Empirical Asset Pricing

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Research in Finance - 104

Overview

- **Lesson 1**: Historical Summary of Asset Pricing, Portfolio Theory, CAPM, Empirical Tests, the Stochastic Discount Factor and GMM.
- **Lesson 2**: Working with accounting data and prices, Anomalies and The Factor Zoo.
- **Lesson 3**: Return Predictability
- **Lesson 4**: ICAPM, and Consumption Based Asset Pricing.
- **Lesson 5**: Production Based Asset Pricing
- **Lesson 6**: Liquidity and Intermediary Asset Pricing
- **Lesson 7**: Future Research

Evaluation

- Empirical Project (30%)
- Final Exam (70%)

A Fast derivation of the CAPM

We start in the world of **Markowitz (1952)**: Markowitz, H.M. (1952). "Portfolio Selection". The Journal of Finance.

$$\begin{aligned} \min_{\boldsymbol{\omega}} \boldsymbol{\omega}' \boldsymbol{V} \boldsymbol{\omega} \\ \text{s.t} \\ \boldsymbol{\omega}' \boldsymbol{\mu} = \mu_0 \\ \boldsymbol{\omega}' \mathbf{1} = 1 \end{aligned} \tag{1}$$

Lagrangian

$$\mathcal{L}(\boldsymbol{\omega}; \lambda, \gamma) = \min_{\boldsymbol{\omega}} \boldsymbol{\omega}' \boldsymbol{V} \boldsymbol{\omega} + \lambda(\boldsymbol{\omega}' \boldsymbol{\mu} - \mu_0) + \gamma(\boldsymbol{\omega}' \mathbf{1} - 1) \tag{2}$$

$$2\boldsymbol{V}\boldsymbol{\omega} + \lambda\boldsymbol{\mu} + \gamma\mathbf{1} = \mathbf{0} \tag{3}$$

A Fast derivation of the CAPM

To solve fast numerically

$$\begin{pmatrix} 2\boldsymbol{V} & \boldsymbol{\mu} & \mathbf{1} \\ \boldsymbol{\mu}' & 0 & 0 \\ \mathbf{1}' & 0 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\omega} \\ \lambda \\ \gamma \end{pmatrix} = \begin{pmatrix} 0 \\ \mu_0 \\ 1 \end{pmatrix} \tag{4}$$

Closed form solution

$$\begin{aligned} 2\boldsymbol{V}\boldsymbol{\omega} + \lambda\boldsymbol{\mu} + \gamma\mathbf{1} &= 0 \\ \boldsymbol{\omega} &= -\frac{1}{2}(\lambda\boldsymbol{V}^{-1}\boldsymbol{\mu} + \gamma\boldsymbol{V}^{-1}\mathbf{1}) \end{aligned} \tag{5}$$

Pre-multiply by $\boldsymbol{\mu}'$ and $\mathbf{1}'$ and replace the constraints

$$\begin{aligned} \mu_0 &= -\frac{1}{2}(\lambda\boldsymbol{\mu}'\boldsymbol{V}^{-1}\boldsymbol{\mu} + \gamma\boldsymbol{\mu}'\boldsymbol{V}^{-1}\mathbf{1}) \\ 1 &= -\frac{1}{2}(\lambda\mathbf{1}'\boldsymbol{V}^{-1}\boldsymbol{\mu} + \gamma\mathbf{1}'\boldsymbol{V}^{-1}\mathbf{1}) \end{aligned} \tag{6}$$

A Fast derivation of the CAPM

Solve for the Lagrange multipliers, define $a_{\boldsymbol{x},\boldsymbol{y}} = \boldsymbol{x}'\boldsymbol{V}^{-1}\boldsymbol{y}$

$$
\begin{aligned}
-\frac{1}{2}\begin{pmatrix} a_{\boldsymbol{\mu}\boldsymbol{\mu}} & a_{\boldsymbol{\mu}\boldsymbol{1}} \\ a_{\boldsymbol{1}\boldsymbol{\mu}} & a_{\boldsymbol{1}\boldsymbol{1}} \end{pmatrix}\begin{pmatrix} \lambda \\ \gamma \end{pmatrix} &= \begin{pmatrix} \mu_0 \\ 1 \end{pmatrix} \\
\begin{pmatrix} \lambda \\ \gamma \end{pmatrix} = \frac{1}{a_{\boldsymbol{\mu}\boldsymbol{\mu}}a_{\boldsymbol{1}\boldsymbol{1}} - a_{\boldsymbol{1}\boldsymbol{\mu}}a_{\boldsymbol{\mu}\boldsymbol{1}}}\begin{pmatrix} a_{\boldsymbol{1}\boldsymbol{1}} & -a_{\boldsymbol{\mu}\boldsymbol{1}} \\ -a_{\boldsymbol{1}\boldsymbol{\mu}} & a_{\boldsymbol{\mu}\boldsymbol{\mu}} \end{pmatrix} & \begin{pmatrix} -2\mu_0 \\ -2 \end{pmatrix} \\
\lambda &= -2\left(\frac{\mu_0 a_{\boldsymbol{1}\boldsymbol{1}} - a_{\boldsymbol{\mu}\boldsymbol{1}}}{a_{\boldsymbol{\mu}\boldsymbol{\mu}}a_{\boldsymbol{1}\boldsymbol{1}} - a_{\boldsymbol{1}\boldsymbol{\mu}}a_{\boldsymbol{\mu}\boldsymbol{1}}}\right) \\
\gamma &= -2\left(\frac{-\mu_0 a_{\boldsymbol{1}\boldsymbol{\mu}} + a_{\boldsymbol{\mu}\boldsymbol{\mu}}}{a_{\boldsymbol{\mu}\boldsymbol{\mu}}a_{\boldsymbol{1}\boldsymbol{1}} - a_{\boldsymbol{1}\boldsymbol{\mu}}a_{\boldsymbol{\mu}\boldsymbol{1}}}\right)
\end{aligned} \tag{7}
$$

