State Estimation using Sliding Mode Observers

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MP-273: Sliding Mode Control

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Abstract

This document contains the final computational exercise for the Sliding Mode Control (SMC) class. Based on the content given in the lectures, it was requested to formulate an algorithm capable of representing a Mass-Spring-Damper model, whose position and velocity were to be estimated by a Slotine Observer and a Super-Twisting Observer. The estimators were implemented in Matlab scripts[1] and plots were generated to illustrate the errors between the real states and their estimated values.

1 Problem Formulation

Consider a Mass-Spring-Damper system as illustrated by Figure 1. The translational dynamics of this system can be described by

$$m\ddot{y} + b\dot{y}|\dot{y}| + k_0 y + k_1 y = d, \tag{1}$$

where $y \in \mathbb{R}$ represents the block's position (consider $y = 0$ the undeformed spring position), $m \in \mathbb{R}_+$ is the block's mass, $b \in \mathbb{R}_+$ is the damper's friction coeficient, $k_0, k_1 \in \mathbb{R}_+$ are the spring constants, and $d \in \mathbb{R}$ is a disturbance input. Also, assume that $|d| < \rho$, with known $\rho < \infty$.

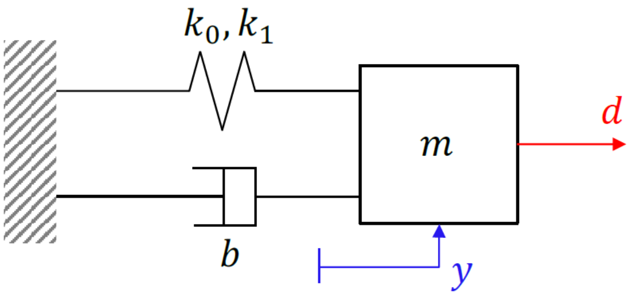


Figure 1: Graphic representation of a Mass-Spring-Damper system.

Let us represent the system dynamics, described in equation (1), in State Space regular form for non-linear systems.

$$\dot{\mathbf{x}}_1 = \mathbf{f}_1\left(\mathbf{x}_1, \mathbf{x}_2\right), \tag{2}$$

$$\dot{\mathbf{x}}_2 = \mathbf{f}_2\left(\mathbf{x}_1, \mathbf{x}_2\right) + \mathbf{B}\left(\mathbf{x}_1, \mathbf{x}_2\right)(\mathbf{u} + \mathbf{d}), \tag{3}$$

where $\mathbf{x}_1 \triangleq y$, $\mathbf{x}_2 \triangleq \dot{y}$, $\mathbf{u} \triangleq 0$, and $\mathbf{d} \triangleq d$.

[1] All scripts can be found in: https://github.com/jfilipe33/ExCompMP273

From (1) and (2)-(3), we obtain $\mathbf{B} = \frac{1}{m}$ and:

$$\mathbf{f}_1(\mathbf{x}) = x_2, \tag{4}$$
$$\mathbf{f}_2(\mathbf{x}) = \frac{1}{m}\left(-bx_2|x_2| - k_0x_1 - k_1x_1^3\right). \tag{5}$$

The problem studied in this assignment is the estimation of the states $\mathbf{x}$ using the measurements of y.

2 Sliding Mode Observers

Consider a n-th order system described by

$$x^{(n)} = f(\mathbf{x}) + b(\mathbf{x})u + d, \tag{6}$$

where $\mathbf{x} \equiv (x_1, x_2, \ldots, x_n) \triangleq \left(x, \dot{x}, \ldots, x^{(n-1)}\right) \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}$ is the input, $y \triangleq x \in \mathbb{R}$ is the (measured) output, f and b are known smooth functions, and d is the disturbance input. Also, assume that $|d| < \rho$, with known $\rho < \infty$.

With this system defined, let us study the sliding-mode observers that will be used in this assignment.

