Estimator Analysis and Comparison

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Abstract

This document contains the first simulation exercise for the Optimal Filtering with Aerospace Applications class. Based on the content given in the lectures, it was requested to formulate an algorithm capable of representing the measurement model of an ultrasonic sensor mounted on a MAV. Said model contained a vector of normally distributed parameters, to be estimated by a Least Squares and a Maximum a Posteriori estimators. The sensor and the estimators were implemented in Matlab scripts[1] and histograms were generated to illustrate the estimate realizations.

1 Problem Statement

Consider a MAV equipped with an ultrasonic sensor that provides a set $y_{1:N}$ of N measures of some physical quantity. Assume that $y_i \in \mathbb{R}$ is modeled by:

$$y_i = \mathbf{h_i}\theta + v_i, \quad i = 1, \dots, N \tag{1}$$

where $\mathbf{h_i} \triangleq [i \ 1]$, $\theta \triangleq [\theta_1 \ \theta_2]^T \in \mathbb{R}^2$ is a realization of a random variable $\boldsymbol{\Theta} \sim \mathcal{N}(\mathbf{m_\Theta}, \mathbf{P_\Theta})$, v_i is the realization of a random variable $V_i \sim \mathcal{N}(0, R)$, $\{V_i\}$ is a white sequence, and V_i and $\boldsymbol{\Theta}$ are uncorrelated. Consider that $\mathbf{m_\Theta} = [1 \ 2]^T$,

$$\mathbf{P_\Theta} = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.04 \end{bmatrix}$$

and $R = 0.01$.

The goal of this exercise is to present the estimation of θ from the measurements $y_{1:N}$, using a Least-Squares estimator and a Maximum a Posteriori estimator.

2 Least-Squares Estimator (LS)

The LS estimator $\hat{\theta}_\mathbf{N} \in \mathbb{R}^p$ of θ from $y_{1:N}$ is given by

$$\hat{\theta}_\mathbf{N} = \arg\min_\theta J_N(\theta) \tag{2}$$

where

$$J_N(\theta) \triangleq \sum_{i=1}^{N} (\mathbf{y}_i - \mathbf{h}_i(\theta))^T \mathbf{W}_i (\mathbf{y}_i - \mathbf{h}_i(\theta)) \tag{3}$$

[1]All scripts can be found in: https://github.com/jfilipe33/ExCompMP208

and $\mathbf{W}_i \in \mathbb{R}^{m \times m}$ is a weighting matrix.

Considering (1) a linear model in the form

$$y_i = \mathbf{H}_i \theta + \mathbf{v}_i, \quad i = 1, \ldots, N \tag{4}$$

the LS estimator defined in (2) is given explicitly by

$$\hat{\theta}_{\mathbf{N}} = \left(\sum_{i=1}^{N} \mathbf{H}_i^T \mathbf{W}_i \mathbf{H}_i \right)^{-1} \sum_{i=1}^{N} \mathbf{H}_i^T \mathbf{W}_i \mathbf{y}_i \tag{5}$$

2.1 LS Simulations

For $\mathbf{W}_i = 1$ and conducting a Monte Carlo simulation with M = 100 realizations of $\hat{\boldsymbol{\Theta}}_N$, the histograms obtained is illustrated in Figure 1.

