EFK Estimator Analysis

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MP-208: Optimal Filtering with Aerospace Applications

November 23, 2020

Abstract

This document contains the third simulation exercise for the Optimal Filtering with Aerospace Applications class. Based on the content given in the lectures, it was requested to formulate an algorithm capable of representing a non-linear dynamic system model whose states were to be estimated by a Extended Kalman Filter estimator. The estimator was implemented in Matlab scripts[1] and plots were generated to illustrate the estimation errors.

1 Problem Statement

Consider a system described by:

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{f}\left(\mathbf{x}(t), u(t)\right) + \mathbf{w}(t) \\ y_{k+1} &= h\left(\mathbf{x}_{k+1}\right) + v_{k+1} \end{aligned} \tag{1}$$

where $\mathbf{x}(t) \triangleq [x_1(t)\ x_2(t)]^T \in \mathbb{R}^2$, $y_k \in \mathbb{R}$ is the state vector at continuous time t, $\mathbf{x}_k \triangleq \mathbf{x}(t_k)$ is the state vector at the discrete time k, $u(t) \in \mathbb{R}$ is the control input, $\{\mathbf{w}_k \in \mathbb{R}^2\}$ is the state noise, $y_k \in \mathbb{R}$ is the measured output, $\{v_k \in \mathbb{R}\}$ is the measurement noise and,

$$\mathbf{f}\left(\mathbf{x}(t), u(t)\right) \triangleq \begin{bmatrix} -x_1(t) + x_2(t) \\ -0.1x_1(t)^2 - 1 + u(t) \end{bmatrix}, \tag{2}$$

$$h(\mathbf{x}) = x_1 \tag{3}$$

For the sake of convenience, consider the control input

$$u(t) = -10y(t) + 10 \tag{4}$$

where $y(t) \in \mathbb{R}$ is the continuous-time counterpart of y_k.

Adopt the parameters presented in Table 1.

The goal of this exercise is to present the estimation errors of the states $\mathbf{x}$, using an Extended Kalman Filter.

[1] All scripts can be found in: https://github.com/jfilipe33/ExCompMP208

Table 1: System Parameters

Description	Value
Covariance of the state noise	$\mathbf{Q}(t) = 0.01\mathbf{I}_2$
Covariance of the measurement noise	$R_k = 0.01$
Statistics of the Initial state	$\bar{\mathbf{x}} = \mathbf{0}_2, \quad \mathbf{P} = \mathbf{I}_2$
Sensor sampling time	$T_s = 0.1s$

2 Extended Kalman Filter (EKF)

A Kalman Filter uses a Minimum Mean Square Error estimator to estimate the state at the current instant k, given the measurements obtained since the start of the simulation until the instant k. For non-linear systems and noises not necessarily normally distributed, an approximately optimal estimator can be calculated using the Extended Kalman Filter. Equations (1)-(3) show a non-linear continuous system, with $\mathbf{f}$ and h differentiable.

Approximate the non-linear functions $\mathbf{f}$ and $\mathbf{h}$ by Taylor series expansion and truncation:

$$\mathbf{f}\left(\mathbf{x}(t), \mathbf{u}(t)\right) \approx \mathbf{f}\left(\hat{\mathbf{x}}(t), \mathbf{u}(t)\right) + \mathbf{F}(t)\left(\mathbf{x}(t) - \hat{\mathbf{x}}(t)\right) \tag{5}$$

$$\mathbf{h}_{k+1}(\mathbf{x}_{k+1}) \approx \mathbf{h}_{k+1}\left(\hat{\mathbf{x}}_{k+1|k}\right) + \mathbf{H}_{k+1}\left(\mathbf{x}_{k+1} - \hat{\mathbf{x}}_{k+1|k}\right) \tag{6}$$

where $\hat{\mathbf{x}}(t) \triangleq E(\mathbf{x}(t)|\mathbf{Y}_{1:k})$, $\hat{\mathbf{x}}_{k+1|k} \triangleq E(\mathbf{x}_{k+1}|\mathbf{Y}_{1:k})$, and

$$\mathbf{F}(t) \triangleq \frac{\partial \mathbf{f}(\hat{\mathbf{x}}(t), u(t))}{\partial \mathbf{x}}, \quad \mathbf{H}_{k+1} \triangleq \frac{d\mathbf{h}_{k+1}(\hat{\mathbf{x}}_{k+1|k})}{d\mathbf{x}} \tag{7}$$

