Lesson 4: GMM in Practice

Objectives

1. Estimate a Consumption Based Asset Pricing Model using GMM
2. Face the challenges of estimating a model in practice with real data

Benchmark, Mehra and Prescott (1985)

Consider the classical consumption based asset pricing model with power utility and habit formation

$$\max \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\gamma}}{1-\gamma}$$

$$\text{s.t.} \quad C_t + A_{t+1} = R_t A_t + Y_t$$

where Y_t is the endowment process, R_t is the return on the risky asset, and A_t is the asset holdings. The pricing condition is

$$\mathbb{E}_t[M_{t+1} R_{t+1}] = 1$$

where $M_{t+1} = \beta\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}$ is the stochastic discount factor.

Benchmark, Mehra and Prescott (1985) continued

- How would you estimate this model with real data (regardless of the methodology)?
- One option: Log-linearize the model.

A log-linearized version of the model is

$$m_{t+1} = \log \beta - \gamma \Delta c_{t+1}$$

if a variable has a lognormal distribution, then

$$\log \mathbb{E}[X] = \mathbb{E}[\log X] + \frac{1}{2}\mathrm{Var}[\log X]$$

continued

Using the fundamental asset pricing equation

$$
\begin{aligned}
1 &= \mathbb{E}_t[M_{t+1}R_{t+1}] \\
0 &= \mathbb{E}_t \log(M_{t+1}R_{t+1}) + \frac{1}{2}\mathrm{Var}_t[\log(M_{t+1}R_{t+1})] \\
0 &= \mathbb{E}_t m_{t+1} + \mathbb{E}_t r_{t+1} + \frac{1}{2}\mathrm{Var}_t[m_{t+1} + r_{t+1}] \\
0 &= \mathbb{E}_t m_{t+1} + \mathbb{E}_t r_{t+1} + \frac{\sigma_m^2}{2} + \frac{\sigma_r^2}{2} + \sigma_{rm} \\
0 &= \log\beta - \gamma\Delta\mathbb{E}_t c_{t+1} + \mathbb{E}_t r_{t+1} + \frac{\gamma^2\sigma_c^2}{2} + \frac{\sigma_r^2}{2} - \gamma\sigma_{rc}
\end{aligned}
$$

Predictions

$$\mathbb{E}_t r_{t+1} = -\log\beta + \gamma\Delta\mathbb{E}_t c_{t+1} - \frac{\gamma^2\sigma_c^2}{2} - \frac{\sigma_r^2}{2} + \gamma\sigma_{rc}$$

$$r_{f,t+1} = -\log\beta + \gamma\Delta\mathbb{E}_t c_{t+1} - \frac{\gamma^2\sigma_c^2}{2}$$

$$\mathbb{E}_t[r_{t+1} - r_{f,t+1}] = \gamma\sigma_{rc} - \frac{\sigma_r^2}{2}$$

The whole Equity Premium Puzzle is about the last equation. We need an unrealistic γ to match the data. Consumption growth does not covary that much with returns.

The risk-free puzzle (risk free rates in the data are too low)

$$r_{f,t+1} = -\log \beta + \gamma \Delta \mathbb{E}_t c_{t+1} - \frac{\gamma^2 \sigma_c^2}{2}$$

When estimating this model in the data, we find that that investors should have a very low risk aversion and a large β. Totally the opposite of the conclusion of the equity risk premium puzzle.

Solving the puzzles

- The equity premium puzzle has many different ***solutions***, including habit formation, long-run risk, rare disasters, measurement error, etc.
- The first attempt to solve the risk-free puzzle was to think about Habit.