2.1 Slotine Observer

Consider the observer [1]:

$$\begin{aligned}
\dot{\hat{x}}_1 &= a_1\tilde{x}_1 + \hat{x}_2 + \kappa_1 \operatorname{sign}(\tilde{x}_1), \\
\dot{\hat{x}}_2 &= a_2\tilde{x}_1 + \hat{x}_3 + \kappa_2 \operatorname{sign}(\tilde{x}_1), \\
&\vdots \\
\dot{\hat{x}}_n &= a_n\tilde{x}_1 + f(\hat{\mathbf{x}}) + b(\hat{\mathbf{x}})u + \kappa_n \operatorname{sign}(\tilde{x}_1),
\end{aligned} \tag{7}$$

where the scalars $a_i > 0$ and $\kappa_i > 0$ are design parameters, $\tilde{x}_1 \triangleq x_1 - \hat{x}_1$, $\hat{x}_i$ denotes the estimate of x_i, and $f(\hat{\mathbf{x}})$ and $b(\hat{\mathbf{x}})u$ are the mimetization of the non-linear functions of the system, taking the states estimates as arguments.

For $n = 2$, equation (7) can be rewritten as

$$\dot{\hat{x}}_1 = a_1\tilde{x}_1 + \hat{x}_2 + \kappa_1 \operatorname{sign}(\tilde{x}_1), \tag{8}$$
$$\dot{\hat{x}}_2 = a_2\tilde{x}_1 + f(\hat{\mathbf{x}}) + b(\hat{\mathbf{x}})u + \kappa_2 \operatorname{sign}(\tilde{x}_1). \tag{9}$$

Similarly, equation (6) can be rewritten as

$$\dot{x}_1 = x_2, \tag{10}$$
$$\dot{x}_2 = f(\mathbf{x}) + b(\mathbf{x})u + d \tag{11}$$

Knowing that $\tilde{x}_i \triangleq x_i - \hat{x}_i$, the dynamics of the estimation error can be described, from (8)-(11), by

$$\dot{\tilde{x}}_1 = -a_1\tilde{x}_1 + \tilde{x}_2 - \kappa_1 \operatorname{sign}(\tilde{x}_1), \tag{12}$$
$$\dot{\tilde{x}}_2 = -a_2\tilde{x}_1 + \delta - \kappa_2 \operatorname{sign}(\tilde{x}_1), \tag{13}$$

where

$$\delta \triangleq (f(\mathbf{x}) - f(\hat{\mathbf{x}})) + (b(\mathbf{x}) - b(\hat{\mathbf{x}}))u + d. \tag{14}$$

To check if there is sliding-mode for the estimation error $\tilde{x}_1$, let us consider a Lyapunov-function candidate $V \triangleq \tilde{x}_1^2/2$. Differentiating it with respect to time, we have:

$$
\begin{aligned}
\dot{V} &= \tilde{x}_1\dot{\tilde{x}}_1, \\
&= \tilde{x}_1\left(-a_1\tilde{x}_1 + \tilde{x}_2 - \kappa_1 \operatorname{sign}(\tilde{x}_1)\right), \\
&= \tilde{x}_1\tilde{x}_2 - a_1\tilde{x}_1^2 - \kappa_1|\tilde{x}_1|, \\
\dot{V} &\leq -\left(\kappa_1 - |\tilde{x}_2|\right)|\tilde{x}_1|.
\end{aligned}
\tag{15}
$$

From the definition of V, we have that $V^{1/2} = \sqrt{2}|\tilde{x}_1|$. Thus, from the latter result and equation (15)

$$
\dot{V} \leq -\sqrt{2}\left(\kappa_1 - |\tilde{x}_2|\right)V^{1/2} \tag{16}
$$

Therefore, from [2], if $\kappa_1 > |\tilde{x}_2|$, then $\tilde{x}_1 = 0$ is finite-time stable.

When the first state estimation error reaches sliding-mode, $\tilde{x}_1 = \dot{\tilde{x}} = 0$ and equations (12)-(13) can be rewritten as

$$
0 = \tilde{x}_2 - \kappa_1 \operatorname{sign}(\tilde{x}_1), \tag{17}
$$

$$
\dot{\tilde{x}}_2 = \delta - \kappa_2 \operatorname{sign}(\tilde{x}_1). \tag{18}
$$

From (17), $\operatorname{sign}(\tilde{x}_1) = \tilde{x}_2/\kappa_1$. By substituting this result in (18), we obtain

$$
\dot{\tilde{x}}_2 = \delta - \frac{\kappa_2}{\kappa_1}\tilde{x}_2. \tag{19}
$$

