Estimator Analysis and Comparison

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MP-208: Optimal Filtering with Aerospace Applications

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Abstract

This document contains the second simulation exercise for the Optimal Filtering with Aerospace Applications class. Based on the content given in the lectures, it was requested to formulate an algorithm capable of representing a second-order dynamic system model whose states were to be estimated by a Kalman Filter and an Information Filter estimators. The estimators were implemented in Matlab scripts[1] and plots were generated to illustrate the estimation errors.

1 Problem Statement

Consider a discrete-time second-order dynamic system described, with a sampling time of $T = 0.1$ s, by:

$$\begin{aligned} \mathbf{x}_{k+1} &= \mathbf{A}\mathbf{x}_k + \mathbf{B}u_k + \mathbf{w}_k \\ y_{k+1} &= \mathbf{C}\mathbf{x}_{k+1} + v_k \end{aligned} \tag{1}$$

where $\mathbf{x}_k \in \mathbb{R}^2$, $y_k \in \mathbb{R}$ is the measured output, $\{\mathbf{w}_k \in \mathbb{R}^2\}$ is a realization of the SP $\{\mathbf{W}_k\}$, with $\mathbf{W}_k \sim \mathcal{N}(0, \mathbf{Q})$ and $\mathbf{Q} = diag(1\times10^{-2}; 4\times10^{-2})$, $\{v_k \in \mathbb{R}\}$ is a realization of the SP $\{V_k\}$, with $V_k \sim \mathcal{N}(0, R)$ and $R = 1 \times 10^{-2}$, $\mathbf{x}_1 \in \mathbb{R}^2$ is a realization of the RV $\mathbf{X}_1 \sim \mathcal{N}\left(\bar{\mathbf{x}}, \bar{\mathbf{P}}\right)$, with $\bar{\mathbf{x}} = [1\ 0]^T$ and $\bar{\mathbf{P}} = diag(1 \times 10^{-4}; 1 \times 10^{-8})$, the sequence $\{\{V_k\}, \{\mathbf{w}_k\}, \mathbf{X}_1\}$ is uncorrelated, $u_k \in \mathbb{R}$ is a control input

$$u_k = 10\left(\bar{y}_k - \mathbf{e}_1^T\mathbf{x}_k\right) - 2\mathbf{e}_2^T\mathbf{x}_k \tag{2}$$

in which $\bar{y}_k \in \mathbb{R}$ is the command input, consider $\bar{y}_k = 5, \forall k$, and finally,

$$\mathbf{A} = \begin{bmatrix} 1 & 0.1 \\ 0 & 1 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 0.005 \\ 0.1 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix} \tag{3}$$

The goal of this exercise is to present the estimation errors of the states $\mathbf{x}$, using a Kalman Filter and an Information Filter.

2 Kalman Filter (KF)

A Kalman Filter uses a Minimum Mean Square Error estimator to estimate the state at the current instant k, given the measurements obtained since the start of the simulation until the instant k. Its estimate $\hat{\mathbf{x}}_{k|k} \in \mathbb{R}^{n_x}$ of $\mathbf{x}_k$ from $\mathbf{y}_{1:k}$, $\mathbf{u}_{1:k-1}$ and the system model (1) is given by

[1] All scripts can be found in: https://github.com/jfilipe33/ExCompMP208

$$\hat{\mathbf{x}}_{k|k} = \arg\min_{\bar{\mathbf{x}}_k} E\left((\mathbf{X}_k - \bar{\mathbf{x}}_k)^T (\mathbf{X}_k - \bar{\mathbf{x}}_k) \,|\, \{\mathbf{Y}_{1:k} = \mathbf{y}_{1:k}\} \right) \tag{4}$$

.

Structuring the solution as a recursive algorithm, each iteration will be divided in two steps:

- **Prediction:** the use of (1) to obtain a predictive estimate;
- **Update:** The fusion of a new measure with the predictive estimate to obtain a filtered estimate.