Abel 1990

$$u(C_t, C_{t-1}) = \frac{(C_t/X_t)^{1-\gamma}}{1-\gamma}$$

where $X_t = C_{t-1}^{\kappa}$ is the consumption Habit.

$$M_{t+1} = \beta\left(\frac{C_t}{C_{t-1}}\right)^{\kappa(\gamma-1)}\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}$$

$$m_{t+1} = \log\beta + \kappa(\gamma-1)\Delta c_t - \gamma\Delta c_{t+1}$$

The extra parameter lets us match the level of the risk free rate better

$$r_{f,t+1} = -\log\beta - \underbrace{\kappa(\gamma-1)\Delta c_t}_{\text{Extra term}} + \gamma\Delta\mathbb{E}_t c_{t+1} - \frac{\gamma^2\sigma_c^2}{2}$$

Can it help us resolve the equity premium puzzle?

Applying GMM to estimate the model

$$M_{t+1} = \delta\left(\frac{C_t}{C_{t-1}}\right)^{\kappa(\gamma-1)}\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}$$

we do not need to log-linearize the model to estimate it using GMM. Let's use the pricing condition

$$\mathbb{E}_t[M_{t+1}R_{t+1}] = 1$$

Take expectations on both sides, and include instruments z_t to get more moments.

$$\begin{aligned}
\mathbb{E}_t[M_{t+1}R_{t+1} - 1] &= 0 \\
\mathbb{E}_t[M_{t+1}R_{t+1} - 1]z_t &= 0 \\
\mathbb{E}_t[(M_{t+1}R_{t+1} - 1)z_t] &= 0 \\
\mathbb{E}[\mathbb{E}_t[(M_{t+1}R_{t+1} - 1)z_t]] &= 0 \\
\mathbb{E}[(M_{t+1}R_{t+1} - 1)z_t] &= 0
\end{aligned}$$

GMM

$$g(\theta) = \mathbb{E}[(M_{t+1}(\theta)R_{t+1} - 1)z_t] = 0$$

E.g.

$$g([\gamma, \kappa, \delta]) = \mathbb{E}[\begin{pmatrix} \left(\delta\left(\frac{C_t}{C_{t-1}}\right)^{\kappa(\gamma-1)}\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}R_{t+1} - 1\right) \\ \left(\delta\left(\frac{C_t}{C_{t-1}}\right)^{\kappa(\gamma-1)}\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}R_{t+1} - 1\right)C_t \\ \left(\delta\left(\frac{C_t}{C_{t-1}}\right)^{\kappa(\gamma-1)}\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}R_{t+1} - 1\right)C_{t-1} \\ \left(\delta\left(\frac{C_t}{C_{t-1}}\right)^{\kappa(\gamma-1)}\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}R_{t+1} - 1\right)R_t \end{pmatrix}] = 0$$

GMM continued

$$g_T([\gamma,\kappa,\delta]) = \frac{1}{T}\sum_{t=1}^{T}\begin{pmatrix} \left(\delta\left(\frac{C_t}{C_{t-1}}\right)^{\kappa(\gamma-1)}\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}R_{t+1}-1\right) \\ \left(\delta\left(\frac{C_t}{C_{t-1}}\right)^{\kappa(\gamma-1)}\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}R_{t+1}-1\right)C_t \\ \left(\delta\left(\frac{C_t}{C_{t-1}}\right)^{\kappa(\gamma-1)}\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}R_{t+1}-1\right)C_{t-1} \\ \left(\delta\left(\frac{C_t}{C_{t-1}}\right)^{\kappa(\gamma-1)}\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}R_{t+1}-1\right)R_t \end{pmatrix} = 0$$

Data

Use quarterly data

- R_t is the return on the US market portfolio, obtain it here from Kenneth French's website https://mba.tuck.dartmouth.edu/pages/faculty/ken.french/ftp/F-F_Research_Data_Factors_CSV.zip returns are at a monthly frequency, so you need to compute the quarterly returns. (Compound the monthly returns to get the quarterly return). Recall that R is already a gross return.
- C_t is the Real Personal Consumption Expenditures, obtain it here from FRED https://fred.stlouisfed.org/series/PCECC96.

This is Problem Set 3 due March 23 the day before the exam (groups of max 2)