By solving the differential equation in (19), we have

$$
|\tilde{x}_2| \leq e^{-\frac{\kappa_2}{\kappa_1}(t-t_s)}\left(|\tilde{x}_2| - \delta_{max}\frac{\kappa_1}{\kappa_2}\right) + \delta_{max}\frac{\kappa_1}{\kappa_2}, \tag{20}
$$

where δ_{max} is the upper bound of δ. Therefore, for $t \to \infty$, the first term of equation converges to zero and $|\tilde{x}_2|$ is ultimately bounded by

$$
\lim_{t\to\infty} |\tilde{x}_2| \leq \delta_{max}\frac{\kappa_1}{\kappa_2}. \tag{21}
$$

With this, it is possible to conclude that, when using the Slotine Observer, only the first state estimation error will have finite time convergence. All the other states estimation errors will converge to a region bounded by the chosen gains and the maximum value of δ.

2.2 Super-Twisting Observer

Consider the Super-Twisting Algorithm, which is represented by the following dynamic system

$$
\dot{\sigma}_1 = -k_1\left|\sigma_1\right|^{1/2} \operatorname{sign}(\sigma_1) + \sigma_2, \tag{22}
$$

$$
\dot{\sigma}_2 = -k_2 \operatorname{sign}(\sigma_1) + \delta, \tag{23}
$$

where $\sigma_1 \in \mathbb{R}$ and $\sigma_2 \in \mathbb{R}$ are its state variables, $\kappa_1, \kappa_2 > 0$ are design parameters, and $\delta \in \mathbb{R}$ is a disturbance term. As the differential equations in (22)-(23) have discontinuous right hand sides, their solutions must be understood in the Filippov sense [3]. From the theorems 1 and 2 of reference [4], we know that the Super-Twisting Algorithm is finite-time stable, if the gains

κ_1, κ_2 respect equation 11 of the same reference. However, the suggestion of gains in the work of Shtessel *et. al* (see [5] p.253) provides better results, and its given by

$$\kappa_1 = 1.5\sqrt{\delta_{max}}, \tag{24}$$
$$\kappa_2 = 1.1\,\delta_{max}. \tag{25}$$

Knowing that the aforementioned algorithm provides finite-time stability for both of its arguments, the dynamics in (22)-(23) can be applied to the estimation error $\tilde{\mathbf{x}} \triangleq \mathbf{x} - \hat{\mathbf{x}}$ as

$$\dot{\tilde{x}}_1 = -\kappa_1 \left|\tilde{x}_1\right|^{1/2} \operatorname{sign}\left(\tilde{x}_1\right) + \tilde{x}_2, \tag{26}$$
$$\dot{\tilde{x}}_2 = -\kappa_2 \operatorname{sign}\left(\tilde{x}_1\right) + \delta, \tag{27}$$

with δ equivalent to (14).

From (10)-(11), (14), and (26)-(27), we obtain the following observer:

$$\dot{\hat{x}}_1 = \hat{x}_2 + \kappa_1 \left|\tilde{x}_1\right|^{1/2} \operatorname{sign}\left(\tilde{x}_1\right), \tag{28}$$
$$\dot{\hat{x}}_2 = f(\hat{x}) + b(\hat{x})u + \kappa_2 \operatorname{sign}\left(\tilde{x}_1\right). \tag{29}$$

Therefore, by assuming that $|\delta| \leq \delta_{max} < \infty$, we can use (24)-(25) to compute κ_1 and κ_2 that ensures robust finite-time stability of $\tilde{\mathbf{x}} = \mathbf{0}$.

3 Simulations and Results

We begin this section by defining the simulation parameters that will be used for all the experiments.

