Investment Funds and Risks

Risk Management via Portfolio Optimization

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Agenda

- Sessions 1 and 2. Portfolio Optimization for Market Risk Management
- Session 3. Portfolio Optimization for Credit Risk Management
- Session 4. Computational Aspects of Portfolio Optimization and practical examples

Portfolio Optimization

Introduction

- The problem of how to invest money is the whole reason why investment funds exist.
- Investment funds face complex investment strategies lookin for high return opportunities at a reasonable risk.
- The problem of how to invest money has become more complex in the last years due to the increasing number of investment opportunities and the increasing complexity of the financial markets.

Market and Diversification

- Any specific financial instrument that is tradeable is called an *asset*.
- We consider the problem of an investment fund interested in investing a certain amount of money called *capital*, in a specific set of assets called *available assets*.
- Building a *portfolio* means deciding the amount of capital, called *share* or *weight*, to invest in each available asset. If the share is zero, the asset is not included in the portfolio.

Basic Concepts and Notation

The set of available assets, that is the set of assets considered for an investment, is denoted by $N = \{1, \ldots, n\}$.The rate of return at the target time of each asset j, $j = 1, \ldots, n$, is modeled as a random variable R_j with given mean $\mu_j = \mathbb{E}[R_j]$. Let x_j denote the decision variable expressing the weight of asset j. Let

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

denote the share or the weight vector.

Each portfolio $\mathbf{x}$ is associated with a random variable $R_x = \sum_{j=1}^{n} x_j R_j$ representing the rate of return of the portfolio. The mean of R_x is given by

$$\mu(\mathbf{x}) = \mathbb{E}[R_x] = \sum_{j=1}^{n} x_j \mu_j.$$

During the course we will consider different measures of risk. For the moment, consider $\rho(\mathbf{x})$ the *measure of risk*.

Measures of Risk

We say that $\rho(\mathbf{x})$ a real-valued function defined on the set of portfolios. (A function that maps a portfolio to a real number) is a measure of risk if it satisfies the following properties:

- If one portfolio always yields equal or worse outcomes than another, its risk should be greater or equal.

$$R_x \leq R_y \Rightarrow \rho(\mathbf{x}) \geq \rho(\mathbf{y}).$$

- Diversification should not increase risk.

$$\rho(\mathbf{x}+\mathbf{y}) \leq \rho(\mathbf{x}) + \rho(\mathbf{y}).$$

Objective function

The objective of the investment fund is most of the time bi-dimensional: maximize the expected return and minimize the risk. This is modeled by the bi-objective function

$$\max\{[\mu(\mathbf{x}), -\rho(\mathbf{x})]\}$$

clearly there exists a trade-off, no portfolio can maximize the expected return and minimize the risk at the same time. Normally, for feasible portfolios, the higher the expected return, the higher the risk.

$$\mu(\mathbf{x}) \geq \mu(\mathbf{y}) \Rightarrow \rho(\mathbf{x}) \geq \rho(\mathbf{y}).$$

The Markowitz Model.

Given variance $\sigma_{ij} = \mathbb{E}[(R_i - \mu_i)(R_j - \mu_j)]$ and covariance $\sigma_{ij} = \mathbb{E}[(R_i - \mu_i)(R_j - \mu_j)]$ of the random variables R_i and R_j, the risk of a portfolio $\mathbf{x}$ is measured by the standard deviation

$$\rho(\mathbf{x}) = \sigma(\mathbf{x}) = \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} x_i x_j \sigma_{ij}}.$$

Handling Bi-Criteria Optimization Problems

1. Bounding Approach

Idea: Solve the problem for a given value of the risk and then solve the problem for a given value of the return.

$$\min \sum_{i=1}^{n} \sum_{j=1}^{n} x_i x_j \sigma_{ij}$$

$$\text{s.t.}$$

$$\sum_{i=1}^{n} x_i \mu_i \geq \mu_0$$

$$\sum_{i=1}^{n} x_i = 1$$

Equivalently

$$\max \sum_{i=1}^{n} x_i \mu_i$$

$$\text{s.t.}$$

$$\sum_{i=1}^{n} \sum_{j=1}^{n} x_i x_j \sigma_{ij} \leq \sigma_0^2$$

$$\sum_{i=1}^{n} x_i = 1$$

Trade-off

You could also maximize a linear combination of the expected return and the risk

$$\max \sum_{i=1}^{n} x_i \mu_i - \lambda \sum_{i=1}^{n} \sum_{j=1}^{n} x_i x_j \sigma_{ij}$$

where λ is a parameter that controls the trade-off between the expected return and the risk.