A Fast derivation of the CAPM

$$
\begin{aligned}
\boldsymbol{\omega} &= -\frac{1}{2}\Big(\lambda \boldsymbol{V}^{-1}\boldsymbol{\mu} + \gamma \boldsymbol{V}^{-1}\mathbf{1}\Big) \\
\boldsymbol{\omega} &= \frac{1}{a_{\boldsymbol{\mu\mu}}a_{\mathbf{11}} - a_{\mathbf{1}\boldsymbol{\mu}}a_{\boldsymbol{\mu}\mathbf{1}}}\Big((\mu_0 a_{\mathbf{11}} - a_{\boldsymbol{\mu}\mathbf{1}})\boldsymbol{V}^{-1}\boldsymbol{\mu} + (a_{\boldsymbol{\mu\mu}} - \mu_0 a_{\mathbf{1}\boldsymbol{\mu}})\boldsymbol{V}^{-1}\mathbf{1}\Big) \\
\boldsymbol{\omega} &= \frac{1}{a_{\boldsymbol{\mu\mu}}a_{\mathbf{11}} - a_{\mathbf{1}\boldsymbol{\mu}}a_{\boldsymbol{\mu}\mathbf{1}}}\Big(a_{\mathbf{1}\boldsymbol{\mu}}(\mu_0 a_{\mathbf{11}} - a_{\boldsymbol{\mu}\mathbf{1}})\frac{\boldsymbol{V}^{-1}\boldsymbol{\mu}}{a_{\mathbf{1}\boldsymbol{\mu}}} + a_{\mathbf{11}}(a_{\boldsymbol{\mu\mu}} - \mu_0 a_{\mathbf{1}\boldsymbol{\mu}})\frac{\boldsymbol{V}^{-1}\mathbf{1}}{a_{\mathbf{11}}}\Big) \\
\boldsymbol{\omega} &= \theta_1 \frac{\boldsymbol{V}^{-1}\boldsymbol{\mu}}{a_{\mathbf{1}\boldsymbol{\mu}}} + \theta_0 \frac{\boldsymbol{V}^{-1}\mathbf{1}}{a_{\mathbf{11}}}
\end{aligned}
\tag{8}
$$

Recall that

$$
\frac{\mu_0 a_{\mathbf{11}}a_{\mathbf{1}\boldsymbol{\mu}} - a_{\mathbf{1}\boldsymbol{\mu}}a_{\boldsymbol{\mu}\mathbf{1}} + a_{\mathbf{11}}a_{\boldsymbol{\mu\mu}} - \mu_0 a_{\mathbf{1}\boldsymbol{\mu}}a_{\mathbf{11}}}{a_{\boldsymbol{\mu\mu}}a_{\mathbf{11}} - a_{\mathbf{1}\boldsymbol{\mu}}a_{\boldsymbol{\mu}\mathbf{1}}} = 1 \tag{9}
$$

$$
\boldsymbol{\omega} = (1 - \theta_0(\mu_0))\frac{\boldsymbol{V}^{-1}\boldsymbol{\mu}}{\mathbf{1}\boldsymbol{V}^{-1}\boldsymbol{\mu}} + \theta_0(\mu_0)\underbrace{\frac{\boldsymbol{V}^{-1}\mathbf{1}}{\mathbf{1}\boldsymbol{V}^{-1}\mathbf{1}}}_{\text{M.V.P.}} \tag{10}
$$

A Fast derivation of the CAPM

Refer to

$$\boldsymbol{\omega}_U = \frac{\boldsymbol{V}^{-1}\mathbf{1}}{\mathbf{1}\boldsymbol{V}^{-1}\mathbf{1}} \tag{11}$$

$$\boldsymbol{\omega}_{T_0} = \frac{\boldsymbol{V}^{-1}\mu}{\mathbf{1}\boldsymbol{V}^{-1}\mu} \tag{12}$$

Any efficient portfolio satisfies

$$\boldsymbol{\omega} = \theta_0(\mu_0)\boldsymbol{\omega}_U + (1-\theta_0(\mu_0))\boldsymbol{\omega}_{T_0} \tag{13}$$

We can think of this economy as composed of 2 assets with expected return and variances

$$\begin{aligned} \tilde{\boldsymbol{\mu}} &= \begin{pmatrix} \boldsymbol{\mu}'\boldsymbol{\omega}_U \\ \boldsymbol{\mu}'\boldsymbol{\omega}_{T_0} \end{pmatrix} = \begin{pmatrix} \tilde{\mu}_U \\ \tilde{\mu}_{T_0} \end{pmatrix} \\ \tilde{\boldsymbol{V}} &= \begin{pmatrix} \boldsymbol{\omega}_U'\boldsymbol{V}\boldsymbol{\omega}_U & \boldsymbol{\omega}_{T_0}'\boldsymbol{V}\boldsymbol{\omega}_U \\ \boldsymbol{\omega}_U'\boldsymbol{V}\boldsymbol{\omega}_{T_0} & \boldsymbol{\omega}_{T_0}'\boldsymbol{V}\boldsymbol{\omega}_{T_0} \end{pmatrix} = \begin{pmatrix} \tilde{\boldsymbol{V}}_{UU} & \tilde{\boldsymbol{V}}_{UT_0} \\ \tilde{\boldsymbol{V}}_{T_0U} & \tilde{\boldsymbol{V}}_{T_0T_0} \end{pmatrix} \end{aligned} \tag{14}$$

A Fast Derivation of the CAPM

Introduce a risk free asset, and consider an investor with mean variance preferences, coefficient of risk aversion a, such that he/she invests in a portfolio of the 3 assets. The investor can buy or short sell any of the three assets. (Borrow to invest more on another asset).

$$\max_{\theta_U,\theta_{T_0}} \tilde{\mu}_U\theta_U + \tilde{\mu}_{T_0}\theta_{T_0} + r_f(1-\theta_U-\theta_{T_0}) - \frac{a}{2}\left(\theta_U^2\tilde{\boldsymbol{V}}_{UU} + \theta_{T_0}^2\tilde{\boldsymbol{V}}_{T_0T_0} + 2\theta_U\theta_{T_0}\tilde{\boldsymbol{V}}_{UT_0}\right) \tag{15}$$

F.O.C.

$$\begin{aligned} \tilde{\mu}_i - r_f &= a(\theta_i\tilde{\boldsymbol{V}}_{ii} + \theta_j\tilde{\boldsymbol{V}}_{ij}) \\ \theta_i &= \frac{\frac{\tilde{\mu}_i - r_f}{a} - \theta_j\tilde{\boldsymbol{V}}_{ij}}{\tilde{\boldsymbol{V}}_{ii}} \end{aligned} \tag{16}$$

Change indexes and replace

$$\tilde{\mu}_i - r_f = a\left(\theta_i\tilde{\boldsymbol{V}}_{ii} + \left(\frac{\frac{\tilde{\mu}_j - r_f}{a} - \theta_i\tilde{\boldsymbol{V}}_{ij}}{\tilde{\boldsymbol{V}}_{jj}}\right)\tilde{\boldsymbol{V}}_{ij}\right) \tag{17}$$