Table 1: Simulation Parameters

Symbol	Description	Value
t_f	Simulation Time	10s
m	Block Mass	1 kg
b	Damper Friction Coeficient	$0.1\text{N}/(\text{m/s})^2$
k_0	Spring Constant	0.05N/m
k_1	Spring Constant	0.05N/m^3
$y(0)$	Block Initial Position	0.1m
$\dot{y}(0)$	Block Initial Velocity	-0.1m/s

Also, through every experiment, the system will be under the effects of a disturbance input described by:

$$d \triangleq 0.5 * (0.8\sin(2\pi 2t) + 0.2\sin(2\pi 5t)) \tag{30}$$

3.1 Slotine Observer

For the Mass-Spring-Damper system described in equations (2)-(5), the Slotine Observer is modeled as:

$$\dot{\hat{x}}_1 = a_1\tilde{x}_1 + \hat{x}_2 + \kappa_1 \operatorname{sign}\left(\tilde{x}_1\right), \tag{31}$$
$$\dot{\hat{x}}_2 = a_2\tilde{x}_1 + \kappa_2 \operatorname{sign}\left(\tilde{x}_1\right) + \frac{1}{m}\left(-b\hat{x}_2|\hat{x}_2| - k_0\hat{x}_1 - k_1\hat{x}_1^3\right). \tag{32}$$

For the following simulations, $a_1 = a_2 = k_2 = 1$, $k_1 = 0.3$ and $\delta_{max} = 0.5$. The observer states were both initialized at zero.

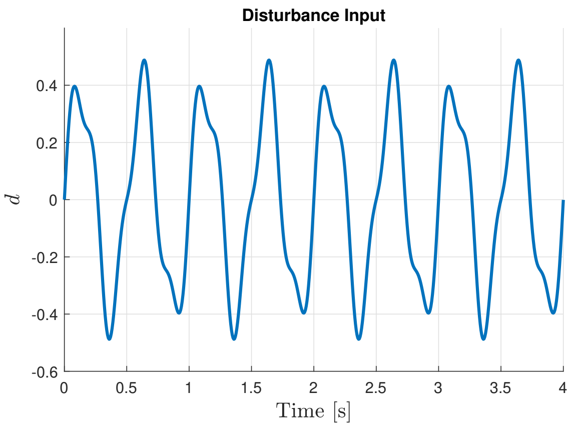
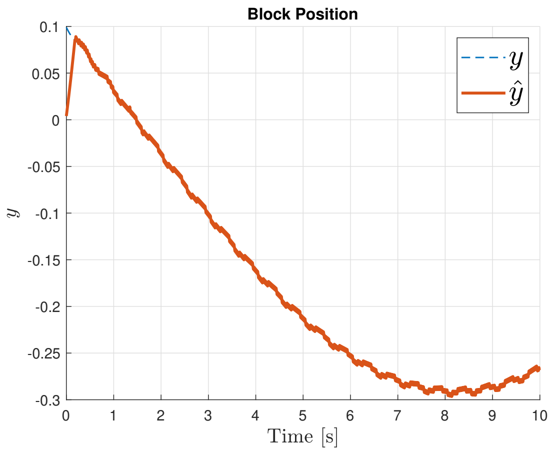
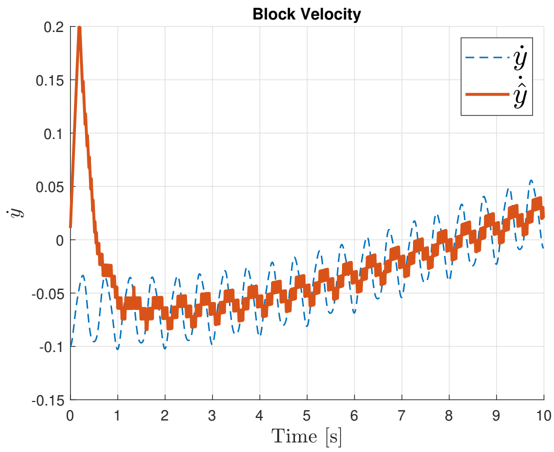


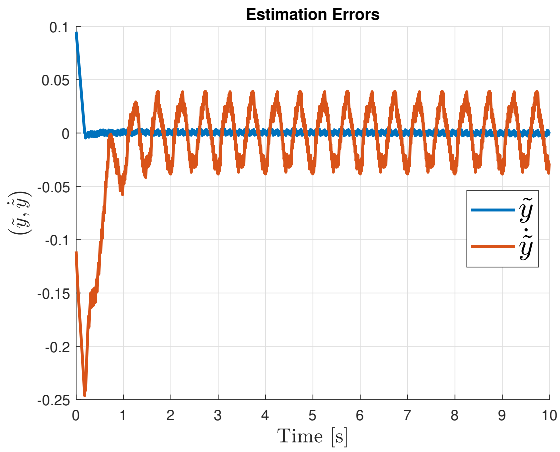
Figure 2: Disturbance input.



