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Providing Incentives for Better Cost Forecasting

By

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PROVIDING INCENTIVES FOR BETTER COST FORECASTING

By

Kent Harold Osband

Richard J Gilbert

Abstract

The dissertation examines contractual schemes for eliciting information about an uncertain outcome. Such information is requested regularly in the course of central planning, government contracting, risk-analysis, exchange-rate prediction, weather forecasting, and in a variety of other contexts. In the basic framework, the agent, usually a manager, knows the probability distribution of a random variable (say "project cost", or "achievable output"), or perceives an unknown outcome as if it were a random variable with this distribution. The planner "principal" wants to obtain some summary measures of this information. After the reports are made, the outcome will be observed by both parties. The planner's problem is to devise a compensation scheme providing positive incentives to the manager to reveal his beliefs truthfully. Such a scheme is called a proper scoring rule.

The dissertation provides a virtually complete characterization of proper scoring rules for a broad class of probability indicators, thereby unifying and extending previous work on scoring rules. Some practical complications confronting potential users of these schemes are explored, including conditional project acceptance, risk-aversion, cost-padding dangers, multi-attribute evaluation, multi-agent bidding, and multi-period contracting. In addition, agent learning costs are incorporated into the model, allowing for the derivation of optimal contracts and explicit consideration of the costs of agency.

Acknowledgements

The seeds of the present work were sown in a seminar taught by Stefan Reichelstein in the spring of 1983. In searching for a final project, I more or less stumbled onto the solution of a problem he had been working on. This led to one joint paper, and a second one, and eventually to this dissertation. I thank Stefan for encouraging me to pursue these topics and for arranging financial support.

I had not originally intended to write such a "mathematical" dissertation, preferring instead to try to apply the incentive literature to analysis of comparative economic institutions. However, I was not capable of accomplishing in a year or two the unrealistic plans I had set for myself. At issue was not just my dissertation but also my illusions about myself. I thank my original advisor, Gregory Grossman, for tactfully helping me to see this. He has been very helpful to me over the last five years.

Richard Gilbert graciously took me on as advisee when I changed topics. As a student in his and Steve Goldman's graduate micro theory class and later as a teaching assistant for the same, we had fought "tooth and nail". Paradoxically, through fighting we became friends. I am glad he stuck with me.

I have had so many professors I could turn to for help, including some outside the department, that I am at a loss over whom to thank next. So let me put in a few kind words for U.C. Berkeley, which I

find to be a fantastic place. The atmosphere here is combination think tank, carnival, and battleground. Berkeley's tolerance for the ideas I brought with me helped open me up to their limitations.

Numerous friends and relatives have given me more emotional support than I deserve. Out of all these people, I would like to single out my father, Dr. Richard Osband, because in the past I wasn't very appreciative. This dissertation is dedicated to him, and to my mother Shirley Kanter Osband, although I don't expect them to follow the equations.

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CHAPTER 1: INTRODUCTION

This dissertation examines contracts for eliciting information about an uncertain outcome. Such information is requested regularly in the course of central planning, government contracting, risk-analysis, exchange-rate prediction, weather forecasting, and in a variety of other contexts. In the basic framework, the agent, usually a manager, knows the probability distribution of a random variable (say "project cost", or "achievable output"), or perceives an unknown outcome as if it were a random variable with this distribution. The planner "principal" wants to obtain some summary measures of the agent's beliefs, and therefore requests some reports. After the reports are received, the outcome will be observed by both parties. The planner's problem is to design a compensation scheme that encourages the agent manager to make sincere reports no matter what the manager's underlying beliefs are. In other words, the problem is one of providing incentives for forecasters.

The research discussed in these pages does not fall neatly into any one field. It draws on the theories of both economic planning and statistical decision-making. Unfortunately, the statisticians and economists have each as a group tended to be only vaguely familiar with the other group's work. As a result, major propositions have been independently discovered or rediscovered in each camp, without that knowledge crossing professional boundaries.

Perhaps the classic example of probabilistic forecasting occurs in meteorology. The future event is not necessarily deemed random, but insufficient data and limited data-processing capabilities make exact projections impossible. Since weather forecasters cannot influence the eventual outcome, there is rarely any incentive for them to withhold their true opinions; Hence elicitation per se is not a significant problem. However, some sort of evaluation or "scoring" rule is needed in order to rank different forecasters or forecasting methods. Any scoring rule in turn an be easily turned into an elicitation scheme, by offering the forecaster a payment based on the score.

Beginning in the 1950's, a number of authors (including Brier (1950), Holloway and Woodbury (1955), Gringorten (1958), Sanders (1958), Miller (1962), Epstein (1969), Murphy (1970), and Stael von Holstein (1970)) devised useful scoring rules for meteorology and explored their properties. Murphy and Epstein (1967) provide a brief review of prior literature. Inspiration for much of this work came from Bayesian statistical theory, whose stock in trade is precisely subjective probability.

Naturally the question of eliciting subjectively held statistics aroused a broader interest. McCarthy (1956) demonstrated the use of convex functions in devising scoring rules. According to Savage (1971), de Finetti independently rediscovered this insight and brought it to Savage's attention in 1960. The two of them worked on various aspects of the problem over the next decade (de Finetti (1962, 1965, 1970), Savage (1971)). In the last-mentioned article, Savage looked at elicitation schemes for expectations of arbitrary random variables (not just indicator variables on a partition, as previous studies had done), and explored necessary as well as sufficient conditions for a given scheme to be incentive compatible. Brown (1974), Matheson and Winkler (1976), and Haim (1982) used Savage's results to devise shemes for eliciting an entire continuous probability distribution.

Savage was aware of many economic applications for elicitation schemes. Unfortunately, few economists were aware of Savage's work. However, in the 1970's a parallel and in many respects overlapping literature on elicitation did emerge in economics, motivated by consideration of planning problems.

Economic planning shares with meteorology the problem of incomplete data and limited data-processing capabilities. However, an additional problem arises, as the person or organization called upon to make the forecast can often influence the event in question. Consider the endemic problem in developed capitalist countries of cost overruns in government contracts, particularly military procurement contracts. The source of the problem is easily identified: the advocates and/or beneficiaries of most government projects have strong incentives to understate anticipated costs, and much weaker incentives to keep costs down after the project is begun. In the widely used "cost-plus" contract, the firm will earn greater profit the higher its costs, as long as the bid is low enough to garner the project in the first place.

In Soviet-type economies the problem takes another form, but is no less severe. To encourage Soviet managers to achieve plan targets, bonuses are offered for fulfillment and overfulfillment.

Unfortunately, these bonuses have the side effect of encouraging managers to hoard supplies and understate production capabilities. Anticipating this reaction, Soviet planners tend to adjust managers' submitted targets upwards. This in turn prompts further hoarding and underreporting by managers. So, paradoxically, one of the greatest strengths of a centrally-planned economy -- its potential for easy transmission of information across enterprise boundaries -- is turned into a glaring weakness.

On account of these and similar problems, increasing attention in economic planning has been directed to questions of information acquisition and transfer. Malinvaud (1967), Hurwicz (1972, 1973), and Reiter (1974), among others, explored possibilities of using decentralization to reduce the amount of information transfer needed. Marschak and Radner (1972) examined the converse problem of coordinating actions given restricted or "noisy" communication. A limitation of these treatments is that they assumed superiors and subordinates share the same objectives. Dropping this assumption, Groves (1973), Groves and Loeb (1975), and Green and Laffont (1977) characterized mechanisms for inducing individuals to reveal their preferences for public goods. Hildebrandt and Tyson (1979), Reichelstein (1980), and Roberts (1979) examined the impact of various incentive considerations on decentralized planning procedures. Myerson (1979, 1981) characterized mechanisms for some classic bargaining problems using notions of Bayesian incentive compatibility.

Naturally, questions of eliciting probabilistic information were bound to arise. Weitzman (1976) demonstrated the theoretical validity of a proposed Soviet scheme for eliciting a specified quantile (e.g. the median) of a probability distribution. Bonin (1976) arrived at essentially the same schemes, by generalizing earlier work of Fan (1975). Thomson (1979) derived the most general class of scoring rules that would work under these conditions. Conn (1979) examined the possible use of Thomson rules in central planning. Bonin and Marcus (1979) and Miller and Murrell (1981) explored limitations of Bonin's previously derived schemes when reporting agents are able to influence outcomes. As yet unaware of Savage's work, Reichelstein and Osband (1984) turned to the elicitation of expectations. In Osband and Reichelstein (1984), they provided the first rigorous proof that incentive-compatible scoring rules for vector expectations must take the form indicated by Savage.

The present work begins where the last two articles leave off. The next chapter provides a virtually complete characterization of incentive-compatible scoring rules for any indicator of a distribution, thereby unifying and extending previous work on elicitation of particular subclasses of statistics. Chapter 3 explores some practical complications confronting potential users of these schemes: conditional contracting, risk-aversion, cost-padding possibilities, multi-attribute evaluation, and multi-agent bidding for contracts. Their impact on previously derived contracts is examined; where possible, contracts are modified to restore incentive compatibility. Elicitation in multi-period contracting is the subject of Chapter 4, where some curious non-existence results are derived and discussed. In Chapter 5, the assumption of costless information-gathering by the agent about the underlying distribution is relaxed. Expected costs to planners of information-gathering are compared against benefits, and optimal contracts are derived. Chapter 6 briefly summarizes the results and suggests avenues for future research.

In general, planners' informational requirements are taken as given, in order to focus on derivation and analysis of elicitation schemes. Therefore this work should be seen as complementary to work focusing on the value of communication, e.g. Melumad and Reichelstein (1985a, 1985b). Ultimately, one would like to determine both the optimal information needs and the optimal means of eliciting that information. These problems can probably not be solved independently, since different types of information may be more or less costly to elicit. It is hoped that the present work will provide a useful starting point for such research, by characterizing in detail the classes of scoring rules that planners can choose from.

CHAPTER 2: A GENERAL CHARACTERIZATION OF SCORING RULES

In this chapter, a general theory of the elicitation of information about a probability distribution is developed. Section 1 presents the general model. In Section 2, a special case of this model is examined -- elicitation of the mean -- in an effort to provide the reader with some intuitive understanding of the problem and methods for its solution. Core results for the general model are formally derived in Section 3, and applied to the derivation of a variety of scoring rules in Section 4. Section 5 considers elicitation of message functions not covered in Section 3. In Section 6, scoring rules are derived for restricted families of distributions.

1. The Model

Suppose a government principal is interested in financing a major R&D project, for which only one firm possesses the requisite expertise. Denote eventual project cost by x. We assume the agent firm views x as the outcome of a random variable, whose cumulative probability distribution $F(\cdot)$ it knows very well but cannot control. To clarify, let D be the cost range, assumed compact, with lowest feasible cost D_- and highest feasible cost D^+. When the agent is said to "know" the distribution of x, what is meant is that the agent's beliefs about cost can be described by a function of the form:

$$F: D \to [0,1], \quad F(D^+)=1, \quad F \text{ non-decreasing and right continuous}$$

where $F(x)$ is the perceived probability that cost will not exceed x. Implicit here are assumptions of careful government supervision and auditing, so that the firm cannot pad its costs or withhold effort.

To help plan the budget and estimate the funds remaining for other projects, the principal requests finitely many reports from the agent about $F(\cdot)$. Each of the principal's requests specifies a category of information (such as the mean), within which the agent is supposed to supply the appropriate value. Formally, the compression of broader knowledge into a limited number of categories can be described by a "message" function M, which maps any F in the set of distribution functions $\mathfrak{F}$ to a value in a subset Γ of R^n. By assumption, Γ is compact. M is said to induce a **partition** of $\mathfrak{F}$, since every member of $\mathfrak{F}$ is associated with one and only one member of $M(\mathfrak{F})$. *The set of F in $\mathfrak{F}$ for which $M(F)=W$ is called an* equivalence class and will be denoted $M^{-1}(W)$.

To get the agent's help in selecting $M(F)$, the principal offers her a payoff H that depends on the report W and on the future mutually observed outcome x.

$$H: \Gamma \times D \rightarrow R$$

The agent is assumed to be risk-neutral and to maximize expected payoff. Formally, the agent's problem is

$$\text{(2.1)} \qquad \text{choose } W \in \Gamma \text{ to maximize } \int_D H(W,x)\, dF(x),$$

where the integral is interpreted in the Riemann-Stieltjes sense to allow for discontinuities in $F(\cdot)$. [1] For notational convenience we will denote the expectation of $g(\cdot)$ under $F(\cdot)$ by $E^F[g(\cdot)]$, so that (2.1) can be rewritten as "choose $W \in \Gamma$ to maximize $E^F[H(W,x)]$."

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[1]For the integral to be well-defined, $H(W,\cdot)$ must be continuous almost everywhere in x. See Rudin (1976).

Note that any scheme $H(\cdot,\cdot)$ induces through (2.1) a correspondence M^o between $\mathfrak{F}$ and R^n. The principal's problem is to design $H(\cdot,\cdot)$ so as to infer $M(F)$ from the agent's revelation. Without loss of generality, we can confine ourselves to designs that induce $M(\cdot)$ directly. [2] Such contracts, or scoring rules as they are known in the statistical literature, will be called **proper**. Formally, $H:\Gamma\times D\to R$ is said to be proper for $M(\cdot)$ if

$$\text{(2.2)} \qquad \forall F\in\mathfrak{F}, \quad M(F) \in \arg\max_{\{W\in\Gamma\}} E^F[H(W,x)].$$

This definition conveniently assumes that the agent will report $M(F)$ when indifferent between $M(F)$ and other values. If argmax is unique for every $F(\cdot)$, the potential complication is avoided and the contract will be said to be **strictly proper**.

The central task of this paper is to characterize the set of proper scoring rules for a broad class of message functions.

2. A Special Case: Elicitation of the Mean

To clarify the nature of the problem and convey some intuition for its solution, let us begin by analyzing a special case: elicitation of the mean of a distribution; i.e., $M(F) = E^F[x]$. As mentioned in the previous chapter, this problem was first solved by

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[2]Suppose there exist functions $g:\Gamma\to\Gamma$ and $M^o:\mathfrak{F}\to\Gamma$ such that $g(M^o(F))=M(F)$ for all F in $\mathfrak{F}$. Then any $H^o(\cdot,\cdot)$ inducing $M^o(\cdot)$ could be used by the principal to infer $M(\cdot)$, simply by applying the mapping $g(\cdot)$ to the report. However, in such a case the principal could induce M directly with a scheme offering $H^o(W,x)$ whenever $g(W)$ is reported. This is one version of the "revelation principle", with $M(F)$ as the "true" characteristic.

Savage (1971). Let us see, hypothetically, how planners unaware of previous solutions might informally arrive at the same results . Such discussion, it is hoped, will make the formal and more general treatment in the next section much easier to follow.

Suppose our planners begin by examining a relatively simple set of contracts: those which are linear in ex-post cost. Instead of writing these in the usual way, let them express $H(W,x)$ in the form $R(W)-S(W)(W-x)$. Economically, the contract can now be viewed as the sum of two components: an ex-ante reward $R(W)$, and an ex-post fine amounting to $S(W)$ per unit difference between report and outcome. For true mean W^*, expected payoff is just $R(W)-S(W)(W-W^*)$.

If the agent, faced with this contract, were to report W^* truthfully, ex-ante reward would be $R(W^*)$ and expected ex-post fine would be zero. For the sake of comparison, suppose the agent exaggerated the report by h. The ex-ante reward would increase to $R(W^*+h)$, a gain of approximately $R'(W^*)h$ for small h, where ′ denotes the first derivative. A cost underrun of h would now be anticipated, however, imposing an expected ex-post fine of exactly $S(W^*+h)h$. For W^* to be the agent's best report, $S(W^*)$ must equal, and $S(W^*+h)$ must exceed, $R_w(W^*)$. Since this relationship must hold for all W^*, $S(\cdot)$ must be the derivative of $R(\cdot)$, and since $S(\cdot)$ is increasing $R(\cdot)$ must be convex.

As Savage (1971) showed, the same idea can be established by direct appeals to convexity. Expected payoff for report W and true mean W^* equals $R(W)-S(W)(W-W^*)$. For this to be maximized at $W=W^*$

for all W^*, it must be true that $R(W^*)-R(W)\geq S(W)(W^*-W)$ for all W^* and W. This can occur if and only if $S(\cdot)$ is everywhere a subgradient of a convex $R(\cdot)$.

Another way of seeing this is the following: By making its report, the agent chooses from a "menu" of contracts, each of which, $H(W,\cdot)$, can be viewed as a straight line in the (x,H) plane. For a distribution with all its weight on W^*, contract expected value is represented by the intersection of the contract line with the line $x=W^*$. Naturally, the agent chooses the contract line with the highest point of intersection. For incentive compatibility, this must be attained by the contract $H(W^*,\cdot)$, where the value is $R(W^*)$. Hence $R(W^*)$ is a point on the upper envelope -- convex by construction -- of all these contract lines. Since the choice of W^* was arbitrary, the upper envelope must be exactly $R(\cdot)$, so that $R(\cdot)$ must be convex. Furthermore, the optimal contract at $x=W^*$ must be tangent to the envelope there, which establishes its slope as a subgradient of $R(\cdot)$ at W^*. (See Figure 2.1.).

Insert Figure 2.1

Our hypothetical planners could easily generate more proper scoring rules by adding an arbitrary function $T(\cdot)$ of x to the proper scoring rules described above, since this addition will in no way affect incentive compatibility. However, should they try to enlarge this class further, they would not get very far. [5] Recall that the

Figure 2.1:

Constructing a Proper Scoring Rule

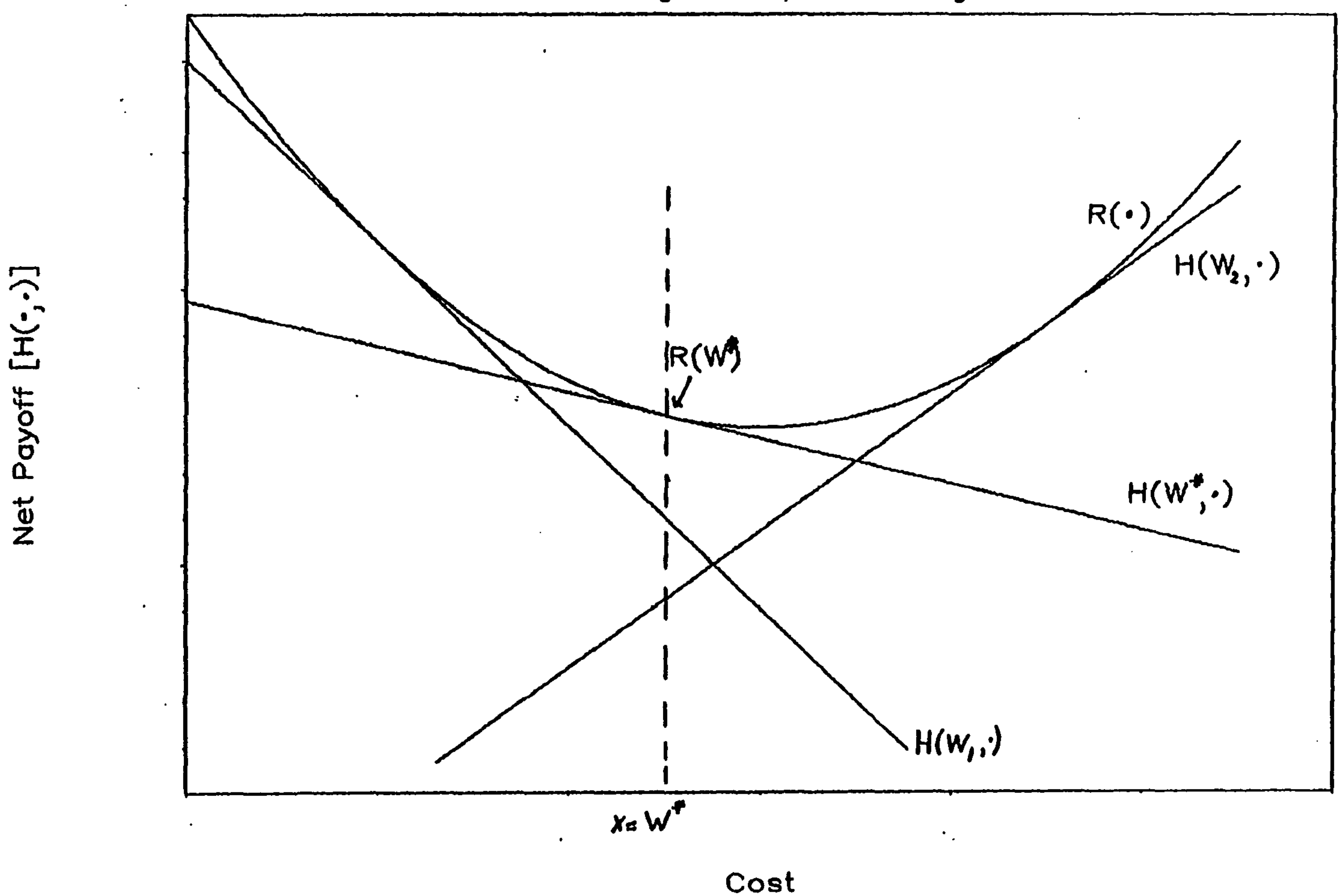
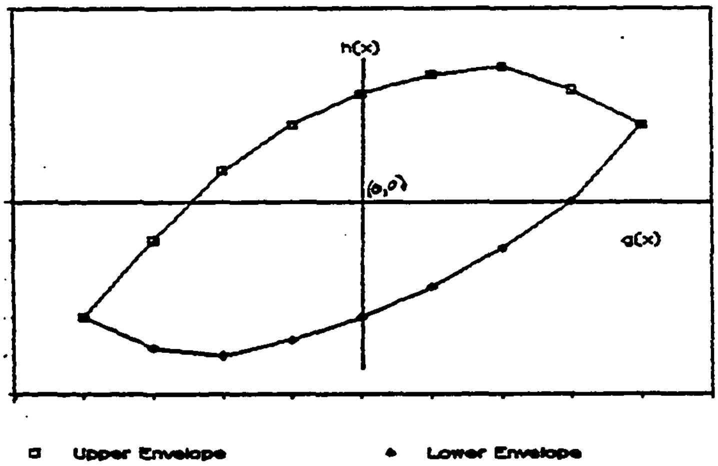
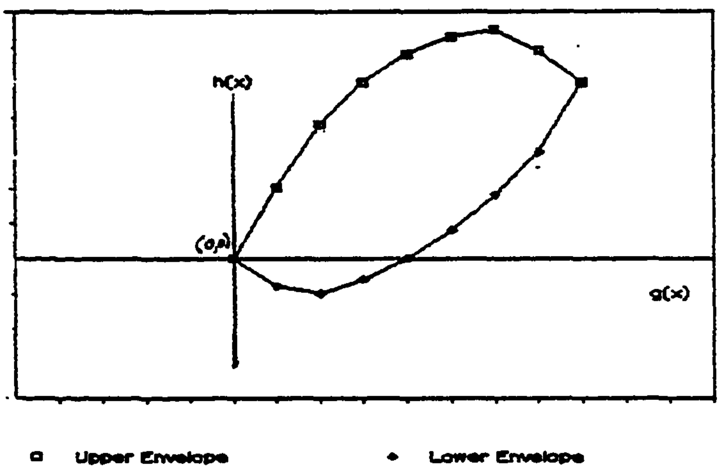