A Fast Derivation of the CAPM

$$
\begin{aligned}
\theta_i &= \frac{1}{a}\frac{\left((\mu_i - r_f)\tilde{\boldsymbol{V}}_{jj} - (\mu_j - r_f)\tilde{\boldsymbol{V}}_{ij}\right)}{\tilde{\boldsymbol{V}}_{ii}\tilde{\boldsymbol{V}}_{jj} - \tilde{\boldsymbol{V}}_{ij}{}^2} \\
\theta_U &= \frac{1}{a}\frac{\left((\tilde{\mu}_U - r_f)\tilde{\boldsymbol{V}}_{T_0T_0} - (\tilde{\mu}_{T_0} - r_f)\tilde{\boldsymbol{V}}_{UT_0}\right)}{\tilde{\boldsymbol{V}}_{UU}\tilde{\boldsymbol{V}}_{T_0T_0} - \tilde{\boldsymbol{V}}^2_{Ut_0}} \\
\theta_{T_0} &= \frac{1}{a}\frac{\left((\tilde{\mu}_{T_0} - r_f)\tilde{\boldsymbol{V}}_{UU} - (\tilde{\mu}_U - r_f)\tilde{\boldsymbol{V}}_{UT_0}\right)}{\tilde{\boldsymbol{V}}_{UU}\tilde{\boldsymbol{V}}_{T_0T_0} - \tilde{\boldsymbol{V}}^2_{Ut_0}} \\
\theta_{rf} &= 1 - \theta_U - \theta_{T_0}
\end{aligned} \tag{18}
$$

A Fast Derivation of the CAPM

Two Fund Separation Theorem

$$
\begin{aligned}
\tilde{r} &= \theta_U \tilde{r}_U + \theta_{T_0} \tilde{r}_{T_0} + (1 - \theta_U - \theta_{T_0}) r_f \\
\tilde{r} &= (\theta_U + \theta_{T_0}) \left(\frac{\theta_U}{\theta_U + \theta_{T_0}} \tilde{r}_U + \frac{\theta_U}{\theta_U + \theta_{T_0}} \tilde{r}_{T_0} \right) + (1 - \theta_U - \theta_{T_0}) r_f \\
\tilde{r} &= \frac{\Lambda}{a} (\tilde{\theta_U} \tilde{r}_U + \tilde{\theta_{T_0}} \tilde{r}_{T_0}) + (1 - \frac{\Lambda}{a}) r_f \\
\tilde{r} &= \frac{\Lambda}{a} \tilde{r}_T + (1 - \frac{\Lambda}{a}) r_f
\end{aligned}
\tag{19}
$$

where

$$
\begin{aligned}
\tilde{\theta}_i &= \frac{(\tilde{\mu}_i - r_f) \tilde{\boldsymbol{V}}_{jj} - (\tilde{\mu}_j - r_f) \tilde{\boldsymbol{V}}_{ij}}{(\tilde{\mu}_U - r_f)(\tilde{\boldsymbol{V}}_{T_0 T_0} - \tilde{\boldsymbol{V}}_{U T_0}) + (\tilde{\mu}_{T_0} - r_f)(\tilde{\boldsymbol{V}}_{UU} - \tilde{\boldsymbol{V}}_{U T_0})} \\
\Lambda &= \frac{(\tilde{\mu}_U - r_f)(\tilde{\boldsymbol{V}}_{T_0 T_0} - \tilde{\boldsymbol{V}}_{U T_0}) + (\tilde{\mu}_{T_0} - r_f)(\tilde{\boldsymbol{V}}_{UU} - \tilde{\boldsymbol{V}}_{U T_0})}{\tilde{\boldsymbol{V}}_{UU} \tilde{\boldsymbol{V}}_{T_0 T_0} - \tilde{\boldsymbol{V}}_{U t_0}^2}
\end{aligned}
\tag{20}
$$

A Fast Derivation of the CAPM

- The composition of portfolio $\omega_T = [\tilde{\theta}_U, \tilde{\theta}_{T_0}]$ does not depend on a. Investors will disagree on the optimal portfolio but they will all hold risky securities in the same proportion ω_T.
- If $r_f = 0 \rightarrow \omega_T = \omega_{T_0}$

$$\tilde{\theta_U} \propto \tilde{\mu}_U \tilde{V}_{T_0 T_0} - \tilde{\mu}_{T_0} \tilde{V}_{U T_0}$$

$$\frac{\boldsymbol{\mu}' \boldsymbol{V}^{-1} \mathbf{1}}{\mathbf{1}' \boldsymbol{V}^{-1} \mathbf{1}} \left(\frac{\boldsymbol{V}^{-1} \mu}{\mathbf{1}' \boldsymbol{V}^{-1} \mu} \right)' \boldsymbol{V} \left(\frac{\boldsymbol{V}^{-1} \mu}{\mathbf{1}' \boldsymbol{V}^{-1} \mu} \right) - \frac{\boldsymbol{\mu}' \boldsymbol{V}^{-1} \mu}{\mathbf{1}' \boldsymbol{V}^{-1} \mu} \left(\frac{\boldsymbol{V}^{-1} \mu}{\mathbf{1}' \boldsymbol{V}^{-1} \mu} \right)' \boldsymbol{V} \left(\frac{\boldsymbol{V}^{-1} \mathbf{1}}{\mathbf{1}' \boldsymbol{V}^{-1} \mathbf{1}} \right) \tag{21}$$

$$\frac{a_{\boldsymbol{\mu\mu}}}{a_{\mathbf{11}} a_{\mathbf{1}\boldsymbol{\mu}}} - \frac{a_{\boldsymbol{\mu\mu}}}{a_{\mathbf{11}} a_{\mathbf{1}\boldsymbol{\mu}}} = 0$$

A Fast Derivation of the CAPM

Closed form tangent portfolio (Maximize Sharpe Ratio).

$$\max_{\boldsymbol{\omega}} \frac{\boldsymbol{\omega}'\boldsymbol{\mu} - r_f}{(\boldsymbol{\omega}'\boldsymbol{V}\boldsymbol{\omega})^{\frac{1}{2}}} \quad \text{s.t.} \quad \boldsymbol{\omega}'\mathbf{1} = 1 \tag{22}$$

