" is the velocity when $t=2$. The forward-back example has $v(2)=+V$ and $v(4)=-V$. The function contains the whole history, like a memory bank that has a record of v at each t.

It is simple to convert forward-back motion into a formula. Here is $v(t)$:

$$v(t)=\begin{cases}+V & \text{if} \quad 0<t<3\\ \;\;? & \text{if} \quad t=3\\ -V & \text{if} \quad 3<t<6\end{cases}$$

The right side contains the instructions for finding $v(t)$. The input t is converted into the output $+V$ or $-V$. The velocity $v(t)$ depends on t. In this case the function is "discontinuous," because the needle jumps at $t=3$. *The velocity is not defined at that instant*. There is no $v(3)$. (You might argue that v is zero at the jump, but that leads to trouble.) The graph of f has a corner, and we can't give its slope.

The problem also involves a second function, namely the distance. The principle behind $f(t)$ is the same: $f(t)$ ***is the distance at time*** t. It is the net distance forward, and again the instructions change at $t=3$. In the forward motion, $f(t)$ equals Vt as before. In the backward half, a calculation is built into the formula for $f(t)$:

$$f(t)=\begin{cases}Vt & \text{if} \quad 0\leqslant t\leqslant 3\\ V(6-t) & \text{if} \quad 3\leqslant t\leqslant 6\end{cases}$$

At the switching time the right side gives two instructions (one on each line). This would be bad except that they agree: $f(3)=3V$.† The distance function is "*con-*

†A function is only allowed *one value* $f(t)$ or $v(t)$ at each time t.

tinuous." There is no jump in f, even when there is a jump in v. After $t = 3$ the distance decreases because of $-Vt$. At $t = 6$ the second instruction correctly gives $f(6) = 0$.

Notice something more. The functions were given by graphs before they were given by formulas. The graphs tell you f and v at every time t—sometimes more clearly than the formulas. The values $f(t)$ and $v(t)$ can also be given by tables or equations or a set of instructions. (In some way all functions are instructions—the function tells how to find f at time t.) Part of knowing f is knowing all its inputs and outputs—its ***domain*** and ***range***:

The domain of a function is the set of inputs. The range is the set of outputs.

The domain of f consists of all times $0 \leqslant t \leqslant 6$. The range consists of all distances $0 \leqslant f(t) \leqslant 3V$. (The range of v contains only the two velocities $+V$ and $-V$.) We mention now, and repeat later, that every "linear" function has a formula $f(t) = vt + C$. Its graph is a line and v is the slope. The constant C moves the line up and down. It adjusts the line to go through any desired starting point.

SUMMARY: MORE ABOUT FUNCTIONS

May I collect together the ideas brought out by this example? We had two functions v and f. One was *velocity*, the other was *distance*. Each function had a *domain*, and a *range*, and most important a *graph*. For the f-graph we studied the slope (which agreed with v). For the v-graph we studied the area (which agreed with f). Calculus produces functions in pairs, and the best thing a book can do early is to show you more of them.

$$\begin{array}{c} \textbf{\textit{in}} \\ \textbf{\textit{the}} \\ \textbf{\textit{domain}} \end{array} \left\{ \begin{array}{llllll} \textbf{\textit{input}}\ t & \rightarrow & \textbf{\textit{function}}\ f & \rightarrow & \textbf{\textit{output}}\ f(t) \\ \textbf{\textit{input}}\ 2 & \rightarrow & \textbf{\textit{function}}\ v & \rightarrow & \textbf{\textit{output}}\ v(2) \\ \textbf{\textit{input}}\ 7 & \rightarrow & f(t) = 2t + 6 & \rightarrow & f(7) = 20 \end{array} \right\} \begin{array}{c} \textbf{\textit{in}} \\ \textbf{\textit{the}} \\ \textbf{\textit{range}} \end{array}$$

Note about the definition of a function. The idea behind the symbol $f(t)$ is absolutely crucial to mathematics. Words don't do it justice! By definition, a function is a "rule" that assigns one member of the range to each member of the domain. Or, a function is a set of pairs $(t, f(t))$ with no t appearing twice. (These are "ordered pairs" because we write t before $f(t)$.) Both of those definitions are correct—but somehow they are too passive.