Figure 2.2: Convex Hull of <g(x),h(x)>

Case 1: Origin Interior to Hull



Case 2: Origin on Boundary of Hull



Case 3: Hull = Line Through Origin

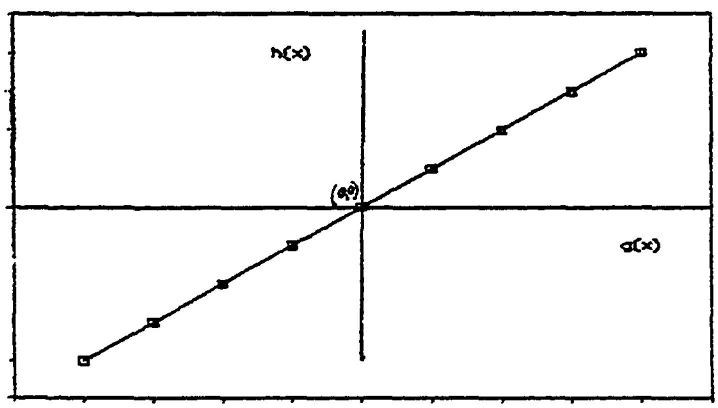
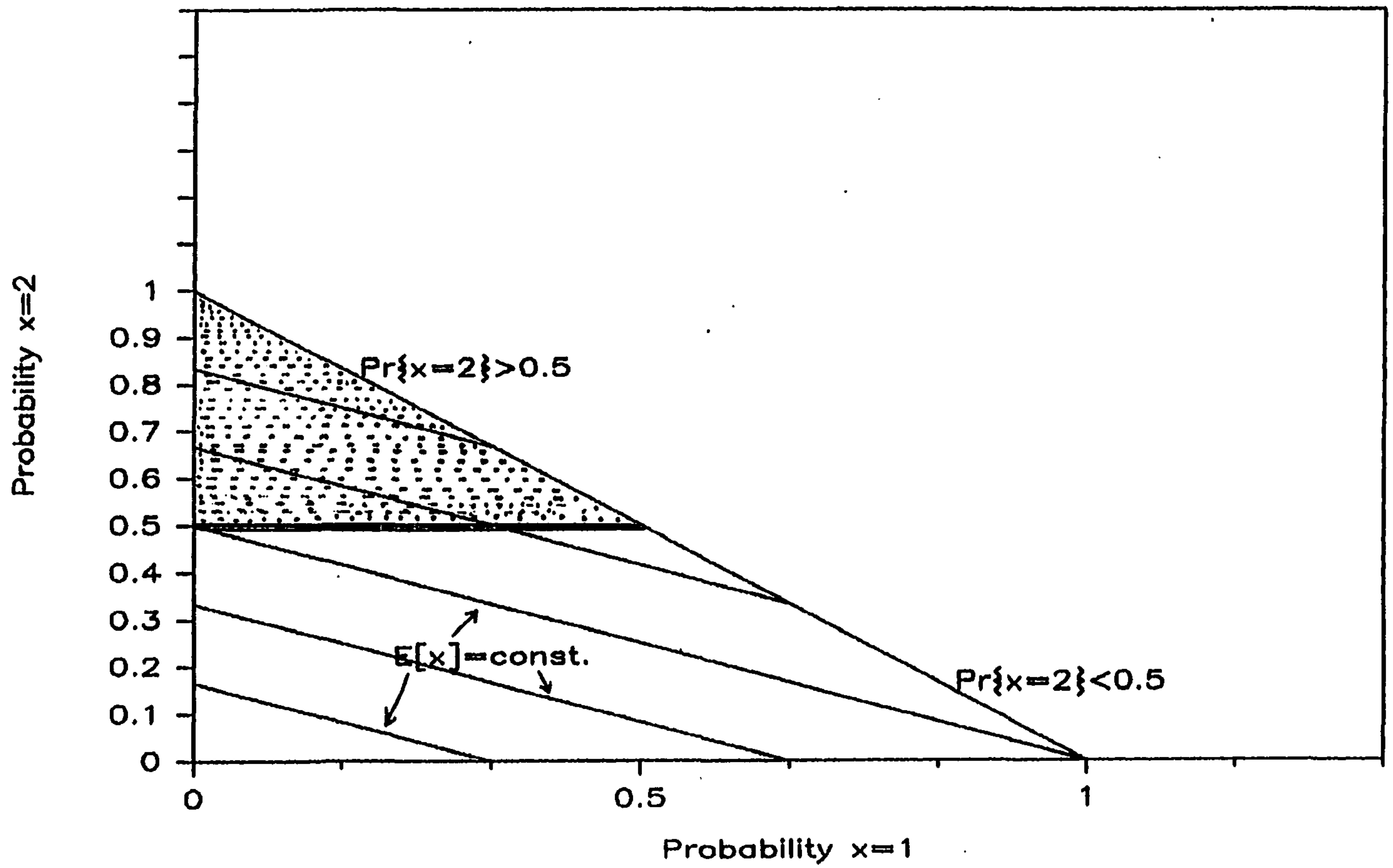


Figure 2.3:

Equivalence Classes when D={0,1,2}



agent picks W to maximize $E^F[H(W,x)]$. The principal wants $E^F[x]$ -- equivalently, the W^* satisfying $E^F[W^*-x]=0$ -- to be picked. If payoffs are sufficiently well behaved under differentiation, first order conditions for a proper contract can be written

$$(2.3)\qquad \frac{\partial}{\partial W}E^F[H(W^*,x)] = E^F[\frac{\partial H}{\partial W}(W^*,x)] = 0 \quad \forall F\in\Phi \text{ s.t. } E^F[W^*-x] = 0.$$

Treating W^* as a parameter, denote $\frac{\partial H}{\partial Y}(W^*,x)$ and W^*-x by $h(x)$ and $g(x)$ respectively. (2.3) says that the expected value of $h(\cdot)$ must be zero for all distributions such that the expected value of $g(\cdot)$ is zero. To see this, graph $h(x)$ against $g(x)$ for all x in D, so that each point on the graph represents a $\langle h(x),g(x)\rangle$ pair. The portion of the h-axis lying within the convex hull of these points represents all values of $E^F[h(x)]$ consistent with $E^F[g(x)]=0$. From (2.3), this portion includes at most the origin. Hence, if $g(\cdot)$ takes both positive and negative values, the convex hull must be a straight line; that is, $h(\cdot)$ must be a multiple of $g(\cdot)$. See Figure 2.2.

Insert Figure 2.2

Applying this result to the case at hand, we see that $\frac{\partial H}{\partial W}(W,x)$ must be a multiple $k(W)$ of $W-x$, as long as W is interior to D. Integration establishes that $H(\cdot,\cdot)$ is restricted to the form

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[3]Exceptions occur when the mean coincides with one of the endpoints, say, D_-. The only distribution compatible with this puts all its weight on D_-. Hence $H(D_-,x)$ is only weakly specified for $x\neq D^-$, with an upper bound to prevent misreporting but no lower bound.

described above. Summing up, we have that a differentiable $H(\cdot,\cdot)$ is proper if and only if it can be expressed in the form

$$(2.4) \qquad H(W,x) = R(W) - R'(W)(W - x) + T(x), \quad \text{with } R(\cdot) \text{ convex.}$$

3. A Generalization

The (informal) derivation of (2.4) suggests a generalization to a broader class of message functions. Suppose that, given a message function $M(\cdot)$, we could find a function $A(\cdot,\cdot)$ of W and x such that $M(F)=W$ if and only if $E^F[A(W,x)]=0$. If $H(\cdot,\cdot)$ were a proper scoring rule for $M(\cdot)$, the analogue of (2.3) would be "$E^F[H(W,x)]=0$ for all $F\in\mathfrak{F}$ such that $E^F[A(W,x)]=0$." Assuming $A(\cdot,\cdot)$ takes both positive and negative values, we could apply the arguments of the preceding section to show that $\frac{\partial H}{\partial W}$ has to be a W-based multiple of $A(W,x)$. Perhaps we could even extend the results to encompass a vector W and a vector function $A(\cdot,\cdot)$. This section provides such an extension.

Let us denote by Ω the family of message functions which can be be associated with an $A(\cdot,\cdot)$ in the manner described above. Formally,

$$(2.5) \qquad \Omega = \{M:\mathfrak{F}\to\Gamma \mid \exists A:\Gamma\times D\to R^n \text{ s.t. } E^F[A(M(F),x)] = 0 \ \forall F\in\mathfrak{F}\},$$

where the expectation of a vector function is defined as the vector of the expectation of its components. $M^{-1}(W)$, the equivalence class of W, consists of those $F\in\mathfrak{F}$ for which $\int_D A(W,x)dF(x) = 0$; that is, of all distributions with "densities" in the kernel of $A(W,\cdot)$. Occasionally the notation M_A will be used to denote an M associated with A. Clearly $A(W,\cdot)$ is not uniquely defined for members of Ω: one could as well use $L(W)A(W,\cdot)$ for any non-zero-valued function $L(\cdot)$.

While Ω does not include every message function, it does include functions for the two most commonly used indicators: quantiles (e.g. the median) and expectations. If the desired message is the expectation of a vector function $a:D\to R^n$, define $A(W,x) \equiv W-a(x)$. The expectation of $A(W,\cdot)$ under $F(\cdot)$ is then $W-E^F[a(x)]$, which is zero if and only if $E^F[a(x)]=W$. For example, the message function associated with $A(W,x) = [W_1-x \quad W_2-x^2]^T$ separates probability distributions into equivalence classes according to their first and second moments.[4] If the message is the "q-quantile", the W for which $F(W)=q$, define $A(W,x) \equiv I_W(x)-q$, where the indicator function $I_W(x)$ equals 1 if $x\leq W$ and 0 otherwise. Here the expectation of $A(W,\cdot)$ under $F(\cdot)$ equals $F(W)-q$. (For the "q-quantile" message function to be well-defined, one must exclude from Φ those distributions for which $F^{-1}(q)$ is nonexistent or nonunique.). Ω includes other types of message functions as well. For example, $A(W,x) \equiv r(x)-Ws(x)$ corresponds (assuming $s(\cdot)$ takes only positive or only negative values) to a mapping of $F(\cdot)$ onto the ratio of the expectations of $r(\cdot)$ and $s(\cdot)$.

The next step is to characterize proper scoring rules for members of Ω. Suppose that the partial derivative of $H(\cdot,\cdot)$ with respect to W_i, the i-th component of W, is sufficiently well-behaved to allow the order of differentiation and integration to be reversed; i.e., the derivative of the expected payoff equals the derivative of the

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[4]The same equivalence classes would be induced by $A(W,x) = [W_1-x \quad W_2-W_1^2-x^2]$, only in this case the classes would be indexed according to mean and variance instead of mean and second moment.

expected payoff. Incentive compatibility then requires that $E^F[\frac{\partial H}{\partial W_i}(W,x)]=0$ for all distributions $F(\cdot)$ such that $E^F[A(W,x)]=0$. The following lemmas and theorem establish that $\frac{\partial H}{\partial W}(W,x)=K(W)\bullet A(W,x)$, for some vector function $K:\Gamma\to R^n$, provided the class of distributions for which the expectation of $A(W,x)$ is zero is sufficiently rich. Algebraically, "sufficiently rich" requires that for every F_1 in $\mathfrak{F}$, $\alpha F_1+(1-\alpha)F_2$ belongs to $M^{-1}(W)$ for some F_2 in $\mathfrak{F}$ and some α in $[0,1]$. For a geometric interpretation, consider the convex hull of $\{A(W,x)|x\in D\}$. The condition states that the origin must not merely lie in the hull (required for $M^{-1}(W)$ to be non-empty), but within its interior.

Lemma 2.1: Let C be the convex hull of $x_1,\ldots,x_t\in R^n$, and let Δ^t be the unit simplex in R^t. If z is interior to C, there exists a strictly positive $\theta=(\theta_1,\ldots,\theta_t)\in\Delta^t$ such that $\sum_1^t\theta_i x_i=z$.

Proof: For $\epsilon>0$, define w to satisfy $(1-t\epsilon)w+\epsilon\sum_1^t x_i=z$. For ϵ sufficiently small, w belongs to C, so that $w=\sum_1^t\mu_i x_i$ for some $\mu=(\mu_1,\ldots,\mu_t)\in\Delta^t$. Then θ defined by $\theta_i=(1-t\epsilon)\mu_i+\epsilon$ for i=1 to t possesses the desired properties.∎

Lemma 2.2: For $g:D\to R^n$, suppose $0\in$ int co $g(D)$. If $h:D\to R$ satisfies $E^F[h(x)]=0$ $\forall F\in\mathfrak{F}$ such that $E^F[g(x)]=0$, then $h(x)\equiv k\bullet g(x)$ for some vector $k\in R^n$.

Proof (Adapted from Osband and Reichelstein (1984)): Imbed 0 in the interior of a simplex whose vertices all lie in co g(D). By Caratheodory's theorem (see Rockafellar (1970)), each vertex in turn must lie in another simplex with vertices in $g(D)$. By construction, the convex hull of all these $(n+1)^2$ or fewer points in $g(D)$ contains 0 in its interior. Denote these points $g(x_1),..,g(x_s)$, and define G an $n\times s$ matrix with (i,j) element $g_i(x_j)$. G must have rank n; otherwise the interior of the convex hull of $\{g(x_1),..,g(x_s)\}$ would be empty. Furthermore, by lemma 1, there exists a strictly positive $\theta^{oo}\in\Delta^s$ such that $G\cdot\theta^{oo}=0$.

Add an arbitrary point $z\in D$ and consider the $n\times(s+1)$ matrix $B_z = [g(z)\ G]$. Applying Lemma 1 again, there exists a strictly positive $\theta^o\in\Delta^{s+1}$ such that $B_z\cdot\theta^o=0$. Finally, define the $(n+1)\times(s+1)$ matrix

$$C_z = \begin{vmatrix} h(z) & h(x_1)\cdots h(x_s) \\ & B_z & \end{vmatrix}$$

By hypothesis, $C_z\cdot\theta=0$ whenever $B_z\cdot\theta=0$ for $\theta\in\Delta^{s+1}$. Indeed, this must be true for all positive θ in R^n, since $C_z\cdot\theta=0$ implies $C_z\cdot\alpha\theta=0$ for arbitrary α. Hence $P\in\text{Kernel}(B_z)$ implies $P\in\text{Kernel}(C_z)$, because $\theta^o+\epsilon P\gg0$ for ϵ sufficiently small and $C_z\cdot(\theta^o+\epsilon P)=0$. Conversely, $C_z\cdot P=0$ implies $B_z\cdot P=0$. Therefore rank C_z = rank B_z = rank $G = n$, so that $h(z)-k\cdot g(z) = h(x_i)-k\cdot g(x_i) = 0$ for some row vector $k\in R^n$. Since z is arbitrary and k is unique (otherwise rank $G < n$), the lemma is shown.∎

Note that Lemma 2.2 must remain true when ξ is restricted to distributions with finite support, since only the latter were used in the proof given above. It is conjectured that the Lemma holds for almost all x if ξ is restricted to the class of continuous distributions. It is conjectured that the proof can be further simplified by applying a functional space extension of the Kuhn-Tucker Theorem, with the caveat that $h(x)$ and $k\cdot g(x)$ may differ on a set of measure zero. The basic idea is as follows. Consider the problem

(2.6) Choose $f(\cdot)$ to minimize $\int_D h(x)f(x)dx$

subject to

$\int_D g_i(x)f(x)dx = 0$ for $i=1,..,n$ and $f(x)\geq 0$ for all x in D.

This can be formulated as a Lagrangean, with multipliers k_i $(i=1,..,n)$ on the first n constraints and $m(x)$ on each constraint $f(x)\geq 0$. First-order conditions for interior solution require that $h(x) = k\cdot g(x)+m(x)$, with complementary slackness constraints $m(\cdot) \geq 0$, $f(\cdot) \geq 0$, and $\int_D m(x)f(x)dx = 0$. In general, (2.6) will not have an interior solution, but it must for the conditions of Lemma 2.2 to be satisfied. Indeed, every feasible $f(\cdot)$ must be a solution. If $0 \in \text{int co } g(D)$, there exist many feasible, strictly positive $f(\cdot)$. For these $f(\cdot)$, $m(\cdot)$ must be zero almost everywhere (everywhere if $h(\cdot)$ is continuous), which proves the lemma.

Theorem 2.1: A proper scoring rule $H(\cdot,\cdot)$ for $M_A(\cdot)$ in $\mathfrak{A}$ must satisfy, wherever it is differentiable in W,

(2.7) $\frac{\partial H}{\partial W_i}(W,x) = K_i(W)\cdot A(W,x)$ for $i=1,..,n$

for some vector functions $K_i:\Gamma\to R^n$, provided the convex hull of $\{A(W,x)|x\in D\}$ includes 0 in its interior. Furthermore, the nxn matrix with i,j-component

(2.8) $K_i(W)\cdot\frac{\partial}{\partial W_j}E^F[A(W,x)]$

must be negative semi-definite for all F in $M_A^{-1}(W)$. If the matrix defined by (2.8) is negative definite and $K(\cdot)$ is smooth, the scoring rule defined by (2.7) is proper.

Proof: For a differentiable scoring rule to be proper, first-order conditions require that

$$(2.9) \qquad \frac{\partial}{\partial W_i}\left[\int_D H(W,x)dF(x)\right] = 0 \text{ for } i=1,..,n,\ \forall F \in M^{-1}(W)$$

If $F(\cdot)$ has finite support, the integral is a finite sum, so that the order of differentiation and integration can be reversed. (If $F(\cdot)$ is continuous, the order can be reversed provided $\frac{\partial H}{\partial W_i}$ is uniformly continuous on the feasible set. See Rudin (1976), pp. 236-7.). (2.9) then becomes

$$(2.10) \qquad \int_D \frac{\partial H}{\partial W_i}(W,x)dF(x) = E^F\left[\frac{\partial H}{\partial W_i}(W,x)\right] = 0 \quad \text{for } i=1,..,n,\ \forall F \in M^{-1}(W)$$

Condition (2.7) is obtained by applying Lemma 2.2 to (2.10). (2.8) evaluates the expected payoff Hessian at its maximum $W=M_A(F)$. Second-order conditions require the matrix to be negative semi-definite. Negative definiteness ensures a local maximum; it also ensures that $K(\cdot)$ has full rank. Hence $\int_D H(W,x)dF(x)=0$ has only one stationary point, and the local maximum is global.∎

4. Applications

To solve the differential equations given by Theorem 1, more information is needed about the $A(\cdot,\cdot)$ associated with the particular message function. The following applications indicate the versatility and power of the theorem. In every case, the distribution $F(\cdot)$ is presumed to be such that the convex hull of $\{A(M(F),x) \mid x \in D\}$ contains 0 in its interior, and $H(\cdot,\cdot)$ is presumed to be continuous and piecewise differentiable in W. We begin with a proposition first proved in Osband and Reichelstein (1984).

Proposition 2.1: $H(\cdot,\cdot)$ is proper for eliciting the expectation of a vector $a(x)$ if and only if it takes the form

$$(2.11) \qquad H(W,x) = R(W) - \nabla R(W) \cdot [W - a(x)] + T(x),$$

where $R(\cdot)$ is convex and $\nabla R(W)$ is the gradient of $R(\cdot)$ at W. (with $\nabla R(W)$ interpreted as any subgradient of $R(\cdot)$ at W in case of kinks).

Proof: From Theorem 2.1, $\frac{\partial H}{\partial W_1}(W,x)$ takes the form $K_1(W) \cdot (W-a(x))$. Integrate $K_1(W) \cdot (W-a(x))$ with respect to W_1 to show that $H(\cdot,\cdot)$ takes the form $L(W) + M(W) \cdot a(x) + N(x,W_{-1})$, where $N(x,W_{-1})$ denotes the constant of integration. Differentiate this expression with respect to W_2 and equate to $K_2(W) \cdot (W-a(x))$. The equality takes the form:

$$(2.12) \qquad P(W) + Q(W) \cdot a(x) = \frac{\partial N}{\partial W_2}(x,W_{-1}).$$

The right-hand side of (2.12) is independent of W_1, and since the vectors $\{a(x)\}_{x \in D}$ lie in a set of dimension n, it follows that both $P(\cdot)$ and $Q(\cdot)$ are independent of W_1. Integration of (2.12) with respect to W_2 shows that $N(x,R_{-1})$ can be expressed $P^o(W)+Q^o(W) \cdot a(x)+N^o(x,W_{-1,2})$. Repeating this argument with respect to $W_3,\ldots,W_n$ yields the form $R(W)-S(W) \cdot [W-a(x)]+T(x)$ for $H(\cdot,\cdot)$. Incentive compatibility requires that $R(W^*) \geq R(W)-S(W) \cdot (W-W^*)$ for all W and W^*, implying that $R(\cdot)$ is convex and $S(W)$ is a subgradient of $R(\cdot)$ at W. Conversely, every scoring rule satisfying these conditions is proper.∎

Corollary 2.2.1: If the feasible set D consists of $t+1$ distinct values, then $H(\cdot,\cdot)$ is proper for eliciting the probabilities $P\equiv(P_1,..,P_t)$ of values 1 through t occurring if and only if it takes the form

$$\begin{aligned} (2.13)\qquad H(P,x_i) &= R(P) - \nabla R(P)\bullet P + \frac{\partial R}{\partial P_i}(P) \quad \text{for } i=1,..,t \\ &= R(P) - \nabla R(P)\bullet P \qquad\qquad \text{for } i=t+1 \end{aligned}$$

with $R(\cdot)$ convex.

Proof: Define $J(\cdot)$ as a t-vector of indicator functions such that $J_i(x)$ equals 1 if x is the i-th value in D and 0 otherwise. Eliciting the expectation of $J(\cdot)$ amounts to eliciting the entire vector of probabilities. From Proposition 2.2, these schemes must take the form $R^o(P)-\nabla R(P)\bullet(P-J(x))+T(x)$, where $R^o(\cdot)$ is convex. Define $R(P) \equiv R^o(P) + \sum_1^t (T(x_i)-T(x_{t+1}))P_i + T(x_{t+1})$. Substitution yields (2.13).∎

Complete elicitation in the case of interval D requires infinitely many reports and hence falls beyond the scope of this investigation. The general problem is solved in Haim (1982). The solution, not surprisingly, preserves the basic features of the scoring rule above, with the convex function defined on a space of bounded measurable functions instead of Euclidean t-space.

We now turn to elicitation of quantiles, providing a simple proof of a proposition first discovered by Thomson (1979).

Proposition 2.2: $H(\cdot,\cdot)$ is proper for eliciting the q-quantile of a distribution if and only if it takes the form, for almost all x, of

$$(2.14)\qquad \begin{aligned} H(W,x) &= -qS(W) + S(W) + T(x) \quad \text{for } x\leq W \\ &= -qS(W) + S(x) + T(x) \quad \text{for } x>W \end{aligned}$$

where $S(\cdot)$ is non-increasing.

Proof: From Theorem 2.1 (modifying for the case of continuous distributions), $\frac{\partial H}{\partial W}(W,x) = K(W)[I_W(x)-q]$ for almost all x, wherever the partial derivative of W exists, with $K(\cdot)\leq 0$. Integrating for $x\leq W$, we have $H(W,x) = (1-q)S(W) + T(x)$ where $S(\cdot)$ is an indefinite integral of $K(\cdot)$; for $x>W$ we have $H(W,x) = -qS(W) + T_1(x)$. Expected payoff $E^F[H(W,x)]$ equals

$$S(W)[F(W)-q] + \int^W T(x)dF(x) + \int_W T_1(x)dF(x),$$

which is maximized for almost all W at the W satisfying

$$K(W)[F(W)-q] + S(W)f(W) + T(W)f(W) - T_1(W)f(W) = 0.$$

For incentive-compatibility, F(W) must equal q, implying $T_1(x) = T(x)+S(x)$ for almost all x. Substitution yields (2.14).∎

We now advance into new terrain. Suppose it is desired to elicit both the q-quantile and some other message $M_A(F)$, where $M_A(\cdot)$ belongs to Ω (e.g. the mean, or another quantile). One obvious way to do this would be to add together proper rules for eliciting each of the messages separately. But we might also expect to find some non-separable rules, analogous to those generated in the case of vector expectations by non-separable convex functions. It turns out, however, that no such rules exist.

Proposition 2.3: Proper scoring rules for eliciting the q-quantile and any other message generated by a member of Ω must be separable in the two reports.

Proof: Let W be the report of the q-quantile, and Y be the report of another message, associated with some function A(Y,x). Integrate $\frac{\partial H}{\partial Y} = K(W,Y)(I_w(x)-q) + L(W,Y)A(Y,x)$ to obtain

$$\begin{aligned} H(W,Y,x) &= (1-q)M(W,Y) + \int L(W,Y)A(Y,x)dY + N_1(W,x) \quad \text{for } x \le W \\ &= \quad - qM(W,Y) + \int L(W,Y)A(Y,x)dY + N_2(W,x) \quad \text{for } x > W; \end{aligned}$$

where $\frac{\partial M}{\partial Y}(W,Y) = K(W,Y)$ and $N(W,x)$ denotes the constant of integration. Taking the partial derivative of expected payoff with respect to W, we have at the true reports

$$\frac{\partial}{\partial W}E^F[H(W,Y,x)] = [M(W,Y) - N_1(W,W) + N_2(W,W)]f(W) + \int \frac{\partial L}{\partial W}(W,Y)A(Y,x)dY = 0.$$

For this to be zero for at least two different values of f(W) and for all values of W and Y, we must have both $M(W,Y) = N_2(W,W)-N_1(W,W)$ and $\frac{\partial L}{\partial W}(W,Y)=0$. That is, $M(W,\cdot)$ must be independent of Y and $L(\cdot,Y)$ must be independent of W. ∎

Some messages, like quantiles and expectations, have closed form expressions. Others do not. This need not be an obstacle to elicitation, however. For loss function $\ell(W,x)$, suppose planners are interested in minimizing expected loss. One obvious way to elicit the desired message is to simply fine the agent some multiple of $\ell(W,x)$.