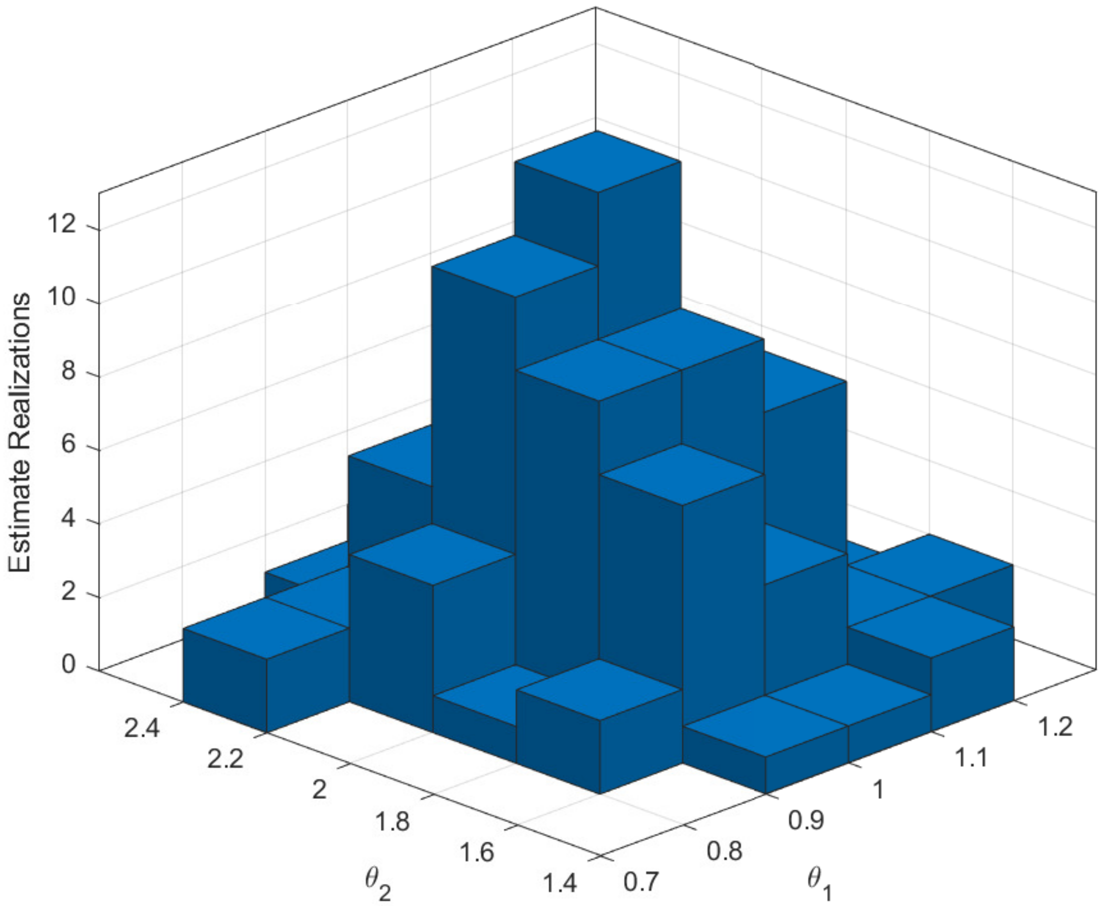

Figure 1: Histogram of the LS estimator realizations of $\hat{\boldsymbol{\Theta}}_{\mathbf{M}}$

The mean of the estimate realizations was calculated as:

$$\mathbf{m}_{\hat{\boldsymbol{\Theta}}_{\mathbf{N}}}\mathbf{LS} = \frac{1}{N} \sum_{j=1}^{N} \hat{\boldsymbol{\Theta}}_{\mathbf{N}}^{\mathbf{j}} = \begin{bmatrix} 1.0179 \\ 2.0038 \end{bmatrix} \tag{6}$$

The Root Mean Square Error, which compares the estimate and the real parameter, was calculated as:

$$\mathbf{RMS_{LS}} = \sqrt{\frac{1}{N-1}\sum_{j=1}^{N}\left(\hat{\mathbf{\Theta}}_{\mathbf{N}}^{\mathbf{j}} - \theta\right)\left(\hat{\mathbf{\Theta}}_{\mathbf{N}}^{\mathbf{j}} - \theta\right)^{T}} = \begin{bmatrix} 0.0000 & -0.0000 \\ -0.0000 & 0.0031 \end{bmatrix} \tag{7}$$

which means the error between the estimate and the real parameter was close to zero. Therefore, the estimate converges to the real θ. From the histogram, it is possible to see that most realizations are close to the mean of θ and within the standard deviation given by the $\sqrt{\mathbf{P_\Theta}}$.

