Structural Models of Credit Risk

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Brownian Motion - Intro

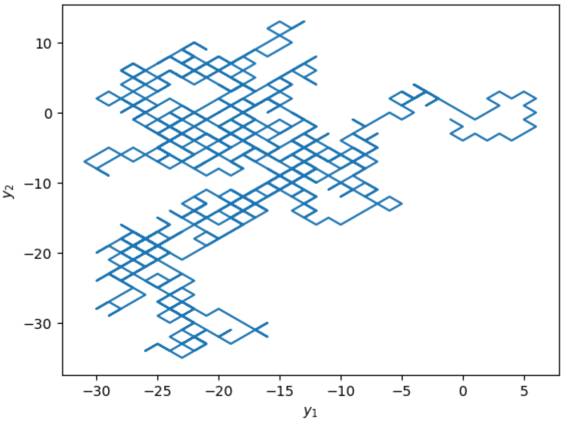
Brownian Motion - Origin in Physics

- Brownian motion is the random moving of particles suspended in a fluid (a liquid or a gas) resulting from their bombardment by the fast-moving atoms or molecules in the gas or liquid. The movement was discovered by botanist Robert Brown.
- Albert Einstein published a paper in 1905 that explained in precise detail how the motion that Brown had observed was a result of the pollen being moved by individual water molecules.
- The first mathematical rigorous construction of Brownian motion is due to Wiener in 1923.

A simple model - Random Walks

- $S_n = \sum_{i=1}^{n} Y_i$ where Y_i is i.i.d. on $\{-1, 1\}^d$.
- $\mathbb{E}[S_n] = \sum_{i=1}^{n} \mathbb{E}[Y_i] = 0$
- $Var[S_n] = \sum_{i=1}^{n} Var[Y_i] = n$

$d = 2$ and $n = 1000$



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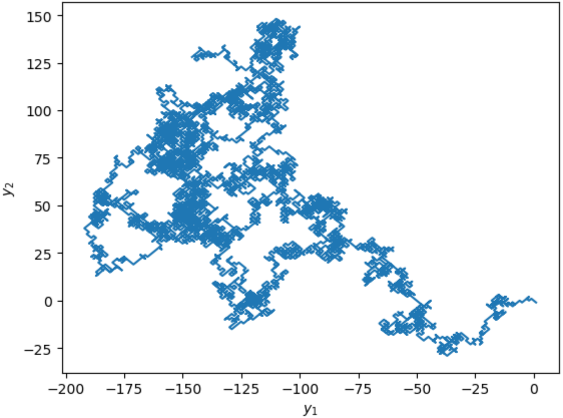


Quick brush-up of continuous time stochastic processes

Consider a p

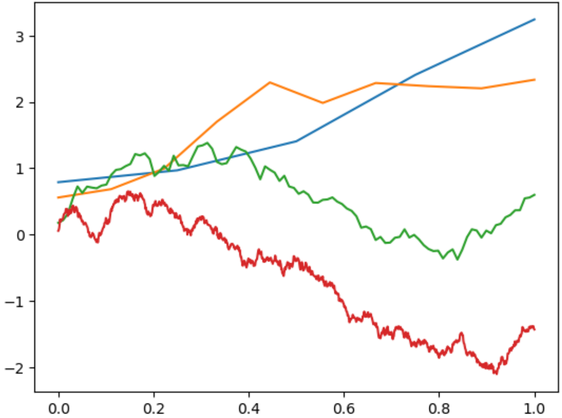
- $S_n = Y_1 + ... + Y_n$ with Y_i being i.i.d. on $\{0, 1\}$.

In the limit

- What occurs when the time steps become smaller?
- Brownian motion (Assume 1D for simplicity):
  - $t \in [0, \infty)$
  - $Z_0 = 0$
  - $Z_t - Z_s \sim N(0, t - s)$