To uncover the whole class, note that the desired W satisfies $\int_D \frac{\partial \eta}{\partial W}(W,x)dF(x) = 0$, provided $\eta(\cdot,\cdot)$ is sufficiently well-behaved. We can then apply Theorem 2.1, with $A(W,x) = \frac{\partial \eta}{\partial W}(W,x)$.

For example, suppose planning losses are incurred of

$$a_1(W-x)^2 + b_1(W-x) \quad \text{if } x \leq W$$

$$a_2(W-x)^2 + b_2(x-W) \quad \text{if } x > W$$

where $a_2>a_1>0$ and $b_2>b_1>0$, so that cost overruns are more harmful than cost underruns. It is easily shown that proper scoring rules must take the form:

$$2a_1R(W) - 2a_1R'(W)(W-x) - b_1R'(W) + T(x) \quad \text{for } x \leq W$$

$$2a_1R(W) - 2a_1R'(W)(W-x) + b_2R'(W) + T^o(x) \quad \text{for } x>W;$$

with $R(\cdot)$ convex. From the continuity of net payoffs at $x=W$, it follows that $T^o(x)=T(x)+2(a_1-a_2)R(x)-(b_1+b_2)R'(x)$. These conditions ensure incentive-compatibility. The payoff $-\eta(W,x)$ can be generated by $R(W)=0.5W^2$, $T(x)=-a_1x^2$.

5. Message functions outside Ω

While Ω is a very broad class of message functions, it is by no means all-inclusive. In this section we explore possibilities of developing proper scoring rules for message functions outside Ω. Fortunately, our task is simplified by the following two propositions.

Proposition 2.4: *If expected payoff under $F(\cdot)$ of a smoothly differentiable proper scoring rule $H(\cdot,\cdot)$ never has a stationary point other than at $W=M(F)$, for all F, $M(\cdot)$ must belong to Ω.*

Proof: First-order conditions require that $E^F[\frac{\partial H}{\partial W_i}(W,x)] = 0$ for $i=1$ to n, for all F in $M^{-1}(W)$ and for all W in Γ. Define $A_i(W,x) \equiv \frac{\partial H}{\partial W_i}(W,x)$. Since the stationary point is unique, $E^F[A(W,x)]=0$ if and only if $W=M(F)$.∎

Proposition 2.5: No strictly proper scoring rules exist for message functions with non-convex equivalence classes.

Proof: A strictly proper scoring rule $H(\cdot,\cdot)$ for message function $M(\cdot)$ must satisfy

$$(2.15) \qquad \int_D [H(W,x)-H(W^o,x)]\, dF(x) > 0 \quad \forall F\in M^{-1}(W),\ \forall W^o\neq W,\ \forall W\in\Gamma.$$

If (2.15) holds for F_1 and F_2 in $M^{-1}(W)$, it holds for $\alpha F_1+(1-\alpha)F_2$ given any $\alpha\in[0,1]$, so $M^{-1}(W)$ must be convex.∎

As an application of Proposition 2.5, consider the problem of eliciting the variance of a distribution. If we are allowed two reports, this is easily done: for example, apply Proposition 2.1 to the case $a(x) = [x\ x^2]$, and use the reports of the first and second moments to calculate the variance. Suppose, however, we are restricted to a single report. [1] Consider two distributions $F_1,F_2\in\Phi$ with identical variance σ^2 but different means μ_1,μ_2. The convex combination $0.5(F_1+F_2)$ will have mean $0.5(\mu_1+\mu_2)$ and second moment $0.5(\mu_1^2+\mu_2^2+2\sigma^2)$, so that its variance equals

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[1]The term "a single report" is used here in the conventional sense of one symbol per equivalence class. If this restriction is not imposed, a single report might convey multiple bits of information, e.g. have the odd-numbered digits indicate the first moment and the even-numbered ones the second moment.

$\sigma^2+0.25(\mu_1-\sigma_2)^2\neq\sigma^2$. We have thus established:

Corollary 2.5.1: No strictly proper scoring rules exist for eliciting the variance of a distribution by means of a single report.

For an example of a message function outside Ω with convex equivalence classes, consider the partition of $\mathfrak{F}$ into two classes "0" and "1" according to whether or not $F(x_o)$ exceeds 0.5, for some $x_o \in D$. One proper scoring rule for this message function sets $H(0,x)=0$ for all x, $H(1,x)=1$ for $x\leq x_o$, and $H(1,x)=-1$ for $x>x_o$.

Algebraically, the difference between the partitions induced by members of Ω and those induced by *elicitable functions outside* Ω is the following. For elicitable functions outside Ω, F_1 and F_2 belonging to the same equivalence class implies $\alpha F_1+(1-\alpha)F_2$ also belongs to that class, for all α between 0 and 1. For members of Ω, this property holds <u>regardless</u> of α, provided $\alpha F_1+(1-\alpha)F_2$ belongs to $\mathfrak{F}$. Another way of expressing this is that for members of Ω, the convex combination of two distributions in different equivalence classes must belong to a third class. Clearly this need not be true for convex equivalence classes generally.

For a feasible set D consisting of three points -- say, 0, 1, and 2 -- the distinction can be made graphic. Let the horizontal coordinate indicate the probability p_1 of $x=1$ occurring, and the vertical coordinate indicate the probability p_2 of $x=2$ occurring. The space of all probability distributions $\mathfrak{F}$ is represented by a triangle with axes (0,0), (1,0), and (0,1), with the probability of $x=0$ occurring represented by the distance from $\langle p_1,p_2\rangle$ to the line

$p_1+p_2=1$. The message function mapping $F(\cdot)$ onto its expectation forms equivalence classes according to the value p_1+2p_2. Hence, the partition slices the triangle into straight line segments of slope -1/2. Now consider an additional partition dividing line segments into two sections according to whether or not p_2 exceeds 1/2. The resulting equivalence classes are still convex, but now their linear extensions overlap within the triangle. Hence the message function inducing this partition cannot belong to Ω.

Insert Figure 2.3

Fortunately, the techniques developed for elicitation of members of Ω can still be of use. Consider any refinement of the partition (preferably the coarsest) that reestablishes equivalence classes appropriate for a member of Ω. Using Theorem 2.1, derive general scoring rules for the new message function. Then eliminate all those which distinguish scores on the refinement. The remaining rules, and only those, are proper. Formally, M^* is a refinement for M if there exists a function $s:\Gamma\rightarrow\Gamma$ such that $s(M^*(F))=M(F)$ for all F in $\mathfrak{F}$. It follows that $H(s(\cdot),\cdot)$ is proper for M if and only if (a) $H(\cdot,\cdot)$ is proper for M^*, and (b) $H(W,x)=H(W^o,x)$ for all $W,W^o\in\Gamma$ such that $s(W)=s(W^o)$.

In the example above, the only suitable refinement separates the individual points within the triangle. Let $J(x)$ be the indicator function taking a value of 1 when $x=2$ and 0 otherwise; let Y denote the reported $E^F[J(x)]$ and W denote the reported $E^F[x]$. Scoring rules for W and Y take the form

$$S(W,Y) - \frac{\partial S}{\partial W}(W,Y)(W - x) - \frac{\partial S}{\partial Y}(W,Y)(Y - J(x)) + N(x)$$

for $S(\cdot,\cdot)$ convex. The score must be the same for all $P>1/2$, implying that

$$(2.16) \qquad \frac{\partial^2 S}{\partial W \partial Y}(W,Y)(W - x) + \frac{\partial^2 S}{\partial Y^2}(W,Y)(Y - J(x)) = 0 \quad \forall W \in \Gamma, x \in D.$$

Comparing the value of (2.16) for different values of x, $S(\cdot,\cdot)$ is seen to be separable in W and Y and linear in Y. Substitution establishes that $H(W,x)$ takes the form

$$R(W) - R'(W)(W-x) + T(x) \quad \text{for } Y \leq 1/2$$

$$R(W) - R'(W)(W-x) + T^o(x) \quad \text{for } Y > 1/2$$

where $R(\cdot)$ is convex. For Y to be selected correctly, $T^o(x)$ must equal $T(x)-s$ for $x = 0$ or 1 and $T(x)+s$ for $x=2$, where $s \geq 0$.

6. Scoring Rules for Restricted Families of Distributions

When the principal has additional information about $F(\cdot)$ other than that reported by the agent, a richer class of proper scoring rules can often be found. The reason for this is that proper scoring rules no longer need be incentive-compatible for all members of $\mathfrak{F}$, but only for members of a restricted family. Clearly, not every restriction will serve to widen the class of proper scoring rules. Consider, for example, the exclusion of distributions supported by less than k points: Lemmas 2.1 and 2.2 can easily be re-proved using only distributions supported by k points or more, so that Theorem 2.1 goes through as before. At the other extreme, knowledge that $F(\cdot)$

puts all its weight on one point opens up an enormous class of proper scoring rules, as the outcome provides complete information. What about intermediate cases, such as elicitation of the mean when the variance σ^2 is known? As we shall see, Theorem 2.1 and its supporting lemmas prove useful here too.

To pursue our example, recall that, with $F(\cdot)$ *unrestricted,* first-order conditions for elicitation of a mean require that

$$(2.17)\qquad \forall W,\quad E^F[\tfrac{\partial H}{\partial W}(W,x)] = 0 \quad \forall F\in\mathfrak{F} \text{ s.t. } E^F[W-x] = 0.$$

With $F(\cdot)$ restricted to distributions with variance σ^2, (2.17) is replaced by

$$(2.18)\qquad \forall W,\ E^F[\tfrac{\partial H}{\partial W}(W,x)]=0 \quad \forall F\in\mathfrak{F} \text{ s.t. } E^F[W-x]=E^F[W^2+\sigma^2-x^2]=0.$$

Applying Lemma 2.2, $\frac{\partial H}{\partial W}(W,x)$ must take the form $K_1(W)(W-x) + K_2(W)(W^2+\sigma^2-x^2)$. From here, derivation is straightforward: integrate with respect to W, establish second-order conditions, and check the global properties of local maxima. The following theorem generalizes this approach.

Theorem 2.2: Suppose, as in Theorem 2.1, that elicitation of $M_A(F)$ is desired, for some $M_A(\cdot)$ in Ω. However, let the permissible $F(\cdot)$ be restricted to those probability distributions satisfying $M_B(F)=C(M_A(F))$, for some $M_B(\cdot)$ in Ω and some function $C:\Gamma\to\Gamma$. Then a differentiable proper scoring rule must be expressable in the form

$$(2.19)\qquad \frac{\partial H}{\partial W_1}(W,x) = K(W)\bullet A(W,x) + L(W)\bullet B(C(W),x)$$

for all W such that $0 \in \text{int co } \{\langle A(W,x),B(C(W),x)\rangle \mid x\in D\}$. Furthermore, for those W, the nxn matrix with $\langle i,j\rangle$-element

$$(2.20) \qquad K_1(W)\ \frac{\partial}{\partial W_j}E^F[A(W,x)] + L_1(W)\ \frac{\partial}{\partial W_j}E^F[B(C(W),x)]$$

must be negative semi-definite.

Proof: First-order conditions for properness require that $\frac{\partial H}{\partial W}(W,x)$ vanish for all $F\in\mathfrak{F}$ satisfying

$$(2.21) \qquad E^F[A(W,x)]=0 \text{ and } E^F[B(C(W)),x]=0.$$

Invoke Lemma 2.2 to show that $\frac{\partial H}{\partial W}$ takes the form $K(W)A(W,x) + L(W)B(C(W),x)$. Negative semi-definiteness of the expected payoff Hessian at its maximum $W=M_A(F)$ together with (2.21) implies (2.20]).∎

Corollary 2.2.1: If W is a scalar and C is differentiable and strictly monotonic, proper scoring rules take the form

$$(2.22) \qquad H^1(W,x) + H^2(C(W),x),$$

where $H^1(\cdot,\cdot)$ and $H^2(\cdot,\cdot)$ satisfy first-order conditions for scoring rules for $M_A(\cdot)$ and $M_B(\cdot)$ respectively over unrestricted distributions. If $H^1(\cdot,\cdot)$ and $H^2(\cdot,\cdot)$ are proper for $M_A(\cdot)$ and *$M_B(\cdot)$ without restriction, scoring rule (2.22) is proper for M_A with restriction.*

Proof: Define $H^1(W,x)=\int K(W)A(W,x)dW$; clearly it satisfies the first-order conditions for a proper scoring rule for $M_A(\cdot)$ over unrestricted distributions. Define $H^2(\cdot,\cdot)$ as a solution to $H^2(C(W),x)=\int L(W)B(C(W),x)dW$. In other words, for $Y=C(W)$,

$$H^2(Y,x) = \int K_2(C^{-1}(Y))\ B(Y,x)\ d(C^{-1}(Y)),$$
$$= \int K_2(C^{-1}(Y))\ B(Y,x)\ C^{-1\prime}(Y)\ dY;$$

so $H^2(\cdot,\cdot)$ is indeed well-defined up to addition of a function in x. Furthermore, $\frac{\partial H^2}{\partial Y}$ is a Y-based multiple of $B(Y,x)$ and thus satisfies first-order conditions for an unrestricted scoring rule for $M_B(\cdot)$. The last claim follows from the fact that if two functions are each maximized at the same point, their sum must be maximized there as well.∎

Let us apply Corollary 2.2.1 to the example cited above. Define $A(W,x)$ as $W-x$, $B(W,x)$ as $W-x^2$, and $C(W)$ as $W^2+\sigma^2$. Then the permissible $F(\cdot)$ are those for which $E[x^2]=(E[x])^2+\sigma^2$; i.e., those for which the variance is σ^2. Hence proper scoring rules for the example must be expressable in the form

$$R_1(W) - R_1'(W)(W-x) + R_2(W^2+\sigma^2) - R_2'(W^2+\sigma^2)(W^2+\sigma^2-x^2) + T(x),$$

Second-order conditions require $R_1''(W)+R_2''(W^2+\sigma^2)$ to be non-negative for all feasible W. If both $R_1(\cdot)$ and $R_2(\cdot)$ are convex, $H(\cdot,\cdot)$ is the sum of two proper scoring rules for the first and second moments respectively and hence is proper under the restriction.

Corollary 2.2.2: If $C(W)$ is a constant C_0, proper scoring rules can be expressed in the form

$$H(W,x) = H^1(W,x) + N(W)\ B(C_0,x), \tag{2.23}$$

where $H^1(\cdot,\cdot)$ satisfies first-order conditions for proper scoring rules for $M_A(\cdot)$ over unrestricted distributions. As in the previous corollary, properness of $H^1(\cdot,\cdot)$ without restriction implies properness of $H(\cdot,\cdot)$ with restriction.

Proof: Define $H^1(W,x)$ as before and $N(W)$ such that $N'(W)=L(W)$; the rest of the claims are self-evident. ∎

As an illustration of Corollary 2.2.2, keep $A(W,x)$ and $B(W,x)$ the same as in the previous example, but redefine $C(W)\equiv C_0$. Now the permissible $F(\cdot)$ are those with a second moment of C_0. Proper scoring rules for these distributions can be written

$$R_1(W) - R_1'(W)(W-x) + N(W)(C_0-x^2) + T(x),$$

Convexity of $R(\cdot)$ is necessary and sufficient for properness. In this case the "widening" of the class of proper scoring rules permitted by the restriction on $F(\cdot)$ amounts to adding multiples of a fair x-based lottery onto scores proper without restriction.

7. Summary

This chapter has investigated incentive-compatible schemes -- otherwise known as proper scoring rules -- for eliciting a finite number of messages about a perceived probability distribution. The methods described here permit a virtually complete characterization of proper scoring rules, for virtually any message. To be sure, we generally do not address the possibility of nowhere-differentiable scoring rules (as, for example, Thomson (1979) did for quantiles). For the price of this small narrowing of vision, however, we purchase an enormous simplification in derivations. Previously separate treatments can be unified, and new ground can be charted.

CHAPTER 3: PRACTICAL MODIFICATIONS

In order to focus on issues of truthful revelation, the previous chapter made some strong simplifying assumptions about the world in which contracts are used. Actual expenditures are easily monitored. The agent firm and the principal are neutral with respect to risk. All project attributes other than cost are completely specified, and the principal neither caters to the firm nor interferes with its work once the project is begun. No competitors vie with the firm for the project in question, and no aspect of project performance will affect the firm's access to future projects.

In real-life contracting, it is unlikely that all these conditions are ever simultaneously met. What happens when they are not? Will the contracts described above retain their incentive compatibility? If not, what sorts of deviations are likely to occur? And of these, which can be corrected through contract modifications? These are the questions we shall address in this chapter. Whereas in the previous chapter we examined elicitation for a variety of message functions, in this chapter we shall deal only with expected cost.

1. Threshold Level (Conditional) Contracting

In many real-life situations, planners want to undertake a project only so long as expected cost is below a certain threshold level L. [1] If the project is not undertaken, the firm is assumed to

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[1]L is assumed to be within the interior of the feasible region D for

obtain a profit of z elsewhere, without being paid or penalized additionally by the planners. How can the schemes described before be modified to allow for conditional implementation? This question was first posed by Savage (1971), and solved by Reichelstein and Osband (1983).

Here a simpler derivation of the Reichelstein-Osband result is provided, based on the following insight: from a scoring standpoint, it is irrelevant whether (a) the project is rejected for $W>L$ or (b) it is never rejected but payoffs satisfy the restriction

$$H(W,x) = z \qquad \text{for } W > L, \tag{3.1}$$

where $H(W,x)$ as before is the net payoff based on reported mean* W and audited costs x. (The principal's gross payments are $H(W,x)+x$ for W less than or equal to L and 0 for W greater than L.). Hence, we do not need to concoct the appropriate mechanisms from scratch, but can start with the general, unrestricted class of mechanisms and use restriction (3.1) as a "sieve".

Proposition 3.1: If projects are rejected for $W>L$, proper scoring rules must be expressable in the form

$$H(W,x) = R(W) - R'(W)(W - x), \tag{3.2}$$

with $R(\cdot)$ convex and non-increasing and $R(L)=z$.

Proof: The desired scoring rules amount to those which

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*. Otherwise the problem is trivial.

simultaneously satisfy (3.1) and

$$(2.4) \qquad H(W,x) = R(W) - R'(W)(W-x) + T(x), \quad \text{with } R(\cdot) \text{ convex.}$$

Substitute (3.1) into (2.4) and rearrange terms to obtain $R'(W)x = -R(W)+R'(W)W-T(x)+z$ for $W>L$. Since the right-hand side of the equation is additively separable in W and x, the left-hand side must be too, so that $R'(W)$ is constant for $W>L$. It follows that, for $W>L$, $R(W)=bW+c$ for some constants b and c, and $T(x)=-c-bx+z$ for all x. Define a new function $R^{\circ}(W) \equiv R(W)-bW-c+z$. Then

$$R^{\circ}(W) - R^{\circ\prime}(W)(W - x) = R(W) - bW - c + z - (R'(W) - b)(W - x)$$

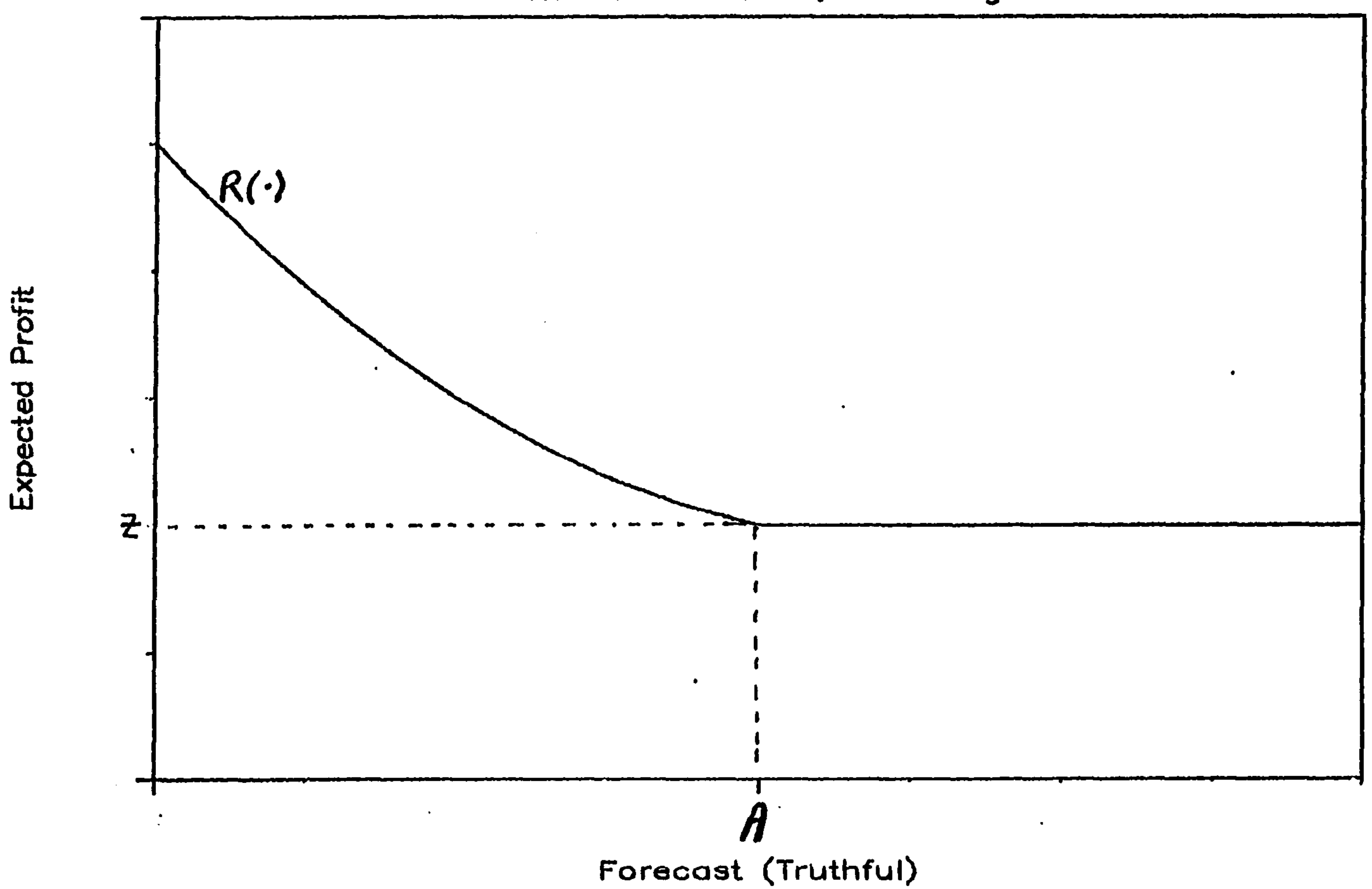
$$= R(W) - R'(W)(W - x) - c - bx - z = R(W) - R'(W)(W - x) + T(x).$$

Furthermore, $R^{\circ}(\cdot)$ will be convex if and only if the section of $R^{\circ}(\cdot)$ for $W\leq L$ is (weakly) monotone-decreasing and convex. See Figure 3.1. (Assuming differentiability, $R^{\circ\prime\prime}(\cdot)\geq 0$ and $R^{\circ\prime}(W)=0$ for $W>L$ implies $R^{\circ\prime}(W)\leq 0$ for $W\leq L$.).∎

Insert Figure 3.1

Proposition 3.1 may be informally interpreted as "implying" $R(\cdot)$ is z and $T(\cdot)$ is 0 on the rejection region $W>L$. Technically, this is not true: as we have seen, given any pair $\langle R(W),T(x)\rangle$ generating a proper scoring rule, the pair $\langle R(W)+bW+c,T(x)-bx-c\rangle$ will generate the same scoring rule. Instead, what Proposition 1 says is that any scoring rule with $H(W,x)=z$ on the rejection region can be generated by the pair described. $T(x)=0$ for all $W>L$ in turn implies that $T(x)=0$

Figure 3.1:
Threshold Value Contracting



everywhere; $R(W)=z$ for all $W>L$ together with the convexity of $R(\cdot)$ implies that $R(W)$ is non-increasing and $R(L)=z$.

