Online Appendix: Learning from Quant (Qual)-itative Information

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1. Mathematical Appendix

Proof of Theorem (1). The extrema of a functional of the form $J[y] = \int_a^b H(y(x), y'(x), x)dx$ with boundary conditions $y(a) = y_a$ and $y(b) = y_b$ is given by the solution to the differential equation $H_y(y(x), y'(x), x) - \frac{d}{dx}H_{y'}(y(x), y'(x), x) = 0$ where $H_y, H_{y'}$ are the partial derivatives of H with respect to y and y' (See Dacorogna (1992, Theorem 2.1)). For the particular case in which the functional is of the form of equation (3) the Euler-Lagrange equation can be written as:

$$\frac{d^2G}{dx^2}\frac{dF}{dx} - \frac{dG}{dx}\frac{d^2F}{dx^2} = 0$$

or for a normal prior with mean μ and variance σ^2

$$\frac{d^2G}{dx^2} + (\frac{x-\mu}{\sigma^2})\frac{dG}{dx} = 0$$

The integrating factor $\exp\{\frac{x^2-2\mu x}{2\sigma^2} + C\}$ where C is a constant can be rewritten as $C_1 \exp\{\frac{1}{2}(\frac{x-\mu}{\sigma})^2\}$ where $C_1 = \exp\{-(\frac{\mu^2}{2\sigma^2} + C)\}$. This leads to the following general solution:

$$\frac{dG}{dx} = C_2 \exp\{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2\} \tag{1}$$

The value of the constant C_2 can be calculated based on the boundary conditions. There is no guarantee for the function $\frac{dG}{dx}$ to be smooth, however we can

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split the differential equation into two differential equations with boundaries at $(-\infty, \tilde{\alpha}]$ and $(\tilde{\alpha}, \infty)$ as follows:

For the interval $(-\infty, \tilde{\alpha}]$

$$C_2 \int_{-\infty}^{\tilde{\alpha}} \exp\{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2\}dx = \frac{1}{2} \rightarrow C_2 = \frac{1}{2\sqrt{2\pi\sigma}\Phi(\frac{\tilde{\alpha}-\mu}{\sigma})}$$

and the interval $(\tilde{\alpha}, \infty)$

$$C_2 \int_{-\infty}^{\tilde{\alpha}} \exp\{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2\}dx = \frac{1}{2} \rightarrow C_2 = \frac{1}{2\sqrt{2\pi\sigma}(1-\Phi(\frac{\tilde{\alpha}-\mu}{\sigma}))}$$

replacing both solutions to C_2 in equation (1) gives us the p.d.f. in Theorem (1), finally integrating from $(-\infty, x)$ gives us the c.d.f.

Lemma 1. $z\Phi(z) + \phi(z) \geqslant 0$

Proof.

$$z\Phi(z) + \phi(z) = \int_{-\infty}^{z} z\phi(x)dx + \phi(z) \geqslant \int_{-\infty}^{z} x\phi(x)dx = -\phi(x)|_{-\infty}^{x} + \phi(x) = 0$$

[1]

Proof. of Theorem (2): If x has the p.d.f. in **??**

$$\begin{aligned}
\Psi(t) &= \mathbb{E}\Big(e^{tx}\Big) \\
&= \int_{-\infty}^{\infty} e^{tx}\frac{dG(x)}{dx}dx \\
&= \frac{1}{\sqrt{8\pi\sigma}}\left(\int_{-\infty}^{\bar{\alpha}} \frac{1}{\Phi(\frac{\bar{\alpha}-\mu}{\sigma})} e^{tx-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}dx + \int_{\bar{\alpha}}^{\infty} \frac{1}{(1-\Phi(\frac{\bar{\alpha}-\mu}{\sigma}))} e^{tx-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}dx\right)
\end{aligned}$$

[1] I thank Dilip Sarwate for his elegant proof available in math.stackexchange.com