The predictive estimate and its covariance are given by:

$$\hat{\mathbf{x}}_{k+1|k} = \mathbf{A}\hat{\mathbf{x}}_{\mathbf{k|k}} + \mathbf{B}u_k \tag{5}$$
$$\mathbf{P}_{k+1|k} = \mathbf{A}\mathbf{P}_{k|k}\mathbf{A}^T + \mathbf{Q} \tag{6}$$

It is also possible to obtain the predictive measure, its corresponding conditional covariance, and the conditional cross-covariance between $\mathbf{X}_{k+1}$ and $\mathbf{Y}_{k+1}$ by calculating:

$$\hat{\mathbf{y}}_{k+1|k} = \mathbf{C}\hat{\mathbf{x}}_{\mathbf{k+1|k}} \tag{7}$$
$$\mathbf{P}^{Y}_{k+1|k} = \mathbf{C}\mathbf{P}_{k+1|k}\mathbf{C}^T + \mathbf{R} \tag{8}$$
$$\mathbf{P}^{XY}_{k+1|k} = \mathbf{P}_{k+1|k}\mathbf{C}^T \tag{9}$$

Now, for the Update step, the filtered estimate and the estimation error covariance are calculated by:

$$\hat{\mathbf{x}}_{k+1|k+1} = \hat{\mathbf{x}}_{\mathbf{k+1|k}} + \mathbf{K}_{k+1}\left(\mathbf{y}_{k+1} - \hat{\mathbf{y}}_{k+1|k}\right) \tag{10}$$
$$\mathbf{P}_{k+1|k+1} = \mathbf{P}_{k+1|k} - \mathbf{P}^{XY}_{k+1|k}\left(\mathbf{P}^{Y}_{k+1|k}\right)^{-1}\left(\mathbf{P}^{XY}_{k+1|k}\right)^{-1} \tag{11}$$

where $\mathbf{K}_{k+1}$ is the Kalman gain biven by

$$\mathbf{K}_{k+1} = \mathbf{P}^{XY}_{k+1|k}\left(\mathbf{P}^{Y}_{k+1|k}\right)^{-1} \tag{12}$$

2.1 KF Simulations

With the system model in (1), 10 realizations $\{y_k\}$ of the measured output were obtained via digital simulation, assuming that the states initial values are a realization of $\mathbf{x}_1$. The realizations were plotted in a graph, alongside the plot of the input command $\{\bar{y}_k\}$, and these results are illustrated in Figure 1.

Next, the Kalman Filter was implemented on all the realizations previously obtained. Through equations (5)-(12), assuming the initial estimation states $\mathbf{x}_{1|1} = \bar{\mathbf{x}}$ and initial estimation error covariance $\mathbf{P}_{1|1} = \bar{\mathbf{P}}$, the estimation errors obtained are illustrated in Figures 2 and 3.

The mean of the estimate error in the ten realizations was calculated as:

$$\mathbf{m}_{\tilde{\mathbf{x}}_{\mathbf{k|k}}}\mathbf{KF} = \frac{1}{N}\sum_{k=1}^{N}\tilde{\mathbf{x}}_{\mathbf{k|k}} =$$

$$\begin{bmatrix} -0.0023 & 0.0085 & -0.0054 & -0.0094 & -0.0109 & 0.0115 & -0.0004 & 0.0058 & 0.0035 & 0.0038 \\ -0.0461 & -0.0379 & -0.0659 & -0.0080 & -0.1287 & 0.0366 & 0.1717 & -0.2619 & -0.1145 & -0.0270 \end{bmatrix}$$