It is easy to understand why $R(\cdot)$ must equal or exceed z: a contractor expecting a net payoff less than z from going through with the project would prefer rejection to acceptance, and so would not truthfully reveal a mean less than L. Setting $R(L)$ equal to zero keeps the contractor indifferent at the threshold between acceptance and rejection, thus matching the contractor's preferencess with those of the government. $R'(W)$, we recall, is just the unit fine on the budget variance. Since it is zero or negative ($R(W)$ decreasing), the contractor is penalized for cost overruns.

That the per unit fine decreases with W is just the convexity requirement discussed earlier; we shall not attempt here to justify it further. It carries with it two very interesting implications. First, the sizes of ex-ante rewards and ex-post unit fines go hand in hand. By reporting low expected cost, the contractor reaps a large ex-ante bonus, but exposes herself to greater unit penalties on cost overruns. Second, the government faces a tradeoff between providing incentives for accurate reporting and keeping its expected payments low. As we saw in Figure 2.1, $H(W,\cdot)$ can be regarded as a subgradient to $R(\cdot)$ at W. Expected payoff for report W and true mean W^* is given by the height at W^* of this subgradient. Hence incentives for accurate reporting depend on the curvature of the expected payoff function. The more curved $R(\cdot)$ is, the greater the incentives for accurate reporting are, but greater too is the anticipated contract cost to the government.

2. Cost Padding

"Pure" cost-padding, that is, overpayments to suppliers without compensating kickbacks, can be prevented in principle by requiring net payoffs to be decreasing in ex-post cost. As we have seen, this requirement is redundant with threshold value contracting: net payoffs **cannot** be increasing in cost. Project perquisites ("perks") such as added staff and acquisition of expertise or equipment for future projects can be more difficult to handle. For incentive-compatibility, $H(W,x)$ should represent the **net** payoff to the contractor. In practice, however, it can be very difficult for auditors to judge where pure project costs end and project perquisites begin. Suppose the firm accumulates perks at the rate of s dollars per dollar spent. Then when the government pays the firm $H(W,x)+x$ for a project costing x dollars, net payoff to the firm is not $H(W,x)$ but $H(W,x)+sx$.

In order to retain incentive compatibility, we have to ensure that $H(W,x)+sx$ is decreasing in x, and that the firm remains indifferent at the threshold between project acceptance and rejection. To accomplish this, have the government deduct the value of perks from its gross payment; that is, have it pay the reporting firm $H(W,x)+(1-s)x$ if the project is accepted and 0 otherwise.

To make this discussion more concrete, let us consider the simplest case, in which $R(\cdot)$ is quadratic over the acceptance region. For $R(W) = aW^2+bW+c$ to satisfy the conditions of (2), we must have $2a\geq 0$, $2aW+b\leq 0$ for all $W\leq L$, and $aL^2+bL+c=z$. Defining $d=-2aL-b$ and rearranging terms, we have, for $W \leq L$,

$$(3.3) \qquad R(W) = a(W - L)^2 - d(W - L) + z, \quad \text{where } a,d \geq 0.$$

Gross government payments to the firm then amount to:

$$(3.4) \qquad \begin{cases} -aW^2 + (2aW - d - 2aL + 1 - s)x + aL^2 + dL + z & \text{for } W \leq L \\ 0 & \text{for } W > L \end{cases}$$

We then have to decide on particular values for a and d. The coefficient **a** is what defines the curvature. Suppose the contractor knew the expected cost was W^* but reported W^*+1 instead. The difference between the expected payoffs in the two cases is readily shown to be **a**. The coefficient d, on the other hand, helps determine the unit penalty on the firm for cost overruns, since $R'(W) = 2a(W-L)-d$. The greater d is, the more eager the firm will be to trim its expenditures, but, again, the more the government can expect to pay for this. The cheapest contract for the government is obtained by setting d at its minimum 0, in which case the firm will have only weak incentives for cost restraint at the threshold L (Net payoff for report L will be z regardless of outcome.).

These considerations shed some light on the controversial subject of profit limits in weapons contracts. Popular resentment of what was seen as excessive supplier profits led Congress to impose profit ceilings on weapons projects, ceilings which the Pentagon and the weapons industry have been trying to repeal. Without rendering judgment, let us make a few observations. To the extent that high profits accrue to firms not shouldering much cost risk, they represent not-so-hidden subsidies to the weapons industry. However, any plan to penalize firms more for cost overruns relative to estimates must also offer firms more profits in case of cost underruns. Hence, as we have seen, profits must be even higher in case of accurate prediction of low (relative to threshold) costs. That a contractor could reap superprofits simply for admitting that direct project costs will not amount to much may strike many as unfair. But it is a price the public must be willing to pay in order to provide sufficient incentives for reporting higher estimates truthfully and trying to keep overall costs low. At least this is so when there is only one bidder: in the next section we shall see how competition can be used to reduce expected contract costs.

For a numerical example, let the opportunity profit be 1 million dollars, let the threshold level for project acceptance be 40 million dollars, and let perks be accrued at the rate of 3% of audited costs. Finally, suppose it is worth \$10,000 to the government to have the contractor report the expected cost accurately instead of deviating by a million dollars. Then the incentive-compatible contract with the lowest expected payoff for the government will offer gross payments of $17-0.01W^2+(0.02W+0.17)x$ million dollars for W , with expected profit of $.01(40-W)^2+1$ million dollars. If the contractor reports expected costs of, say, \$30 million, gross payments work out to \$8 million plus 77% of audited costs. Expected net profit given truthful reporting would be \$2 million, but the contractor would bear 23% (20% net of perks) of any cost overrun.

A practical complication is that s is generally not known by the government. If s in (3.4) is set too low, firms which accumulate perks at a higher rate than that will be inclined to submit bids just under the threshold and pad costs. If on the other hand s is set higher than the actual rate, firms with estimates close to but still under the threshold cannot expect to earn z or greater by truthful reporting. They will then submit bids higher than L, or, equivalently, not bid at all. Given the relative magnitudes of potential losses, caution suggests the government set s on the high side, depending on the government's perception of the firm's likely costs and perks and on the importance of the project. One approach the government might take would be to audit firms with reported expected costs close to the threshold more carefully than those with very low reports, since the former are penalized less for cost overruns than the latter.

To clear up a potential source of confusion, let us note that enforcement of threshold level project acceptance and deterrence of cost-padding do not by themselves require use of non-zero a. If a is zero, government payments to the firm (3.4) amount to $(1-s)x + d(L-x) + z$ if the firm decides to undertake the project and 0 if it does not, so that the precise value of W (other than which side of the threshold it lies on) is irrelevant. The net expected advantage to the firm from undertaking the project is $d(L-W)$, with a net penalty of d per dollar cost overrun.

This naturally raises the question of why strictly proper scoring rules should be used at all. In many cases they need not be. Government contracting tends to suffer simultaneously from excessive information reporting requirements and insufficient checks on the reliability of that information. Reductions in the numbers of indicators could often benefit both firms and planners. However, some information will generally still be useful, not so much for regulating the project in question but in setting budget limits for other projects. In the models we are presently considering, such information can still be acquired costlessly without providing positive incentives. However, when agent learning costs or nonlinear cost-padding possibilities are introduced, this will no longer be true. Nontrivial optimal contracts can then be selected from a class of proper scoring rules.

3. Competition

In threshold value contracting, expected payoff rises with the difference between the true mean W^* and the threshold L. From the government's point of view, therefore, it is tempting to look for ways to reduce L while still keeping it equal to or greater than W^*. One possibility would be to have the firm submit W first, and the government to set L=W afterwards. Expected payoff if the firm reports truthfully would be z no matter what the actual mean was. But of course the firm would have no incentive to report truthfully in this situation: better to submit a high bid and aim for a bonus for cost underruns. In general, the government can never use the information submitted by the firm to reduce L, unless the firm is naive.

To formalize this argument, consider any net payoff scheme

$$(3.5) \qquad R(W-L) - R'(W-L)\,(W - x) + z, \quad \text{with } R(\cdot)\geq 0,\ R''(\cdot)\geq 0, \text{ and } R(0)=0.$$

Provided L is independent of W, (3.5) is incentive-compatible for contracting with threshold value L. For dependent L, first-order conditions for expected payoff maximization require

$$R''(W-L)\ (1 - \frac{\partial L}{\partial W})\ (W - W^*) = - R'(W-L)\ \frac{\partial L}{\partial W}$$

where $\frac{\partial L}{\partial W}$ denotes the conjectural variation of L in response to changes in W. Since $R''(\cdot)$ is and $R'(W-L)$ is negative, $W-W^*$ is positive for $0<\frac{\partial L}{\partial W}<1$. That is, if L rises with W, but not so much as to make W-L fall, the firm will tend to inflate its report above the mean.

To lower the threshold value, then, the government must rely on some third party's information. Here is where competition can be very useful. Offer every firm a contract of the form (3.5), with W as the firm's own bid and L as the lowest bid from other firms. [2] As long as firms do not collaborate, each will have an incentive to tell the truth, as its report affects only the threshold values of other firms and not its own. Ties for low bid can be handled arbitrarily, since the firm's net expected payoff for W=L will be z whether it receives the project or not. Total expected government payment (project cost plus net firm payoff) will be $R(W-L)+z+W$, where W is the lowest and L is the second-lowest bid.

The approach recalls that of the sealed-bid auction proposed by

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More generally, offer a contract of the form $R(W,L) - \frac{\partial R}{\partial W}(W,L)(W-x)$, where $R(W,L)$ is convex in W and equals z for all $W>L$.

Vickrey (1961), where the highest bidder wins the object and pays the amount of the second-highest bid. A crucial assumption in both results is the independence of bidders' evaluations. If evaluations are not independent, the above contracts are no longer incentive-compatible due to the so-called "winner's curse": a firm is more likely to win the project when it underestimates costs than when it overestimates them (Capen, Clapp, and Campbell (1971)). To restore incentive compatibility, firms should report both prior expected cost and their rule for revising expected cost given other firms' reports. Let W_i be firm i's report of prior expected cost, and let $C_i(W_1,W_2,...)$ be its rule for reformulating expected cost given all the other reports. The imputed value of C_i should serve as the bid and the lowest bidder should be awarded the project; net payoffs should satisfy (3.2) with the second-lowest bid serving as the threshold value. [3]

Let us explore further the merits of competition, assuming that bidders' evaluations are independent. Suppose that, before bidding begins, the principal considers no firm more likely than any other to submit a given bid. Let g(W) represent the principal's subjectively-held probability density that any given firm will submit a (truthful) estimate of W, and let G(·) be the cumulative

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[3]If other agents' rules $C_j(\cdot)$ themselves convey useful information to the firm about expected cost, this approach may not be valid. Rules for revising information based on other rules might need reporting, and so on. If information requests are not to multiply ad infinitum, firms' uncertainties must be based on what is known in game theory as "common knowledge". A rigorous examination of this concept and its application to auctions falls beyond the scope of this dissertation.

distribution function for $g(\cdot)$. The probability density for the first bid being W_1, the second bid $W_2 > W_1$, and the remaining bids greater than or equal to W_2 is $g(W_1)g(W_2)[1-G(W_2)]^{N-2}$. Multiplying this by the number of permutations of two elements out of N, the probability density for a lowest bid of W_1 and second-lowest of W_2 is found to be $N(N-1)g(W_1)g(W_1)[1-G(W_2)]^{N-2}$. Hence, before bidding begins, the principal expects total payments of

$$(3.6) \qquad \iint^{W_2}[R(W_1-W_2)+W_1+z]N(N-1)[1-G(W_2)]^{N-2}dG(W_1)dG(W_2).$$

Below (3.6) is calculated for the case when $R(W)$ is the quadratic aW^2, and $g(\cdot)$ is the uniform density $(T-S)^{-1}$ between S and T. Carrying out the integration with respect to W_1 yields

$$(3.7) \qquad \int_{[S,T]} \tfrac{1}{3}a(W_2 - S)^3N(N - 1)(T - W_2)^{N-2}(T - S)^{-N}\, dW_2$$
$$+ \;\; z \;\; + \int_{[S,T]} \tfrac{1}{2}(W_2^2 - S^2)N(N - 1)(T - W_2)^{N-2}(T - S)^{-N}\, dW_2.$$

Making the substitution $Y=(W_2-S)(T-S)^{-1}$, (3.7) becomes

$$(3.8) \qquad \int_{[0,1]}a(T - S)^2\tfrac{1}{3}N(N - 1)Y^3(1 - Y)^{N-2}\, dY \;+\; z$$
$$+ \int_{[0,1]}(T - S)\tfrac{1}{2}N(N - 1)Y^2(1 - Y)^{N-2}\, dY$$
$$+ \int_{[0,1]}N(N - 1)SY(1 - Y)^{N-2}\, dY.$$

Each of the integrals above contains the expression $Y^j(1-Y)^k$, where j and k are integers. To evaluate, consider a random number generator with output uniformly distributed between 0 and 1. Out of j+k+1 numbers, the probability density that the j+1-th lowest number (j lower, k higher) equals Y is $(j+k+1)!(j!)^{-1}(k!)^{-1}Y^j(1-Y)^k$. (This is calculated as the probability that the first j numbers are less than Y times the probability density that the j+1-th number equals Y times the probability that the last k numbers exceed Y times the number of ways that j+k+1 objects can be divided into piles of 1,j, and k objects each.). Since the probability density must integrate to 1, we have

$$(3.9) \qquad \int_{[0,1]} Y^j (1-Y)^k \, dY = \frac{j!\,k!}{(j+k+1)!} \quad .$$

Subsitution of (3.9) into (3.8) yields

$$(3.10) \qquad \frac{2a(T-S)^2}{(N+2)(N+1)} + \frac{T-S}{N+1} + S + z \quad .$$

If there is only one agent and the threshold value is T (project acceptance is guaranteed), it is easily checked that the formula still applies.

The first term in (3.10), which is the only one containing the coefficient a, indicates the anticipated expense from providing strictly positive incentives for truthful revelation. The sum of the second and third terms is the anticipated direct project cost. Clearly, the revelation expense falls more rapidly with N than direct project cost does. Hence, the need for accurate forecasting makes competition even more desirable than it would be otherwise. Furthermore, since anticipated revelation expense rises more rapidly with T-S than anticipated direct project cost does, forecasting needs especially favor competition when the degree of initial uncertainty is high.

4. Risk-Averse Agent

Incentive-compatibility of contracts (2.4) or (3.2) hinges on the assumption that the reporting firm strives to maximize expected net payoff. Among other things, the firm must be risk-neutral. In the general case, one must make sure that the utility of the firm's net payoff, rather than its monetary value, meets the specifications of (2.4) or (3.2).

Proposition 3.2: If $U:R\to R$ is the firm's utility function, with inverse $U^{-1}(\cdot)$, a contract is incentive-compatible if and only if net payoffs take the form

$$(3.11) \qquad U^{-1}(R(W)-R'(W)(W-x)+T(x)) \quad \text{for } R(\cdot) \text{ convex}$$

in the general case, and

$$(3.12) \qquad U^{-1}(R(W)-R'(W)(W-x)) \quad \text{for } R(\cdot) \text{ convex and } R(W)=L \ \forall W>L$$

with threshold value contracting.

Proof: Immediate by substitution of the respective expressions into (2.4) or (3.2).∎

If the firm is a risk-averse utility-maximizer, $U(\cdot)$ is increasing and concave while $U^{-1}(\cdot)$ must be increasing and convex. (This follows from $U^{-1\prime}(\cdot) = [U'(U^{-1}(\cdot))]^{-1}$, and $U^{-1\prime\prime}(\cdot) = -U''(U^{-1}(\cdot))[U'(U^{-1}(\cdot))]^{-3}$.). The second derivative with respect to x of (3.11) equals $U''[R'+T']^2+U'T''$, where $U(\cdot)$, $R(\cdot)$, and $T(\cdot)$ are evaluated at $U^{-1}(R(W)-R'(W)(W-x)+T(x))$, W, and x respectively. Hence net monetary payoffs will be convex in x provided $T(\cdot)$ is convex in x.

The difficulty with implementing (3.11) or (3.12) of course lies in determining the firm's utility function. One possibility might be to elicit the firm's utility function simultaneously with the expected cost. If $H^0(V(\cdot),W,x)$ denotes the firm's net monetary payoff based on reported utility function $V(\cdot)$, reported expectation W, and eventual cost x, incentive-compatibility requires that

$$(3.13)\quad V(H^{o}(V(\cdot),W,x)) = R(V(\cdot),W) - \frac{\partial R}{\partial W}(V(\cdot),W)(W - x) + T(V(\cdot),x),$$

with

$$(3.14)\quad E^{F}[U(H^{\bullet}(U(\cdot),W^{*},x))] \geq E^{F}[U(H^{o}(V(\cdot),W,x))]$$

for all $F(\cdot)$ such that $E^{F}[W^{*}-x]=0$, and for all W^{*}, W, $U(\cdot)$, and $V(\cdot)$. It turns out far from easy to construct a contract satisfying (3.13) and (3.14). Indeed, if $T(V(\cdot),\cdot)$ is restricted to be zero -- as occurs, for example, with threshold value contracting -- no such contracts exist.

Proposition 3.3: *If* $T(V(\cdot),\cdot)$ equals zero for all $V(\cdot)$, no strictly proper contract simultaneously satisfies (3.13) and (3.14).

Proof: Suppose the firm is known to take one of two types: risk-averse with utility function $U(\cdot)$, or risk-neutral. From (3.13), net payoffs take the form $U^{-1}(R_1(W)-R_2'(W)(W-x))$ if the firm claims to be risk-averse, and $R_2(W)-R_2'(W)(W-x)$ otherwise. Suppose outcome W^{*} is certain. Then (3.14) requires that $R_1(W^{*})\geq U(R_2(W^{*}))$ and $R_2(W^{*})\geq U^{-1}(R_1(W^{*}))$, or $R_1(\cdot)\equiv U(R_2(\cdot))$. Intuitively, when cost is determinate, attitudes towards risk are inconsequential, so that the firm must be indifferent about which utility function to claim. Now suppose the mean remains W^{*}, but the outcome is no longer certain. A risk-neutral agent could earn $R_2(W^{*})$ by completely truthful reporting, and $\int_D U^{-1}(R_1(W^{*})-R_1'(W^{*})(W^{*}-x))dF(x)$ by reporting W^{*} but claiming to be risk-averse. $U^{-1}(\cdot)$ is strictly convex; by Jensen's inequality (which is geometrically self-evident) the expectation of $U^{-1}(\cdot)$ of a random Y exceeds $U^{-1}(\cdot)$ of the expectation of Y. Hence

$$\int_D U^{-1}(R_1(W^*) - R_1'(W^*)(W^* - x))\, dF(x)$$
$$> U^{-1}(\int_D [R_1(W^*) - R_1'(W^*)(W^* - x)]\, dF(x))$$
$$= U^{-1}(R_1(W^*)) = R_2(W^*)$$

So the agent could do better claiming to be risk-averse, which violates (3.14).∎

Let us reflect briefly on the philosophical implications of Proposition 3. The classic way to determine attitude towards risk is to see how much the agent is willing to pay for a small gamble of a given perceived probability of success. How the principal comes to know the agent's probability perception is left unspecified. Presumably, the agent has formed an opinion based on the records of past gambling experiments and others' assurances, and has revealed this opinion or the thought process leading up to it to the principal. For practical purposes, where approximations are useful and necessary, this approach may suffice. But theoretically, the principal must always retain some doubt about the accuracy of the communication, given the possibility of misreporting or misinterpretation. Another way to determine the agent's perception of a gamble is to calculate it on the basis of information about the agent's utility function and the agent's willingness to pay for it. But this approach takes us in circles: to determine the utility function, the agent's perception must be known, but the latter is determined from information about the utility function.

Proposition 3.3 suggests that the dilemma may be unresolvable. When an agent reports a relatively low willingness to pay for a gamble, perhaps we can never be quite sure whether the agent is highly risk-averse or whether the perceived probability of success is very low. However, it must be emphasized that Proposition 3 falls short of a general impossibility result, since $T(\cdot,V(\cdot))$ is assumed to be zero. So far I have neither been able to extend it nor come up with a counter-example. The question must be left to future research (or past research that I am unaware of).

A related problem concerns the effect of misjudging attitudes towards risk. Suppose a risk-averse firm is offered a threshold value contract (3.2) suitable for a risk-neutral agent. First-order conditions for expected utility maximization require that

$$\int_D (x - W)\, U'(R(W)-R'(W)(W-x))\, dF(x) = 0. \tag{3.15}$$

A higher cost x implies lower net payment implies higher marginal utility of payment. Hence $(x-W)U'(R(W)-R'(W)(W-x)) > (x-W)U'(R(W))$ for all $x>W$, and the same is true for $x<W$. This in turn implies that $\int_D (x-W)U'(R(W)-R'(W)(W-x))dF(x) \geq U'(R(W))(W^*-W)$, where W^* is the mean of $F(\cdot)$. If W is less than W^*, the right-hand side is positive and the first-order conditions cannot be met. Therefore the firm should submit a bid which is higher than the true mean.

By making such a bid, the firm sacrifices some expected profit in order to lessen the levy $R'(W)$ on report/outcome discrepancies, thereby obtaining a "safer" contract. Thus risk aversion would in some ways alleviate the cost overrun problem, but only by making estimates unduly pessimistic and by hazarding rejection of some worthwhile projects.

The more risk-averse the firm is or the riskier the project is, the greater is the exaggeration of reports. To see this, substitute the Taylor's series approximation

$$U'(R(W) - R'(W)(W - x)) \approx U'(R(W)) + U''(R(W))R'(W)(x - W)$$

into (3.15), to yield approximate first-order conditions

$$(3.16)\quad (W^*-W)U'+E^F[(W-x)^2]U''R' = (W^*-W)U' + (W^*-W)^2U''R' + \sigma^2U''R' = 0,$$

where W^* is the mean and σ^2 the variance of $F(\cdot)$, $R'(\cdot)$ is evaluated at W, and $U'(\cdot)$ and $U''(\cdot)$ are evaluated at $R(W)$. By treating (3.16) as a quadratic in W^*-W, we obtain the solution

$$(3.17)\qquad W - W^* = (-2eR')^{-1} - [(4e^2R'^2)^{-1} - \sigma^2]^{1/2}$$

where $e = -\frac{U''}{U'}$ is the coefficient of absolute risk-aversion at the optimum.[4]

Let us assume the firm has constant absolute risk-aversion, and treat e as a parameter. The comparative statics exercises are easier to follow if we define $B \equiv (-2eR')^{-1}$, so that $W-W^* = B-(B^2-\sigma^2)^{1/2}$. Then $\frac{\partial W}{\partial B} = 1-B(B^2-\sigma^2)^{-1/2} < 0$. $\frac{\partial B}{\partial e} = C_1 + C_2\frac{\partial W}{\partial e}$, where $C_1 = 0.5e^{-2}R'^{-1}$ is negative , and $C_2 = 0.5e^{-1}R'^{-2}R''$ is positive. Hence

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[4]In (3.17), the choice of the minus sign before the square root instead of the plus sign was made by checking the second-order condition for maximization that the left-hand side of (18) be nonincreasing in W at the optimum. For this to be true, $W-W^*$ must be less than $(-2eR')^{-1}$, where R', as we recall, is negative.

$$\frac{\partial W}{\partial e} = \frac{\partial W}{\partial B}\cdot\frac{\partial B}{\partial e} = \frac{\partial W}{\partial B}\cdot(C_1 + C_2\frac{\partial W}{\partial e}),$$

or

$$\frac{\partial W}{\partial e} = C_1\frac{\partial W}{\partial B}[1 - C_2\frac{\partial W}{\partial B}]^{-1} > 0.$$

Similarly,

$$\frac{\partial W}{\partial \sigma^2} = 0.5(B^2 - \sigma^2)^{-1/2} + \frac{\partial W}{\partial B}C_2\frac{\partial W}{\partial \sigma^2},$$

or

$$\frac{\partial W}{\partial \sigma^2} = 0.5(B^2 - \sigma^2)^{-1/2}[1 - C_2\frac{\partial W}{\partial B}]^{-1} > 0,$$

As claimed, the spread between W and W^* rises with the riskiness of the project and the agent's aversion to risk.