(a) Position vs Estimated Position



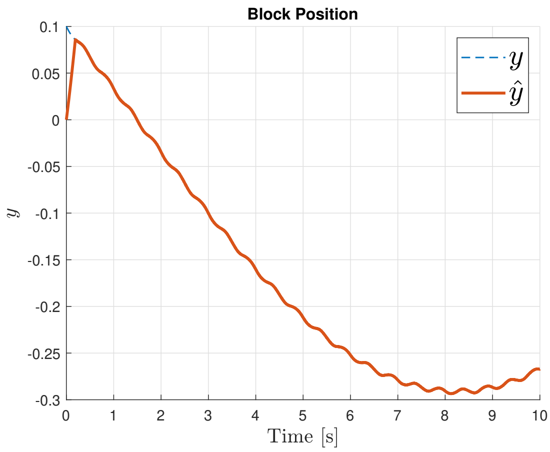
(b) Velocity vs Estimated Velocity



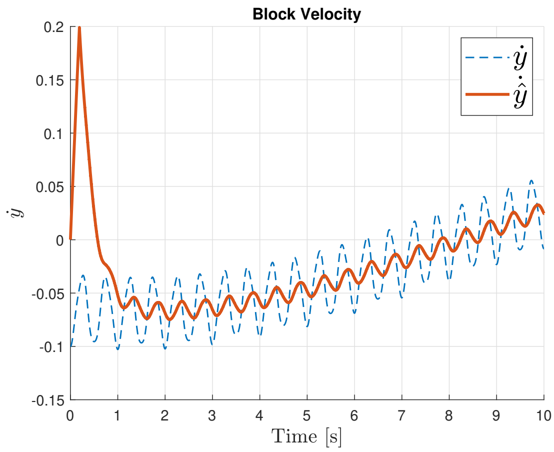
(c) Estimation Errors

Figure 3: Slotine Observer performance - $Ts = 0.01$s

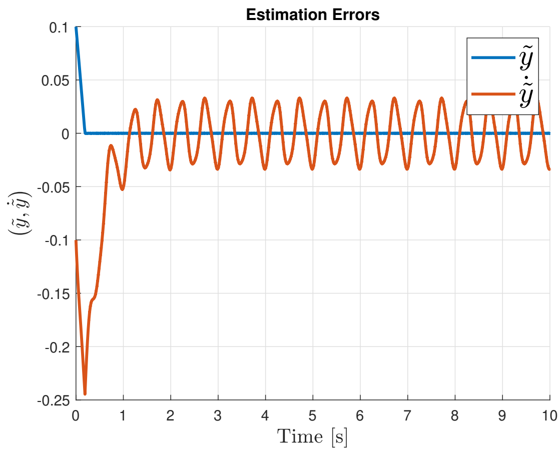
For a sampling time of 0.01s, Figure 3 represents the behaviour of the Slotine SMO. As discussed in the previous section, the position estimation error converges to zero in finite time, and the velocity estimation error converges to a bounded region around zero in finite time. The chosen gain κ_1 is greater than the maximum value of $|\tilde{x}_2|$, which satistfies the finite-time stability requirement from equation (16). From equation (21), the velocity estimation error is supposed to be contained in a bounded region $|\tilde{x}_2| \leq \delta_{max}\kappa_1/\kappa_2 = 0.15$. Figure 3c shows that the error is inside a region of $|\tilde{x}_2| < 0.05$, which satisfies equation (21).



(a) Position vs Estimated Position



(b) Velocity vs Estimated Velocity



(c) Estimation Errors

Figure 4: Slotine Observer performance - $Ts = 0.0001$s

For a sampling time of 0.0001, Figure 4 represents the behaviour of the Slotine SMO. Similarly to the previous simulation, the position estimation error converges to zero in finite time, and the velocity estimation error converges to a bounded region around zero in finite time. The estimation errors in Figure 4c are smaller than the ones in Figure 3c, from which it is possible to conclude that at least part of the estimation errors is caused by the sampling process.