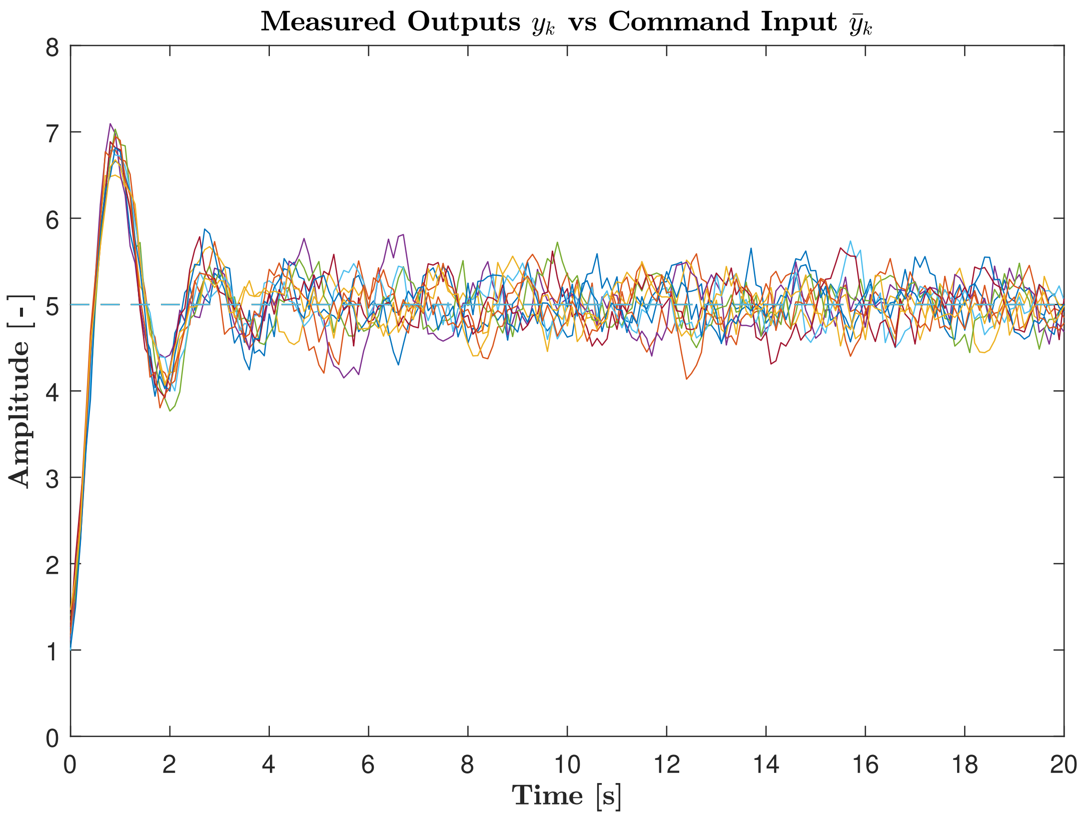

Figure 1: Visual representation of 10 realizations of y_k and the reference command input.

The Root Mean Square Error, which also compares the estimate and the model states, was calculated as:

$$\mathbf{RMS}(\tilde{\mathbf{x}}_{\mathbf{k}|\mathbf{k_i}}) = \sqrt{\frac{1}{N-1}\sum_{j=1}^{N} \tilde{x}_i \tilde{x}_i^T} =$$

$$\begin{bmatrix} 0.0827 & 0.0819 & 0.0822 & 0.0834 & 0.0817 & 0.0839 & 0.0889 & 0.0776 & 0.0881 & 0.0847 \\ 0.5218 & 0.4558 & 0.3841 & 0.5541 & 0.5147 & 0.5203 & 0.6120 & 0.5671 & 0.4279 & 0.5366 \end{bmatrix}$$

where $i = \{1, 2\}$ represents each component of $\tilde{x}_{k|k}$, and each column represents a realization $\{y_k\}$. It is noticeable that the error in estimating x_2 is often considerably greater than the error in estimating x_1. This happens because the systems only provides measurements of x_1, which improves this state's estimation precision. Also, the disturbance $\mathbf{w}_k$ inflicts a greater variation in x_2 than in x_1, according to the disturbance covariance matrix Q.

The matrix $P_{k|k}$, updated along with the states estimation every iteration, represents the covariance matrix of the estimation error. Therefore, its square root represents the standard deviation matrix of the estimation error. The calculation the matrix square root of $P_{k|k}$, for each of the 10 realizations, gives the same matrix:

$$\sqrt{P_{k|k}} = \begin{bmatrix} 0.0802 & 0.0197 \\ 0.0197 & 0.4918 \end{bmatrix} \tag{13}$$

From Figure 2, it is possible to see that error estimations of x_1 have mean near zero and vary within the standard deviation given by the first value of matrix (13). A similar analysis can be done with Figure 3, x_2 and the last value of matrix (13).