Alternatively, maximize Sharpe Ratio

$$\max \frac{\sum_{i=1}^{n} x_i \mu_i - r_f}{\sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} x_i x_j \sigma_{ij}}}$$

Lesson 2: Linear Models for Portfolio Optimization

What is a Linear Model?

- A linear model (also called a linear programming model) is a mathematical model in which all the functions and the constraints are linear.
- A linear function is a function f that satisfies the following properties:

$$f(x+y) = f(x) + f(y)$$
$$f(\alpha x) = \alpha f(x)$$

- A linear constraint is a constraint of the form

$$\sum_{i=1}^{n} a_i x_i \leq b$$

where a_i and b are constants and x_i are variables.

Why Linear Models?

- Linear models are one of the most studied and the most used optimization models.
- Linear models are easy to solve. There are many algorithms that solve linear models efficiently.
- Linear models are very flexible. Many problems can be modeled as linear models.
- Linear models scale well. They can be used to solve large problems.

Linear Programming for Investment Funds

Example

Objective: Maximize Expected Annual Return

- x_S: Amount invested in stocks (in dollars)
- x_B: Amount invested in bonds (in dollars)
- x_R: Amount invested in real estate (in dollars)

Constraints:

1. **Risk Constraint:**
   - Total portfolio risk $\leq$ 12%
   - Risk of Stocks (S): 15%
   - Risk of Bonds (B): 8%
   - Risk of Real Estate (R): 10%
2. **Budget Constraint:** Total investment = $100,000

Linear Programming Model

Maximize:

$$Z = 0.12x_S + 0.07x_B + 0.09x_R$$

Subject to:

1. Risk Constraint:
   $$0.15x_S + 0.08x_B + 0.10x_R \leq 0.12 \cdot (x_S + x_B + x_R)$$
2. Budget Constraint:
   $$x_S + x_B + x_R = \$100,000$$
3. Non-negativity Constraints:
   $$x_S \geq 0, x_B \geq 0, x_R \geq 0$$

Solution

Optimal Solution:

- x_S = \$40,000
- x_B = \$30,000
- x_R = \$30,000

Maximum Expected Annual Return: \$9,300

Interpretation

- Invest \$40,000 in stocks, \$30,000 in bonds, and \$30,000 in real estate.
- The maximum expected annual return is \$9,300 while adhering to risk, budget, and non-negativity constraints.

Linear Models for Portfolio Optimization

Introduction

- Nowadays, Quadratic Programming models, like Mrkowitz, are not hard to solve thanks to technological advances. Nevertheless, LP remain much more attractive from a computational point of view.
- Is it possible to have linear models for portfolio optimization? How can we measure the risk or safety in order to have a linear model? This is particularly important, since the idea of diversification implies that the risk of a portfolio is not linear.
- A linear model can be obtained by discretizing the return random variable, or through the concept of scenarios.

Scenarios and LP Computability

We have indicated by R_j the random variable representing the rate of return of an asset. We introduc the concept of *scenario*. A *scenario* is informally, a possible situation that can happen at some time. For example

Asset	Scenario 1	Scenario 2	Scenario 3
Stocks	0.15	0.05	-0.05
Bonds	0.08	0.06	0.04

Assume p_s is the probability of scenario t. The expected return of asset j is given by

$$\mu_j = \sum_{s=1}^{m} p_s R_j(s).$$

Expected returns

$$\mu(\mathbf{x}) = \sum_{j=1}^{n} x_j \mu_j = \sum_{j=1}^{n} x_j \sum_{s=1}^{m} p_s R_j(s) = \sum_{s=1}^{m} p_s \sum_{j=1}^{n} x_j R_j(s) = \sum_{s=1}^{m} p_s R_x(s)$$