In practice what matters is the active part. The number $f(t)$ is produced from the number t. We read a graph, plug into a formula, solve an equation, run a computer program. The input t is "mapped" to the output $f(t)$, which changes as t changes. Calculus is about the ***rate of change***. This rate is our other function v.

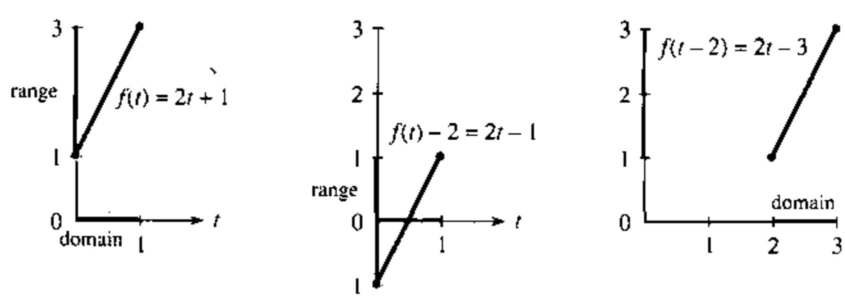


Fig. 1.5 Subtracting 2 from f affects the range. Subtracting 2 from t affects the domain.

It is quite hard at the beginning, and not automatic, to see the difference between $f(t) - 2$ and $f(t-2)$. Those are both new functions, created out of the original $f(t)$. In $f(t) - 2$, we subtract 2 from all the distances. That moves the whole graph *down*. In $f(t-2)$, we subtract 2 from the time. That moves the graph over *to the right*. Figure 1.5 shows both movements, starting from $f(t) = 2t + 1$. The formula to find $f(t-2)$ is $2(t-2)+1$, which is $2t-3$.

A graphing calculator also moves the graph, when you change the viewing window. You can pick any rectangle $A \leqslant t \leqslant B$, $C \leqslant f(t) \leqslant D$. The screen shows that part of the graph. But on the calculator, *the function $f(t)$ remains the same*. It is the axes that get renumbered. In our figures the axes stay the same and the function is changed.

There are two more basic ways to change a function. (We are always creating new functions—that is what mathematics is all about.) Instead of subtracting or adding, we can *multiply* the distance by 2. Figure 1.6 shows $2f(t)$. And instead of shifting the time, we can *speed it up*. The function becomes $f(2t)$. Everything happens twice as fast (and takes half as long). On the calculator those changes correspond to a "***zoom***"—on the f axis or the t axis. We soon come back to zooms.

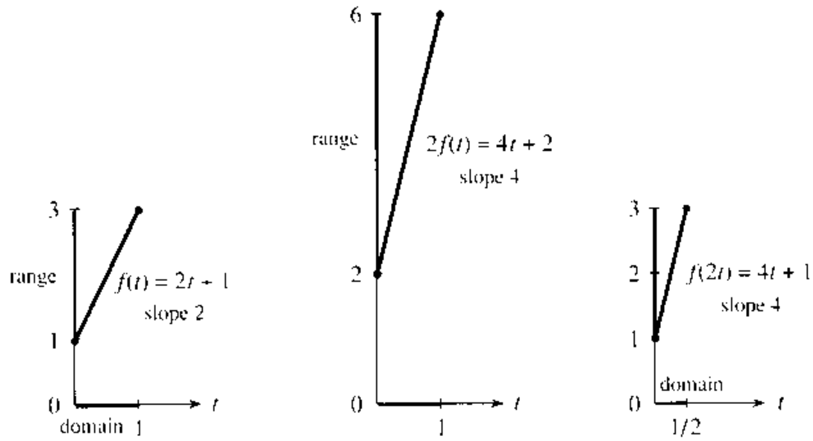


Fig. 1.6 Doubling the distance or speeding up the time doubles the slope.

1.1 EXERCISES

Each section of the book contains read-through questions. They allow you to outline the section yourself—more actively than reading a summary. This is probably the best way to remember the important ideas.