5. Multiple Planning Criteria

In our discussion until now, we have assumed that all project attributes other than cost are "etched in stone" -- that is, completely and irrevocably specified by the principal before contracting begins. Obviously this is a gross simplification. In real-life contracting, the principal is often willing to make adjustments in the original specifications to realize cost savings or quality improvements. With military jets, for example, some peak speed capabilities might be sacrificed to reduce cost or improve reliability. Since it is difficult to anticipate adjustments in advance, new specifications and compensation schemes are generally negotiated in the midst of the project. This poses some serious problems for R&D contracting. On the one hand, the firm tries to get the renegotiated compensation pay for previously incurred cost overruns, and may even understate its initial estimate in anticipation of later adjustments. On the other hand, the principal is tempted to renege on past promises and withhold payment.

One potential remedy is to have the principal assign a value to each potential project outcome, with lower values denoting more desirable outcomes, and base the contract on the expected and actual values of the outcome. Let x be cost as before, y be a vector of other attributes, and $v(\cdot,\cdot)$ be the principal's "disutility" evaluation of the project based on x and y, so that a lower $v(\cdot,\cdot)$ signifies a higher planner utility. For $R(\cdot)$ convex, a contract offering net payoffs of the form

$$R(W) - R'(W)(W - v(x,y)) + T(v(x,y)) \tag{3.18}$$

will induce a risk-neutral firm to report a W equal to the expected value of $v(x,y)$. Indeed, from the preceding chapter we see that (3.18) is the only way to induce truthful revelation. If the project is rejected for W above a threshold L, then from Proposition 3.1 we know that the contract must be expressable in the form

$$R(W) - R'(W)(W - v(x,y)), \text{ with } R''(\cdot)\geq 0,\ R'(\cdot)\leq 0,\ R(\cdot)\geq R(L)=z. \tag{3.19}$$

A potential complication is that the firm may not choose x and y to minimize $v(x,y)$, so that W, while truthful, is higher than it need be. This is the multi-attribute counterpart to the problem of cost padding discussed earlier, and the remedy is much the same: make sure net payoffs are decreasing in $v(\cdot,\cdot)$. Again, this desirable property is always present in threshold value contracts.

Use of such contracts can permit each side to take responsibility for what it knows best. The planners concentrate on formulating their needs, while the firm concentrates on developing technology to meet those needs in a cost-efficient way. Better work may get done, with less mid-project intervention. However, some mid-project intervention may still be appropriate, if new information is revealed to the principal in the course of the project. This complex topic will be examined in a separate chapter.

Contracts (3.18) and (3.19) may also be used when all project attributes other than cost are specified, but costs themselves are evaluated nonlinearly because of planner risk-aversion, extra benefits accruing to low-cost discoveries, or some other factor. One approach is to request additional measures of the cost spread besides the mean: say, the first n moments, or the probabilities of falling within specified cost ranges. [5] An alternative approach is to inform the firm of the cost valuation function $v(\cdot)$, and elicit the expected value of it via (3.18) or (3.19) with y fixed.

Returning to the case of variable y, it may be wondered whether the focus on eliciting the expectation of $v(x,y)$, even a minimized $v(x,y)$, is not too narrow. Since budgets in market economies are generally formulated in terms of cost, there remains a potential use for estimates of x by itself. To elicit the expectations W^* and Y^* of x and $v(x,y)$ respectively, net payoffs can take the form:

$$(3.20)\quad R(W,Y) - \frac{\partial R}{\partial W}(W,Y)(W - x) - \frac{\partial R}{\partial Y}(W,Y)(Y - v(x,y)) + T(x,v(x,y)).$$

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[5] To elicit the probability of falling within a cost range C, and elicit the expectation of the indicator variable $I(\cdot)$ defined by $I(x)=1$ for $x \in C$ and 0 otherwise.

One question of interest is whether any contract of the form (3.20) can provide positive incentives for both revealing expected cost truthfully and for minimizing $v(x,y)$. Suppose an expenditure of s is capable of reducing $v(x,y)$ by $C(s)$. The agent will choose s to maximize expected payoff of

$$(3.21) \qquad R(W^*+s, Y^*-C(s)) + E^F[T(x+s, v(x,y)-C(s))],$$

where $F(\cdot)$ here is the joint distribution of x and y. First-order conditions require

$$(3.22) \qquad \frac{\partial R}{\partial W} - \frac{\partial R}{\partial Y} C'(s) + E^F[T_1 - T_2 C'(s)] = 0 ,$$

where T_1 and T_2 represent the partial derivatives of $T(x+C(s), v(x,y)-s)$ with respect to the left-hand and right-hand variables respectively, and the partial derivatives of $R(\cdot,\cdot)$ are evaluated at $(W^*+s, Y^*-C(s))$. For $C'(s)$ to be zero as required for minimization of $v(x,y)$, it must be that

$$(3.23) \qquad \frac{\partial R}{\partial W}(W^*+s, Y^*-C(s)) = - E^F[T_1(x+s, v(x,y)-C(s)] ,$$

for all $F(\cdot)$ such that $E^F[x]=W^*$ and $E^F[v(x,y)]=Y^*$. Hence if $T(\cdot,v(\cdot,\cdot))$ is zero as in threshold value contracting, $R(W,Y)$ must be independent of W, so that there are only weak incentives for reporting costs accurately.

If this restriction is not imposed on $T(\cdot,v(\cdot,\cdot))$, matters are more complicated. In (3.23), make the substitutions $p \equiv x+s$, $q = v(x,y)+s$, $P \equiv W^*+s$, and $Q \equiv Y^*-C(s)$, and apply Lemma 2.2 to show that

$$(3.24) \qquad T_1(p,q) = - \frac{\partial R}{\partial P}(P,Q) - 2j(P - p) - k(Q - q)$$

for some constants 2j and k. Hence $T_1(p,q)-2kp-1q = -\frac{\partial R}{\partial P}(P,Q)-2kP-1Q$, which in turn must equal a constant i, since the left-hand and right-hand sides depend on different variables. Solving for $T_1(\cdot,\cdot)$ and $\frac{\partial R}{\partial W}(\cdot,\cdot)$ and integrating, we have $T(x,v(x,y)) = jx^2 + kxv(x,y) + ix + m(v(x,y))$ and $R(W,Y) = - jW^2 - kWY - iW + n(Y)$, for some $m(\cdot)$ and $n(\cdot)$. Substituting into (3.20) and simplifying, net payoffs are seen to equal

$$(3.25)\quad j(W-x)^2 + k(W-x)(Y-v(x,y)) + n(Y) - n'(Y)(Y-v(x,y)) + m(v(x,y))$$

Second-order conditions require $j \leq 0$ and $jn''(Y)+k^2 \leq 0$ for all Y. The constant k must be zero if x and v(x,y) are correlated. Given truthful reporting, the expected value of (3.25) as a whole (regardless of the value of k) is

$$(3.26)\qquad -j\sigma_F^2 + n(Y) + E^F[m(v(x,y))] ,$$

where σ_F^2 is the variance of x under $F(\cdot)$. Adding s to x will not affect its variance; hence the agent will adjust s solely with an eye to minimizing v(x,y). Thus we have found in (3.25) scoring rules which provide positive incentives for both revealing expected cost truthfully and for minimizing v(x,y).

6. Summary

In this chapter we have examined a number of practical modifications of contracts for eliciting expected values. We have made allowances for or incorporated threshold value contracting, cost-padding, competition, risk-averse agents or principals, and multi-attribute projects. Two general areas remain to be considered: first, the implementation of sequential (multi-period) contracts; second, the selection of optimal contracts out of an incentive-compatible class. It is to these topics that we now turn.

CHAPTER 4: ELICITATION IN MULTI-PERIOD CONTRACTING

Most expensive R&D projects require more than one budget period to complete. Accordingly, planners are generally interested in the likely timing of project outlays as well as total expenditures. One straightforward way to elicit cost estimates for n periods is to string together n separate one-period incentive-compatible contracts. Here each period's compensation depends only on the report and the outcome for that period. While convenient for elicitation, such schemes can conflict with other planning objectives. For example, if planners want to reject projects with excessive total expected costs, it is easily shown that no contract which is additively separable in each period's report can provide strictly positive incentives for truthful reporting. The aim of this chapter is to elucidate general classes of scoring rules suitable for multi-period elicitation, with and without additional planning restrictions. To anticipate some results, it turns out that certain "reasonable" budgetary objectives may be essentially irreconcilable.

1. Elicitation When Costs Are Independent

Suppose a project consists of two independent subprojects, with the cost of each subproject highly uncertain. Denote actual subproject costs by x and z respectively. They are viewed by the firm as outcomes of two independent random variables outside the firm's control, with cumulative probability distributions $F(\cdot)$ and $G(\cdot)$ respectively over identical intervals D. [1]

Let $\mathfrak{F}$ as before be the set of all probability distributions on D. The planner is assumed to be interested in eliciting scalar messages $M_A(F)$ and $M_B(G)$, where $M_A(\cdot)$ and $M_B(\cdot)$ both belong to the family Ω described in Chapter 2. That is, $M_A, M_B: \mathfrak{F} \to \Gamma$ are defined implicitly by functions $A, B: \Gamma \times D \to R$, such that

$$E^F[A(M_A(F),x)] = E^G[B(M_B(G),z)] = 0 \quad \forall F,G \in \mathfrak{F}.$$

If W is the firm's report of $M_A(F)$ and Y is its report of $M_B(G)$, the firm's net compensation is $H(W,x,Y,z)$. Assuming the firm is risk-neutral, it will choose W and Y to maximize its expected payoff. Formally, it solves

(4.1) choose (W,Y) to maximize $E^G E^F[H(W,x,Y,z)]$.

The planner's problem is to design $H(\cdot,\cdot,\cdot,\cdot)$ so that (4.1) is solved by choosing $W=M_A(F)$ and $Y=M_B(G)$ for all $F,G \in \mathfrak{F}$.

As in our analysis of single-period contracting, characterization proceeds by first establishing the necessary form of functions with mean zero for any distribution within a given equivalence class. For notational convenience, the partial derivative of a function $S(\cdot)$ with respect to a variable V will be written $S_V(\cdot)$. $E^F[\cdot]$ and $E^G[\cdot]$ will denote the expectation operators with respect to $F(\cdot)$ of x and $G(\cdot)$ of z respectively, so that, for example, $M_A(F)=W$ and $E^F[A(W,x)]=0$ are equivalent statements.

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[1]The assumption of identical cost ranges is made purely for notational convenience.

Lemma 4.1: Suppose two functions $a,b:D\to R$ each take both positive and negative values. If $h:D\times D\to R$ satisfies $E^G E^F[h(x,z)] = 0$ for all $F(\cdot)$ and $G(\cdot)$ in $\mathfrak{F}$ such that $E^F[a(x)] = E^G[b(z)] = 0$, then $h(x,z) = k_1(z)a(x) + k_2(x)b(z)$ for some functions $k_1,k_2:D\to R$.

Proof: Define $J(x;F)\equiv E^F[h(x,z)]$. By Lemma 2.2, $J(x;F) = K(F)b(z)$ for some function $K:\mathfrak{F}\to R$, provided $E^F[a(x)] = 0$. Since $J(x;\cdot)$ is linear in F, $K(\cdot)$ must be as well, so that $K(F) = E^F[k_2(x)]$ for some $k_2:D\to R$. Hence $E^F[h(x,z)-k_2(x)b(z)]=0$ for all F such that $E^F[a(x)]=0$. Invoking Lemma 2.2 a second time establishes the claim.∎

Proposition 4.1: If $H:\Gamma\times D\times\Gamma\times D\to R$ is proper for $M_A(\cdot)$ and $M_B(\cdot)$ and everywhere twice-differentiable in the reports W and Y, then for $0\in\text{int co}\{A(W,x)|x\in D\}$ and $0\in\text{int co}\{B(Y,z)|z\in D\}]$, $H(\cdot,\cdot,\cdot,\cdot)$ must satisfy

$$(4.2)\quad H_W(W,x,Y,z) = A(W,x)\ \{S_1(W,Y) + S_2(W,z) + \int S_3(W,Y)B(Y,z)dY\} + S_4(W,Y)A(W,x)B(Y,z) + S_5(W,Y)A_W(W,x)B(Y,z);$$

$$(4.3)\quad H_Y(W,x,Y,z) = S_6(W,Y)A(W,x)B_Y(Y,z) + S_7(W,Y)A(W,x)B(Y,z) + B(Y,z)\ \{S_8(W,Y) + S_9(x,Y) + \int S_{10}(W,Y)A(W,x)dW\}$$

for some scalar-valued functions $S_i(\cdot,\cdot)$, $i=1,..,10$.

Proof: The above conditions on $H(\cdot,\cdot,\cdot,\cdot)$ permit differentiation within the integral sign, so that for all pairs $F\in M_A^{-1}(W)$, $G\in M_B^{-1}(Y)$

$$(W,Y) \in \text{argmax } E^G E^F[H(W,x,Y,z)]$$

$$\Rightarrow \quad \frac{\partial}{\partial V}(E^G E^F[H(W,x,Y,z)]) \text{ for } V = W \text{ or } Y$$

$$\Rightarrow \quad E^G E^F[H_Z(W,x,Y,z)] = 0 \quad \text{for } V = W \text{ or } Y.$$

Application of Lemma 4.2 shows that the partial derivatives of net payoff with respect to the reports take the form

$$H_W(W,x,Y,z) = k(W,Y,z)\ A(W,x) + l(W,x,Y)\ B(Y,z);$$

$$H_Y(W,x,Y,z) = m(W,Y,z)\ A(W,x) + n(W,x,Y)\ B(Y,z).$$

Since the cross-partials are identical, $H_{WY}(W,x,Y,z)$ equals either of the two expressions below:

$$(4.4)\qquad k_Y(W,Y,z)\ A(W,x) + l(W,x,Y)\ B_Y(Y,z) + l_Y(W,x,Y)\ B(Y,z)$$

$$(4.4')\qquad m_W(W,Y,z)\ A(W,x) + m(W,Y,z)\ A_W(W,x) + n_W(W,x,Y)\ B(Y,z)$$

Taking the expectation with respect to x and z, we find that

$$E^F[l(W,x,Y)]\ E^G[B_Y(Y,z)] = E^F[A_W(W,x)]\ E^G[m(W,Y,z)]$$

$\forall F \in M_A^{-1}(W)$, $G \in M_B^{-1}(Y)$. Rearranging terms,

$$(4.5)\qquad E^F[l(W,x,Y)]\{E^F[A_W(W,x)]\}^{-1} = E^G[m(W,Y,z)]\{E^G[B_Y(Y,z)]\}^{-1}$$

The right-hand side of (4.5) is independent of x and the left-hand side is independent of z; each side is therefore a function $q_0(\cdot,\cdot)$ of W and Y alone, with $E^F[l(W,x,Y)-q_0(W,Y)A_W(W,x)] = 0 = E^G[m(W,Y,z)-q_0(W,Y)B_Y(Y,z)]$ for all $F \in M_A^{-1}(W)$, $G \in M_B^{-1}(Y)$. Apply Lemma 2.2 to show that $l(\cdot,\cdot,\cdot)$ takes the form $q_0(W,Y)A_W(W,x)+q_1(W,Y)A(W,x)$ and $m(\cdot,\cdot,\cdot)$ takes the form $q_0(W,Y)B_Y(Y,z)+q_2(W,Y)B(Y,z)$. Substitute into (4.4) and (4.4′) and take the expectation with respect to z, conditional on x and a truthful report of $Y=M_B(G)$. Comparing expressions, $E^G[H_{WY}(W,x,Y,z)]$ equals

$$E^G[k_Y(W,Y,z)]A(W,x) + \{q_0(W,Y)A_W(W,x)+q_2(W,Y)A(W,x)\}E^G[B_Y(Y,z)]A(W,x)$$

$$= q_{0W}(W,Y)E^G[B_Y(Y,z)]A(W,x) + q_0(W,Y)E^G[B_Y(Y,z)]A_W(W,x)$$

Fix x arbitrarily and invoke Lemma 2.2 again to establish that

$k_Y(\cdot,\cdot,\cdot)$ takes the form $q_3(W,Y) + q_4(W,Y)B_Y(Y,z) + q_5(W,Y)B(Y,z)$. Integrating with respect to Y, $k(W,Y,z)$ equals $q_6(W,Y) + q_4(W,Y)B(Y,z) + \int q_7(W,Y)B(Y,z)dY + q_8(W,z)$, where $q_{6Y}(W,Y) \equiv q_3(W,Y)$, $q_7(W,Y) \equiv q_5(W,Y)-q_{4Y}(W,Y)$, and $q_8(W,z)$ is the "constant" of integration. Rearranging and relabeling terms establishes (4.2), and (4.3) follows by symmetry.∎

Proposition 4.2: To elicit the means of x and z, a proper contract must take the form

$$(4.6)\quad H^*(W,x,Y,z) = Q(W,x,Y,z) - Q_W(W,x,Y,z)(W - x) - Q_Y(W,x,Y,z)(Y - z) + Q_0(W,Y)(W - x)(Y - z),$$

with

$$(4.7)\quad Q(W,x,Y,z) \equiv Q_1(W,Y) + Q_2(W,z) + Q_3(x,Y) + Q_4(x,z)$$

for all $W,Y \in \text{int } D$, for some functions $Q_i(\cdot,\cdot)$, $i=0,\ldots,4$. Furthermore, for all $F(\cdot)$ having mean W and $G(\cdot)$ having mean Y, it must be true that

$$(4.8)\quad E^G[Q_{WW}(W,x,Y,z)] \le 0$$

and

$$(4.9)\quad E^G[Q_{WW}(W,x,Y,z)]\, E^F[Q_{YY}(W,x,Y,z)] \ge [Q_{1WY}(W,Y) - Q_0(W,Y)]^2$$

If the inequalities in (4.8) and (4.9) are strict, the above conditions are sufficient for strict local incentive-compatibility.

Proof: Message functions for the mean are generated by $A(W,x)=W-x$ and $B(Y,z)=Y-z$. Substituting into (4.2) and integrating with respect to W, a proper scoring rule $H(\cdot,\cdot,\cdot,\cdot)$ is seen to take the form

$$(4.10) \quad [R_1(W,Y) + R_2(W,z) + R_3(W,Y)(Y - z)]$$
$$- [R_{1w}(W,Y) + R_{2w}(W,z) + R_{3w}(W,Y)(Y - z)](W - x) + R_4(x,Y,z),$$

where

$$R_{1ww}(W,Y) = - S_1(W,Y) - \iint S_3(W,Y)dYdY,$$
$$R_{2ww}(W,z) = - S_3(W,z),$$
$$R_{3ww}(W,Y) = - S_4(W,z) + \int S_3(W,Y)dY,$$

and $R_4(x,Y,z)$ is the "constant" of integration. Since $H_{wxx}(\cdot,\cdot,\cdot,\cdot)=0$, it follows by symmetry that $H_{Yzz}(W,x,Y,z) = R_{4Yzz}(x,Y,z) = 0$ for all x, Y, and z. Hence $R_4(\cdot,\cdot,\cdot)$ takes the form $R_5(x,Y)+R_6(x,Y)z+R_7(x,z)$. First-order conditions require that

$$E^G E^F[H_Y(W,x,Y,z)] = R_{1Y}(W,Y) + R_3(W,Y) + E^F[R_{5Y}(x,Y)+YR_{6Y}(x,Y)] = 0.$$

for all distributions $F(\cdot)$ and $G(\cdot)$ having means W and Y respectively. Apply Lemma 2.2 to show that

$$(4.11) \quad R_{1Y}(W,Y) + R_3(W,Y) + R_{5Y}(x,Y) + R_{6Y}(x,Y)Y = R_8(W,Y)(W - x)$$

for some $R_8(\cdot,\cdot)$.

Now consider an $H^*(\cdot,\cdot,\cdot,\cdot)$ defined by (4.6) and (4.7), with $Q_0(W,Y) = R_{3w}(W,Y) - R_8(W,Y)$, $Q_1(W,Y) = R_1(W,Y)$, $Q_2(W,z) = R_2(W,z)$, $Q_3(x,Y) = R_5(x,Y) + YR_6(x,Y)$, and $Q_4(x,z) = R_7(x,z)$. Expanding, the difference between $H^*(\cdot,\cdot,\cdot,\cdot)$ and the $H(\cdot,\cdot,\cdot,\cdot)$ defined in (4.10) equals (dropping the argument lists)

$$R_1 + R_2 + R_5 + YR_6 + R_7 - R_{1w}(W - x) - R_{2w}(W - x) - R_{1Y}(Y - z)$$
$$- R_{5Y}(Y - z) - R_6(Y - z) - YR_{6Y}(Y - z) + R_{3w}(W - x)(Y - z) - R_1$$
$$- R_2 - R_3(Y - z) - R_{1w}(W - x) - R_{2w}(W - x) - R_{3w}(Y - z)(W - x)$$
$$+ R_8(Y - z)(W - x) - R_5 - R_6z - R_7$$

$$= - R_{1Y}(Y-z) - R_{3Y}(Y-z) - YR_{4Y}(Y-z) - R_{3}(Y-z) + R_{8}(Y-z)(W-x)$$

$$= - (Y - z)\,[R_{1Y} + R_{3Y} + YR_{4Y} + R_{3} - R_{8}(W - x)] = 0,$$

where the last equality follows from (4.11). (4.8) and (4.9) give the second-order conditions for expected payoff maximization at the true reports.∎

Proposition 4.2 can easily be extended to encompass any invertible function of an expectation; that is, any message $M_A(\cdot)$ defined by $A(W,x) = r(W)-s(x)$ for some functions $r(\cdot)$, which must be one-to-one, and $s(\cdot)$.

Corollary 4.2.1: If $M_A(F) = r^{-1}(E^F[s(x)])$ and $M_B(G)=t^{-1}(E^G[u(z)]$, then incentive-compatible scoring rules are characterized by (4.6) and (4.7) with $s(x)$ substituted for x, $r(W)$ for W; $u(z)$ for z, and $t(Y)$ for Y.

Proof: For $A^*(W,x)=W-x$ and $B^*(Y,z)=Y-z$, note that $M_A(\cdot)$ is defined by $A^*(r(W),s(x))$ and $M_B(\cdot)$ by $B^*(t(Y),u(z))$. Hence $H(r(\cdot),s(\cdot),t(\cdot),u(\cdot))$ will be proper for $M_A(\cdot)$ and $M_B(\cdot)$ if and only if $H^*(\cdot,\cdot,\cdot,\cdot)$ is proper for $M_{A*}(\cdot)$ and $M_{B*}(\cdot)$.∎

If either of the messages cannot be expressed in this form, Proposition 4.1 and its corollary are not applicable. However, another sort of simplification is possible in Lemma 4.1, in what amounts to setting $S_1(\cdot,\cdot) \equiv S_3(\cdot,\cdot) \equiv S_5(\cdot,\cdot) \equiv S_6(\cdot,\cdot) \equiv S_8(\cdot,\cdot) \equiv S_{10}(\cdot,\cdot) \equiv 0$. To show this two more lemmas are needed.

Lemma 4.2: Let $M_A(\cdot)$ be a message function defined by $A(\cdot,\cdot)$, and suppose there exists a function $q(\cdot)$ such that $E^F[A_W(W,x)] = q(W)$ for all $F \in M_A^{-1}(W)$ and for all W. Then $M_A(\cdot)$ maps $F(\cdot)$ onto a function of an expectation of $F(\cdot)$.

Proof: Applying Lemma 2.2, we have

$$A_W(W,x) = p'(W)A(W,x) + q(W) \tag{4.12}$$

for some function $p'(\cdot)$. It is easily checked that $t(W) = e^{p(W)}\int^W q(w)e^{-p(w)}dw$ yields one solution to (4.12). If $A^*(\cdot,\cdot)$ is any other solution then

$$A_W^*(W,x) - t'(W) = p'(W)[A^*(W,x) - t(W)]$$

$$\Rightarrow \quad \frac{\partial}{\partial W}\ln[A^*(W,x)-t(W)] = p'(W)$$

$$\Rightarrow \quad \ln[A^*(W,x)-t(W)] = p(W) + \ln[s(x)] \quad \text{for some function } s:D\to R^+$$

$$\Rightarrow \quad A^*(W,x) = s(x)e^{p(W)} + t(W). \tag{4.13}$$

So $M_A(F)$ must equal that W for which $E^F[s(x)] = t(W)e^{-p(W)}$.∎

Lemma 4.3: No non-trivial message function in Ω can satisfy $E^F[A_W(W,x)] = 0$ for all $F \in M_A^{-1}(W)$.