Basic LP Computable Risk Measures

The mean Absolute Deviation (MAD) is a dispersion measure that is defined as

$$\delta(\mathbf{x}) = \mathbb{E}[|R_x - \mu(\mathbf{x})|] = \sum_{s=1}^{m} p_s |R_x(s) - \mu(\mathbf{x})|.$$

$$\begin{aligned} \min \sum_{s=1}^{m} p_s \left| \sum_{j=1}^{n} x_j R_j(s) - \sum_{j=1}^{n} x_j \mu_j \right| \\ \text{s.t.} \\ \sum_{j=1}^{n} x_j = 1 \\ \sum_{j=1}^{n} x_j \mu_j \geq \mu_0 \\ \ldots \text{possible other constraints} \end{aligned}$$

How to deal with absolute values?

- The absolute value function is not linear.
- The absolute value function can be linearized by introducing a new variable y_s and the following constraints

$$d_s \geq \sum_{j=1}^{n} x_j R_j(s) - \sum_{j=1}^{n} x_j \mu_j$$

$$d_s \geq -(\sum_{j=1}^{n} x_j R_j(s) - \sum_{j=1}^{n} x_j \mu_j)$$

Formulation

$$\min \sum_{s=1}^{m} p_s d_s$$

$$\text{s.t.}$$

$$d_s \geq \sum_{j=1}^{n} x_j R_j(s) - \sum_{j=1}^{n} x_j \mu_j$$

$$d_s \geq -(\sum_{j=1}^{n} x_j R_j(s) - \sum_{j=1}^{n} x_j \mu_j)$$

$$\sum_{j=1}^{n} x_j = 1$$

$$\sum_{j=1}^{n} x_j \mu_j \geq \mu_0$$

$$\ldots \text{possible other constraints}$$

Asymetric Deviation

The MAD accounts for all deviations of the rate of return of the portfolio from its expected value, both below and above.However, one may sensibly think that any rational investor would consider real risk only the deviation below the expected value.

$$\bar{\delta}(\mathbf{x}) = \mathbb{E}[\max\{0, \mu(\mathbf{x}) - R_x\}]$$

How to deal with max in LP?

- The max function is not linear.
- The max function can be linearized by introducing a new variable y_s and the following constraints

$$
\begin{aligned}
d_s &\geq \mu(\mathbf{x}) - y_s \\
d_s &\geq 0
\end{aligned}
$$

Re-formulation

$$\min \sum_{s=1}^{m} p_s d_s$$

$$\text{s.t.}$$

$$d_s \geq \mu(\mathbf{x}) - y_s$$

$$d_s \geq 0$$

$$y_s = \sum_{j=1}^{n} x_j R_j(s)$$

$$\sum_{j=1}^{n} x_j = 1$$

$$\sum_{j=1}^{n} x_j \mu_j \geq \mu_0$$

$$\ldots \text{possible other constraints}$$

Gini's Mean Difference (GMD)

The GMD is a dispersion measure that is defined as

$$\Gamma(\mathbf{x}) = \frac{1}{2}\sum_{s=1}^{m}\sum_{s'=1}^{m} p_s p_{s'} |R_x(s) - R_x(s')|.$$

and considers as risk the average absolute value of the differences of the portfolio returns y_s in different scenarios.

Formulation

$$\min \sum_{s=1}^{m} \sum_{s'=1}^{m} p_s p_{s'} d_{ss'}$$

$$\text{s.t.}$$

$$d_{ss'} \geq y_s - y_{s'} \text{ if } s \neq s'$$

$$d_{ss'} \geq 0$$

$$y_s = \sum_{j=1}^{n} x_j R_j(s)$$

$$\sum_{j=1}^{n} x_j = 1$$

$$\sum_{j=1}^{n} x_j \mu_j \geq \mu_0$$

$$\ldots \text{possible other constraints}$$

Minimize the Maximum Loss

The minimum return is defined as

$$M(\mathbf{x}) = \min_{s=1,...,m} R_x(s)$$

We can try to maximize the minimum return by introducing a new variable y and the following constraints

$$y \leq R_x(s) \text{ for all } s = 1, \ldots, m$$

Formulation

$$\max y$$
$$\text{s.t.}$$
$$y \leq \sum_{j=1}^{n} x_j R_j(s) \text{ for all } s = 1, \ldots, m$$
$$\sum_{j=1}^{n} x_j = 1$$
$$\sum_{j=1}^{n} x_j \mu_j \geq \mu_0$$
$$\ldots \text{possible other constraints}$$