3 Maximum a Posteriori Estimator (MAP)

The MAP estimator $\hat{\theta}_{\mathbf{N}} \in \mathbb{R}^p$ of θ from $y_{1:N}$ is given by

$$\hat{\theta}_{\mathbf{N}} = \arg\max_{\theta} f_{\mathbf{\Theta}|\mathbf{Y}_{1:N}}\left(\theta|\mathbf{y}_{1:N}\right) \tag{8}$$

where $f_{\mathbf{\Theta}|\mathbf{Y}_{1:N}}\left(\theta|\mathbf{y}_{1:N}\right)$ is the *a posteriori* pdf given by the Bayes Theorem:

$$f_{\mathbf{\Theta}|\mathbf{Y}_{1:N}}\left(\theta|\mathbf{y}_{1:N}\right) = \frac{f_{\mathbf{Y}_{1:N}|\mathbf{\Theta}}\left(\mathbf{y}_{1:N}|\theta\right) f_{\mathbf{\Theta}}\left(\theta\right)}{f_{\mathbf{Y}_{1:N}}\left(\mathbf{y_{1:N}}\right)} \tag{9}$$

where $f_{\mathbf{Y}_{1:N}|\mathbf{\Theta}}\left(\mathbf{y}_{1:N}|\theta\right)$ is the likelihood function of $\mathbf{Y}_{1:N}$ given $\{\mathbf{\Theta} = \theta\}$, $f_{\mathbf{\Theta}}\left(\theta\right)$ is the *a priori* pdf of $\mathbf{\Theta}$, and $f_{\mathbf{Y}_{1:N}}\left(\mathbf{y_{1:N}}\right)$ is a normalizing factor.

Considering (1) a linear Gaussian model in the form

$$y_i = \mathbf{H}_i\theta + \mathbf{v}_i, \quad i = 1, \ldots, N \tag{10}$$

the MAP estimator defined in (10) is given explicitly by

$$\hat{\theta}_{\mathbf{N}} = \mathbf{P}_N\mathbf{P}_{\mathbf{\Theta}}^{-1}\mathbf{m}_{\mathbf{\Theta}} + \sum_{i=1}^{N}\mathbf{H}_i^T\mathbf{R}_i^{-1}\mathbf{y}_i \tag{11}$$

where

$$\mathbf{P}_N \triangleq \left(\sum_{i=1}^{N}\mathbf{H}_i^T\mathbf{R}_i^{-1}\mathbf{H}_i + \mathbf{P}_{\mathbf{\Theta}}^{-1}\right)^{-1} \in \mathbb{R}_i^{p\times p} \tag{12}$$

3.1 MAP Simulations

For $\mathbf{R}_i = 0.01$ and conducting a Monte Carlo simulation with M $= 100$ realizations of $\hat{\mathbf{\Theta}}_N$, the histograms obtained is illustrated in Figure 2.

The mean of the estimate realizations was calculated as:

$$\mathbf{m}_{\hat{\mathbf{\Theta}}_{\mathbf{N}}}\mathbf{MAP} = \frac{1}{N}\sum_{j=1}^{N}\hat{\mathbf{\Theta}}_{\mathbf{N}}^{\mathbf{j}} = \begin{bmatrix} 1.0076 \\ 1.9910 \end{bmatrix} \tag{13}$$

The Root Mean Square Error, which compares the estimate and the real parameter, was calculated as:

$$\mathbf{RMS_{MAP}} = \sqrt{\frac{1}{N-1}\sum_{j=1}^{N}\left(\hat{\mathbf{\Theta}}_{\mathbf{N}}^{\mathbf{j}} - \mathbf{\Theta}\right)\left(\hat{\mathbf{\Theta}}_{\mathbf{N}}^{\mathbf{j}} - \mathbf{\Theta}\right)^{T}} = \begin{bmatrix} 0.0000 & -0.0000 \\ -0.0000 & 0.0026 \end{bmatrix} \tag{14}$$