To derive the m.g.f. let us focus first on the term:

$$\int \exp\Big\{tx - \frac{1}{2\sigma^2}(x^2 - 2x\mu + \mu^2)\Big\}dx$$

$$\int \exp\Big\{-\frac{1}{2\sigma^2}(x^2 - 2x\mu - 2x\sigma^2\mu + \mu^2)\Big\}dx$$

$$\int \exp\Big\{-\frac{1}{2\sigma^2}(x^2 - 2x(\mu + \sigma^2 t) + \mu^2)\Big\}dx$$

$$\int \exp\Big\{-\frac{1}{2\sigma^2}(x^2 - 2x(\mu + \sigma^2 t) + (\mu + \sigma^2 t)^2 - (\mu + \sigma^2 t)^2 + \mu^2)\Big\}dx$$

$$\int \exp\Big\{-\frac{1}{2\sigma^2}((x - (\mu + \sigma^2 t))^2 - (\mu + \sigma^2 t)^2 + \mu^2)\Big\}dx$$

Moving out of the integral all terms that do not depend on x, and plugging it into the original expression:

$$\frac{e^{-\frac{(\mu^2 - (\mu+\sigma^2 t)^2)}{2\sigma^2}}}{2}\left(\frac{1}{\Phi(\frac{\bar{\alpha}-\mu}{\sigma})}\int_{-\infty}^{\bar{\alpha}} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}(\frac{x-(\mu+\sigma^2 t)}{\sigma})^2} dx + \frac{1}{(1-\Phi(\frac{\bar{\alpha}-\mu}{\sigma}))}\int_{\bar{\alpha}}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}(\frac{x-(\mu+\sigma^2 t)}{\sigma})^2} dx\right)$$

$$\frac{e^{-\frac{(\mu^2 - (\mu+\sigma^2 t)^2)}{2\sigma^2}}}{2}\left(\frac{\Phi(\frac{\bar{\alpha}-(\mu+\sigma^2 t)}{\sigma})}{\Phi(\frac{\bar{\alpha}-\mu}{\sigma})} + \frac{1-\Phi(\frac{\bar{\alpha}-(\mu+\sigma^2 t)}{\sigma})}{(1-\Phi(\frac{\bar{\alpha}-\mu}{\sigma}))}\right)$$

$$\frac{e^{-\frac{\mu^2 - \mu^2 - 2\mu\sigma^2 t + \sigma^4 t^2}{2\sigma^2}}}{2}\left(\frac{\Phi(\frac{\bar{\alpha}-(\mu+\sigma^2 t)}{\sigma})}{\Phi(\frac{\bar{\alpha}-\mu}{\sigma})} + \frac{1-\Phi(\frac{\bar{\alpha}-(\mu+\sigma^2 t)}{\sigma})}{(1-\Phi(\frac{\bar{\alpha}-\mu}{\sigma}))}\right)$$

$$\Psi(t) = \frac{e^{\mu t + \frac{\sigma^2 t^2}{2}}}{2}\left(\frac{\Phi(\frac{\bar{\alpha}-(\mu+\sigma^2 t)}{\sigma})}{\Phi(\frac{\bar{\alpha}-\mu}{\sigma})} + \frac{1-\Phi(\frac{\bar{\alpha}-(\mu+\sigma^2 t)}{\sigma})}{(1-\Phi(\frac{\bar{\alpha}-\mu}{\sigma}))}\right)$$

Proof. of Theorem (3): The first and second derivatives of $\Psi(t)$ are:

$$\frac{d}{dt}\Psi(t) = \frac{\exp\{\mu t + \frac{\sigma^2 t^2}{2}\}}{2} \times$$
$$\Big((\mu + \sigma^2 t)\big(\frac{\Phi(\frac{\bar{\alpha}-(\mu+\sigma^2 t)}{\sigma})}{\Phi(\frac{\bar{\alpha}-\mu}{\sigma})} + \frac{1-\Phi(\frac{\bar{\alpha}-(\mu+\sigma^2 t)}{\sigma})}{(1-\Phi(\frac{\bar{\alpha}-\mu}{\sigma}))}\big) - \sigma\big(\frac{\phi(\frac{\bar{\alpha}-(\mu+\sigma^2 t)}{\sigma})}{\Phi(\frac{\bar{\alpha}-\mu}{\sigma})} - \frac{\phi(\frac{\bar{\alpha}-(\mu+\sigma^2 t)}{\sigma})}{1-\Phi(\frac{\bar{\alpha}-\mu}{\sigma})}\big)\Big)$$