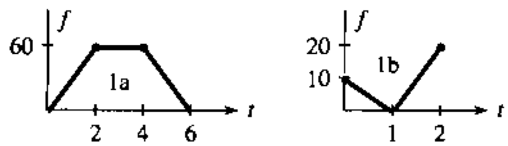
Starting from $f(0) = 0$ at constant velocity v, the distance function is $f(t) =$ ___a___. When $f(t) = 55t$ the velocity is $v =$ ___b___. When $f(t) = 55t + 1000$ the velocity is still ___c___ and the starting value is $f(0) =$ ___d___. In each case v is the ___e___ of the graph of f. When ___f___ is negative, the graph of ___g___ goes downward. In that case area in the v-graph counts as ___h___.

Forward motion from $f(0) = 0$ to $f(2) = 10$ has $v =$ ___i___. Then backward motion to $f(4) = 0$ has $v =$ ___j___. The distance function is $f(t) = 5t$ for $0 \leqslant t \leqslant 2$ and then $f(t) =$ ___k___ (not $-5t$). The slopes are ___l___ and ___m___. The distance $f(3) =$ ___n___. The area under the v-graph up to time 1.5 is ___o___. The domain of f is the time interval ___p___, and the range is the distance interval ___q___. The range of $v(t)$ is only ___r___.

The value of $f(t) = 3t + 1$ at $t = 2$ is $f(2) =$ ___s___. The value 19 equals $f($___t___$)$. The difference $f(4) - f(1) =$ ___u___. That is the change in distance, when $4 - 1$ is the change in ___v___. The ratio of those changes equals ___w___, which is the ___x___ of the graph. The formula for $f(t) + 2$ is $3t + 3$ whereas $f(t+2)$ equals ___y___. Those functions have the same ___z___ as f: the graph of $f(t) + 2$ is shifted ___A___ and $f(t+2)$ is shifted ___B___. The formula for $f(5t)$ is ___C___. The formula for $5f(t)$ is ___D___. The slope has jumped from 3 to ___E___.

The set of inputs to a function is its ___F___. The set of outputs is its ___G___. The functions $f(t) = 7 + 3(t-2)$ and $f(t) = vt + C$ are ___H___. Their graphs are ___I___ with slopes equal to ___J___ and ___K___. They are the same function, if $v =$ ___L___ and $C =$ ___M___.

Draw the velocity graph that goes with each distance graph.

1


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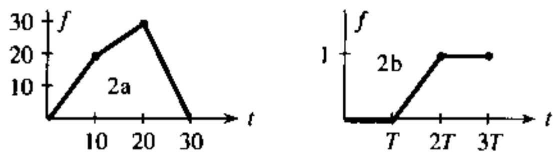


3 Write down three-part formulas for the velocities $v(t)$ in Problem 2, starting from $v(t) = 2$ for $0 < t < 10$.

4 The distance in 1b starts with $f(t) = 10 - 10t$ for $0 \leqslant t \leqslant 1$. Give a formula for the second part.

5 In the middle of graph 2a find $f(15)$ and $f(12)$ and $f(t)$.

6 In graph 2b find $f(1.4T)$. If $T = 3$ what is $f(4)$?

7 Find the *average speed* between $t = 0$ and $t = 5$ in graph 1a. What is the speed at $t = 5$?

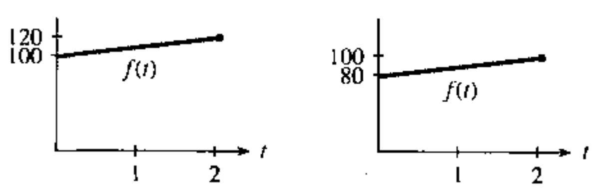
8 What is the average speed between $t = 0$ and $t = 2$ in graph 1b? The average speed is zero between $t = \frac{1}{2}$ and $t =$ ________.

9 (recommended) A car goes at speed $v = 20$ into a brick wall at distance $f = 4$. Give two-part formulas for $v(t)$ and $f(t)$ (before and after), and draw the graphs.