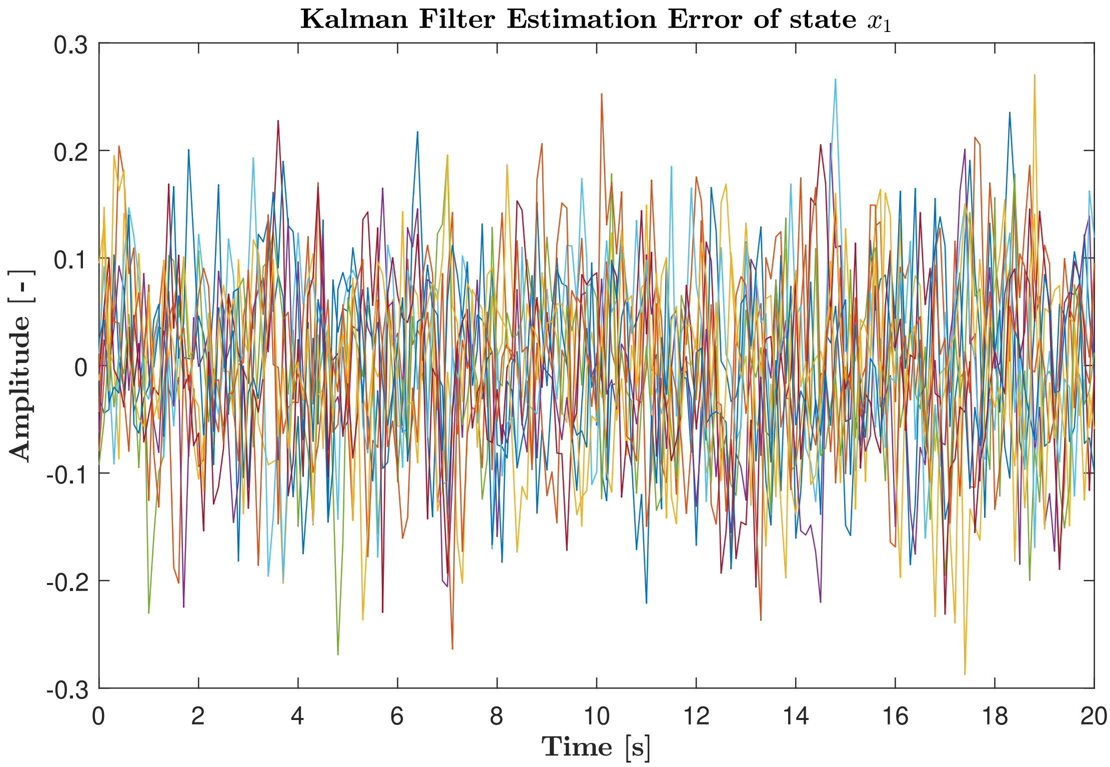

Figure 2: Estimation Error of State x_1 with a Kalman Filter

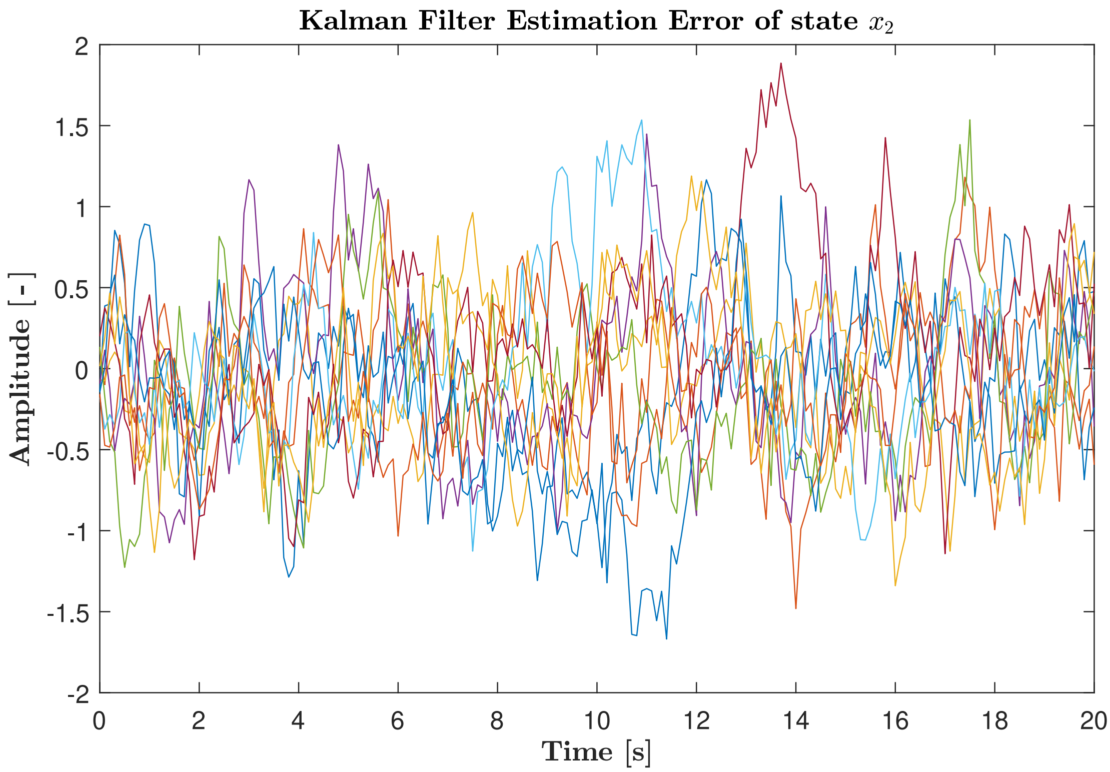

Figure 3: Estimation Error of State x_2 with a Kalman Filter

3 Information Filter (IF)

The information matrix is the inverse of the covariance matrix. That being said, the Kalman Filter can be adapted to a Information Filter by the following variables transformations:

$$\mathbf{L}_{k|k} \triangleq \left(\mathbf{P}_{k|k}\right)^{-1}$$
$$\hat{\mathbf{z}}_{k|k} \triangleq \mathbf{L}_{k|k}\hat{\mathbf{x}}_{k|k}$$

Given $\mathbf{L}_{k|k}$ and $\hat{\mathbf{z}}_{k|k}$, the Information Filter prediction step for this experiment can be calculated by:

$$\mathbf{\Pi}_k = \mathbf{A}^{-T}\mathbf{L}_{k|k}\mathbf{A}^{-1}$$
$$\mathbf{K}^* = \mathbf{\Pi}_k\left(\mathbf{\Pi}_k + \mathbf{Q}^{-1}\right)^{-1}$$
$$\hat{\mathbf{z}}_{k+1|k} = \left(\mathbf{I}_2 - \mathbf{K}^*\right)\mathbf{A}^{-T}\hat{\mathbf{z}}_{k|k} + \left(\mathbf{I}_2 - \mathbf{K}^*\right)\mathbf{\Pi}_k\mathbf{B}_k\mathbf{u}_k$$