$$\frac{d^2}{dt^2}\Psi(t) = \frac{\exp\{\mu t + \frac{\sigma^2 t^2}{2}\}}{2} \times$$
$$\Big(2(\mu + \sigma^2 t)\sigma\big(\frac{\phi(\frac{\bar{\alpha}-(\mu+\sigma^2 t)}{\sigma})}{1-\Phi(\frac{\bar{\alpha}-\mu}{\sigma})} - \frac{\phi(\frac{\bar{\alpha}-(\mu+\sigma^2 t)}{\sigma})}{\Phi(\frac{\bar{\alpha}-\mu}{\sigma})}\big) + \sigma^2\big(\frac{\bar{\alpha} - (\mu + \sigma^2 t)}{\sigma}\big)\big(\frac{\phi(\frac{\bar{\alpha}-(\mu+\sigma^2 t)}{\sigma})}{1-\Phi(\frac{\bar{\alpha}-\mu}{\sigma})} - \frac{\phi(\frac{\bar{\alpha}-(\mu+\sigma^2 t)}{\sigma})}{\Phi(\frac{\bar{\alpha}-\mu}{\sigma})}\big)$$
$$+ \sigma^2\big(\frac{\Phi(\frac{\bar{\alpha}-(\mu+\sigma^2 t)}{\sigma})}{\Phi(\frac{\bar{\alpha}-\mu}{\sigma})} + \frac{1-\Phi(\frac{\bar{\alpha}-(\mu+\sigma^2 t)}{\sigma})}{(1-\Phi(\frac{\bar{\alpha}-\mu}{\sigma}))}\big) + (\mu + \sigma^2 t)^2\big(\frac{\Phi(\frac{\bar{\alpha}-(\mu+\sigma^2 t)}{\sigma})}{\Phi(\frac{\bar{\alpha}-\mu}{\sigma})} + \frac{1-\Phi(\frac{\bar{\alpha}-(\mu+\sigma^2 t)}{\sigma})}{(1-\Phi(\frac{\bar{\alpha}-\mu}{\sigma}))}\big)\Big)$$

evaluating the first derivative at $t = 0$

$$\tilde{\mu} = \mu + \frac{\sigma\phi(\frac{\bar{\alpha}-\mu}{\sigma})}{2}\Big(\frac{1}{1-\Phi(\frac{\bar{\alpha}-\mu}{\sigma})} - \frac{1}{\Phi(\frac{\bar{\alpha}-\mu}{\sigma})}\Big)$$

now to get the second moment we evaluate the second derivative at $t = 0$ which gives us:

$$\mu^2 + \sigma^2 + \mu\sigma\big(\frac{\phi(\frac{\bar{\alpha}-\mu}{\sigma})}{1-\Phi(\frac{\bar{\alpha}-\mu}{\sigma})} - \frac{\phi(\frac{\bar{\alpha}-\mu}{\sigma})}{\Phi(\frac{\bar{\alpha}-\mu}{\sigma})}\big) + \frac{\sigma^2}{2}\big(\frac{\bar{\alpha}-\mu}{\sigma}\big)\big(\frac{\phi(\frac{\bar{\alpha}-\mu}{\sigma})}{1-\Phi(\frac{\bar{\alpha}-\mu}{\sigma})} - \frac{\phi(\frac{\bar{\alpha}-\mu}{\sigma})}{\Phi(\frac{\bar{\alpha}-\mu}{\sigma})}\big)$$

Substracting the square of the first moment $\tilde{\mu}^2$ I obtain

$$\tilde{\sigma}^2 = \sigma^2 + \frac{\sigma^2}{2}\big(\frac{\bar{\alpha}-\mu}{\sigma}\big)\big(\frac{\phi(\frac{\bar{\alpha}-\mu}{\sigma})}{1-\Phi(\frac{\bar{\alpha}-\mu}{\sigma})} - \frac{\phi(\frac{\bar{\alpha}-\mu}{\sigma})}{\Phi(\frac{\bar{\alpha}-\mu}{\sigma})}\big) - \frac{\sigma^2}{4}\big(\frac{\phi(\frac{\bar{\alpha}-\mu}{\sigma})}{1-\Phi(\frac{\bar{\alpha}-\mu}{\sigma})} - \frac{\phi(\frac{\bar{\alpha}-\mu}{\sigma})}{\Phi(\frac{\bar{\alpha}-\mu}{\sigma})}\big)^2$$