10 Draw any reasonable graphs of $v(t)$ and $f(t)$ when

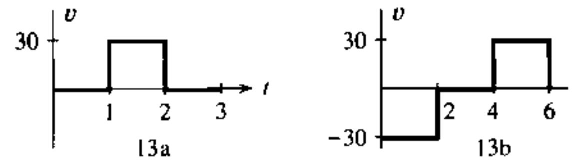
(a) the driver backs up, stops to shift gear, then goes fast;
(b) the driver slows to 55 for a police car;
(c) in a rough gear change, the car accelerates in jumps;
(d) the driver waits for a light that turns green.

11 Your bank account earns simple interest on the opening balance $f(0)$. What are the interest rates per year?



12 The earth's population is growing at $v = 100$ million a year, starting from $f = 5.2$ billion in 1990. Graph $f(t)$ and find $f(2000)$.

Draw the distance graph that goes with each velocity graph. Start from $f = 0$ at $t = 0$ and mark the distance.

13


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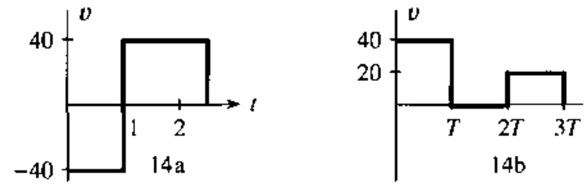

15 Write down formulas for $v(t)$ in Problem 14, starting with $v = -40$ for $0 < t < 1$. Find the average velocities to $t = 2.5$ and $t = 3T$.

16 Give 3-part formulas for the areas $f(t)$ under $v(t)$ in 13.

17 The distance in 14a starts with $f(t) = -40t$ for $0 \leqslant t \leqslant 1$. Find $f(t)$ in the other part, which passes through $f = 0$ at $t = 2$.

18 Draw the velocity and distance graphs if $v(t) = 8$ for $0 < t < 2$, $f(t) = 20 + t$ for $2 \leqslant t \leqslant 3$.

19 Draw rough graphs of $y = \sqrt{x}$ and $y = \sqrt{x-4}$ and $y = \sqrt{x} - 4$. They are "half-parabolas" with infinite slope at the start.

20 What is the break-even point if x yearbooks cost $\$1200 + 30x$ to produce and the income is $40x$? The slope of the cost line is ________ (cost per additional book). If it goes above ________ you can't break even.

21 What are the domains and ranges of the distance functions in 14a and 14b—all values of t and $f(t)$ if $f(0) = 0$?

22 What is the range of $v(t)$ in 14b? Why is $t = 1$ not in the domain of $v(t)$ in 14a?

Problems 23–28 involve *linear functions* $f(t) = vt + C$. Find the constants v and C.

23 What linear function has $f(0) = 3$ and $f(2) = -11$?

24 Find *two* linear functions whose domain is $0 \leqslant t \leqslant 2$ and whose range is $1 \leqslant f(t) \leqslant 9$.

25 Find the linear function with $f(1) = 4$ and slope 6.

26 What functions have $f(t+1) = f(t) + 2$?

27 Find the linear function with $f(t+2) = f(t) + 6$ and $f(1) = 10$.

28 Find the only $f = vt$ that has $f(2t) = 4f(t)$. Show that every $f = \frac{1}{2}at^2$ has this property. To go ________ times as far in twice the time, you must accelerate.

29 Sketch the graph of $f(t)=|5-2t|$ (absolute value) for $|t|\leqslant 2$ and find its slopes and range.

30 Sketch the graph of $f(t)=4-t-|4-t|$ for $2\leqslant t\leqslant 5$ and find its slope and range.

31 Suppose $v=8$ up to time T, and after that $v=-2$. Starting from zero, when does f return to zero? Give formulas for $v(t)$ and $f(t)$.

32 Suppose $v=3$ up to time $T=4$. What new velocity will lead to $f(7)=30$ if $f(0)=0$? Give formulas for $v(t)$ and $f(t)$.

33 What function F(C) converts Celsius temperature C to Fahrenheit temperature F? The slope is ________, which is the number of Fahrenheit degrees equivalent to 1°C.