Solution, the first order condition of the lagrangian of the efficient frontier is zero at every point

$$\begin{aligned}
\boldsymbol{V}\boldsymbol{\omega} + \lambda\boldsymbol{\mu} + \gamma\mathbf{1} &= 0 \\
\boldsymbol{\omega}'\boldsymbol{V}\boldsymbol{\omega} &= -\lambda\mu_0 - \gamma \\
\boldsymbol{\omega}'\boldsymbol{V}\boldsymbol{\omega} &= 2\Big(\frac{\mu_0 a_{\mathbf{11}} - a_{\boldsymbol{\mu}\mathbf{1}}}{a_{\boldsymbol{\mu\mu}}a_{\mathbf{11}} - a_{\mathbf{1}\boldsymbol{\mu}}a_{\boldsymbol{\mu}\mathbf{1}}}\Big)\mu_0 + 2\Big(\frac{-\mu_0 a_{\mathbf{1}\boldsymbol{\mu}} + a_{\boldsymbol{\mu\mu}}}{a_{\boldsymbol{\mu\mu}}a_{\mathbf{11}} - a_{\mathbf{1}\boldsymbol{\mu}}a_{\boldsymbol{\mu}\mathbf{1}}}\Big) \\
\boldsymbol{\omega}'\boldsymbol{V}\boldsymbol{\omega} &= 2\frac{\mu_0(\mu_0 a_{\mathbf{11}} - a_{\boldsymbol{\mu}\mathbf{1}}) - \mu_0 a_{\mathbf{1}\boldsymbol{\mu}} + a_{\boldsymbol{\mu\mu}}}{a_{\boldsymbol{\mu\mu}}a_{\mathbf{11}} - a_{\mathbf{1}\boldsymbol{\mu}}a_{\boldsymbol{\mu}\mathbf{1}}}
\end{aligned} \tag{23}$$

A Fast Derivation of the CAPM

Use the fact that $a_{\mathbf{1}\boldsymbol{\mu}} = a_{\boldsymbol{\mu}\mathbf{1}}$

$$\boldsymbol{\omega}'\boldsymbol{V}\boldsymbol{\omega} = 2\frac{\mu_0^2 a_{\mathbf{11}} - \mu_0 a_{\boldsymbol{\mu}\mathbf{1}} - \mu_0 a_{\boldsymbol{\mu}\mathbf{1}} + a_{\boldsymbol{\mu}\boldsymbol{\mu}}}{a_{\boldsymbol{\mu}\boldsymbol{\mu}} a_{\mathbf{11}} - a_{\boldsymbol{\mu}\mathbf{1}}^2} = 2\frac{a_{\mathbf{11}}\mu_0^2 - 2a_{\boldsymbol{\mu}\mathbf{1}}\mu_0 + a_{\boldsymbol{\mu}\boldsymbol{\mu}}}{a_{\boldsymbol{\mu}\boldsymbol{\mu}} a_{\mathbf{11}} - a_{\boldsymbol{\mu}\mathbf{1}}^2} \tag{24}$$

A Fast Derivation of the CAPM

$$
\begin{aligned}
\max_{\mu_0} \sqrt{\frac{a_{\mu\mu}a_{11} - a_{\mu 1}^2}{2}} \frac{\mu_0 - r_f}{\sqrt{a_{11}\mu_0^2 - 2a_{\mu 1}\mu_0 + a_{\mu\mu}}} \\
\frac{d}{d\mu_0} \frac{\mu_0 - r_f}{\sqrt{a_{11}\mu_0^2 - 2a_{\mu 1}\mu_0 + a_{\mu\mu}}} &= 0 \\
\frac{\sqrt{a_{11}\mu_0^2 - 2a_{\mu 1}\mu_0 + a_{\mu\mu}} - \frac{1}{2}(\mu_0 - r_f)(a_{11}\mu_0^2 - 2a_{\mu 1}\mu_0 + a_{\mu\mu})^{-\frac{1}{2}}(2a_{11}\mu_0 - 2a_{\mu 1})}{a_{11}\mu_0^2 - 2a_{\mu 1}\mu_0 + a_{\mu\mu}} &= 0 \\
a_{11}\mu_0^2 - 2a_{\mu 1}\mu_0 + a_{\mu\mu} - (\mu_0 - r_f)(a_{11}\mu_0 - a_{\mu 1}) &= 0 \\
a_{11}\mu_0^2 - 2a_{\mu 1}\mu_0 + a_{\mu\mu} - a_{11}\mu_0^2 + \mu_0 a_{\mu 1} + a_{11}r_f\mu_0 - r_f a_{\mu 1} &= 0 \\
\mu_0(a_{11}r_f - a_{\mu 1}) &= r_f a_{\mu 1} - a_{\mu\mu} \\
\mu_0 &= \frac{a_{\mu\mu} - r_f a_{\mu 1}}{a_{\mu 1} - r_f a_{11}}
\end{aligned}
\tag{25}
$$

A Fast Derivation of the CAPM

$$
\begin{aligned}
\boldsymbol{\omega} &= \frac{1}{a_{\boldsymbol{\mu\mu}}a_{\mathbf{11}} - a_{\boldsymbol{\mu}\mathbf{1}}^2}\Big((\mu_0 a_{\mathbf{11}} - a_{\boldsymbol{\mu}\mathbf{1}})\boldsymbol{V}^{-1}\boldsymbol{\mu} + (a_{\boldsymbol{\mu\mu}} - \mu_0 a_{\mathbf{1}\boldsymbol{\mu}})\boldsymbol{V}^{-1}\mathbf{1}\Big) \\
\frac{1}{a_{\boldsymbol{\mu\mu}}a_{\mathbf{11}} - a_{\boldsymbol{\mu}\mathbf{1}}^2}&\Big(((\frac{a_{\boldsymbol{\mu\mu}} - r_f a_{\boldsymbol{\mu}\mathbf{1}}}{a_{\boldsymbol{\mu}\mathbf{1}} - r_f a_{\mathbf{11}}})a_{\mathbf{11}} - a_{\boldsymbol{\mu}\mathbf{1}})\boldsymbol{V}^{-1}\boldsymbol{\mu} + (a_{\boldsymbol{\mu\mu}} - (\frac{a_{\boldsymbol{\mu\mu}} - r_f a_{\boldsymbol{\mu}\mathbf{1}}}{a_{\boldsymbol{\mu}\mathbf{1}} - r_f a_{\mathbf{11}}})a_{\mathbf{1}\boldsymbol{\mu}})\boldsymbol{V}^{-1}\mathbf{1}\Big) \\
&\frac{1}{a_{\boldsymbol{\mu}\mathbf{1}} - r_f a_{\mathbf{11}}}\Big(\boldsymbol{V}^{-1}\boldsymbol{\mu} - r_f\boldsymbol{V}^{-1}\mathbf{1}\Big) \\
&\frac{1}{a_{\boldsymbol{\mu}\mathbf{1}} - r_f a_{\mathbf{11}}}\Big(\boldsymbol{V}^{-1}(\boldsymbol{\mu} - r_f\mathbf{1})\Big) \\
&\frac{\boldsymbol{V}^{-1}(\boldsymbol{\mu} - r_f\mathbf{1})}{\boldsymbol{\mu}'\boldsymbol{V}^{-1}\mathbf{1} - r_f\mathbf{1}'\boldsymbol{V}^{-1}\mathbf{1}} \\
&\frac{\boldsymbol{V}^{-1}(\boldsymbol{\mu} - r_f\mathbf{1})}{\mathbf{1}'\boldsymbol{V}^{-1}\boldsymbol{\mu} - r_f\mathbf{1}'\boldsymbol{V}^{-1}\mathbf{1}} \\
&\frac{\boldsymbol{V}^{-1}(\boldsymbol{\mu} - r_f\mathbf{1})}{\mathbf{1}'\boldsymbol{V}^{-1}(\boldsymbol{\mu} - r_f\mathbf{1})}
\end{aligned} \tag{26}
$$