and after some algebra

$$\tilde{\sigma}^2 = \sigma^2\Big(1 + \frac{1}{2}\big(\frac{\phi(\frac{\bar{\alpha}-\mu}{\sigma})}{1-\Phi(\frac{\bar{\alpha}-\mu}{\sigma})} - \frac{\phi(\frac{\bar{\alpha}-\mu}{\sigma})}{\Phi(\frac{\bar{\alpha}-\mu}{\sigma})}\big)\big(\big(\frac{\bar{\alpha}-\mu}{\sigma}\big) - \frac{1}{2}\big(\frac{\phi(\frac{\bar{\alpha}-\mu}{\sigma})}{1-\Phi(\frac{\bar{\alpha}-\mu}{\sigma})} - \frac{\phi(\frac{\bar{\alpha}-\mu}{\sigma})}{\Phi(\frac{\bar{\alpha}-\mu}{\sigma})}\big)\big)\Big)$$

Proof. of Theorem (4): After receiving the quantitative signal r_1 investors posterior at time 1 is normally distributed with mean $(\alpha_0 + \frac{\omega}{\gamma+\omega} r_1)$ and standard deviation $\frac{1}{\sqrt{\gamma+\omega}}$ (See. Berk and Green (2004) Proposition (1)). Replacing this as μ and σ into the equation of the mean gives the desired result.

Proof. of Theorem (5): The sign of $\frac{\partial F}{\partial \tilde{\alpha}} = \frac{1}{a q_0}\frac{\partial \alpha_1}{\partial \tilde{\alpha}}$ is equal to the sign of $\frac{\partial \alpha_1}{\partial \tilde{\alpha}}$, defining $\Phi = \Phi(J(\tilde{\alpha}, r_1, \alpha_0)), \phi = \phi(J(\tilde{\alpha}, r_1, \alpha_0))$ and $z = J(\tilde{\alpha}, r_1, \alpha_0)$

$$\frac{\partial \alpha_1}{\partial \tilde{\alpha}} = \frac{1}{2} \times \Big(\frac{\big(-z\Phi\big(-2\Phi^2 + 3\Phi - 1\big)\phi + \big(2\Phi^2 - 2\Phi + 1\big)\phi^2\big)}{(\Phi-1)^2\Phi^2}\Big)$$

since $\phi > 0$ the sign of the derivative depends on the sign of the following expression:

$$f(z) = \Phi z\big(2\Phi^2 - 3\Phi + 1\big) + \big(2\Phi^2 - 2\Phi + 1\big)\phi$$

which can be factorized as:

$$f(z) = (z\Phi + \phi)(2\Phi^2 - 2\Phi + 1) - z\Phi^2$$

In order to prove that the expression is strictly positive I will prove that the function is (i) continuous,(ii) it tends to zero in the limits $\pm\infty$, (iii) there $\exists z \in R : f(z) > 0$ and (iv) the function has no real roots. This implies that the function never crosses the x-axis which means it is strictly positive everywhere.
(i) f is continuous since it is a composition of continuous functions ϕ, Φ and z.
(ii) Its behaviour approaching $\pm\infty$ is:

$$\begin{aligned}
\lim_{z\to\pm\infty} f(z) &= \lim_{z\to\pm\infty} 2z\Phi^3 - 2z\Phi^2 + z\Phi + 2\phi\Phi^2 - 2\phi\Phi + \phi - z\Phi^2 \\
&= \lim_{z\to\pm\infty} z\Phi(1-\Phi)(1-2\Phi) + 0
\end{aligned}$$

since the limit of the term $2\phi\Phi^2 - 2\phi\Phi + \phi$ is 0 when $z \to \pm\infty$.
I use the fact that if $z \geqslant 1 \to (1-\Phi(z)) \leqslant \phi(z)$ and that $\lim_{z\to\infty} z^k\phi(z) = 0$ for $k > 0$, we know that $\lim_{z\to\infty} z(1-\Phi(z)) = 0$ and since $\Phi(z) = 1-\Phi(-z)$ we know that $\lim_{z\to-\infty} z\Phi = 0$, plugging this limits into the limit of $z\Phi(1-\Phi)(1-2\Phi)$ shows that the limit of $f(z)$ is zero in $\pm\infty$.
(iii, iv) I can show that the function is strictly positive for all $z < 0$ and since the function is even there are no real roots. I start by looking at the sign of the following functions: $z\Phi + \phi \geqslant 0$ from Proposition (1), $2\Phi^2 - 2\Phi + 1 > 0$ since it has no real roots and its convex, and $-z\Phi^2 > 0$ when $z < 0$. Finally we just need to prove that the function is even, we can use the fact that ϕ is even and that $\Phi(-z) = 1 - \Phi(z)$.