$$\mathbf{L}_{k+1|k} = \left(\mathbf{I}_2 - \mathbf{K}^*\right)\mathbf{\Pi}_k$$

The equations for the update step of the Information Filter follow:

$$\hat{\mathbf{z}}_{k+1|k+1} = \hat{\mathbf{z}}_{k+1|k} + \mathbf{C}^T\mathbf{R}^{-1}\mathbf{y}_{k+1}$$
$$\mathbf{L}_{k+1|k+1} = \mathbf{L}_{k+1|k} + \mathbf{C}^T\mathbf{R}^{-1}\mathbf{C}$$

Whenever necessary, it is possible to recover the filtered estimate as well as its corresponding covariance by the following variable transformations:

$$\mathbf{P}_{k+1|k+1} \triangleq \left(\mathbf{L}_{k+1|k+1}\right)^{-1}$$
$$\hat{\mathbf{x}}_{k+1|k+1} \triangleq \mathbf{P}_{k+1|k+1}\hat{\mathbf{z}}_{k+1|k+1}$$

It is interesting to notice the similarity between the equations of the prediction step of the Kalman Filter and the update step of the Information Filter, and vice-versa, caused by the variable transformations. This aspect can be useful when the number of measurements is greater than the number of states, as inverting matrix P in the Kalman Filter process could be difficult in this case.