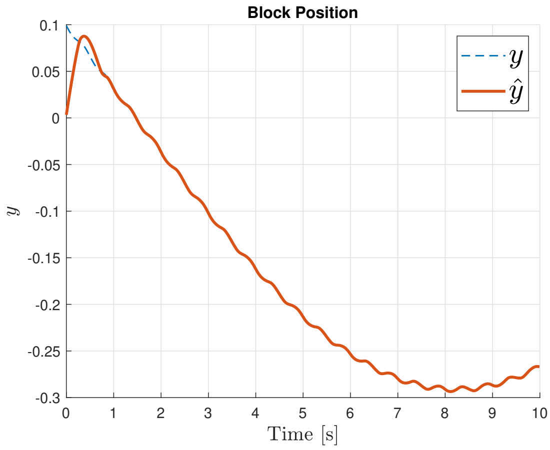
3.2 Super-Twisting Observer

For the Mass-Spring-Damper system described in equations (2)-(5), the Super-Twisting Observer is modeled as:

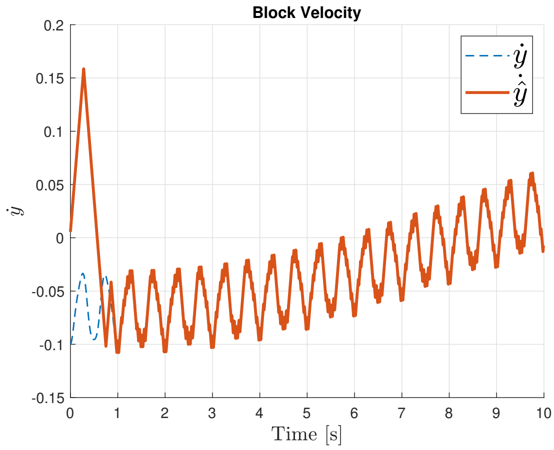
$$\dot{\hat{x}}_1 = \hat{x}_2 + \kappa_1 \left|\tilde{x}_1\right|^{1/2} \operatorname{sign}\left(\tilde{x}_1\right), \tag{33}$$

$$\dot{\hat{x}}_2 = \kappa_2 \operatorname{sign}\left(\tilde{x}_1\right) + \frac{1}{m}\left(-b\hat{x}_2|\hat{x}_2| - k_0\hat{x}_1 - k_1\hat{x}_1^3\right). \tag{34}$$

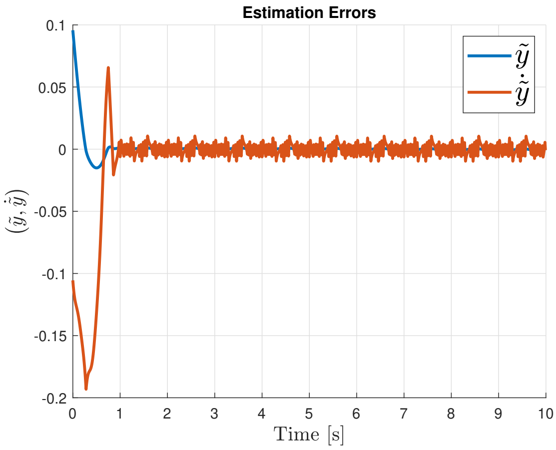
For the following simulations, the observer gains were calculated using equations (22)-(23) and $\delta_{max} = 0.5$. The observer states were both initialized at zero.