Brownian motion as a series of random walks

$n = T/\Delta t$

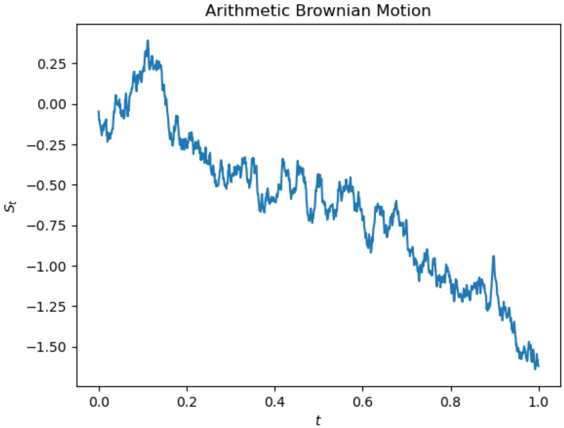


Geometric and Arithmetic Brownian Motions

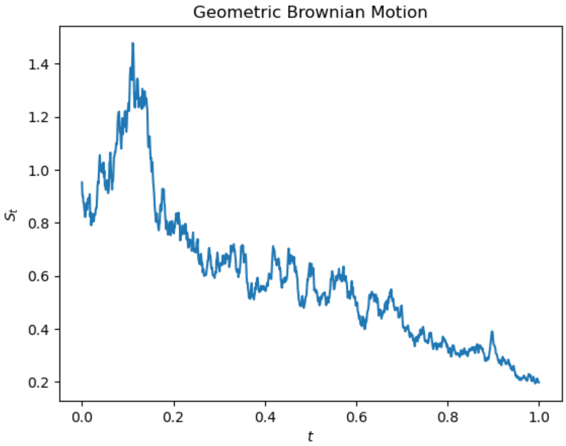
Stochastic Differential Equations, dX stands for a small change in X.

- Geometric - $dS_t = \mu S_t dt + \sigma S_t dZ_t$
- Arithmetic - $dS_t = \mu dt + \sigma dZ_t$

Examples


Arithmetic Brownian Motion
0.25
0.00
−0.25
−0.50
−0.75
−1.00
−1.25
−1.50
S_t
0.0
0.2
0.4
0.6
0.8
1.0
t

Examples


Geometric Brownian Motion
1.4
1.2
1.0
0.8
0.6
0.4
0.2
S_t
0.0
0.2
0.4
0.6
0.8
1.0
t

Girsanov Theorem

- When markets are complete and there are no arbitrage opportunities we can perform a *change* of *measure*. In other words, to express a stochastic process under some other *special* dynamics named $\mathbb{Q}$.
- Given $r > 0$, and imagine a G.B.M. S_t that represents the market value of some asset, how can we make

$$S_t = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[S_T] \tag{1}$$

- We "change" probability measures (distributions) such that
  - $dS_t = rdt + \sigma dZ_t^{\mathbb{Q}}$
  - $S_t = S_0 \exp((r - \frac{1}{2}\sigma^2)t + \sigma Z_t^{\mathbb{Q}})$

Why do we care?

- Because any contract that pays $f(X_T)$ in the future should have the no-arbitrage price

$$P_t = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[f(X_T)] \tag{2}$$

- Which can be solve using Feynmann-Kac's formula (Partial Differential Equation) and the boundary conditions of the nature of $f()$.

Credit Risk - Model the Company

Assets	Liabilities
M_t	D_t
S_t	E_t

Where M_t represents the LIQUID RESERVES of the company att ime t, S_t represents the (market) value of the company productive assets. D_t represents the market value of the company's debt and E_t represents the market value of equity.

Why is Option Pricing Useful in Corporate Finance?

Main model - Basic Assumptions

- Time is continuous $t \in [0, \infty)$.
- Riskless rate $r > 0$ constant.
- Markets are competitive, frictionless and complete.

$$p_t = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[x_T] \tag{3}$$

- Where $\mathbb{Q}$ is the risk-neutral measure.
- The value of the firm's assets (which are supposed to be tradeable at no cost) evolves according to the following stochastic differential equation

$$dS_t = S_t[(r-\beta)dt + \sigma dZ_t] \tag{4}$$

- Under these dynamics

$$\mathbb{E}^{\mathbb{Q}}\left[\frac{dS_t}{S_t} + \beta dt\right] = rdt \tag{5}$$

Pricing corporate debt

- The company's net earnings equal its operating income $\beta S_t dt$ plus its financial income $rM_t dt$ minus the total payment $C_t dt$ to the firm's creditors.
- We are in a position to use option-pricing methods to find the market value of the firm's equity and debt.
- Consider a zero coupon bond with face value B
- Following Merton (1974) consider the simplest case where debt has a zero coupon.

$$dM_t = (\beta S_t + rM_t - C_t)dt - dL_t \tag{6}$$

where dL_t represents the transfer from the company to shareholders. If positive, it represents share buy-backs or dividends. If negative, it represents new equity issues or recapitalizations. Which for the moment are done at no cost.

Pricing corporate debt - cont.