A Fast Derivation of the CAPM

- **Sharpe**, William F. (1964), Capital Asset Prices: A Theory of Market Equilibrium Under Conditions of Risk. The Journal of Finance
- **Lintner** John (1965), The Valuation of Risk Assets and the Selection of Risky Investments in Stock Portfolios and Capital Budgets, The Review of Economics and Statistics
- **Mossin** Jan (1966) Equilibrium in a Capital Asset Market, Econometrica

Since every investor demands the tangent portfolio in equal proportions, the tangent portfolio is the market portfolio. Consider an investor that holds a portfolio between asset i and the tangent portfolio. All possible combinations between these assets satisfy

$$\begin{aligned} \mathbb{E}[r_p] &= \omega r_i + (1-\omega) r_p \\ \sigma_p &= \sqrt{\omega^2 \sigma_i^2 + (1-\omega)^2 \sigma_m^2 + 2\omega(1-\omega)\sigma_{im}} \end{aligned} \tag{27}$$

Parametrize the functions as

$$\begin{aligned} g(\omega) &= \mathbb{E}[r_p(\omega)] \\ h(\omega) &= \sigma_p(\omega) \end{aligned} \tag{28}$$

A Fast Derivation of the CAPM

Imagine the portfolio can be expressed as $\mathbb{E}[r_p] = f(\sigma_p) \rightarrow = g(\omega) = f(h(\omega))$. Using the cahin rule

$$\begin{aligned} g'(\omega) &= f'(\sigma_p)h'(\omega) \\ f'(\sigma_p) &= \frac{g'(\omega)}{h'(\omega)} \end{aligned} \tag{29}$$

We have that

$$\begin{aligned} g'(\omega) &= \mathbb{E}[r_i] - \mathbb{E}[r_m] \\ h'(\omega) &= \frac{\omega\sigma_i^2 - (1-\omega)\sigma_m^2 + (1-2\omega)\sigma_{ij}}{\sqrt{\omega^2\sigma_i + (1-\omega)^2\sigma_m^2 + 2\omega(1-\omega)\sigma_{im}}} \end{aligned} \tag{30}$$

when $\omega = 0$

$$f'(\sigma_p) = [\mathbb{E}[r_i] - \mathbb{E}[r_m]] \times \frac{\sigma_m}{\sigma_{ij} - \sigma_m^2} \tag{31}$$

A Fast Derivation of the CAPM

Efficiency of the market portfolio implies

$$
\begin{aligned}
\frac{\mathbb{E}[r_m]-r_f}{\sigma_m} &= [\mathbb{E}[r_i]-\mathbb{E}[r_m]] \times \frac{\sigma_m}{\sigma_{ij}-\sigma_m^2} \\
(\sigma_{ij}-\sigma_m^2)\frac{\mathbb{E}[r_m]-r_f}{\sigma_m^2} &= \mathbb{E}[r_i]-\mathbb{E}[r_m] \\
\mathbb{E}[r_i] &= \mathbb{E}[r_m]+(\mathbb{E}[r_m]-r_f)(\frac{\sigma_{ij}}{\sigma_m^2}-1) \\
\mathbb{E}[r_i] &= r_f+\frac{\sigma_{ij}}{\sigma_m^2}(\mathbb{E}[r_m]-r_f) \\
\mathbb{E}[r_i] &= r_f+\beta_i(\mathbb{E}[r_m]-r_f)
\end{aligned}
\tag{32}
$$

Testable Implications of the CAPM

Eugene F. **Fama** and James D. **Macbeth** (1973) Risk, Return, and Equilibrium: Empirical Tests. *The Journal of Political Economy*

$$\mathbb{E}[r_{it}^e] = \beta_i \mathbb{E}[r_{mt}^e] \rightarrow \text{ Security Market Line}$$

Test

$$r_{it} = a_i + \beta_i r_{mt} + \epsilon_{it} \quad (33)$$

$$s_i = sd(\hat{\epsilon}_{it}) \quad (34)$$

$$r_{it} = \gamma_0 + \gamma_1 \beta_{it} + \gamma_2 \beta_{it}^2 + \gamma_3 s_{it} + \nu_{it} \quad (35)$$

Hypotheses

- $\gamma_0 = r_f$ (Sharpe-Lintner Hypothesis)
- $\gamma_1 = \mathbb{E}[r_{mt}^e] > 0$ (Return-Rick tradeoff)
- γ_3 = 0 (Linearity)
- γ_4 = 0 (No other systematic effect)

Testable Implications of the CAPM

Econometric challenges

- Error in variables for β_i. $\rightarrow$ Portfolios.
- Cross-sectional correlation in ν_{it}.
  - Sequential Cross-sectional regressions.
  - Modern approaches use GLS and cluster by time.

Fama MacBeth Procedure

- Compute $\hat{\beta_{it}} = \hat{\beta_i}$ and $\hat{s}_{it} = \hat{s_i}$ on a rolling basis for every t using data up to $t-1$.

$$r_{is} = a_i + \beta_i r_{ms} + \epsilon_{is} \; \forall t - w < s < t \tag{36}$$

$$s_{it} = sd(\epsilon_{is}) \; \forall t - w < s < t \tag{37}$$

- Compute a sequence of γ estimates, fix s and estimate cross-sectionally.

$$r_i = \gamma_{0s} + \gamma_{1s}\hat{\beta}_i + \gamma_{2s}\hat{\beta}_i^2 + \gamma_3 \hat{s}_i + \nu_i \; \forall i \tag{38}$$

- Hypothesis testing for any coefficient γ

$$\frac{\bar{\hat{\gamma}}_j}{s(\hat{\gamma}_j)/\sqrt{T}} \sim t_{T-1} \tag{39}$$

A Test of the Efficiency of a Given Portfolio

Gibbons, Ross, and Shanken (1989).
Consider a balanced single-factor model (e.g. CAPM) with N assets, and T periods.

$$r^e_{it} = \alpha_i + \beta_i f_t + \epsilon_{it} \tag{40}$$

and assume residuals are normally distributed, uncorrelated overtime but they can be cross-sectionally correlated with covariance matrix Σ.

$$\begin{aligned} \hat{\beta}_i &= \frac{s_{r^e_i f}}{s^2_f} \\ \hat{\alpha}_i &= \bar{r^e_i} - \hat{\beta}_i \bar{f} \end{aligned} \tag{41}$$

where s_{xy} is the sample but biased covariance between x and y. Unless otherwise notice we will estimate covariances dividing by T and not $T-1$.