3.1 IF Simulations

The Information Filter was implemented on all the realizations previously obtained. Assuming the initial estimation states $\mathbf{x}_{1|1} = \bar{\mathbf{x}}$ and initial estimation error covariance $\mathbf{P}_{1|1} = \bar{\mathbf{P}}$, the variables transformations to $\mathbf{z}$ and $\mathbf{L}$ were made, and the estimation errors were obtained by reversing the transformation in the end of the algorithm. The results are illustrated in Figures 4 and 5.

$$\mathbf{m}_{\tilde{\mathbf{x}}_{\mathbf{k}|\mathbf{k}}}\mathbf{IF} = \frac{1}{N}\sum_{k=1}^{N}\tilde{\mathbf{x}}_{\mathbf{k}|\mathbf{k}} =$$
$$\begin{bmatrix} -0.0023 & 0.0085 & -0.0054 & -0.0094 & -0.0109 & 0.0115 & -0.0004 & 0.0058 & 0.0035 & 0.0038 \\ -0.0461 & -0.0379 & -0.0659 & -0.0080 & -0.1287 & 0.0366 & 0.1717 & -0.2619 & -0.1145 & -0.0270 \end{bmatrix}$$

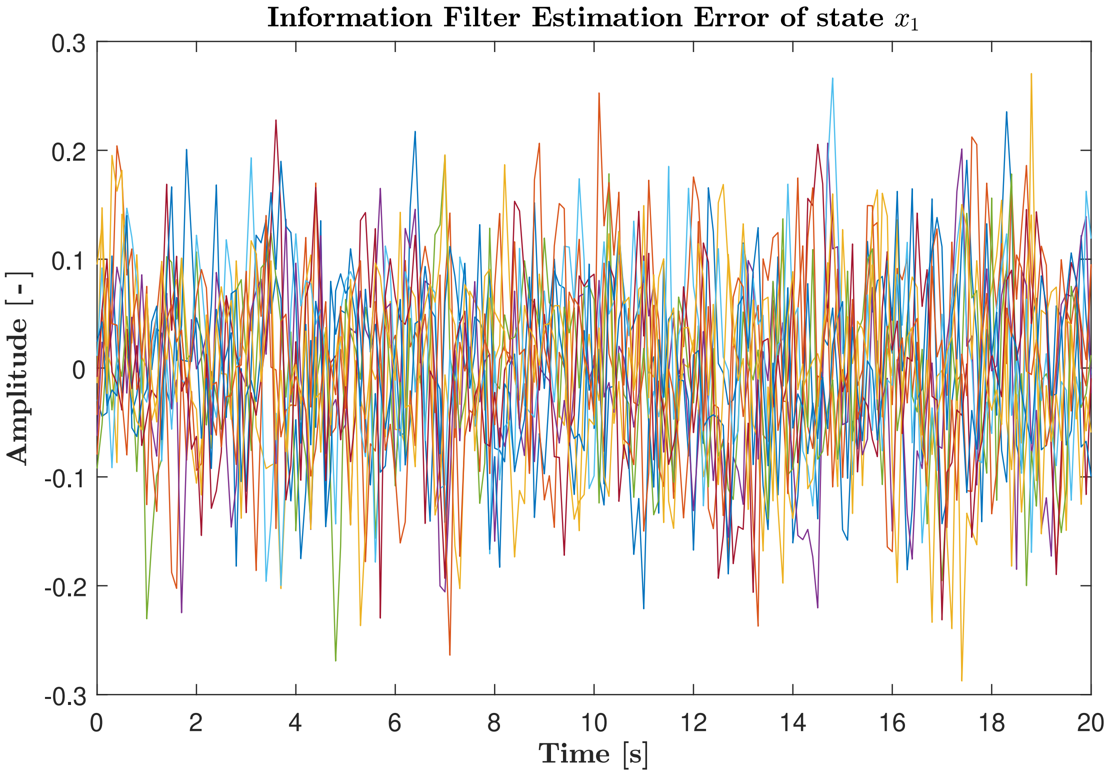

Figure 4: Estimation Error of State x_1 with a Information Filter

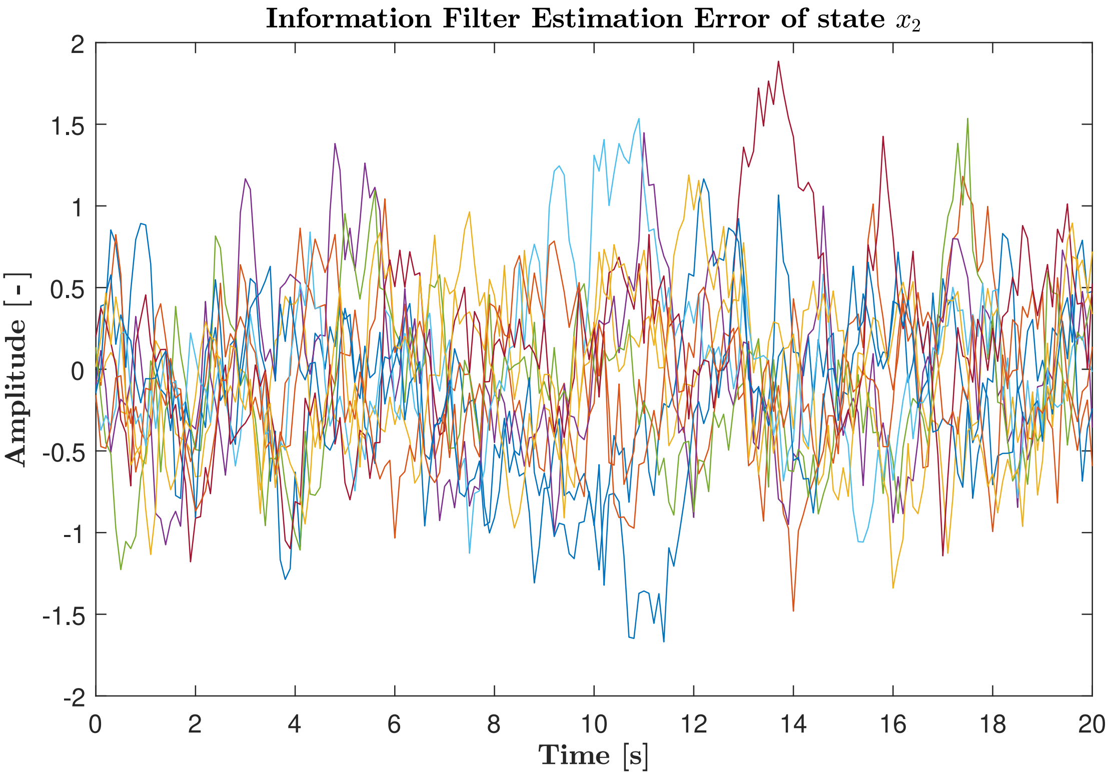

Figure 5: Estimation Error of State x_2 with a Information Filter

$$\mathbf{RMS}(\tilde{\mathbf{x}}_{\mathbf{k}|\mathbf{k_i}}) = \sqrt{\frac{1}{N-1}\sum_{j=1}^{N}\tilde{x}_i\tilde{x}_i^T} =$$

$$\begin{bmatrix} 0.0827 & 0.0819 & 0.0822 & 0.0834 & 0.0817 & 0.0839 & 0.0889 & 0.0776 & 0.0881 & 0.0847 \\ 0.5218 & 0.4558 & 0.3841 & 0.5541 & 0.5147 & 0.5203 & 0.6120 & 0.5671 & 0.4279 & 0.5366 \end{bmatrix}$$

$$\sqrt{L_{k|k}^{-1}} = \sqrt{P_{k|k}} = \begin{bmatrix} 0.0802 & 0.0197 \\ 0.0197 & 0.4918 \end{bmatrix} \tag{14}$$

As expected, all values of sample mean, RMS error and Estimation Error Standard Deviation are the same as the obtained in the Kalman Filter, as both filters are practically identical, differing only by a variable transformation. The variation of the errors in figures 4 and 5 are also mostly inside the standard deviation given in matrix (14).

4 Conclusion

The objective of this exercise was to formulate an algorithm capable of representing a simple second-order system and estimate its states by using both a Kalman and an Information Filters. The simulations were successful and both estimators provided good estimations of the x. Ideally, the mean and variation of the estimation error would both be zero. But the fact that the system only provides measurements of one component of the state vector, along with the existence of added state disturbances and measurement noises in the form of $\mathbf{w}$ and v, respectively, reduces the precision of the estimation. Apart from the disparities in algorithm, no differences were observed whilst comparing the performance of the estimators. When simulated together, trying to estimate the same realization of y_k, both methods provided identical graphs, error means and covariance matrices.