Proof: If $q(W)=0$, then $t(W)=0$ in (4.15), so that $A(W,x)$ is multiplicatively separable in W and x and all $F(\cdot)$ must be mapped onto the same W.∎

We are at last ready for our central theorem, which classifies all locally proper two-variable scoring rules for members of Ω.

Theorem 4.1: Under the conditions of Proposition 4.1, a twice-differentiable proper scoring rule $H(\cdot,\cdot,\cdot,\cdot)$ must take the form

$$(4.14)\quad H(W,x,Y,z) = \int R_1(W,z)A(W,x)dW + \int R_2(x,Y)B(Y,z)dY + \iint R_3(W,Y)A(W,x)B(Y,z)dWdY + R_4(W,Y)A(W,x)B(Y,z)$$

Furthermore, for all $F\in M_A^{-1}(W)$ and $G\in M_B^{-1}(Y)$ it must be true that

$$(4.15)\qquad T_1(F(\cdot),G(\cdot)) \le 0 \qquad \text{and}$$

$$(4.16)\qquad T_1(F(\cdot),G(\cdot))T_2(F(\cdot),G(\cdot)) \ge \{R_4(W,Y)\}^2$$

where

$$T_1(F(\cdot),G(\cdot)) = E^F[A_W(W,x)]\ \{E^G[R_1(W,z)] + E^G[\int R_3(W,Y)B(Y,z)dY]\};$$

$$T_2(F(\cdot),G(\cdot)) = E^G[B_Y(Y,z)]\ \{E^F[R_2(x,Y)] + E^F[\int R_3(W,Y)A(W,x)dW]\}.$$

If the inequalities in (4.15) and (4.16) are strict, the above conditions are sufficient for strict local incentive-compatibility.

Proof: For $M_A(\cdot)$ and $M_B(\cdot)$ denoting expectations, (4.14)-(4.16) follow from the substitutions $R_1(W,z) = -Q_{2WW}(W,z)$, $R_2(x,Y) = -Q_{3YY}(x,Y)$, $R_3(W,Y) = Q_{1WWYY}(W,Y)$, and $R_4(W,Y) = Q_0(W,Y)-Q_{1WY}(W,Y)$. (The term $Q_4(x,z)$ is included implicitly in the indefinite integrals.). This can be extended to functions of expectations as in Corollary 4.2.1.

Otherwise, suppose $M_A(\cdot)$ does not denote the function of an expectation. From (4.2) and (4.3), the expectation of H_{WY} given truthful reporting is:

$$E^G E^F[H_{WY}(W,x,Y,z)] = S_5(W,Y)E^G[B_Y(Y,z)] = S_6(W,Y)E^F[A_W(W,x)]$$

Hence $S_6(W,Y)E^F[A_W(W,x)]$ takes the same value for all $F\in M_A^{-1}(W)$. Applying Lemma 4.2, we see that $S_6(\cdot,\cdot)$ must equal zero. Lemma 4.3 implies that $S_5(\cdot,\cdot)$ is zero as well. Taking the expectation of H_{WY} conditional on z, we have

$$S_7(W,Y)\ E^F[\hat{A}_W(W,x)]\ B(Y,z) + S_{8W}(W,Y)\ B(Y,z) = 0 \quad \forall F \in M_A^{-1}(W),$$

to which Lemma 4.2 is again applied with result $S_7(\cdot,\cdot) = S_{8W}(\cdot,\cdot) = 0$. Since $S_8(\cdot,Y)$ is independent of W, we can "incorporate" it in $S_9(x,Y)$; that is, define a new $S_9(\cdot,\cdot)$ as the sum of the old $S_8(\cdot,\cdot)$ and the old $S_9(\cdot,\cdot)$ and redefine $S_8(\cdot,\cdot)$ as zero. Equality of the cross-partials implies

$$H_{WY}(\cdot,\cdot,\cdot,\cdot) = S_{1Y}A + S_3AB + S_4AB_Y + S_{4Y}AB = S_{10}AB.$$

Solving for $S_{1Y}(W,Y)$ and integrating, we have

$$(4.17)\quad S_1(W,Y) = \int[S_{10}(W,Y)-S_3(W,Y)]B(Y,z)dY - S_4(W,Y)B(Y,z) + S_0(W)$$

for some $S_0(W)$, but this $S_0(W)$ can be subtracted from $S_1(\cdot,\cdot)$ and added to $S_2(\cdot,\cdot)$ without loss of generality and hence taken as zero.

Substitution of (4.17) and $S_0(\cdot,\cdot)=S_5(\cdot,\cdot)=0$ into (4.2) yields

$$(4.18)\qquad H_W(W,x,Y,z) = S_2(W,z)A(W,x) + A(W,x)\int S_{10}(W,Y)B(Y,z)dY;$$

Integrating with respect to W, we obtain

$$H(W,x,Y,z) = \int S_2(W,z)A(W,x)dW + \iint S_{10}(W,Y)A(W,x)B(Y,z)dWdY + T(x,Y,z),$$

where $T(x,Y,z)$ is the integrating "constant". Substituting into (4.3) and recalling that $S_6(\cdot,\cdot) = S_7(\cdot,\cdot) = S_8(\cdot,\cdot) = 0$, we have $T_Y(x,Y,z) = S_9(\cdot,\cdot)B(Y,z)$. Integration and relabeling of terms establishes (4.14), with (4.15) and (4.16) the corresponding second-order conditions.∎

2. Interpretation

The contracts described in Proposition 4.2 are easier to understand if we "construct" them out of contracts for eliciting the mean of a single random variable. Recall that the latter must take the form $R(W)-R'(W)(W-x)+T(x)$, where $R(\cdot)$ is convex. By defining $Q(W,x)\equiv R(W)+T(x)$, we can rewrite this as

(4.19) $\qquad H(W,x) = Q(W,x) - Q_W(W,x)(W - x)$, where

(4.20) $\qquad Q(\cdot,\cdot)$ is additively separable in W and x, and

(4.21) $\qquad Q(\cdot,x)$ is convex in W.

Condition (4.19) by itself is really no restriction at all. Any differentiable function $H(\cdot,\cdot)$ can be written in the form (4.19), by solving the differential equation $H_W(W,x) = Q_{WW}(W,x)(W-x)$ and defining suitable integrating constants. The crucial information comes from (4.20) and (4.21). These conditions can also be formulated in terms of restrictions on $Q_W(\cdot,\cdot)$:

(4.20′) $\qquad Q_W(W,\cdot)$ is independent of x;

(4.21′) $\qquad Q_W(W,\cdot)$ is non-decreasing in W.

In this version the economic interpretation of these conditions is straightforward: the unit fine for report/outcome discrepancies is independent of the outcome and rises with the report.

Let us now try to extend the formulas to encompass elicitation of two random variables. By analogy with (4.19), we begin with contracts of the form

(4.22) $\qquad Q(W,x,Y,z) - Q_W(W,x,Y,z)(W - x) - Q_Y(W,x,Y,z)(Y - z)$.

Applying condition (4.20), terms in $Q(\cdot,\cdot,\cdot,\cdot)$ which are non-separable in W and x or non-separable in Y and z are eliminated. That leaves us with four combinations -- W with Y, W with z, x with Y, and x with z -- so that $Q(\cdot,\cdot,\cdot,\cdot)$ can be written $Q_1(W,Y)+Q_2(W,z)+Q_3(x,Y)+Q_4(x,z)$ as in (4.7). We then check the second-order conditions for further restrictions -- something akin to convexity, we suspect -- on the $Q_i(\cdot,\cdot)$.

It is tempting to stop here, and conclude that only scoring rules of the form (4.22) can be proper. Such a conclusion would be premature. Second-order conditions for multivariable maximization problems are much more complex than those for single-variable maximization, allowing for "twists" in functions that have no one-dimensional analogy. Consider, for example, the function $(W-x)(Y-z)$. Both its expected value and its expected marginal value equal zero at the true report. Adding it to one of the scoring rules described above would not affect either first-order conditions or local expected payoffs. To assure local incentive-compatibility, only the second-order conditions would need to be checked.

As for other functions with both an expected value and an expected marginal value always equal to zero at the true reports, clearly any multiple $Q_0(W,Y)$ of $(W-x)(Y-z)$ will do. There are no others, however. [2] For a heuristic explanation, consider that only non-w-based multiples of $W-x$ will have zero expected value whenever W is the true first-period mean. If marginal expected value is also

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[2]If $E^F E[q(W,x,Y,z)] = 0$, then by Lemma 4.1 $q(W,x,Y,z)$ takes the form $k(W,Y,z)(W-x)+l(W,x,Y)(Y-z)$. Zero marginal expected value requires $E^G[k(W,Y,z)] = E^F[l(W,x,Y)] = 0$, so that $k(W,Y,z) = m(W,Y)(Y-z)$ and $l(W,x,Y) = n(W,Y)(W-x)$ for some $m(\cdot,\cdot)$ and $n(\cdot,\cdot)$, and $q(W,x,Y,z) = [m(W,Y)+n(W,Y)](W-x)(Y-z)$.

zero, small errors in reporting of W are insignificant, suggesting that the multiplying factor itself has expected second-period value of zero at a true report. This in turn implies the factor is a non-w-or-y-based multiple of $Y-z$. By adding $Q_o(W,Y)(W-x)(Y-z)$ to (4.22), we obtain the form (4.8).

It might still be wondered whether a non-zero $Q_o(\cdot,\cdot)$ inevitably violates global incentive compatibility requirements. After all, although the expected value of $(W-x)(Y-z)$ is close to zero for (W,Y)-reports near the true means, it is a very large positive or negative number for outlandish reports. The following example shows that $Q_o(\cdot,\cdot)$ need not be zero.

Let $Q(W,x,Y,z) \equiv W^2+Y^2$. For local incentive compatibility, all that Proposition 4.1 requires is for $Q_o(\cdot,\cdot)$ to be bounded by -2 and +2. For global incentive compatibility, consider the expected payoff for a false report (W_o,Y_o) where the true means are W and Y. It needs to be checked that

(4.23) $W^2 + Y^2 \geq - W_o^2 + 2W_oW - Y_o^2 + 2Y_oY + Q_o(W_o,Y_o)(W_o-W)(Y_o-Y)$.

Rearranging terms and defining $c \equiv W_o-W$, $d \equiv Y_o-Y$ allows (4.23) to be rewritten

(4.24) $c^2 + d^2 - Q_o(W_o,Y_o)cd \geq 0$.

The left-hand side of (4.24) is bounded by $c^2+d^2-2cd = (c-d)^2$ and $c^2+d^2+2cd = (c+d)^2$, both of which are nonnegative, so that (4.24) is indeed satisfied.

For an economic interpretation of Proposition 4.2, it is helpful to recall the earlier discussion of an incentive-compatible contract as a system of compensating ex-ante and ex-post payments. Let us consider terms in each $Q_1(\cdot,\cdot)$ separately. $Q_1(W,Y)$ is an ex-ante reward based on the agent's reports, while $Q_{1w}(W,Y)$ and $Q_{1w}(W,Y)$ are unit fines for discrepancies between the reports and the outcomes. If the first-period [second-period] report exceeds the true mean by 1, the ex-ante reward will increase by approximately $Q_{1w}(W,Y)$ [$Q_{1v}(W,Y)$], but the agent expects to lose this amount in ex-post fines due to cost underruns relative to the report. $Q_2(W,z)$ rewards the agent for the first-period report, with the precise amount rewarded dependent on a lottery in y, the second-period outcome. No matter what the outcome of this lottery is, however, the ex-post unit fine $Q_{2w}(W,z)$ on discrepancies between first-period report and outcome is calibrated to ensure truthful revelation of first-period mean. The same logic applies for $Q_3(x,Y)$. $Q_4(x,z)$ is independent of reports and hence does not influence incentive compatibility. Finally, $Q_0(W,Y)$ is a unit fine on the product of the two discrepancies. If either period's mean is truthfully reported, the expected value of this term will be zero regardless of the other report.

It should be clear, then, why truthful reporting yields a stationary value for expected payoffs. But a stationary point will not be a local maximum unless certain curvature conditions are met, and this is the point of (4.8) and (4.9). If the contract consisted of terms in $Q_1(\cdot,\cdot)$ alone, we would know from our previous discussion of single-period contracting that $Q_1(W,Y)$ must be jointly convex in W and Y. Similarly, a contract in terms of $Q_2(\cdot,\cdot)$ alone must have

$Q_2(W,z)$ convex in W, and a contract in terms of $Q_3(\cdot,\cdot)$ alone must have $Q_2(x,Y)$ convex in Y. As noted above, a contract in terms of $Q_0(\cdot,\cdot)$ alone must have $Q_0(W,Y)$ equal to zero. With multiple $Q_i(\cdot,\cdot)$ terms, more leeway is allowed, as it is the "combined" curvature that is important. For example, if both $Q_1(\cdot,\cdot)$ and $Q_0(\cdot,\cdot)$ are allowed to be non-zero, conditions (4.8) and (4.9) amount to $Q_{1YY}\geq 0$ and $Q_{1WW}Q_{1YY}-[Q_{1WY}-Q_0]^2\geq 0$ for all (W,Y) pairs. In this case $Q_1(W,Y)$ must be convex in each of W and Y alone, but joint convexity is neither necessary nor sufficient for local incentive-compatibility: $Q_0(W,Y)$ can either "aggravate" or "alleviate" the influence of the cross-partial. Such cross-partial effects could obviously not arise in the single-variable, single-message case. Indeed, if Y and z are taken as identical constants, so that second-period reporting and observation is not an issue, the two-period contract collapses (or "projects", to use the geometric term) to the single-period one.

As a generalization of Proposition 4.2 to arbitrary message functions in Ω, Theorem 4.1 is more difficult to interpret economically. Still, the solution preserves some basic features of expectation-based scoring rules. Like scoring rules for expectations, all Ω-type scoring rules are sums of terms in four W-or-Y-dependent functions, plus another term depending only on x and z. The first term, $\int R_1(W,z)A(W,x)dW$, defines a proper scoring rule for $M_A(\cdot)$ for any z, provided second-order conditions are satisfied. If W^* is the true report of $F(\cdot)$, the expected loss incurred from exaggerating W^* by 1 is approximately $R_1(W^*,z)E^F[A_W(W^*,x)]$. The only impact of second-period cost is to select which scoring rule is applied. Similarly, $\int R_2(x,Y)B(Y,z)dY$ is a second-period scoring rule (proper if the second-order conditions are satisfied) selected according to first-period cost. $\iint R_3(W,Y)A(W,x)B(Y,z)dWdY$ is a joint non-separable scoring rule for $M_A(\cdot)$ and $M_B(\cdot)$. The final term $R_4(W,Y)A(W,x)B(Y,z)$ appears to leave marginal incentives for truthful reporting unaffected. However, like $Q_0(\cdot,\cdot)$ in the case of expectations, a non-zero $R_4(\cdot,\cdot)$ can significantly alter second-order conditions.

3. Threshold Value Contracting

We now proceed to apply Theorem 4.1 to analysis of three "threshold value" contracting situations. The first situation is analogous to threshold value contracting in the single-variable case. Suppose the government principal does not want to undertake a project if total anticipated costs of the two subprojects are too high. Let ACC and REJ denote the acceptance and rejection regions respectively. For example, when expected costs are being elicited, ACC and REJ might be separated by the frontier W+Y=A, where A is the value of the completed project to the government. But linearity of the frontier is not essential. It is stipulated only that (a) REJ has a non-empty interior in $\Gamma\times\Gamma$ and (b) there is no first- or second-period estimate which guarantees project acceptance regardless of the other estimate.

To save ourselves another derivation from scratch, it is helpful to recall the observation made earlier about single-variable threshold value contracting. From a scoring standpoint, we may as well imagine that the project is never rejected, as long as the firm's net payoff equals its opportunity profit Π_0 for reports outside ACC. (Opportunity profit was was called z in Chapter 2 but that name is reserved here for the second cost variable.). Hence we can begin with the general class described in Theorem 4.1 and impose the following restriction:

$$(4.25) \qquad H(W,x,Y,z) = \Pi_0 \quad \text{for all } (W,Y)\in REJ.$$

The four conditions (4.14)-(4.16) and (4.25) can be substantially simplified. Intuitively, the fact that $H(W,x,Y,z)-\Pi_0$ must vanish over an open region suggests that each $R_i(\cdot,\cdot)$-generated component could be expressed so as to vanish there as well. So let us set $R_1(W,z) = R_2(x,Y) = R_3(W,Y) = R_4(W,Y) = 0$ for all (W,Y) in REJ. What does this imply? Since $R_1(W,z)$ does not depend on Y, it must be zero whenever <u>some</u> Y coupled with W causes project rejection, even if the <u>actual</u> value of Y merits acceptance. Under assumption (b), this could occur with any Y, so that $R_1(\cdot,\cdot)$ must be zero everywhere. [3] Similarly, the term in $R_2(x,Y)$ must drop out completely, while $R_3(\cdot,\cdot)$ and $R_4(\cdot,\cdot)$ need vanish on the rejection region only. The following proposition verifies this conjecture.

Proposition 4.2: If projects are rejected for (W,Y) belonging to REJ, and REJ satisfies conditions (a) and (b), then differentiable proper scoring rules for members of Ω can be expressed in the form

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[3] If (b) is relaxed, $R_1(W,\cdot)$ could conceivably be nonzero for values of W guaranteeing project acceptance.

$$(4.26) \quad H(W,x,Y,z) = \iint R_3(W,Y)A(W,x)B(Y,z)dWdY + R_4(W,Y)A(W,x)B(Y,z),$$

with

$$(4.27) \qquad R_3(W,Y) = R_4(W,Y) = 0 \quad \forall(W,Y)\in REJ$$

and where the integral is normalized to equal the opportunity profit Π_0 on the rejection region.

Proof: Imposing the restriction (4.25) on (4.14), we see that $E^G E^F[H_{WY}(W,x,Y,z)] = R_4(W,Y)E^F[A_W(W,x)]E^G[B_Y(Y,z)] = 0$ for all $(W,Y)\in REJ$, $F\in M_A^{-1}(W)$, and $G\in M_B^{-1}(Y)$. Hence $R_4(\cdot,\cdot)$ must vanish on REJ. This in turn implies for all (W,Y) in REJ that $H_{WY}(W,x,Y,z) = R_3(W,Y)A(W,x)B(Y,z) = 0$ for all x and z. So $R_3(W,Y)$ likewise vanishes on REJ. Furthermore, for $(W,Y)\in REJ$ we have $H_W(W,x,Y,z) = R_1(W,z)A(W,x) = 0$. Hence $R_1(W,z)=0$ for all $(W,Y)\in REJ$, but since Y does not enter explicitly into $R_1(\cdot,\cdot)$ and since there is always some Y which could put (W,Y) into the rejection region, $R_1(\cdot,\cdot)$ must equal zero everywhere. By symmetry, the same applies for $R_2(\cdot,\cdot)$.∎

A second threshold value contracting procedure requires a temporal separation between the two subprojects. Let us assume that the second stage (outcome z) cannot begin until the first stage outcome x is realized. Then it might be the case that the project is begun regardless of the initial reports, but that the principal reserves an option to cancel midway. Suppose, for example, that the project involves constructing a prototype for future mass production, and that the costs of mass production are expected to be directly proportional to the initial R&D cost. The scientific knowledge gained from completing the first stage of the project might be deemed worthwhile regardless of cost. But the second stage might be shelved if the revised expected total R&D cost ($x+Y$, if Y is the reported expectation of z) exceeds a critical value.

Define CONT and CAN as, respectively, the second-period continuation and cancellation regions on (x,Y). As with REJ above, it is stipulated that (c) CAN has a non-empty interior and that (d) no value of x or Y can guarantee project continuation by itself. The cancellation condition amounts to imposing the restriction

(4.28) $H(\cdot,x,Y,\cdot)$ is independent of z for all $(x,Y)\in CAN$.

Pursuing the same intuitive approach as before, suppose that each net payoff component can be shown to be independent of z for all (x,Y) in CAN. The term in $R_1(W,z)$ is, as we have indicated, a lottery in z on a first-period scoring rule. If there are some (x,Y) combinations for which z is not observed, the rule can never depend on z, i.e., $R_1(W,\cdot)$ is independent of z. The second component, a lottery in x on a second-period scoring rule, must have zero fines when the project is cancelled, so that $R_2(x,Y)=0$ for all $(x,Y)\in CAN$. The remaining components are based on multiples of $B(Y,z)$, which can only be independent of z if the multipliers are zero. Hence $R_2(x,Y) = R_3(W,Y) = R_4(W,Y) = 0$ for all $(x,Y)\in CAN$, and this in turn implies that $R_3(\cdot,\cdot)$ and $R_4(\cdot,\cdot)$ vanish everywhere. The conversion of indefinite integrals to definite integrals allows for an extra payoff term $R_5(\cdot,\cdot)$ in x and z, but since this must also be independent of z for $(x,Y)\in CAN$, $R_5(\cdot,\cdot)$ must depend only on x. [4] Below these claims are demonstrated formally.

Proposition 4.3: If projects are cancelled for (x,Y) belonging to CAN, where CAN satisfies (c) and (d), then differentiable proper scoring rules for members of Ω can be expressed as

$$(4.29) \qquad \int R_1(W)A(W,x)dW + \int R_2(x,Y)B(Y,z)dY + R_3(x)$$

where the second integral is normalized to equal zero on the cancellation region (implying $R_1(x,Y)=0$ there). Second-order conditions require

$$(4.30) \qquad R_1(W)E^F[A_w(W,x)]\leq 0 \text{ for all } F\in M_A^{-1}(W), \text{ and}$$

$$R_2(x,Y)E^G[B_Y(Y,z)]\leq 0 \text{ for all } G\in M_B^{-1}(Y).$$

If these conditions are met with strict inequality, then a contract satisfying (4.29) is globally proper.

Proof: Step 1: In the general form (4.14), $R_3(\cdot,\cdot) = 0$.

Condition (4.28) implies that $H_{wYz}(\cdot,x,Y,\cdot)=0$ for all $(x,Y)\in CAN$, or

$$J(W,x,Y)B_z(Y,z) + K(W,x,Y)B_{Yz}(Y,z) = 0,$$

where $J(W,x,Y) = R_3(W,Y)A(W,x) + R_{4Y}(W,Y)A_w(W,x) + R_{4wY}(W,Y)A(W,x)$

and $K(W,x,Y) = R_{4w}(W,Y)A(W,x) + R_4(W,Y)A_w(W,x)$

Rearranging terms,

$$(4.31) \qquad J(W,x,Y)\{K(W,x,Y)\}^{-1} = -B_{Yz}(Y,z)\{B_z(Y,z)\}^{-1}$$

unless $J(\cdot,\cdot,\cdot)=K(\cdot,\cdot,\cdot)=0$. Since the left-hand side of (4.31) is

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[4]If assumption (d) is relaxed, $R_3(\cdot,Y)$ and $R_4(\cdot,Y)$ need not be zero for values of Y guaranteeing project continuation, and $R_0(x,\cdot)$ need not be independent of z for values of Y guaranteeing project continuation.

independent of z and the right-hand side is independent of W and x, the value of (4.31) must be a function $L(\cdot)$ of Y alone. Hence $J(W,x,Y) = L(Y)K(W,x,Y)$. Substituting the expressions for $J(\cdot,\cdot,\cdot)$ and $K(\cdot,\cdot,\cdot)$ into (4.31) and rearranging terms, we have

$$(4.32) \qquad [R_{4Y}(W,Y) - L(Y)R_4(W,Y)]\, A_W(W,x) = [L(Y)R_{4W}(W,Y) - R_3(W,Y) - R_{4WY}(W,Y)]\, A(W,x)$$

For all $F \in M_A^{-1}(W)$, the expectation of the right-hand side of (4.32) with respect to x must be 0. By Lemma 4.3, the expectation of $A_W(W,x)$ cannot vanish everywhere; hence $R_{4Y}(W,Y)$ must equal $L(Y)R_4(W,Y)$. $R_{4WY}(W,Y)$ in turn must equal $L(Y)R_{4W}(W,Y)$. Substituting back into (4.32) establishes that $R_3(W,Y)=0$.