Asymptotics of $\bar{\beta}_i$

$$
\begin{aligned}
\hat{\beta}_i &= \frac{\sum_t \left(r^e_{it} - \bar{r^e_i}\right)\left(f_t - \bar{f}\right)}{\sum_t \left(f_t - \bar{f}\right)^2} \\
\hat{\beta}_i &= \frac{\sum_t \left((\alpha_i + \beta_i f_t + \epsilon_{it}) - (T^{-1} \sum_s (\alpha_i + \beta_i f_s + \epsilon_{is})\right)\left(f_t - \bar{f}\right)}{\sum_t \left(f_t - \bar{f}\right)^2} \\
\hat{\beta}_i &= \frac{\sum_t \left(\beta_i (f_t - \bar{f}) + (\epsilon_{it} - \cancel{\bar{\epsilon_i}})\right)\left(f_t - \bar{f}\right)}{\sum_t \left(f_t - \bar{f}\right)^2}
\end{aligned}
\tag{42}
$$

Asymptotics of $\hat{\beta}_i$

$$Var(\hat{\beta}_i) = \frac{\sigma^2 \sum_t \left(f_t - \bar{f}\right)^2}{\left[\sum_t \left(f_t - \bar{f}\right)^2\right]^2} = \frac{\sigma^2}{T s_f^2}$$
$$\mathbb{E}[\hat{\beta}_i] = \frac{\beta_i \sum_t \left(f_t - \bar{f}\right)^2}{\sum_t \left(f_t - \bar{f}\right)^2} = \beta_i \tag{43}$$

Asymptotics of $\bar{\alpha}_i$

$$\hat{\alpha}_i = \bar{r_i^e} - \bar{\beta}_i \bar{f} = T^{-1} \sum_t (\alpha_i + \beta_i f_t + \epsilon_{it}) - \hat{\beta}_i \bar{f}$$

$$Var(\hat{\alpha}_i) = T^{-1}\sigma^2 + \bar{f}^2 \frac{\sigma^2}{T s_f^2} = \sigma^2 \frac{1}{T}\left[1 + \left(\frac{\bar{f}}{s_f}\right)^2\right] = \sigma^2 \frac{1}{T}\left[1 + \hat{\theta}^2\right] \tag{44}$$

$$\mathbb{E}[\hat{\alpha}_i] = \alpha_i$$

Now for the multivariate distribution,

$$\hat{\boldsymbol{\alpha}} \sim \boldsymbol{\mathcal{N}}\left(\boldsymbol{\alpha}, \boldsymbol{\Sigma}\frac{1}{T}\left[1 + \hat{\theta}^2\right]\right) \tag{45}$$

$$\sqrt{T/\left[1 + \hat{\theta}^2\right]}\,\hat{\boldsymbol{\alpha}} \sim \boldsymbol{\mathcal{N}}\left(\sqrt{T/\left[1 + \hat{\theta}^2\right]}\,\boldsymbol{\alpha}, \boldsymbol{\Sigma}\right) \tag{46}$$

Preliminaries

- The sample estimate $(T-2)\hat{\mathbf{\Sigma}}$ follows a Wishart distribution with parameters $T-2$ and $\mathbf{\Sigma}$.
- A quadratic form of a multivariate normal random variable and a Wishart variable is called a Hotelling's t^2 statistic. (Up to a scalling term to scale the covariance matrix)
- A t^2 statistic has an equivalent F test.

Under the null $\boldsymbol{\alpha} = 0$

$$X = T\frac{\hat{\boldsymbol{\alpha}}'\hat{\mathbf{\Sigma}}^{-1}\hat{\boldsymbol{\alpha}}}{\frac{(T-2)}{T}\left[1+\hat{\theta}^2\right]} \sim t^2(N,T)$$
$$\frac{(T-N-1)}{NT}X = \frac{(T-N-1)T}{N(T-2)}\frac{\hat{\boldsymbol{\alpha}}'\hat{\mathbf{\Sigma}}^{-1}\hat{\boldsymbol{\alpha}}}{\left[1+\hat{\theta}^2\right]} \sim \mathbf{F}_{N,T-N-1} \tag{47}$$

A Geometric Interpretation

Consider the standard portfolio optimization problem for $N+1$ assets with covariance matrix V. (N assets plus the factor as a portfolio)

$$\begin{aligned} \min \boldsymbol{\omega}' \hat{\boldsymbol{V}} \boldsymbol{\omega} \\ \text{s.t.} \\ \boldsymbol{\omega}' \boldsymbol{\mu} = \mu_0 \\ \boldsymbol{\omega}' \mathbf{1}_{N+1} = 1 \end{aligned} \tag{48}$$

where $\boldsymbol{\mu}' = [\bar{f}, \boldsymbol{\mu}_N]'$ The solution to the above problem is

$$\begin{aligned} \boldsymbol{\omega} &= \lambda \hat{\boldsymbol{V}}^{-1} \boldsymbol{\mu} \\ \lambda &= \frac{\mu_0}{\boldsymbol{\mu}' \hat{\boldsymbol{V}}^{-1} \boldsymbol{\mu}} \text{ (Lagrange Multiplier)} \end{aligned} \tag{49}$$

The squared ratio of mean over s.d. equals

$$\left[\frac{\boldsymbol{\omega}' \boldsymbol{\mu}}{\sqrt{\boldsymbol{\omega}' \hat{\boldsymbol{V}} \boldsymbol{\omega}}} \right]^2 = \frac{\mu_0^2}{\left[\frac{\mu_0}{\boldsymbol{\mu}' \hat{\boldsymbol{V}}^{-1} \boldsymbol{\mu}} \right]^2 \boldsymbol{\mu}' \hat{\boldsymbol{V}}^{-1} \boldsymbol{\mu}} = \boldsymbol{\mu}' \hat{\boldsymbol{V}}^{-1} \boldsymbol{\mu} = \hat{\theta}^{*2} \tag{50}$$

A Geometric Interpretation (cont.)