Using (5)-(6), system (1) can be approximated to:

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{F}(t)\mathbf{x}(t) + \mathbf{w}(t) + \left(\mathbf{f}(\hat{\mathbf{x}}(t), u(t)) - \mathbf{F}(t)\hat{\mathbf{x}}(t)\right) \\ y_{k+1} &= \mathbf{H}\mathbf{x}_{k+1} + v_{k+1} + \left(h(\hat{\mathbf{x}}_{k+1|k}) - \mathbf{H}\hat{\mathbf{x}}_{k+1|k}\right) \end{aligned} \tag{8}$$

Structuring the solution as a recursive algorithm, each iteration will be divided in two steps:

- **Prediction:** the use of (8) to obtain a predictive estimate;
- **Update:** The fusion of a new measure with the predictive estimate to obtain a filtered estimate.

Given the filtered mean $\hat{\mathbf{x}}_{k|k}$ and the covariance $\mathbf{P}_{k|k}$, The predictive estimate and its covariance are given by integrating the ODEs:

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{f}(\hat{\mathbf{x}}(t), u(t)) \tag{9}$$

$$\dot{\mathbf{P}}(t) = \mathbf{F}(t)\mathbf{P}(t) + \mathbf{P}(t)\mathbf{F}(t)^T + \mathbf{Q} \tag{10}$$

from t_k to t_{k+1}, considering $u(t) = u_k, \forall t \in [t_k, t_{k+1})$.

Setting $\hat{\mathbf{x}}_{\mathbf{k+1|k}} = \dot{\hat{\mathbf{x}}}(t)$ and $\mathbf{P}_{k+1|k} = \dot{\mathbf{P}}(t)$, it is also possible to obtain the predictive measure, its corresponding conditional covariance, and the conditional cross-covariance between $\mathbf{X}_{k+1}$ and $\mathbf{Y}_{k+1}$ by calculating:

$$\hat{\mathbf{y}}_{k+1|k} = \mathbf{h}(\hat{\mathbf{x}}_{\mathbf{k+1|k}}) \tag{11}$$

$$\mathbf{P}^{Y}_{k+1|k} = \mathbf{H}\mathbf{P}_{k+1|k}\mathbf{H}^T + \mathbf{R} \tag{12}$$

$$\mathbf{P}^{XY}_{k+1|k} = \mathbf{P}_{k+1|k}\mathbf{H}^T \tag{13}$$

Now, for the Update step, the filtered estimate and the estimation error covariance are calculated by:

$$\hat{\mathbf{x}}_{k+1|k+1} = \hat{\mathbf{x}}_{\mathbf{k+1|k}} + \mathbf{K}_{k+1}\left(\mathbf{y}_{k+1} - \hat{\mathbf{y}}_{k+1|k}\right) \tag{14}$$

$$\mathbf{P}_{k+1|k+1} = \mathbf{P}_{k+1|k} - \mathbf{P}^{XY}_{k+1|k}\left(\mathbf{P}^{Y}_{k+1|k}\right)^{-1}\left(\mathbf{P}^{XY}_{k+1|k}\right)^{-1} \tag{15}$$

where $\mathbf{K}_{k+1}$ is the Extended Kalman gain given by

$$\mathbf{K}_{k+1} = \mathbf{P}^{XY}_{k+1|k}\left(\mathbf{P}^{Y}_{k+1|k}\right)^{-1} \tag{16}$$

2.1 EKF Simulations

With the system model in (1), 100 realizations $\{y_k\}$ of the measured output were obtained via digital simulation, assuming that the states initial values are a realization of $\mathbf{x}_1$. The realizations were plotted in a graph, alongside the sample mean $\{\bar{y}_k\}$, and these results are illustrated in Figure 1.