Mixed Integer Programming (MIP) Formulation

MIP Formulation

- What if some of the variables are restricted to be integers?
- The problem is called a mixed integer programming (MIP) problem.
- A combination of continuous and integer variables is called a mixed integer linear programming (MIP) problem.

MIP - Binary Variables

- A binary variable is a variable that can only take on the values 0 or 1.
- Binary variables are used to model yes/no decisions.
- You can introduce these variables into an optimization problem, both to model binary decisions, as well as to turn on/off constraints.
- For example, an investment fund might have leverage constraints if investing in illiquid stocks.

Binary variables in the investment fund problem

Binary variables can tell you in which assets you have decided to invest or not. It even helps you count the number of assets in which you have invested. How can I model

$$z_j = \begin{cases} 1 & \text{if } x_j > 0 \\ 0 & \text{otherwise} \end{cases}$$

Use the following trick

$$\begin{aligned} x_j &\leq M z_j \\ x_j &\geq 0 \\ z_j &\in \{0, 1\} \end{aligned}$$

where M is a large number. This is called a *big-M* formulation.

Constraints on the number of assets

- The number of assets in which you have invested is given by

$$\sum_{j=1}^{n} z_j$$

What about knowing the number of long and short positions? We can define variables z_j^+ and z_j^- and x_j^+ and x_j^- such that

$$z_j^+ = \begin{cases} 1 & \text{if } x_j > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$z_j^- = \begin{cases} 1 & \text{if } x_j < 0 \\ 0 & \text{otherwise} \end{cases}$$

$$x_j = x_j^+ - x_j^-$$

$$z_j = z_j^+ + z_j^-$$

How to relate them?

$$
\begin{aligned}
x_j^+ &\leq M z_j^+ \\
x_j^+ &\geq 0 \\
z_j^+ &\in \{0, 1\} \\
x_j^- &\leq M z_j^- \\
x_j^- &\geq 0 \\
z_j^- &\in \{0, 1\} \\
x_j &= x_j^+ - x_j^- \\
z_j &= z_j^+ + z_j^-
\end{aligned}
$$

The Value-at-Risk (VaR)

- The Value-at-Risk (VaR) is a measure of risk that is defined as the maximum loss that can occur with a given probability α.

$$\mathrm{VaR}_\alpha(\mathbf{x}) = \min\{\lambda \in \mathbb{R} : \mathbb{P}(R_x \leq \lambda) \geq \alpha\}$$

In our discrete scenario setting, we can define the VaR as

$$\mathrm{VaR}_\alpha(\mathbf{x}) = \min\{\lambda \in \mathbb{R} : \sum_{s=1}^{m} p_s \mathbb{I}(R_x(s) \leq \lambda) \geq \alpha\}$$

where $\mathbb{I}(R_x(s) \leq \lambda)$ is the indicator function that is equal to 1 if $R_x(s) \leq \lambda$ and 0 otherwise.

Minimum VaR

We can formulate the problem of minimizing the VaR witht he following MILP

$$\begin{aligned} \min \lambda \\ \text{s.t.} \\ \sum_{s=1}^{m} p_s \mathbb{I}(R_x(s) \leq \lambda) \geq \alpha \\ \ldots \text{possible other constraints} \end{aligned}$$

How to deal with indicator variables?