Step 2: In the general form (4.14), $R_2(x,Y) = 0 \; \forall (x,Y) \in CAN$.

Next, for $F \in M_A^{-1}(W)$, consider $E^F[H_Y(W,x,Y,z)] = E^F[R_2(x,Y)]B(Y,z)$. This must be independent of z whenever $(x,Y) \in CAN$. Hence $E^F[R_2(x,Y)]=0 \;\; \forall F \in M_A^{-1}(W),\; \forall (x,Y) \in CAN$. Since this holds for all W -- and hence for all probability distributions -- $R_2(x,Y)$ must vanish for all $(x,Y) \in CAN$.

Step 3: In the general form (4.14), $R_4(\cdot,\cdot) = 0$.

Since, for all $(x,Y) \in CAN$, $E^G[H_W(W,x,Y,z)] = H_W(W,x,Y,z)$ for all distributions $G(\cdot)$ on z, we must have in particular that for all $G \in M_B^{-1}(Y)$,

$$(4.33) \qquad E^G[R_1(W,z)]A(W,x) = R_1(W,z)A(W,x) + R_{4W}(W,Y)A(W,x)B(Y,z) + R_4(W,Y)A_W(W,x)B(Y,z).$$

Taking the expectation of (4.33) with respect to x, we have $R_4(W,Y)E^F[A_w(W,x)]B(Y,z)=0$ for all $F(\cdot)$ in $M_A^{-1}(W)$. Invoking Lemma 4.3, $R_4(\cdot,\cdot)$ must equal zero.

Step 4: Substituting $R_4(\cdot,\cdot)=0$ into (4.33), $R_1(W,\cdot)$ is seen to be independent of z. Conversion of indefinite to definite integrals establishes form (4.29) except for a possible term in $r(x,z)$. But $r(\cdot,\cdot)$ must be independent of z for all (x,Y) in CAN and hence for all x, so that it can be incorporated in $R_3(x)$. Second-order conditions follow immediately from substitution into (4.15) and (4.16). Since there is only one stationary point, local incentive compatibility suffices for global incentive compatibility.∎

From an economic standpoint, the scoring rules described in Proposition 4.3 are essentially two single-period scoring rules spliced together. The first component is completely unaffected by second-period report or outcome. The second relies on first-period outcome to select a particular payoff scheme, which is then applied without reference to first-period report. Hence the fines for report/outcome discrepancies in either period are independent of the report for the other period. Furthermore, the fine for second-period discrepancies must be zero whenever (x,Y) belongs to CAN, as then the project is cancelled.

To understand why all other component scoring rules are excluded, consider that any component scoring rule taking second-period outcomes into account must be "triggered" by appropriate (x,Y) values: if (x,Y) lies in CAN it cannot be applied. Hence a generating function

$R_1(\cdot,\cdot)$ which was non-separable in (W,Y) would depend indirectly on xw as well, thus violating the stricture on separability of W and x in generating functions. Similarly, generating functions non-separable in (x,z) or (W,z) would depend indirectly on Y, which would violate the stricture on Y,z-separability.

Let us now turn to a third type of threshold value contracting, in which both pre-project rejection and mid-project cancellation are allowed. Suppose, as in the first problem of this section, the government wants to reject the project if estimated costs are too high. But even if the project is begun, the government may wish to reserve an option to cancel later, should first-period cost far exceed the forecast. For example, suppose that the result of the first two stages is a prototype for future mass production, with costs of production anticipated to be proportional to total research and development costs. Then even though first-period costs are sunk, the government may rationally prefer cancellation over continuation should revised expected total costs be too high. (It would however be irrational to impose more stringent criteria for cancellation than for initial acceptance.). How can the government elicit the information it wants?

To derive the set of applicable scoring rules, we simultaneously impose conditions (4.25) of the first problem and (4.28) of the second problem. (4.28), as we have seen, implies the scoring rule is additively separable in W and Y. (4.25) adds that the score is zero whenever a (W,Y) combination induces project rejection. Clearly, these requirements are incompatible, unless the score is uniformly zero. Alternatively, consider that allowing for mid-project cancellation can be viewed as requiring fines for report/outcome discrepancies in either period to be independent of reports for the other period. (4.25) in effect adds that fines must vanish whenever one of the reports is sufficiently high. Hence the fines must always be zero.

Proposition 4.4: If no first-period estimate can by itself guarantee initial project acceptance, and if no second-period estimate can by itself guarantee mid-project continuation, there is no differentiable strictly proper scoring rule for eliciting the messages in Ω.

Proof: *We know that differentiable proper scoring rules for our problem must be a subset of those described in (4.25).* From (4.29), $E^G E^F[H_{WY}(\cdot,\cdot,\cdot,\cdot)] = H_{WY}(\cdot,\cdot,\cdot,\cdot) = 0$. Hence for $W=M_A(F)$ and $Y=M_B(G)$ we have $R_4(W,Y)E^F[A_W(W,x)]E^G[B_Y(Y,z)]=0$, or $R_4(W,Y)=0$. $H_{WY}(W,x,Y,z)$ in turn equals $R_3(W,Y)A(W,x)B(Y,z)$, so that $R_3(\cdot,\cdot)$ must equal zero.∎

4. The Nature of the Threshold Dilemma

The last result is a surprising one, and difficult to grasp intuitively. It is clear enough why a non-zero contract that calls for project rejection in case either report is too high cannot be separable in the two reports. But why should differentiable non-separable contracts forfeit their incentive-compatibility once they allow for mid-project cancellation? Here it may be helpful to trace through an example with elicitation of two means.

Suppose in (4.6) and (4.7) of Proposition 4.2 that all the $Q_j(\cdot,\cdot)$'s except $Q_1(\cdot,\cdot)$ are zero. Denote $Q_1(W,Y)$ by Q, and the partial derivatives of Q with respect to W and Y by Q_W and Q_Y respectively. (From (4.8) and (4.9), these partial derivatives must be nonpositive if very high reports cause project rejection.). If a project is cancelled midway, z will be taken as Y in the formula for net utility payoff, so that the firm gets "the benefit of the doubt" in case of cancellation. In other words,

$$H(W,x,Y,z) = Q - Q_W\,(W - x) - Q_Y\,(Y - z) \text{ if } (x,Y)\in CONT$$

$$= Q - Q_W\,(W - x) \qquad\qquad \text{if } (w,Y)\in CAN$$

Expected payoff Π for reports W and Y is given by

$$\Pi = Q - Q_W\Delta W - PQ_Y\Delta Y,$$

where $\Delta W\equiv W-W^*$ and $\Delta Y\equiv Y-Y^*$ denote the discrepancies between the reported means and their true values (W^*,Y^*), and $P\equiv P(Y)\equiv Pr\{w|(w,Y)\in CONT\}$ is the probability, given second-period report Y, that a started project will be finished. P is a nonincreasing function of Y, and will be presumed to be differentiable. First-order conditions for maximization then require

$$\Pi_W = -Q_{WW}\Delta W - PQ_{YW}\Delta Y = 0, \text{ and}$$

$$\Pi_Y = -Q_{WY}\Delta W + (1-P)Q_Y - PQ_{YY}\Delta Y - P_YQ_Y\Delta Y = 0.$$

These are simultaneous linear equations in ΔW and ΔY, with solution

$$\Delta W = \frac{-Q_{YW}Q_YP(1-P)}{J}; \quad \Delta Y = \frac{Q_{WW}Q_Y(1-P)}{J};$$

where $J\equiv(Q_{WW}Q_{YY}-Q_{WY}Q_{WY})P+Q_{WW}Q_YP_Y$ is assumed to be non-zero. To sign these expressions, note that P can never be negative, while P_Y and Q_Y can never be positive. Second-order conditions (4.8) and

(4.9) require $Q_{ww}\geq 0$, $Q_{YY}\geq 0$, and $Q_{ww}Q_{YY}-Q_{wY}Q_{wY}\geq 0$. So ΔY will be negative, and ΔW will have the same sign as Q_{wY}. Thus the agent underestimates Y and tends to misreport W as well.

For an economic interpretion of the above, recall that the ex-ante reward Q and ex-post fines Q_w and Q_Y are calibrated so that truthful revelation is the agent's best strategy assuming no mid-project cancellation. Note too that Q_Y is negative, so that cost overruns are penalized. Once mid-project cancellation is possible, there is some chance the second-period fine will not be assessed. That lowers the fine's "threat" value, and encourages underestimation of Y. Now some second-period penalty on average is anticipated. To reduce its amount, the agent readjusts W, subject to concern for effects on other components of net payoff. If Q_{Yw} is positive, the negative Q_Y will decrease in magnitude as W increases, so that it pays to overestimate first-period costs. Conversely, a negative Q_{Yw} is associated with underestimation of W.

Since the difficulty appears to be caused by the occasional non-assessment of second-period fines, one might hope to compensate by raising those fines which are assessed. Let us see what happens when this is attempted. Should the second period fine be scaled upward by a factor M, expected payoff π will now be $Q-Q_w\Delta W-MPQ_Y\Delta Y$. This is uniquely maximized by truthful revelation only if MP(Y) happens to equal 1, which cannot be true for all Y. So we see the incentive problem stems not from the need for scaling per se, but uncertainty about what degree of scaling is needed.

This last remark suggests attempting to restore incentive compatibility by eliciting P as well as W and Y. To express the same idea more formally, perhaps the problem can be remedied by expanding the size of the message space. Also, it may be wondered whether allowing for nondifferentiability might help. Certainly between the two there must be at least some improvement, for the government could offer a multiple of its own utility function to the firm as payment and delegate to the firm the task of maximizing its expected value. This would be a nontrivial compensation scheme, though only weakly incentive compatible.

For a strictly proper but not quite smooth scoring rule allowing for both pre-project rejection and mid-project cancellation, consider the following example. Suppose x and z can each take only 2 values 0 and 1, with W and Y their reported expectations and W^* and Y^* the true values. Choose an arbitrary rejection region, and let a started project be cancelled whenever $x=1$ occurs. Define a scoring rule

$$Q(W,Y) - Q_W(W,Y)(W - x) - Q_Y(W,Y)(1 - W)^{-1}(1 - x)(Y - z)$$

where $Q(\cdot,\cdot)$ equals zero on the rejection region. It is readily checked that profit is zero on the rejection region and independent of z in case of cancellation. Dropping the argument list on $Q(\cdot,\cdot)$, first-order conditions for expected profit maximization require

$$-Q_{WW}(W - W^*) - \frac{\partial}{\partial W}[Q_Y(1-W)^{-1}](1-W^*)(Y-Y^*) = 0$$

and

$$-Q_{WY}(W-W^*) + Q_Y - Q_{YY}(1-W)^{-1}(1-W^*)(Y-Y^*) - Q_Y(1-W)^{-1}(1-W^*) = 0$$

with solution $W=W^*$, $Y=Y^*$. Second order conditions require $Q_{WW}Q\leq 0$

and $Q_{WW}Q_{YY} \geq [Q_{WY}-Q_Y(1-W)]^2$. [5] Let us also note that even if we stick to smooth contracts with incomplete reports, in principle the government can, by relaxing its standards for project cancellation ever so slightly, reduce efficiency losses to very low levels. The "relaxation" entails the following: instead of cancelling **every** undesirable project ($(x,Y) \in CAN$), the government announces that **occasionally**, with probability ϵ, such projects will be continued. Given any strictly proper scoring rule contract allowing for pre-project rejection, modify it so that the agent receives 0 whenever $(x,Y) \in CAN$ and the project is cancelled, and $\frac{1}{\epsilon}H(W,x,Y,z)$ when $(x,Y) \in CAN$ and the project is continued. Incentive compatibility will not be affected as expected payoff remains the same. By squeezing ϵ towards zero, so that undesirable projects are almost certain to be cancelled, expected "undesired" second-stage construction expenditures can be made arbitrarily small.

Unfortunately, this plan suffers from a credibility problem. Once W and Y have been reported, the government has little incentive to follow through on its commitment to continue "undesired" projects occasionally. In the random case that continuation of an "undesired" project is advised, the government generally has an incentive to bribe the firm into **not** undertaking the project. However, if the firm anticipates the government trying to buy out of its commitment, it

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[5]This example is a bit perplexing. Is it the smaller distribution space (with two outcomes per variable, the reports provide complete information) or the allowance for nondifferentiability which is crucial? Unfortunately, I have no positive results to report. Resolution awaits future research.

will tend to misreport W and Y. To borrow the corresponding game theory terminology, the announced stochastic strategy is not subgame-perfect.

5. Scoring Rules for Dependent Random Variables

The usefulness of the preceding discussion may be questioned on the grounds that the truly interesting problems of multi-period elicitation involve correlation of outcomes across time. It turns out, however, that the analysis of elicitation of independent variables lends itself quite readily to the analysis of dependent variable elicitation.

Let us consider first the case in which the two outcomes may be correlated, but the principal does not know what the correlation is and must allow for every possibility. It is easy to characterize proper scoring rules for this problem. Simply think of (x,z) as the two-component outcome v of a **single** random variable V. Outcome x is then a projection $r_1(\cdot)$ of v and outcome z is another projection $r_2(\cdot)$ of v. To say that any possible correlation is allowed is equivalent to saying that V can have any distribution. Hence scoring rules for $M_a(\cdot)$ and $M_b(\cdot)$ are fully characterized in terms of first- and second-order conditions by equations (2.7) and (2.8) of Theorem 2.1, with $A_1(W,Y,v)$ defined as $A(W,r_1(v))$, and $A_2(W,Y,v)$ as $B(Y,r_2(v))$. First-order conditions, for example, are

$$H_w(W,x,Y,z) = K_{11}(W,Y)A(W,x) + K_{12}(W,Y)B(Y,z);$$

$$H_y(W,x,Y,z) = K_{21}(W,Y)A(W,x) + K_{22}(W,Y)B(Y,z).$$

If some aspects of the correlation are known, a larger class of proper scoring rules may be found by applying Theorem 2.2. Suppose for example that it is desired to elicit the means of x and z, where the two variables are known to be uncorrelated but not necessarily independent. Using the terminology of Theorem 2.2, $A_1(W,Y,v)$ equals $W-x$, $A_2(W,Y,v)$ equals $Y-z$, and the restriction can be expressed by $B(C(W,Y),v) = (W-x)(Y-z)$. Then we have

$$H_W(W,x,Y,z) = K_{11}(W,Y)(W-x) + K_{12}(W,Y)(Y-z) + L_1(W,Y)(W-x)(Y-z);$$

$$H_Y(W,x,Y,z) = K_{21}(W,Y)(W-x) + K_{22}(W,Y)(Y-z) + L_2(W,Y)(W-x)(Y-z).$$

These first-order conditions can be solved along the lines of the proof of Proposition 2.1 (elicitation of vector expectations) to establish that $H(\cdot,\cdot,\cdot,\cdot)$ takes the form

$$(4.34) \qquad Q_1(W,Y) - Q_{1W}(W,Y)(W - x) - Q_Y(W,Y)(Y - z) + Q_0(W,Y)(W - x)(Y - z) + Q_4(x,z)$$

Comparing (4.34) with the form (4.6)-(4.7) of proper scoring rules for elicitation of independent random variables, we see that the only difference is the elimination of terms in $Q_2(W,z)$ and $Q_3(x,Y)$. This is quite reasonable: the possibility of dependence between the fine $Q_{2W}(W,z)$ and the discrepancy $W-x$ would impair incentive compatibility, and the same for the fine $Q_{3Y}(x,Y)$ and the discrepancy $Y-z$. With any possibility of correlation, the term in $Q_0(W,Y)$ must be eliminated as well, leaving a proper scoring rule for a two-component expectation.

When the principal does not know the correlation but wants to elicit it, matters are more complicated. Suppose that first-period outcome can take one of n values $x_1,..,x_n$ To each x_j there corresponds a perceived distribution $G_j(\cdot)$ of z. For example, the agent may believe that costs in the two subprojects are positively correlated, so that an increase in x shifts the distribution of z to the right. Instead of requesting one summary message $M_B(G)$ about the unconditional distribution of z (that is, about $G(\cdot) = \sum_1^n p_j G_j$, where each p_j is the probability of x_j occurring), the principal requests a message $Y_j = M_{B_j}(G_j)$ for each $G_j(\cdot)$. The agent's net compensation is then a function of W, Y_1 through Y_n, x, and z.

The easiest way to solve this dependent-variable elicitation problem is to treat it as a special case of an **independent**-variable elicitation problem. Define z_j as the second-period realization if x_j occurs, for j=1 to n. These realizations for different values of j can be thought of as independent. Let $H(W,Y_1,..,Y_n,x,z_1,..,z_n)$ be a proper "independent variable" scoring rule for the message functions associated with $A(\cdot,\cdot)$, $B_1(\cdot,\cdot)$,$B_2(\cdot,\cdot)$, etc. Then $H(\cdot)$ is a proper scoring rule for the dependent-variable problem as well, provided it satisfies the following restriction:

(4.35) If $x=x_j$ occurs, net payoff is independent of z_k

$[H_{z_k}(\cdot,x=x_j) = 0]$ for all $k\neq j$ and all j.

(4.35) is simply a formalization of the statement "if x_j occurs, z_k is not observed for $k\neq j$." Conversely, every proper scoring rule for the restricted problem must be a proper scoring rule for the unrestricted problem.

Pursuit of this approach is hampered by our lack of a multi-variable generalization of Theorem 2.1. Conceptually such a generalization does not appear too difficult, but proofs tend to get bogged down in long algebraic manipulations. I shall accordingly confine myself to an observation about proper scoring rules for the restricted dependent-variable problem. Consider scoring rules $H(\cdot)$ of the following form:

$$\int R_1(W)A(W,x)\,dW + \sum_{1}^{Q}\int R_{2j}(x,Y_j)B_j(Y_j,z_j)\,dY_j \tag{4.36}$$

where $R_{2j}(x_k,Y_j) = 0$ for all $k \neq j$ and all j.

Subject to satisfaction of second-order conditions, (4.36) meets both the conditions for independent-variable proper scoring rules and restriction (4.35). As other generalizations of terms in (4.14) appear to invariably violate (4.35), I conjecture that the form (4.36) is necessary as well.

6. Summary

In the general case, with project implementation assured regardless of reports, the class of two-variable proper scoring rules is found to be somewhat broader than the class of single-variable rules. This allows the planner some additional options in drawing up contracts. However, the class rapidly shrinks once other criteria are imposed, such as allowances for initial project rejection or mid-project cancellation. Proper scoring rules for dependent variables can be derived as special cases of proper scoring rules for independent variables.

CHAPTER 5: OPTIMAL ELICITATION WITH AGENT LEARNING COSTS

In the quest to uncover classes of incentive-compatible scoring rules, previous chapters have skirted issues of optimality: the choice of rules within a class. Indeed, the framework presented in those chapters provides few if any insights into that choice. Presumably the planner would be interested in minimizing expected payoff for truthful reports subject to some sort of income or expected income floor for the agent. But this consideration does not by itself lead to a very interesting choice. Let $H(\cdot,\cdot)$ be a strictly proper scoring rule with positive net payoffs. For any positive α and any z, $\alpha H(\cdot,\cdot)+z$ is then strictly proper with payoffs of at least z. By setting z at the agent's minimum income and letting α approach 0, strictly proper scoring rules can be squeezed arbitrarily close to the minimum -- and only weakly proper -- flat fee.

The apparent superfluousness of scoring rules arises for two reasons. First, the agent is assumed to know the outcome distribution without incurring any learning costs, or at least without choosing what those costs will be. Second, the agent is assumed unable to influence that distribution by exerting or withholding effort. Relax either of these assumptions and a real tradeoff arises between low information transfer costs and high incentives for effort.

In this chapter agent learning costs will be explicitly incorporated into the model, while maintaining the assumption of agent-independent outcomes. The agent, who is asked to report the mean of an uncertain or random outcome, will begin with a rough estimate. By making additional effort, the agent can make this estimate more precise, but the effort is not directly observed. Contracts for inducing a desired degree of learning are derived, and their expected administrative costs analyzed. At the optimum, the marginal expected benefit to the planner from more precise estimates must be balanced by the marginal expected administrative cost. By comparing the solution to this problem with that obtained when effort can be observed, costs of agency can be explicitly determined, and their response to changes in various parameters analyzed.

In a further twist, possibilities are explored for reducing administrative costs through direct investigation and through employment of competing agents. It turns out that two heads can indeed be cheaper than one. In a method recalling that of a second-price auction, every agent's report is used to help evaluate the new information content of every other agent's reports and to help target rewards accordingly. That competition can reduce the cost of information transfer has been shown already by a number of economists, including Nalebuff and Stiglitz (1983) and Lazear and Rosen (1981). What is novel here is, first, the extension to a new class of problems, and, second, the explicit relating of the benefits of competition to the number of agents and other environmental parameters.

While this work is motivated by an interest in government R&D contracting problems, the problems described should be familiar to users of econometric services. How can forecasters be motivated to exercise a specified degree of care before making their predictions?

How much can this "motivation" be expected to cost, and, in light of this cost, what amount of care is optimal? What benefits, if any, are to be had by employing more than one forecaster or forecasting agency? This chapter provides some answers to each of these questions, assuming that the the underlying relationship between forecasting cost and forecasting precision is known.

1. *The Model Without Agency Costs*

Suppose a planner must rely on some estimate W of an R&D project's uncertain costs in order to formulate next year's budget. Should actual cost x deviate from the estimate, some opportunity loss L is incurred proportional to the square of the discrepancy:

$$L(W,x) = c(W - x)^2; \quad c > 0.$$

If $F(\cdot)$ is the distribution of x, with mean W^* and variance V^*, expected planning loss conditional on W can be expressed as

$$E^F[c(W - x)^2] = cW^2 + cWW^* + c(W^{*2} + V^*) = c(W - W^*)^2 + cV^*.$$

Clearly this is minimized when $W=W^*$. Its value there, cV^*, is the planning loss anticipated from inherent project riskiness.

Suppose the planner does not know either W^* or V^* exactly, due to uncertainty about the underlying cost distribution. Instead she possesses some initial estimate W_o of W^*, believed to be an unbiased predictor of W^* with variance σ_o^2. [1] Overall expected planning loss is then cV^* plus c times the expectation of $(W_o-W^*)^2$, or $cV^*+c\sigma_o^2$. In Bayesian terminology, the mean squared error equals

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[1]Estimates of V^* cannot affect planning losses in our problem and are therefore ignored.

the variance of x around its mean plus the variance of the mean "around" its predictor.

To reduce expected losses, the planner can try to refine her estimate of W^* by undertaking investigations. By assumption, each investigation i "measures" W^* with an error ε_i, where the ε_i's are independent random variables with mean 0 and variance $\sigma_i^2=\sigma^2$. With initial estimate W_0 and n measurements $W_1,..,W_n$, the best linear unbiased estimate (BLUE) $\hat{W}_n$ of W^* is given by the formula $(\sum_0^n h_i W_i)(\sum_0^n h_j)^{-1}$, where $h_i = \sigma_i^{-2}$. Its variance equals $(\sum_0^n h_j)^{-1}$. (See Lemma 5.1 in the appendix to this chapter for derivations.). In general, the inverse of the variance is known as the **precision.**. Letting $h=\sigma^{-2}$ denote the precision of measurements 1 through n, and $rh=\sigma_0^{-2}$ denote the precision of the initial estimate (where $r=\sigma^2\sigma_0^{-2}$), the precision of $\hat{W}_n$ is just $(r+n)h$. Expected planning loss using $\hat{W}_n$ as an estimate is then $c(r+n)^{-1}h^{-1}+cV^*$.

Suppose each measurement costs the planner b. Let C(n) denote the sum of expected planning losses and investigation costs for n measurements:

$$C(n) = \quad c(r + n)^{-1}h^{-1} + bn + cV^* \quad \text{for } n=0,1,2,...$$

The planner chooses n to minimize C(n). Treating $C(\cdot)$ as if n could vary continously, we derive first-order conditions for a positive solution:

$$(5.1) \qquad C'(n^*) = - c(r + n^*)^{-2}h^{-1} + b = 0,$$

$$\text{or} \quad n^* = - r + c^{1/2}(bh)^{-1/2}.$$

Since $C(\cdot)$ is convex, the optimal number of measurements must be within 1 of n^*. From (5.1), we see that the optimal final estimate precision $(r+n^*)h$ investigations rises with (but proportionately only half as fast as) the cost of prediction error c; It declines with (but proportionately only half as fast as) the measurement cost per extra unit of estimate precision $\frac{b}{h}$. The direction of change accords with economic intuition. Final estimate precision is independent of r, suggesting that having a prior with precision rh amounts to starting out with r free investigations.