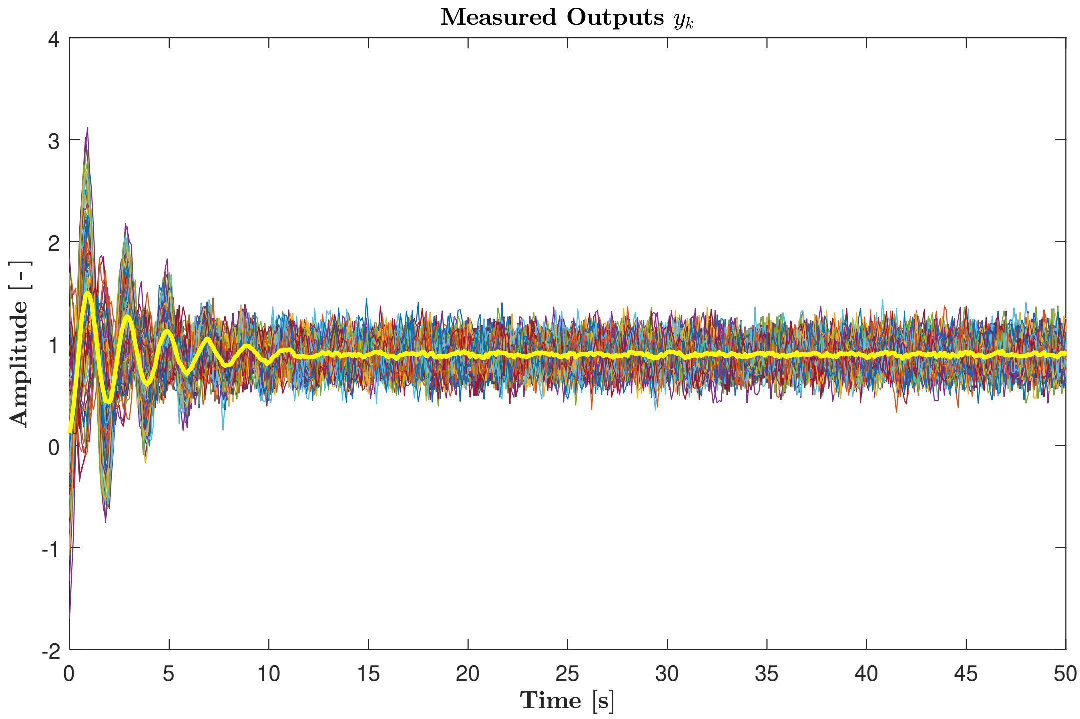

Figure 1: Visual representation of 100 realizations of y_k and its sample mean.

Next, the Extended Kalman Filter was implemented on all the realizations previously obtained. Through equations (9)-(16), assuming the initial estimation states $\mathbf{x}_{1|1} = \bar{\mathbf{x}}$ and initial estimation error covariance $\mathbf{P}_{1|1} = \bar{\mathbf{P}}$, the estimation errors obtained are illustrated in Figures 2 and 3.

The mean of the estimate error in the hundred realizations was calculated as:

$$\mathbf{m}_{\tilde{\mathbf{x}}_{\mathbf{k|k}}}\mathbf{EKF} = \left\{\frac{1}{N}\sum_{j=1}^{N}\tilde{\mathbf{x}}^{(\mathbf{j})}_{\mathbf{k|k}}\right\}_{k=1}^{501} \tag{17}$$

and was plotted with a thick continuous yellow line in figures 2 and 3.