Exploit the factor structure

$$\begin{aligned} r^e_{it} &= \alpha_i + \beta_i f_t + \epsilon_{it} \\ \text{Cov}(r^e_{it}, r^e_{jt}) &= \text{Cov}(\alpha_i + \beta_i f_t + \epsilon_{it}, \alpha_j + \beta_j f_t + \epsilon_{jt}) \\ &= \beta_i \beta_j s_f^2 + \text{Cov}(\epsilon_{it}, \epsilon_{jt}) \end{aligned} \tag{51}$$

or in Matrix form

$$\hat{\boldsymbol{V}}_N = \hat{\boldsymbol{\beta}}\hat{\boldsymbol{\beta}}' s_f^2 + \hat{\boldsymbol{\Sigma}} \text{ For the } N \text{ testing assets} \tag{52}$$

$$\begin{aligned} \hat{\boldsymbol{V}} &= \begin{pmatrix} s_f^2 & \hat{\boldsymbol{\beta}}' s_f^2 \\ \hat{\boldsymbol{\beta}} s_f^2 & \hat{\boldsymbol{\beta}}\hat{\boldsymbol{\beta}}' s_f^2 + \hat{\boldsymbol{\Sigma}} \end{pmatrix} \\ \hat{\boldsymbol{V}}^{-1} &= \begin{pmatrix} s_f^{-2} + \hat{\boldsymbol{\beta}}' \hat{\boldsymbol{\Sigma}}^{-1} \hat{\boldsymbol{\beta}} & -\hat{\boldsymbol{\beta}}' \hat{\boldsymbol{\Sigma}}^{-1} \\ -\hat{\boldsymbol{\Sigma}}^{-1} \hat{\boldsymbol{\beta}} & \hat{\boldsymbol{\Sigma}}^{-1} \end{pmatrix} \end{aligned} \tag{53}$$

A Geometric Interpretation (cont)

$$\boldsymbol{\mu}'\hat{\boldsymbol{V}}^{-1}\boldsymbol{\mu} = \boldsymbol{\mu}' \begin{pmatrix} s_f^{-2} + \hat{\boldsymbol{\beta}}'\hat{\boldsymbol{\Sigma}}^{-1}\hat{\boldsymbol{\beta}} & -\hat{\boldsymbol{\beta}}'\hat{\boldsymbol{\Sigma}}^{-1} \\ -\hat{\boldsymbol{\Sigma}}^{-1}\hat{\boldsymbol{\beta}} & \hat{\boldsymbol{\Sigma}}^{-1} \end{pmatrix} \boldsymbol{\mu}$$

$$= \hat{\theta}^2 + (\boldsymbol{\mu_N} - \hat{\boldsymbol{\beta}}\bar{f})'\hat{\boldsymbol{\Sigma}}^{-1}(\boldsymbol{\mu_N} - \hat{\boldsymbol{\beta}}\bar{f}) \tag{54}$$

$$\hat{\theta}^{*2} - \hat{\theta}^2 = \hat{\boldsymbol{\alpha}}'\hat{\boldsymbol{\Sigma}}^{-1}\hat{\boldsymbol{\alpha}} \tag{55}$$

The t^2 statistic can be rewritten as

$$\frac{\hat{\theta}^{*2} - \hat{\theta}^2}{1 + \hat{\theta}^2} = \left[\frac{\sqrt{1 + \hat{\theta}^{*2}}}{\sqrt{1 + \hat{\theta}^2}}\right]^2 - 1 = \psi^2 - 1 \tag{56}$$

The S.D.F. as the Modern Asset Pricing Framework

- General assumption that asset markets do not permit the ”persistent” existence of arbitrage opportunities. (Riskless large profits)
- In the absence of arbitrage opportunities there exists a ”stochastic discount factor” that maps payoffs to prices in the economy (state prices).
- If the SDF is a linear function to shocks, then returns can be expressed via a linear factor model.
- This paradigm encompasses partial equilibrium economies focusing on the supply and demand of investment assets, and even recent developments in behavioral finance.

The Stochastic Discount Factor

Assumptions

- Consider S states of nature $s = 1, ..., S$ all of which have a strictly positive probability $\pi(s)$.
- Markets are complete, for every state s there exists a contingent claim that pays 1 in that state. Write its price $q(s)$

The Law of one price indicates that any two assets with exactly the same payoffs in every state of nature must have the same price. Given a payoff function X the price of this claim must satisfy

$$P(X) = \sum_{s=1}^{S} q(s)X(s) = \sum_{s=1}^{S} \pi(s)\frac{q(s)}{\pi(s)}X(s) = \sum_{s=1}^{S} \pi(s)M(s)X(s) = \mathbb{E}[MX] \tag{57}$$

And for a riskless asset that pays 1 in every state.

$$B = \sum_{s=1}^{S} q(s) = \mathbb{E}[M] \rightarrow 1 + r_f = \frac{1}{\mathbb{E}[M]} \tag{58}$$

The Stochastic Discount Factor: some structure

Consider a two period economy with a risky asset and a representative agent. The agent must decide how much to consume today and tomorrow to maximize the present value of her expected separable life-time utility.

$$
\begin{aligned}
\max_{c_t, c_{t+1}} u(c_t) + \beta \mathbb{E}[u(c_{t+1})] \\
\text{s.t.} \\
c_t = W_t - \eta_t P_t \\
c_{t+1} = W_{t+1} + \eta_t X_{t+1}
\end{aligned}
\tag{59}
$$

First Order Condition

$$
\begin{aligned}
\mathcal{L}(\eta_t; \lambda_t, \lambda_{t+1}) &= u(W_t - \eta_t P_t) + \beta \mathbb{E}[u(W_{t+1} + \eta_t X_{t+1})] \\
\mathcal{L}_{\eta_t} &= -u'(c_t) P_t + \beta \mathbb{E}[u'(c_{t+1}) X_{t+1}] = 0 \\
P_t &= \mathbb{E}\left[\beta \frac{u'(c_{t+1})}{u'(c_t)} X_{t+1}\right] = \mathbb{E}\left[M_{t+1} X_{t+1}\right]
\end{aligned}
\tag{60}
$$

The Stochastic Discount Factor and the Risk Neutral Measure

$$
\begin{aligned}
P_t &= \mathbb{E}\Big[M_{t+1}X_{t+1}\Big] \\
P_t &= \beta\int_{-\infty}^{\infty}\frac{u'(c_{t+1})}{u'(c_t)}f(X_{t+1})X_{t+1}dX_{t+1} \\
P_t = \beta\int_{-\infty}^{\infty}q(X_{t+1})X_{t+1}dX_{t+1} = \beta\mathbb{E}^Q\Big[X_{t+1}\Big] &= \frac{1}{1+r}\mathbb{E}^Q\Big[X_{t+1}\Big]
\end{aligned}
\tag{61}
$$