$$\begin{aligned}
f(-z) &= (-z(1-\Phi) + \phi)(2(1-\Phi)^2 - 2(1-\Phi) + 1) + z(1-\Phi)^2 \\
&= (z\Phi + \phi - z)(2(1 - 2\Phi + \Phi^2) + 2\Phi - 2 + 1) + z(1 - 2\Phi + \Phi^2) \\
&= (z\Phi + \phi - z)(2\Phi^2 - 4\Phi + 2 + 2\Phi - 2 + 1) + z(1 - 2\Phi + \Phi^2) \\
&= (z\Phi + \phi - z)(2\Phi^2 - 2\Phi + 1) + z(1 - 2\Phi + \Phi^2 + \Phi^2 - \Phi^2) \\
&= (z\Phi + \phi)(2\Phi^2 - 2\Phi + 1) - z\Phi^2 \\
&= f(z)
\end{aligned}$$

Which means that there are no real roots, and the function is strictly positive also for $z > 0$, which concludes the proof.

PROOF. of Proposition 2. Defining:

$$\begin{aligned}
\mu &= W_1 + X_1\alpha_1 \\
\sigma^2 &= X_1^2(\sigma_1^2 + \sigma_\epsilon^2) \\
\bar{\alpha} &= W_1 + X_1\tilde{\alpha} \\
\sigma_\alpha^2 &= (\sigma_1^2 + \sigma_\epsilon^2)
\end{aligned}$$

And replacing $t = -\gamma$ we have that the expected utility of the investor is equal to:

$$\frac{\exp\{-(W_1 + X_1\alpha_1)\gamma + \frac{X_1^2\sigma_\alpha^2\gamma^2}{2}\}}{2}\left(\frac{\Phi(\frac{\tilde{\alpha}-\alpha_1}{\sigma_\alpha} + X_1\sigma_\alpha\gamma)}{\Phi(\frac{\tilde{\alpha}-\alpha_1}{\sigma_\alpha})} + \frac{1-\Phi(\frac{\tilde{\alpha}-\alpha_1}{\sigma_\alpha} + X_1\sigma_\alpha\gamma)}{(1-\Phi(\frac{\tilde{\alpha}-\alpha_1}{\sigma_\alpha}))}\right)$$

Taking the derivative with respect to X_1 gives us:

$$\begin{aligned}
&\left(\frac{\gamma\sigma_\alpha\phi(\frac{\tilde{\alpha}-\alpha_1}{\sigma_\alpha} + X_1^*\gamma\sigma_\alpha)}{\Phi(\frac{\tilde{\alpha}-\alpha_1}{\sigma_\alpha})} - \frac{\gamma\sigma_\alpha\phi(\frac{\tilde{\alpha}-\alpha_1}{\sigma_\alpha} + X_1^*\gamma\sigma_\alpha)}{1-\Phi(\frac{\tilde{\alpha}-\alpha_1}{\sigma_\alpha})}\right) \\
&+ (X_1^*\sigma_\alpha^2\gamma^2 - \alpha_1\gamma)\left(\frac{1-\Phi(\frac{\tilde{\alpha}-\alpha_1}{\sigma_\alpha} + X_1^*\sigma_\alpha\gamma)}{1-\Phi(\frac{\tilde{\alpha}-\alpha_1}{\sigma_\alpha})} + \frac{\Phi(\frac{\tilde{\alpha}-\alpha_1}{\sigma_\alpha} + X_1^*\sigma_\alpha\gamma)}{\Phi(\frac{\tilde{\alpha}-\alpha_1}{\sigma_\alpha})}\right) = 0 \\
&\phi(\frac{\tilde{\alpha}-\alpha_1}{\sigma_\alpha} + X_1^*\gamma\sigma_\alpha)\left(\frac{1}{\Phi(\frac{\tilde{\alpha}-\alpha_1}{\sigma_\alpha})} - \frac{1}{1-\Phi(\frac{\tilde{\alpha}-\alpha_1}{\sigma_\alpha})}\right) \\
&+ (X_1^*\sigma_\alpha\gamma - \frac{\alpha_1}{\sigma_\alpha})\left(\frac{1-\Phi(\frac{\tilde{\alpha}-\alpha_1}{\sigma_\alpha} + X_1^*\sigma_\alpha\gamma)}{1-\Phi(\frac{\tilde{\alpha}-\alpha_1}{\sigma_\alpha})} + \frac{\Phi(\frac{\tilde{\alpha}-\alpha_1}{\sigma_\alpha} + X_1^*\sigma_\alpha\gamma)}{\Phi(\frac{\tilde{\alpha}-\alpha_1}{\sigma_\alpha})}\right) = 0
\end{aligned}$$