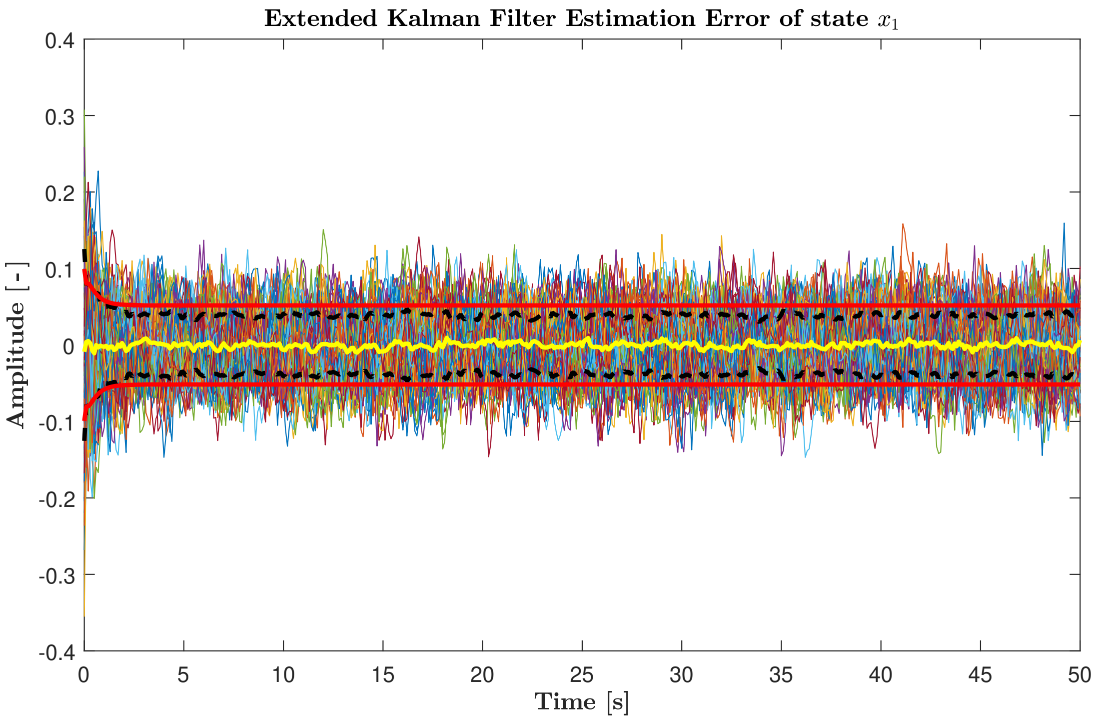

Figure 2: Estimation Error of State x_1 with an Extended Kalman Filter

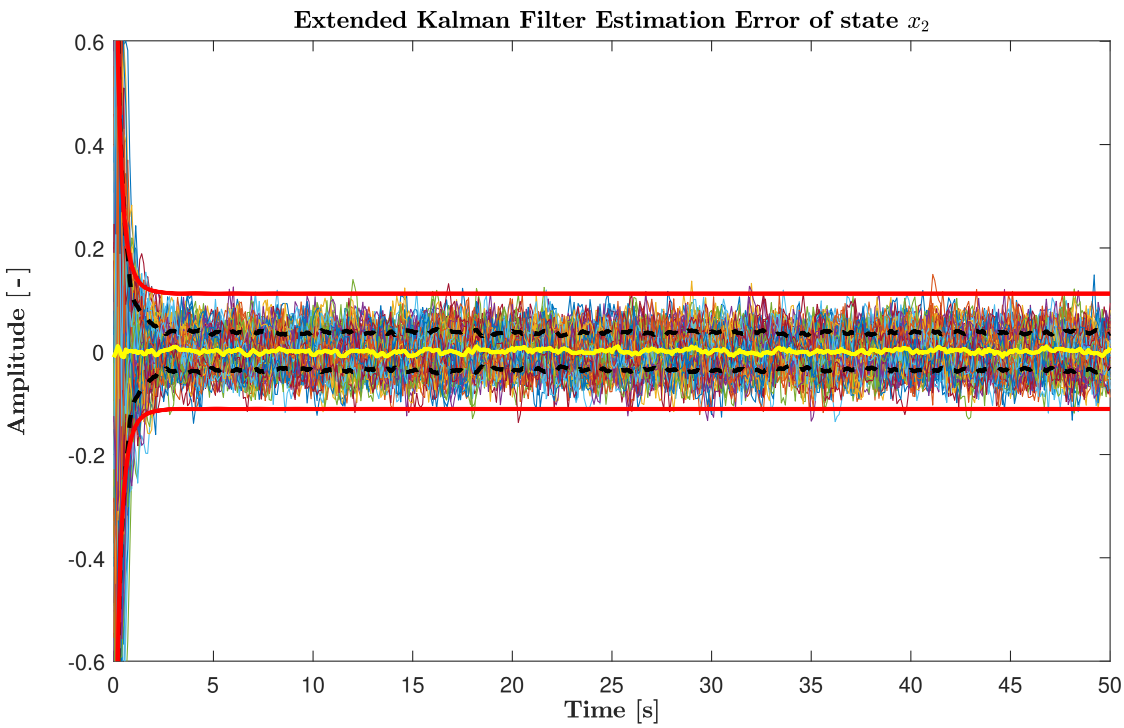

Figure 3: Estimation Error of State x_2 with an Extended Kalman Filter

The Root Mean Square Error, which also compares the estimate and the model states, was calculated as:

$$\mathbf{RMS}(\tilde{\mathbf{x}}_{\mathbf{k|k_i}}) = \sqrt{\left\{ \frac{1}{N-1} \sum_{j=1}^{N} \tilde{x}_i \tilde{x}_i^T \right\}_{k=1}^{501}} \tag{18}$$

where $i = \{1, 2\}$ represents each component of $\tilde{x}_{k|k}$. The calculated values were plotted with a thick dashed black line in Figures 2 and 3. The lines were plotted both positive and negative, to form an envelope around the majority of error values. This lines are supposed to be approximate to the theoretical standard deviation of the signals.