Beta Representation of the SDF

$$
\begin{aligned}
P_t &= \mathbb{E}\Big[M_{t+1}X_{t+1}\Big] \\
1 &= \mathbb{E}\Big[M_{t+1}R_{t+1}\Big] \\
1 &= Cov(M_{t+1}R_{t+1}) + \mathbb{E}\Big[M_{t+1}\Big]\mathbb{E}\Big[R_{t+1}\Big] \\
\frac{1}{\mathbb{E}\Big[M_{t+1}\Big]} &= \frac{Cov(M_{t+1}R_{t+1})}{\mathbb{E}\Big[M_{t+1}\Big]} + \mathbb{E}\Big[R_{t+1}\Big] \\
\mathbb{E}\Big[R_{t+1}\Big] &= R_f - \frac{Cov(M_{t+1}R_{t+1})}{Var(M_{t+1})}\frac{Var(M_{t+1})}{\mathbb{E}\Big[M_{t+1}\Big]} \\
\mathbb{E}\Big[R_{t+1}\Big] &= R_f + \beta_x\lambda_t
\end{aligned}
\tag{62}
$$

Early Work

Relevant Early Work

- **Arrow-Debreu model**: Arrow, K. J., Debreu, G. (1954). ”Existence of an equilibrium for a competitive economy”. Econometrica
- **Options in discrete time**: John C. Cox, Stephen A. Ross, Mark Rubinstein (1979) Option pricing: A simplified approach.
- **Arbitrage Pricing Theory**: Stephen A Ross, The arbitrage theory of capital asset pricing (1976), Journal of Economic Theory.
- **Exchange Economy**: Lucas, R. E. (1978). Asset Prices in an Exchange Economy. Econometrica
- **Books**: Ingersoll (1987), Duffie (1992), Cochrane (1999).

Modern Asset Pricing: Structure + Data

What is the SDF?

- Without structure the SDF can be anything. But the structure we impose could be wrong.
- **SDFs have to be volatile enough**: Hansen, Lars Peter; Jagannathan, Ravi (1991). Implications of Security Market Data for Models of Dynamic Economies. Journal of Political Economy.

$$\mathbb{E}_t[R_{t+1} - R_f] = -\frac{Cov(M_{t+1}, R_{t+1} - R_f)}{\mathbb{E}[M_{t+1}]} = -\frac{\rho\sigma(M_{t+1})\sigma(R_{t+1} - R_f)}{\mathbb{E}[M_{t+1}]} \tag{63}$$

$$-\frac{\mathbb{E}_t[R_{t+1} - R_f]}{\sigma(R_{t+1} - R_f)} = \rho\frac{\sigma(M_{t+1})}{\mathbb{E}[M_{t+1}]} \geq -\frac{\sigma(M_{t+1})}{\mathbb{E}[M_{t+1}]} \rightarrow \frac{\mathbb{E}_t[R_{t+1} - R_f]}{\sigma(R_{t+1} - R_f)} \leq \frac{\sigma(M_{t+1})}{\mathbb{E}[M_{t+1}]} \tag{64}$$

The Joint Hypothesis

- What does it mean that markets are efficient? They ”fully” reflect available information.
- What does fully mean? We need to make a normative statement and specify the process of price formation.

$$\begin{aligned} \mathbb{E}[1 + R_{i,t+1}] = (1 + R_{f,t+1})(1 - Cov_t(M_{t+1}, R_{i,t+1})) = Z_{it} \\ 1 + R_{i,t+1} = Z_{it} + u_{i,t+1} \end{aligned} \tag{65}$$

- We can test if $u_{i,t+1}$ is unpredictable only if we correctly specified $Cov_t(M_{t+1}, R_{i,t+1})$.

The Generalized Method of Moments (Fast Course)

- Hansen (and others) exploit the joint hypothesis realizing that unpredictable abnormal returns can be used to estimate model parameters.

Consider a parametrization $M_{t+1}(\theta)$, e.g. $M_{t+1} = a + bR_{m,t+1}$

$$\mathbb{E}[M_{t+1}(\theta)(1 + R_{t+1}) - 1] = \mathbb{E}[u_{t+1}(\theta)] \tag{66}$$

The idea is to find parameters θ that make linear combinations of the sample counterpart

$$g_T(\theta) = \frac{1}{T}\sum_{t=1}^{T} u_t(\theta) \tag{67}$$

equal to zero using a quadratic form

$$\min g_T' W g_T \tag{68}$$

for some weighting matrix W. The efficiency and finite sample properties of $\hat{\theta}$ will depend on the choice of W and a sequence of re-estimations.

The GMM Estimator

Hansen (1982) Large Sample Properties of Generalized Method of Moments, Econometrica. Two-stage procedure, for any positive semidefined matrix W, e.g. I

$$\begin{aligned} \hat{\theta}_1 &= \arg\min_{\theta} g_T(\theta)' W g_T(\theta) \\ \frac{\partial g_T(\theta)}{\partial \theta} W g_T(\theta) &= a g_T(\theta) = 0 \end{aligned} \tag{69}$$

This estimator is consistent and asymptotically normal but not always efficient, the efficient estimator is obtained by estimating S as the covariance of moments u_t, re-estimate

$$\hat{\theta}_2 = \arg\min_{\theta} g_T(\theta)' \hat{S}^{-1} g_T(\theta) \tag{70}$$

Asymptotics, define $a = \frac{\partial g_T(\theta)'}{\partial \theta} \hat{S}^{-1}$, $d = \frac{\partial g_T(\theta)}{\partial \theta'}$

$$\sqrt{T}(\theta - \hat{\theta}) \rightarrow \mathcal{N}\left[0, (d' S^{-1} d)^{-1}\right] \tag{71}$$

The GMM Estimator

How good is the estimator? (How close to zero are the moment conditions?)

$$T \underbrace{g_T(\hat{\theta})' \hat{S}^{-1} g_T(\hat{\theta})}_{\text{Hansen's } J} \sim \chi^2_{\text{moments-parameters}} \tag{72}$$

Problem Set

Using monthly return data on the 30 industry portfolios obtained from Kenneth French's website test the CAPM. (Code everything from scratch)

- Plot the Security Market Line and compare the realized vs implied expected returns of the 30 portfolios. What can you say about the SML?
- Implement the Fama Mcbeth procedure and test each hypothesis individually.
- Test the null that the α's of all 30 portfolios implied by the CAPM are equal to zero using the GRS test. Compare graphically the ex-ante and ex-post efficient portfolios.