(a) Position vs Estimated Position



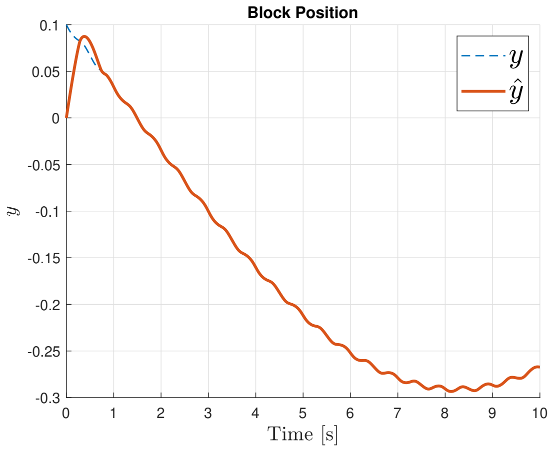
(b) Velocity vs Estimated Velocity



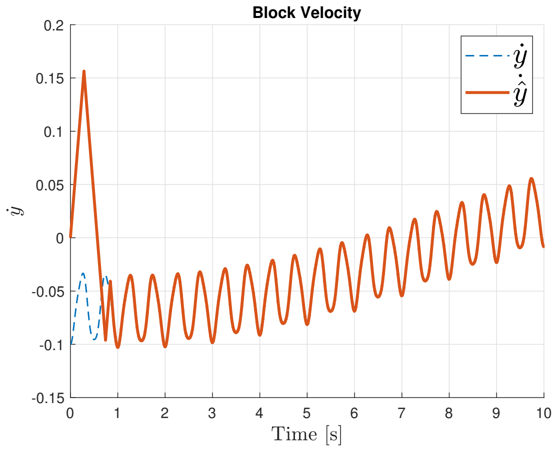
(c) Estimation Errors

Figure 5: Super-Twisting Observer performance - $Ts = 0.01$s

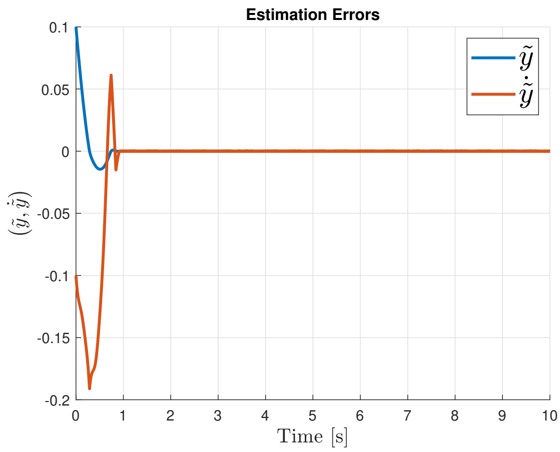
For a sampling time of 0.01s, Figure 5 represents the behaviour of the Super-Twisting SMO. As discussed in the previous section, both the position and velocity estimation errors converge to zero in finite time. The calculated gains $\kappa_1 = 1.061$ and $\kappa_2 = 0.55$, from equations (23)-(24), provide the desired finite-time stability. However, the estimation errors have a small variation around zero caused by the sampling process, as stated in the Slotine Observer analysis. To mitigate this variation, let us reduce the sampling time for the next simulation.



(a) Position vs Estimated Position



(b) Velocity vs Estimated Velocity



(c) Estimation Errors

Figure 6: Super-Twisting Observer performance - $Ts = 0.0001$s

For a sampling time of 0.0001s, Figure 6 represents the behaviour of the Super-Twisting SMO. Similarly to the previous simulation, both the position and velocity estimation errors converge to zero in finite time. The reduction of the sampling time also reduced estimation errors in Figure 6c, making the variation around zero considerably smaller, but not null. Perfect reduction to zero would require a sampling time of 0s, which is impossible to simulate or obtain in practical experimentation.

4 Conclusion

The objective of this assignment was to estimate the position and velocity of a Mass-Spring-Damper system using the position measurements and two types of Sliding-Mode Observers. The Slotine Observer provided finite-time stability of the position estimation error, but a ultimately bounded stability of the velocity estimation error. On the other hand, the Super-Twisting Observer provided finite-time stability of both states estimation error. In both cases, it was possible to see that the sampling process, which is intrinsic to numeric simulation, causes estimation errors that can be mitigated by reducing the sampling time.

References

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[2] S. P. Bhat, D. S. Bernstein. Finite-Time Stability of Continuous Autonomous Systems. SIAM J. of Cont. and Optim., 38(3), 2000.

[3] Filippov, A.F., 1989. Differential equations with discontinuous righthand sides. Kluwer Academic Pubr.

[4] Moreno, J.A. and Osorio, M., 2012. Strict lyapunov functions for the super-twisting algorithm".IEEE Transactions on Automatic Control, Vol. 57, No. 4, p. 1035–1040.

[5] Shtessel, Y., Edwards, C., Fridman, L. and Levant, A., 2014.Sliding mode control and observation. Birkhäuser.