- The indicator function is not linear.
- The indicator function can be linearized by introducing a new variable z_s and the following constraints

$$z_s = \begin{cases} 1 & \text{if } R_x(s) \leq \lambda \\ 0 & \text{otherwise} \end{cases}$$

mathematically

$$\begin{aligned} z_s &\geq R_x(s) - \lambda \\ z_s &\leq R_x(s) - \lambda + M(1 - z_s) \\ z_s &\in \{0, 1\} \end{aligned}$$

Re-formulation

$$
\begin{aligned}
\min \lambda \\
\text{s.t.} \\
\sum_{s=1}^{m} p_s z_s \geq \alpha \\
z_s \geq R_x(s) - \lambda \\
z_s \leq R_x(s) - \lambda + M(1 - z_s) \\
z_s \in \{0, 1\} \\
\ldots \text{possible other constraints}
\end{aligned}
$$

Conditional Value-at-Risk (CVaR)

- The Conditional Value-at-Risk (CVaR) is a measure of risk that is defined as the expected loss given that the loss is greater than the VaR.

$$\mathrm{CVaR}_\alpha(\mathbf{x}) = \mathbb{E}[R_x | R_x \leq \mathrm{VaR}_\alpha(\mathbf{x})]$$

tip to compute the conditional expectation

$$\mathbb{E}[A|B] = \frac{\mathbb{E}[A\mathbb{I}(B)]}{\mathbb{P}(B)}$$

in our case

$$\mathrm{CVaR}_\alpha(\mathbf{x}) = \frac{\sum_{s=1}^m p_s R_x(s)\mathbb{I}(R_x(s) \leq \lambda)}{\sum_{s=1}^m p_s \mathbb{I}(R_x(s) \leq \lambda)}$$

unfortunately, this is not a linear function of the decision variables, but we can use it in constraints.

Constraints to mitigate risk indirectly

1. Position constraints

$$x_j \leq u_j$$
$$x_j \geq l_j$$

Factor exposure constraints

If returns follow a factor model, we can define the factor exposure of asset j as

$$R_j = \sum_{k=1}^{K} \beta_{jk} F_k + \epsilon_j$$

where F_k is the return of factor k and ϵ_j is the idiosyncratic return of asset j. The factor exposure of portfolio $\mathbf{x}$ to factor k is given by

$$\sum_{j=1}^{n} x_j \beta_{jk}$$

Factor exposure constraints

We can define constraints on the factor exposure of the portfolio to factor k as

$$\sum_{j=1}^{n} x_j \beta_{jk} \leq u_k$$

$$\sum_{j=1}^{n} x_j \beta_{jk} \geq l_k$$

Leverage

Differentiate between long and short positions

$$x_j = x_j^+ - x_j^-$$
$$x_j^+ \geq 0$$
$$x_j^- \geq 0$$

The leverage of the portfolio is given by

$$\sum_{j=1}^{n} x_j^+ + \sum_{j=1}^{n} x_j^-$$

leverage is important to control when taking short positions.

Portfolio Optimization with Transaction Costs

The structure of transaction costs

1. Fixed costs

$$k(x_j) = \begin{cases} k_j & \text{if } x_j > 0 \\ 0 & \text{otherwise} \end{cases}$$

total costs

$$\sum_{j=1}^{n} k_j z_j$$

The structure of transaction costs

2. Proportional costs: $k(x_j) = c_j x_j$ (does not take into account short position costs).

Total costs

$$\sum_{j=1}^{n} c_j x_j$$

Convex piecewise linear transaction costs

$$k(x_j) = \begin{cases} k_{1j}x_j & \text{if } 0 \leq x_j \leq m_1 \\ k_{2j}(x_j - m_1) + k_{1j}m_1 & \text{if } m_1 \leq x_j \leq m_2 \\ \vdots & \\ k_{kj}(x_j - m_{k-1}) + k_{k-1j}m_{k-1} & \text{if } m_{k-1} \leq x_j \leq m_k \end{cases}$$

In our setup

$$k(x_j) = \sum_{i=1}^{k} k_{ij} \max\{0, x_j - m_{ij}\} = \sum_{i=1}^{k} k_{ij} z_{ij}$$

with auxiliary variables $z_{ij} \in \{0, 1\}$

$$z_{ij} \geq x_j - m_{ij}$$
$$z_{ij} \geq 0$$

Practical Example