- Merton (1974) assumes a case where $\beta = 0$.
- Given limited-liability, the company has the *option* to default at T.

$$E_t = \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)} \max\{S_T - B, 0\}\right] \tag{7}$$

- $D_t = \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)} \min\{S_T, B\}\right]$
- Using the results from Black-Scholes and Merton

$$D_t = N(-x_1)S_t + Be^{-r(T-t)}N(x_2) \tag{8}$$

where

$$x_1 = \frac{1}{\sigma\sqrt{T}}\left[\log\frac{S_t}{B} + (r + \frac{\sigma^2}{2})T\right] \tag{9}$$
$$x_2 = x_1 - \sigma\sqrt{T}$$

Understanding the equation

- The first term corresponds to the expected present value of what creditors receive in case of default.
- The second one is the present value of the nominal debt, times the risk-adjusted probability that the firm does not default.
- If we denote by R the yield to maturity of the debt, then $D_0 = Be^{-RT}$. Therefore the ratio between the market value of debt ant its nominal value is

$$\frac{D_0}{B^{-rT}} = e^{-(R-r)T} = N(-x_1)\frac{S}{B^{-rT}} + N(x_2) \tag{10}$$

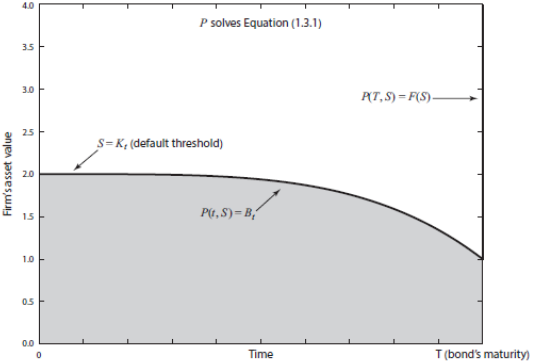
- $R - r = -\frac{1}{T}\log\left[N(-x_1)\frac{S}{B^{-rT}} + N(x_2)\right]$
- This formula is commonly used by financial analysts and fixed-income traders. However it has a crucial drawback when $T = 0$.

Endogenous default date

- Based on Black and Cox (1976), we model the firm defaulting before maturity.
- Consider a default boundary K_t, such that liquidation happens the first time $S_t \leq K_t$.
- Consider a debt contract that pays $C(S_t, t)$ and final payoff $F(S_T)$ at maturity.
- Denote by B_t the liquidation value of the firm should the boundary K_t be hit at date t.

Solution (Advanced Feynmann-Kac's formula)

$$
\begin{aligned}
rP(t,S) = \frac{\partial P}{\partial t} + (r-\beta)S\frac{\partial P}{\partial S} + \frac{\sigma^2}{2}S^2\frac{\partial^2 P}{\partial S^2} + C(t,S) \\
P(t,K_t) = B_t \\
P(T,S) = F(S)
\end{aligned} \tag{11}
$$



Pricing a consol bond

A consol bond satisfies $C(t,S) = C$ for all t. Consider a boundary condition S_B chosen by the shareholders.

- $rD(S) = (r-\beta)SD'(S) + \frac{\sigma^2}{2}S^2D''(S) + C$
- This equation is also called Euler's differential equation.

How to solve it?

- Guess a solution e.g. S^γ, note that the exponential function is not a good guess.

$$\begin{aligned} r(S^\gamma + \kappa) &= (r-\beta)S\gamma S^{\gamma-1} + \frac{\sigma^2}{2}S^2\gamma(\gamma-1)S^{\gamma-2} + C \\ r(S^\gamma + \kappa) &= [(r-\beta)\gamma + \frac{\sigma^2}{2}\gamma(\gamma-1)]S^\gamma + C \\ r &= (r-\beta)\gamma + \frac{\sigma^2}{2}\gamma(\gamma-1) \\ r\kappa &= C \end{aligned} \tag{12}$$

Which has two solutions $\gamma_1 < 0\gamma_2$. The general solution is

$$f(S) = \kappa_1 S^{\gamma_1} + \kappa_2 S^{\gamma_2} \tag{13}$$

The solution to a PDE is the general plus a particular solution. e.g. $f(S) = C/r$.

$$D(S) = \kappa_1 S^{\gamma_1} + \kappa_2 S^{\gamma_2} + C/r \tag{14}$$

Constant terms - use boundary conditions

- When $S_t \to \infty$ the probability of default tends to zero.
- $\lim S \to \infty D(S) = C/r$ implies $\kappa_2 = 0$.
- $D(S_B) = S_B \to \kappa_1 S_B^{\gamma_1} + C/r = S_B$
- $\kappa_1 = S_B^{-\gamma_1} - (C/r) S_B^{-\gamma_1}$

Value of debt

$$D(S) = \frac{C}{r} - S_B^{-\gamma_1} \left(\frac{C}{r} - S_B \right) S^{\gamma_1} \tag{15}$$

Value of equity $E(S) = S - D(S)$

Endogenous default

What is the optimal S_B? Maximize $S_B^{-\gamma_1}\left(\frac{C}{r} - S_B\right)S^{\gamma_1}$

$$S_B^* = \frac{\gamma_1}{\gamma_1 - 1}\frac{C}{r} \tag{16}$$