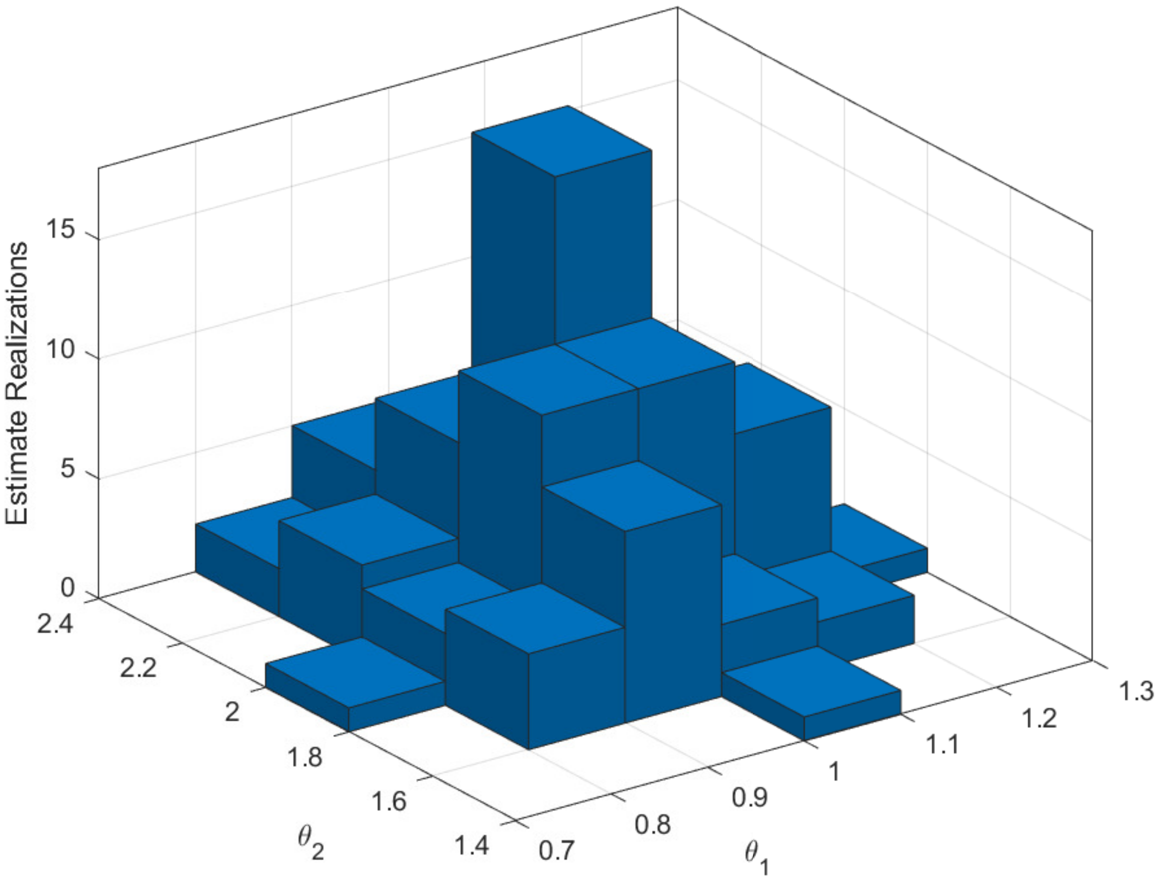

Figure 2: Histogram of the MAP estimator realizations of $\hat{\mathbf{\Theta}}_{\mathbf{M}}$

which means the error between the estimate and the real parameter was close to zero. Therefore, the estimate converges to the real θ. Similarly to the previous estimator, it is possible to see on the histogram that most realizations are close to the mean of θ and within the standard deviation given by the $\sqrt{\mathbf{P}_{\mathbf{\Theta}}}$.

From equation (11), it is possible to calculate the theoretical mean of the MAP estimator.

$$
\begin{aligned}
\mathbb{E}\left[\hat{\mathbf{\Theta}}_N\right] &= \mathbb{E}\left[\mathbf{P}_N\mathbf{P}_{\mathbf{\Theta}}^{-1}\mathbf{m}_{\mathbf{\Theta}} + \mathbf{P}_N\sum_{i=1}^{N}\mathbf{H}_i^T\mathbf{R}_i^{-1}\mathbf{y}_i\right] \\
&= \mathbf{P}_N\mathbf{P}_{\mathbf{\Theta}}^{-1}\mathbf{m}_{\mathbf{\Theta}} + \mathbf{P}_N\sum_{i=1}^{N}\mathbf{H}_i^T\mathbf{R}_i^{-1}\mathbf{H}_i\cancelto{\mathbf{m}_{\mathbf{\Theta}}}{\mathbb{E}\left[\mathbf{\Theta}\right]} + \mathbf{P}_N\sum_{i=1}^{N}\mathbf{H}_i^T\mathbf{R}_i^{-1}\cancelto{0}{\mathbb{E}\left[\mathbf{V}_i\right]} \\
&= \mathbf{P}_N\cancelto{I^{2\times 2}}{\left(\mathbf{P}_{\mathbf{\Theta}}^{-1} + \sum_{i=1}^{N}\mathbf{H}_i^T\mathbf{R}_i^{-1}\mathbf{H}_i\right)}\mathbf{m}_{\mathbf{\Theta}} \\
&= \mathbf{m}_{\mathbf{\Theta}} = \begin{bmatrix}1\\2\end{bmatrix}
\end{aligned}
\tag{15}
$$