34 What function C(F) converts Fahrenheit to Celsius (or Centigrade), and what is its slope?

35 What function converts the weight w in grams to the weight $f(w)$ in kilograms? Interpret the slope of $f(w)$.

36 (Newspaper of March 1989) Ten hours after the accident the alcohol reading was .061. Blood alcohol is eliminated at .015 per hour. What was the reading at the time of the accident? How much later would it drop to .04 (the maximum set by the Coast Guard)? The usual limit on drivers is .10 percent.

Which points between $t=0$ and $t=5$ can be in the domain of $f(t)$? With this domain find the range in 37–42.

37 $f(t)=\sqrt{t-1}$ **38** $f(t)=1/\sqrt{t-1}$

39 $f(t)=|t-4|$ (absolute value) **40** $f(t)=1/(t-4)^2$

41 $f(t)=2^t$ **42** $f(t)=2^{-t}$

43 (a) Draw the graph of $f(t)=\frac{1}{2}t+3$ with domain $0\leqslant t\leqslant 2$. Then give a formula and graph for

(b) $f(t)+1$ (c) $f(t+1)$
(d) $4f(t)$ (e) $f(4t)$.

44 (a) Draw the graph of $U(t)=$ step function $=\{0$ for $t<0$, 1 for $t\geqslant 0\}$. Then draw

(b) $U(t)+2$ (c) $U(t+2)$
(d) $3U(t)$ (e) $U(3t)$.

45 (a) Draw the graph of $f(t)=t+1$ for $-1\leqslant t\leqslant 1$. Find the domain, range, slope, and formula for

(b) $2f(t)$ (c) $f(t-3)$ (d) $-f(t)$ (e) $f(-t)$.

46 If $f(t)=t-1$ what are $2f(3t)$ and $f(1-t)$ and $f(t-1)$?

47 In the forward-back example find $f(\frac{1}{2}T)$ and $f(\frac{3}{2}T)$. Verify that those agree with the areas "under" the v-graph in Figure 1.4.

48 Find formulas for the outputs $f_1(t)$ and $f_2(t)$ which come from the input t:

(1) inside = input*3 (2) inside ← input + 6
output = inside + 3 output ← inside*3

Note BASIC and FORTRAN (and calculus itself) use = instead of ←. But the symbol ← or := is in some ways better. The instruction $t\leftarrow t+6$ produces a new t equal to the old t plus six. The equation $t=t+6$ is not intended.

49 Your computer can add and multiply. Starting with the number 1 and the input called t, give a list of instructions to lead to these outputs:

$$f_1(t)=t^2+t \quad f_2(t)=f_1(f_1(t)) \quad f_3(t)=f_1(t+1).$$

50 In fifty words or less explain what a *function* is.

The last questions are challenging but possible.

51 If $f(t)=3t-1$ for $0\leqslant t\leqslant 2$ give formulas (with domain) and find the slopes of these six functions:

(a) $f(t+2)$ (b) $f(t)+2$ (c) $2f(t)$
(d) $f(2t)$ (e) $f(-t)$ (f) $f(f(t))$.

52 For $f(t)=vt+C$ find the formulas and slopes of

(a) $3f(t)+1$ (b) $f(3t+1)$ (c) $2f(4t)$
(d) $f(-t)$ (e) $f(t)-f(0)$ (f) $f(f(t))$.

53 (hardest) The forward-back function is $f(t)=2t$ for $0\leqslant t\leqslant 3$, $f(t)=12-2t$ for $3\leqslant t\leqslant 6$. Graph $f(f(t))$ and find its *four-part* formula. First try $t=1.5$ and 3.

54 (a) Why is the letter **X** not the graph of a function?
(b) Which capital letters are the graphs of functions?
(c) Draw graphs of their slopes.

1.2 Calculus Without Limits

The next page is going to reveal one of the key ideas behind calculus. The discussion is just about numbers—functions and slopes can wait. The numbers are not even special, they can be any numbers. The crucial point is to look at their differences:

Suppose the numbers are $f=$ 0 2 6 7 4 9
Their differences are $v=$ 2 4 1 −3 5

The differences are printed in between, to show $2-0=2$ and $6-2=4$ and $7-6=1$.