Defining $\frac{\tilde{\alpha}-\alpha_1}{\sigma_\alpha} = z_\alpha$, $\Phi^\alpha = \frac{1}{\Phi(z_\alpha)} - \frac{1}{1-\Phi(z_\alpha)}$, $\Phi_1^\alpha = \frac{1}{\Phi(z_\alpha)}$, and $\Phi_2^\alpha = \frac{1}{1-\Phi(z_\alpha)}$ the first order condition becomes:

$$\phi(z_\alpha + X_1\gamma\sigma_\alpha)\Phi^\alpha + (X_1\sigma_\alpha\gamma - \frac{\alpha_1}{\sigma_\alpha})(\Phi_2^\alpha(1-\Phi(z_\alpha + X\sigma_\alpha\gamma)) + \Phi_1^\alpha(\Phi(z_\alpha + X\sigma_\alpha\gamma))) = 0$$

or

$$\phi(z_\alpha + X_1\gamma\sigma_\alpha)\Phi^\alpha + (X_1\sigma_\alpha\gamma - \frac{\alpha_1}{\sigma_\alpha})(\Phi_2^\alpha + \Phi^\alpha\Phi(z_\alpha + X\sigma_\alpha\gamma)) = 0$$

re arranging terms the optimal holding is the solution to the following system of equations

$$\phi(y)\Phi^\alpha + (y - \frac{\tilde{\alpha}}{\sigma_\alpha})(\Phi_2^\alpha + \Phi^\alpha\Phi(y)) = 0$$
$$y = z_\alpha + X_1\sigma_\alpha\gamma$$

Proof. of Theorem 6: The Nash equilibrium is the solution to the equations:

$$qF(r_1, \tilde{\alpha} + \Delta) + (1 - q)F(r, \tilde{\alpha}) = F(r, \tilde{\alpha})$$

and,

$$pEU(W_2(X(r_1, \tilde{\alpha} + \Delta)), r_1, \tilde{\alpha}) + (1 - p)EU(W_2(X(r_1, \tilde{\alpha})), r_1, \tilde{\alpha}) = EU(W_2(X(r_1, \tilde{\alpha})) - C, r_1, \tilde{\alpha})$$

solving for p and q gives the desired result.

Proof. of Theorem 7:

- $\frac{\partial q}{\partial K} = \frac{F(r_1,\tilde{\alpha}+\Delta)-F(r,\tilde{\alpha})}{(F(r_1,\tilde{\alpha}+\Delta)-F(r,\tilde{\alpha})+K)^2} > 0$
- The sign of $\frac{\partial p}{\partial C}$ can be inferred by observing that the larger C is, the difference

  $$EU(W_2(X(r_1, \tilde{\alpha})), r_1, \tilde{\alpha}) - EU(W_2(X(r_1, \tilde{\alpha})) - C, r_1, \tilde{\alpha})$$

  becomes larger and therefore the derivative is positive.
- $\frac{\partial q}{\partial \tilde{\alpha}} = \frac{K(\frac{\partial F(r_1,\tilde{\alpha})}{\partial \tilde{\alpha}} - \frac{\partial F(r_1,\tilde{\alpha}+\Delta)}{\partial \tilde{\alpha}})}{(F(r_1,\tilde{\alpha}+\Delta)-F(r,\tilde{\alpha})+K)^2} < 0$ if F is convex on $\tilde{\alpha}$ (e.g. $\frac{\partial F(r_1,\tilde{\alpha})}{\partial \tilde{\alpha}} - \frac{\partial F(r_1,\tilde{\alpha}+\Delta)}{\partial \tilde{\alpha}} < 0$) and 0 if F is linear on $\tilde{\alpha}$.

References

Dacorogna, B. (1992). *Introduction to the Calculus of Variations*.