The matrix $P_{k|k}$, updated along with the states estimation every iteration, represents the covariance matrix of the estimation error. Therefore, its square root represents the standard deviation matrix of the estimation error. The calculation the matrix square root of $P_{k|k}$, for each of the 100 realizations, gives the same matrix:

$$\sqrt{P_{k|k}} = \begin{bmatrix} 0.0491 & 0.0160 \\ 0.0160 & 0.1104 \end{bmatrix}, \tag{19}$$

which were also plotted in Figures 2 and 3 as red thick continuous lines. As mentioned before, the RMS lines are supposed vary around the standard deviation. However, as these are approximately optimal estimations, this result might demand the tuning of matrices $\mathbf{Q}$ and $\bar{\mathbf{P}}$.

From Figure 2, it is possible to see that error estimations of x_1 have mean near zero and vary mostly within the standard deviation given by the RMS error calculated in (18). A similar analysis can be done with Figure 3, x_2 and the RMS estimation error of x_2.

2.2 EKF with tuned matrices

Through extensive trial and error, altering matrix $\mathbf{Q}$ used to calculate the filter (see equation (10)) to $\mathbf{Q}_{ekf} = diag([1 \times 10^{-5},\ \ 5 \times 10^{-3}])$, provided the plots illustrated in figures 4 and 5.

For this altered matrix $\mathbf{Q}$, the square root of the estimation error covariance matrix $P_{k|k}$ stabilizes at:

$$\sqrt{P_{k|k}} = \begin{bmatrix} 0.0275 & 0.0208 \\ 0.0208 & 0.0651 \end{bmatrix} \tag{20}$$

The tuning of $\mathbf{Q}$ approximated the theoretical standard deviation $\sqrt{P_{k|k}}$ of $\tilde{x}_1$ (red continuous lines) to its RMS value (black dashed lines). Unfortunately, although closer than before, the theoretical standard deviation $\sqrt{P_{k|k}}$ of $\tilde{x}_2$ could not be matched to its RMS value.

This discrepancy is an effect of the approximation made, due to the linearization of the non-linear second equation in (2). Another effect of this linearization is that it makes the estimations behave non-gaussianly, turning invalid some desired properties like the grouping of 68% of the samples inside the standard deviation bounds.