For the MAP estimator, it is also possible to calculate the theoretical RMS Error. Considering the random variable $\tilde{\mathbf{\Theta}}_N \triangleq \hat{\mathbf{\Theta}}_N - \mathbf{\Theta}$, the RMS_t is given by:

$$
\begin{aligned}
RMS_t = \mathbb{E}\left(\tilde{\boldsymbol{\Theta}}_N \tilde{\boldsymbol{\Theta}}_N^T\right) = {} & \mathbf{P}_N \mathbf{P}_{\boldsymbol{\Theta}}^{-1} \mathbf{m}_{\boldsymbol{\Theta}} \mathbf{m}_{\boldsymbol{\Theta}}^T \mathbf{P}_{\boldsymbol{\Theta}}^{-1} \mathbf{P}_N + \mathbf{P}_1 \left(\mathbf{P}_{\boldsymbol{\Theta}} + \mathbf{m}_{\boldsymbol{\Theta}} \mathbf{m}_{\boldsymbol{\Theta}}^T\right) \mathbf{P}_1 \\
& \mathbf{P}_N \mathbf{P}_{\boldsymbol{\Theta}}^{-1} \mathbf{m}_{\boldsymbol{\Theta}} \mathbf{m}_{\boldsymbol{\Theta}}^T \mathbf{P}_1 + \mathbf{P}_1 \mathbf{m}_{\boldsymbol{\Theta}} \mathbf{m}_{\boldsymbol{\Theta}}^T \mathbf{P}_{\boldsymbol{\Theta}}^{-1} \mathbf{P}_N \\
& \mathbf{P}_N \sum_{i=1}^{N} \mathbf{H}_i^T \mathbf{R}_i^{-1} \mathbf{H}_i \mathbf{P}_N
\end{aligned} \tag{16}
$$

where

$$
\mathbf{P}_1 \triangleq \left(\mathbf{P}_N \sum_{i=1}^{N} \mathbf{H}_i^T \mathbf{R}_i^{-1} \mathbf{H}_i - I_p\right). \tag{17}
$$

Substituting all the respective values, the theoretical RMS results in:

$$
\mathbf{RMS_t} = \begin{bmatrix} 0.0000 & -0.0003 \\ -0.0003 & 0.0200 \end{bmatrix} \tag{18}
$$

Equation (15) shows that the mean of $\hat{\boldsymbol{\Theta}}_N$ is the mean of the estimated RV $\boldsymbol{\Theta}$. Also, from equation (16), it is possible to conclude that, when $N \to \infty$, $RMS_t \to 0$. Which means that with more measurements $y_{1:N}$, the difference between $\hat{\boldsymbol{\Theta}}_N$ and $\boldsymbol{\Theta}$ will reduce and the estimation distribution will converge to the real parameter distribution.

4 Conclusion

The objective of this exercise was to formulate an algorithm capable of representing the measurement model of an ultrasonic sensor mounted on a MAV and estimate the sensor's parameters based on its measurement model and a batch of measurements. The simulations were successful and both estimators provided good estimations of the parameter θ. The sample means given by the Least-Squares and the Maximum a Posteriori estimators converged to the mean of the real parameter, which was realized at each iteration of the estimation loop. Apart from the disparities in algorithm, little differences were observed whilst comparing the performance of the estimators. When simulated together, trying to estimate the same realization of $\boldsymbol{\Theta}$ at each iteration of the estimation loop, both methods provided nearly identical histograms.