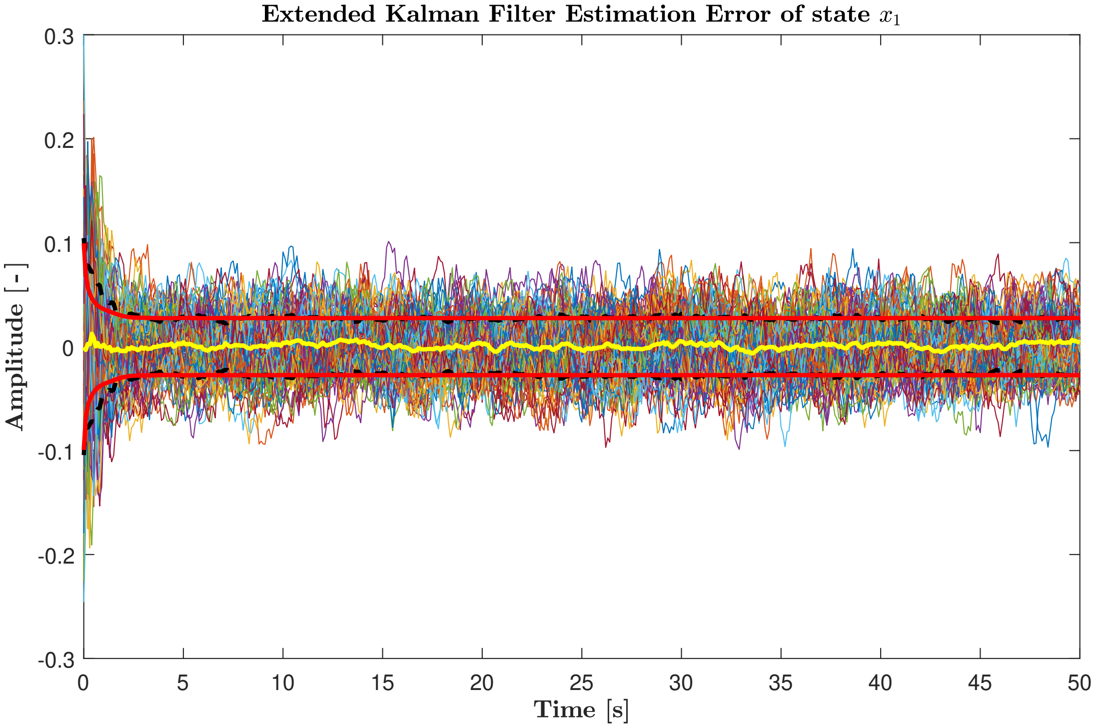

Figure 4: Estimation Error of State x_1 with an Extended Kalman Filter and a modified $\mathbf{Q}$ matrix.

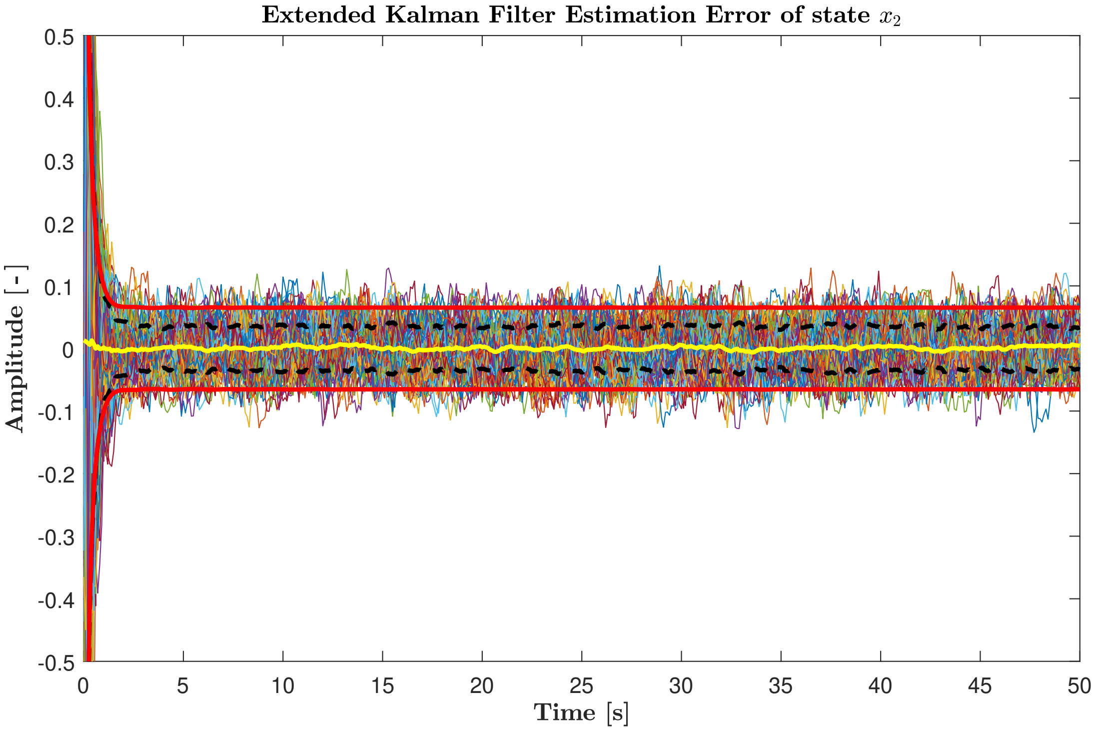

Figure 5: Estimation Error of State x_2 with an Extended Kalman Filter and a modified $\mathbf{Q}$ matrix.

3 Conclusion

The objective of this exercise was to formulate an algorithm capable of representing a non-linear system and estimate its states by using an Extended Kalman Filter. The simulations were successful and the estimator provided good estimations of x, which was proved by the estimation error $\tilde{x}$ having low variation around its zero mean. Ideally, the mean and variation of the estimation error would both be zero. But the fact that the system had to be linearized to fit in the Extended Kalman Filter procedure, along with the existence of added state disturbances and measurement noises in the form of $\mathbf{w}$ and v, respectively, reduces the precision of the estimation. Although the calculations provided a converging filter, the theoretical standard deviation did not match the RMS error calculated from the samples. This disparity was attenuated by tuning Q in the filter calculations, but it could not be cancelled.