For integer n^*, total planning costs at the interior optimum amount to:

$$C(n^*) = c[c^{-1/2}(bh)^{1/2}]^{-1}h^{-1} - b(r + c^{1/2}(bh)^{-1/2}) + cV^*$$
$$= 2c^{1/2}b^{1/2}h^{-1/2} - br + cV^* = 2bn^* - br + cV^*.$$

For n^* not an integer, total planning costs will be slightly higher. Having admitted this qualification, we shall henceforth ignore the integer constraint. Let the expected administrative cost (EAC) denote that part of $C(n^*)$ over and above the "unavoidable" cost cV^*. EAC rises with measurement cost, measurement variance, and the cost of estimate imprecision. As r rises, EAC falls at rate b, which is again consistent with the notion that having a prior with precision rh amounts to having r free investigations. When n^* is much greater than r (that is, the prior has little weight in the final estimate), the ratio of EAC to $2bn^*$ is approximately one. Here EAC rises with (but proportionally only about half as fast as) measurement cost, measurement variance, and the cost of prediction error.

Before moving to analysis of agency costs, let us briefly recast the model in different terms. Consider a forecaster investigating the stochastic relation $x_i = W^* + \epsilon_i$, where W^* is unknown and ϵ_i are independent random variables with mean 0 and variance $\frac{1}{h}$. Initially r observations are available free; additional observations cost b apiece. Ultimately, a prediction W of x_T is desired; discrepancies will be assessed fines of $c(W-x_T)^2$. The forecaster's problem is to determine the optimal forecasting procedure and its expected costs. Formally, the only difference between this description and the preceding one is that now the variance V^* of x is known and equal to the variance of measurement $\frac{1}{h}$.

2. Investigation Via an Agent

When attempting to refine her estimate of W^*, often the planner must rely on someone else to investigate for her. Such is particularly likely to be the case if the planner is removed from the direct production process. Presumably the prospective contracting agent can better formulate and refine cost estimates. The planner then faces a double problem: first, to induce the agent to report his beliefs accurately; second, to ensure that those beliefs are founded on the results of an appropriate investigation. This is the problem we shall explore in this section.

If the planner cannot a priori rule out any cost distributions, we know from before that proper scoring rules must take the form:

(2.4) $$H(W,x) = R(W) - R'(W)(W - x) + T(x) \quad \text{with } R(\cdot) \text{ convex.}$$

But not all $R(\cdot)$ will induce investigation. For example, if $R(\cdot)$ equals 0, the agent's net payoff is independent of his report, so he will not investigate at all. In general, incentives for investigation will vary with the convexity of $R(\cdot)$. Suppose the agent thinks the mean is either $W+\sigma$ or $W-\sigma$ with equal probability. Given (2.4), expected payoff is maximized by reporting W; its value there exclusive of the $T(\cdot)$ term is $R(W)$. Suppose an investigation costing b could determine the mean exactly. If it were carried out, expected payoff exclusive of $T(\cdot)$ and investigation cost would be either $R(W+\sigma)$ or $R(W-\sigma)$, each with probability 1/2. So a risk-neutral agent should undertake investigation if and only if $\frac{1}{2}[R(W-\sigma)+R(W+\sigma)]-R(W) \geq b$. The left-hand expression is just the distance from $\langle W,R(W)\rangle$ to the midpoint of the chord connecting $\langle W-\sigma,R(W-\sigma)\rangle$ with $\langle W+\sigma,R(W+\sigma)\rangle$. The more convex $R(\cdot)$ is, the larger σ is, and the smaller b is, the more attractive investigation will be (See Figure 5.1.).

Insert Figure 5.1

More generally, let W be the current estimated mean, and let $W+\theta$ be the reestimate after another measurement. The estimate correction θ is regarded as a random variable having a distribution $J(\cdot)$ with mean 0 and variance τ. At a cost of b, the measurement will be worthwhile if and only if

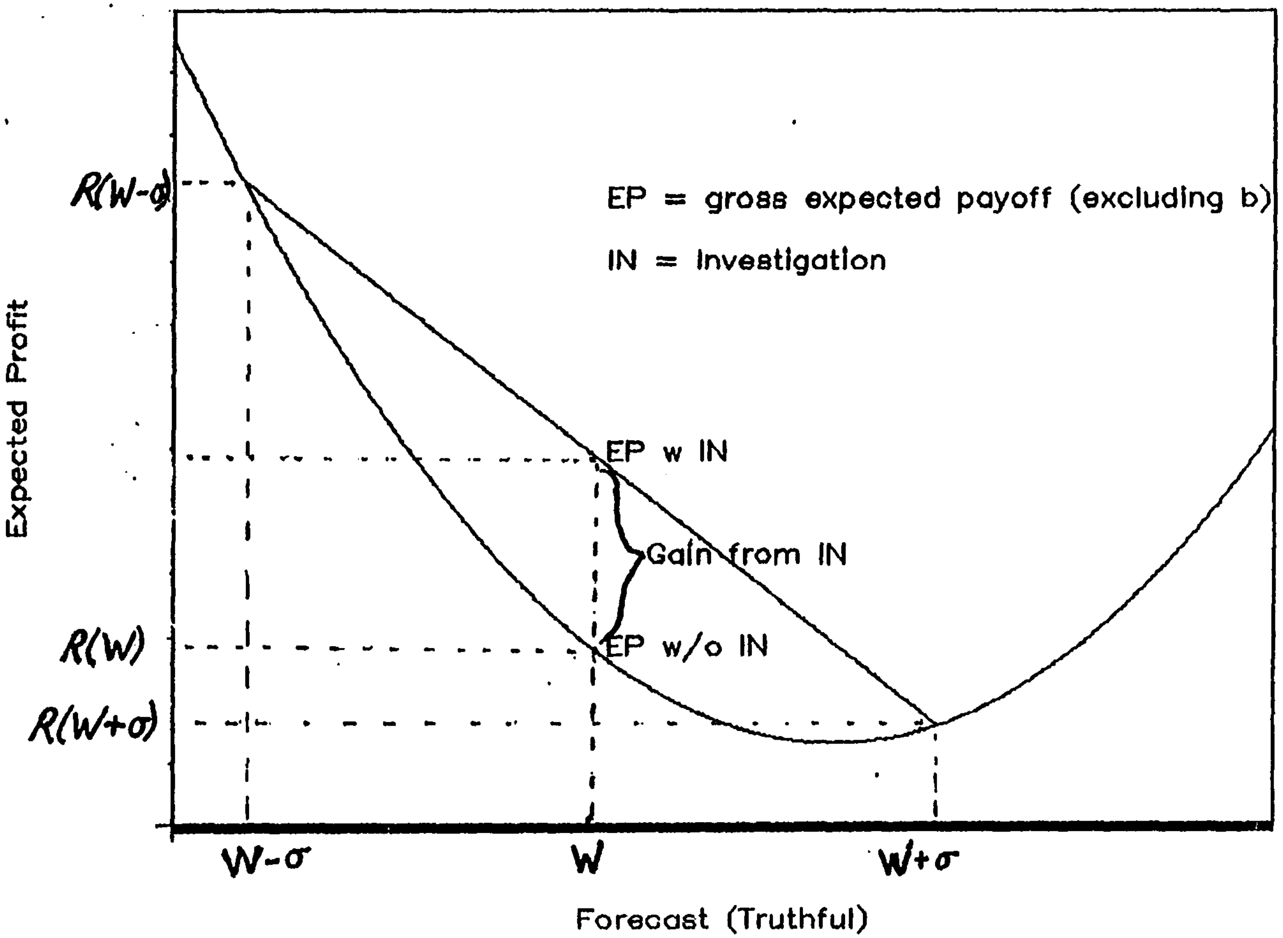
$$(5.2) \qquad E^{J}[R(W+\theta)] \geq R(W) + b.$$

(In case of indifference, the agent is assumed to take the measurement.).

To make the left-hand side of (5.2) more tractable, let us take a

Figure 5.1:

Benefits of Investigation



second-order Taylor series approximation to $R(W+\theta)$. This yields:

$$(5.3) \qquad E^J[R(W+\theta)] \doteq E^J[R(W) + \theta R'(W) + \tfrac{1}{2}\theta^2 R''(W)]$$
$$= R(W) + R'(W)0 + \tfrac{1}{2}R''(W)\tau = R(W) + \tfrac{1}{2}\tau R''(W)$$

Hence the investigation decision turns, to a first approximation, on whether $\frac{\tau}{2}R''(W)$ is greater or less than b. The greater the curvature of $R(\cdot)$, the lower the costs of investigation, or the greater the variance of re-estimation induced by investigation, the more advantageous investigation is for the agent. This accords with the impression given by Figure 1.

The simplest type of proper contract is that generated by quadratic $R(\cdot)$ and zero $T(\cdot)$. In this case the approximation in (5.3) is exact, which enormously simplifies calculations of investigation strategies and expected administrative costs. Leaving aside the tractability issue, however, it may be wondered whether the principal could do better by offering a more complicated type of contract. This is the question we now address.

Starting with a proper contract (2.4), suppose the principal wants to ensure that any investigation costing b and inducing an estimate correction of variance τ will be undertaken, regardless of its other characteristics. Formally, $R(\cdot)$ must satisfy:

$$(5.4) \qquad E^J[R(W+\theta)] \geq R(W) + b$$

for all W and all probability distributions $J(\cdot)$ such that

$$(5.5) \qquad E^J[\theta] = 0 \text{ and } E^J[\theta^2] = \tau.$$

Furthermore, suppose the agent will withhold the results of investigation unless the marginal expected return from doing so is at least 0. This "individual-rationality" constraint is formalized as:

$$(5.6) \quad R(E^F[x]) + E^F[T(x)] \geq 0, \text{ for all } F\in D.$$

In other words, R(W) plus the expectation of T(•) is at least 0 for all distributions having mean W and for all W. [2] Finally, suppose the principal's uncertainty about the results of the investigation is at least as great as the agent's. Then, under these conditions, the simplest contract is in fact the best.

Theorem 5.1: Suppose the principal wants to minimize the expected contract cost with respect to her prior distribution K(•) over reports, given that the contract satisfies (2.4) and (5.4)-(5.6). If the variance of K(•) is at least τ, the optimal contract is generated by

$$(5.7) \quad R(W) = \frac{b}{\tau}(W-W_o)^2 \text{ and } T(x) = 0,$$

where W_o is the mean of K(•).

Proof: Step 1: T(•) can be taken as zero.

Consider any contract (2.4). For every W, define Q(W) as the minimum value of $E_x[T(x)]$ for all distributions on x such that $E_x[x] = W$. Geometrically, Q(•) is just the lower envelope of the convex hull of T(•), and hence is a convex function. (This lower envelope must exist or else expected payoffs for some distributions are unbounded.).

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[2] Assumption (5.6) will be examined more closely in a later section.

Define $R^o(W) \equiv R(W)+Q(W)$, and consider the proper contract $H^o(W,x) = R^o(W)-R^{o\prime}(W)(W-x)$. Given truthful reporting, expected payoff for $H^o(\cdot,\cdot)$ equals $R^o(\cdot)$, which in turn equals the minimum expected payoff for $H(\cdot,\cdot)$ given truthful reporting. So $R^o(\cdot)$ can never be negative. Furthermore, $H^o(\cdot,\cdot)$ will induce any investigation induced by $H(\cdot,\cdot)$. For if $R(\cdot)$ satisfies (5.2), then

$$E^J[R^o(W+\theta)] = E^J[R(W+\theta)] + E^J[Q(W+\theta)]$$

$$\geq b + R(W) + E^J[Q(W+\theta)] \geq b + R(W) + Q(W) = b + R^o(W);$$

where the next-to-last step follows from the convexity of $Q(\cdot)$. Hence $H^o(\cdot,\cdot)$ is at least as good a contract for the principal as $H(\cdot,\cdot)$.

Step 2: Given any feasible initial estimate Y, there must exist constants s_1, s_2, and s_3 such that $R(Y+\theta) \geq s_1 + s_2\theta + s_3\theta^2$ for all θ,

where $s_1 \leq R(Y)$ and $s_1 + s_3\tau \geq R(Y) + b$.

The minimum value of $E^J[R(Y+\theta)]$ -- the expected gross payoff after investigation -- over the class of continuous distributions satisfying (5.5) is found by solving the following Lagrangean:

$$\int R(Y+\theta)j(\theta)d\theta - s_1(\int j(\theta)d\theta - 1) - s_2\int \theta j(\theta)d\theta$$
$$- s_3(\int \theta^2 j(\theta)d\theta - \tau) - \int t(\theta)j(\theta)d\theta;$$

where $j(\cdot)$ is the density of $J(\cdot)$. [3] First-order conditions require

(5.8) $$R(Y+\theta) = s_1 + s_2\theta + s_3\theta^2 + t(\theta),$$

----

[3]To allow for discrete jumps in the probability distribution, the expression '$j(\theta)d\theta$' should be replaced by '$dJ(\theta)$' and the integral interpreted in the Riemann-Stieltjes sense, i.e., as an ordinary Riemann integral plus a discrete sum. The analysis is the same.

with $t(\cdot)\geq 0$, $j(\cdot)\geq 0$, and $\int t(\theta)j(\theta)d\theta = 0$.

In particular, $R(Y)=R(Y+0)\geq s_1$. Substitution of (5.8) establishes that the constrained minimum for $\int R(Y+\theta)j(\theta)d\theta$ is $s_1+s_3\tau$, which from (5.4) must be at least $R(W)+b$.

Step 3: The principal's expected contract cost $E^G[R(W)]$ is at least $\frac{b}{\tau}\Gamma$, where Γ is the variance of $G(\cdot)$.

Applying Step 2 with $Y=W_o$ and $\theta=W-W_o$,

$$E^G[R(W)] \geq E^G[s_1 + s_2(W - W_o) + s_3(W - W_o)^2] = s_1 + s_3\Gamma .$$
$$\geq s_1 + \frac{\Gamma}{\tau}(b + R(W_o) - s_1) = \frac{b}{\tau}\Gamma + (\frac{\Gamma}{\tau} - 1)(R(W_o) - s_1) + R(W_o) \geq \frac{b}{\tau}\Gamma,$$

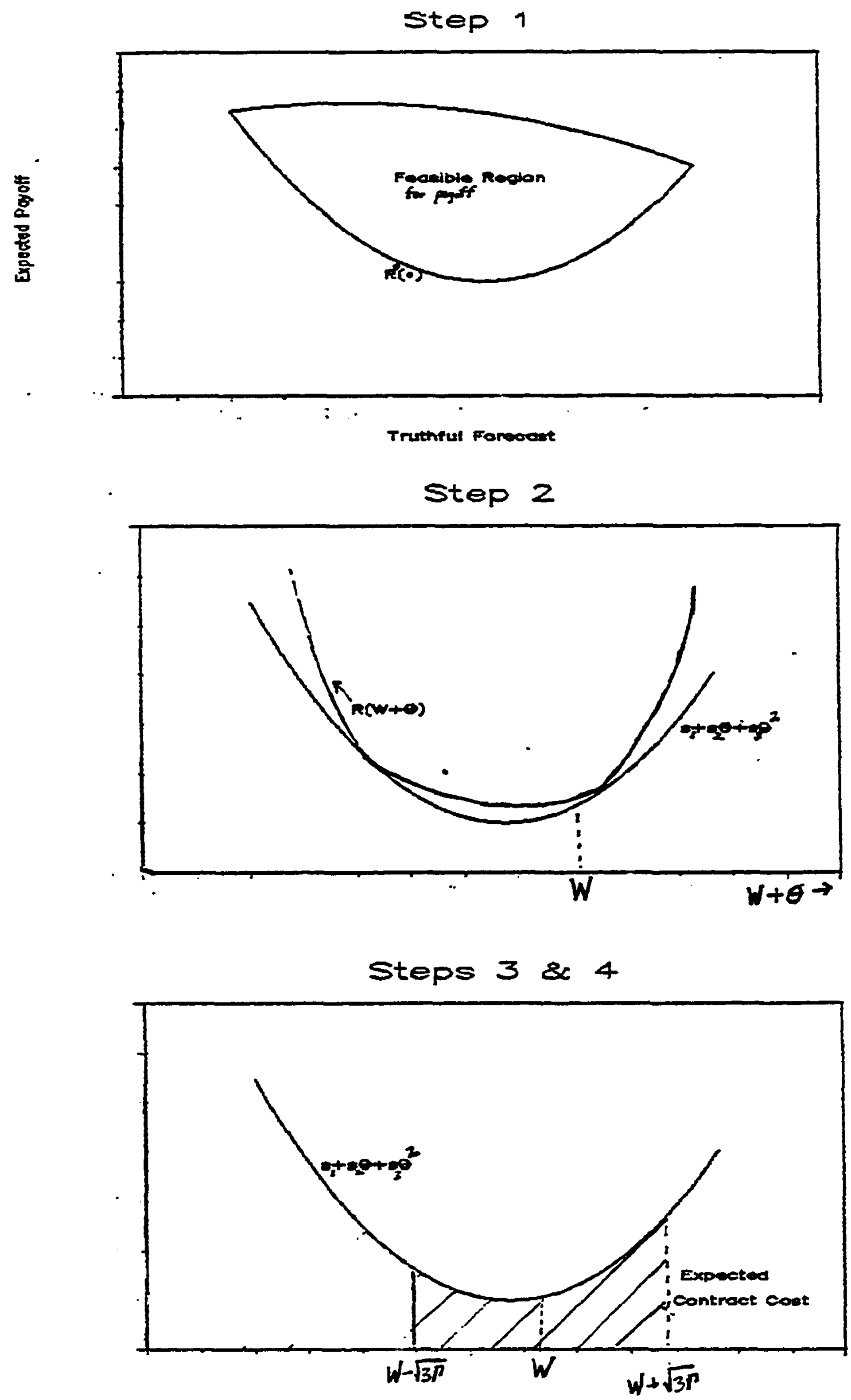
where the last inequality follows from the facts that $R(\cdot)\geq 0$ (required by (5.6) for $T(\cdot)=0$) and $\Gamma\geq\tau$.

Step 4: $R(W)=\frac{b}{\tau}(W-W_o)^2$ is optimal.

For anticipated mean payoff to never exceed its lower bound, $t(\cdot)$ in (5.8) must equal zero along with $s_1=0$ and $s_3=\frac{b}{\tau}$. $s_2=0$ is in turn required. to ensure that $R(\cdot)$ never falls below zero. Only the $R(\cdot)$ defined by (5.7) meets the requirements.∎

Parts of the above proof lend themselves to geometric interpretation. Given any scoring rule, graph the correspondence between the expected payoff for a truthful agent and the agent's report. For $T(\cdot)$ non-linear in x, the graph will be a convex region rather than a line, as two distributions could have the same mean but yield different expected payoffs. Step 1 shows that the lower envelope of this graph may be taken as $R(\cdot)$ and $T(\cdot)$ may be set at

Figure 5.2: Optimal Scoring Rules



zero with no expected loss to the principal. Step 2 says that to induce an investigation from any given point, $R(\cdot)$ must lie on or above a quadratic of specified curvature. The closer this quadratic comes to $R(\cdot)$ at the initial point, the less curved the quadratic need be, but there is a definite limit to how flat it can become. To interpret the last two steps, it is helpful to suppose the principal's prior distribution is uniform over the interval $D=[W_o-(3r)^{1/2},W_o+(3r)^{1/2}]$, so that it has mean W_o and variance r. Expected contract cost is then the area between $R(\cdot)$ and the W-axis over D. Since $R(\cdot)$ must lie on or above the maximum of the W-axis and one of the specified quadratics, the minimum area of the latter is the minimum expected contract cost. Steps 3 and 4 show that this minimum area is attained by one and only one contract -- a quadratic with the least-allowable curvature and a minimum value of 0 at W_o.

Insert Figure 5.2

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3. Costs of Agency

Let us return to the investigation problem discussed in Section 2, where multiple independent measurements of cost b and precision h are possible and the principal's prior estimate W_o has a precision of rh. This time, however, we assume that all measurements have to be taken by an agent, without the principal observing. Our analysis here will involve three steps. First, we shall apply Theorem 5.1 to determine the optimal contract for inducing n measurements, for any positive integer n. Second, we shall look for the n that minimizes the principal's total expected costs, including both expected planning losses and expected contract costs. Third, we shall examine the properties of the solution, and compare the solutions with and without agency.

If the agent and the principal share the same prior beliefs about the true mean, application of Theorem 5.1 is straightforward. We simply calculate the conditional variance of the estimate after n measurements, given the results of the first n-1 measurements. Use this variance in place of τ to determine the optimal contract for inducing the n-th measurement, assuming that n-1 measurements have been taken. Verify that such a contract will indeed induce the first n-1 measurements and we have our answer.

Theorem 5.2: To ensure that n independent measurements of cost b and precision h are taken and the results truthfully reported, given a shared prior mean W_0 of precision rh, the optimal contract is $R(W)-R'(W)(W-x)$, where

$$R(W) = b(r + n - 1)(r + n)h(W - W_0)^2. \tag{5.9}$$

Proof: Suppose n-1 measurements have been taken, so that the current estimate is $\hat{W}_{n-1}$. Applying Lemma 5.1, its precision is $(r+n-1)h$. An additional measurement W_n with precision h would yield a new estimate $\hat{W}_n$. Define $\theta_n = \hat{W}_n-\hat{W}_{n-1}$. Applying Lemma 5.2, θ_n has conditional mean 0 and conditional variance τ_n = $h(r+n-1)^{-1}h^{-1}((r+n-1)h+h)^{-1} = h^{-1}(r+n-1)^{-1}(r+n)^{-1}$. Since τ_n is decreasing in n, earlier measurements are at least as worthwhile to the agent as later ones, so that a scoring rule able to induce the n-th measurement would induce the first n-1 measurements as well. Substituting τ_n for τ in Theorem 5.1 shows that (5.9) generates the optimal scoring rule.∎

The assumption of shared prior beliefs between principal and agent may not at first appear to be very important. Let the agent's prior estimate be W_a with precision $r_a h$, which may or may not equal W_0 and rh respectively. Proceeding as before, the formula for the variance τ_n of the n-th estimate correction is altered only by the substitution of r_a for r. It is tempting then to conclude that the contract generated by

$$(5.10) \qquad R_a(W) = b(r_a + n - 1)(r_a + n)h(W - W_0)^2$$

is optimal for inducing n investigations.

This conclusion may not be valid, however, if the agent's final estimate does not incorporate all the information available to the principal. To clarify the nature of the difficulty, let us consider the following hypothetical example. A principal and agent get their information from two independent sources. The principal's information is of such good quality that her prior W_0 is as precise as the agent's estimate $\hat{W}_n$ after investigation. To keep that information private until after the agent reveals his estimate, the principal announces the following offer to the agent:

"I will pay you R(W)-R'(W)(W-x) based on your report W and the actual cost x, where R(W) is given by (5.10). I will not tell you W_o now, however. To assure you that W_o will not be altered after hearing your report, I am handing you a sealed envelope with the value of W_o written inside. Break that seal before you make your report and the contract is off."

Suppose the agent reports $\hat{W}_n$ truthfully. From the principal's point of view, expected project cost is $\frac{1}{2}(\hat{W}_n+W_o)$, so for $\alpha = h^{-1}(r^A+n-1)^{-1}(r^A+n)^{-1}$ she expects to pay the agent

$$\alpha(\hat{W}_n - W_o)^2 - 2\alpha(\hat{W}_n - W)(\hat{W}_n - \tfrac{1}{2}(\hat{W}_n + W_o))$$
$$= \alpha(\hat{W}_n - W_o)^2 - \alpha(\hat{W}_n - W_o)^2 = 0,$$

Since the agent has to cover investigation costs out of this sum, this is obviously a very tidy arrangement for the principal. But the agent would have to be very naive to accept. The problem is that given additional information W_o about the mean, $\hat{W}_n$ tends to be a biased estimate. An agent anticipating this should demand that any report he submits be averaged with W_o before being inserted into the payoff function.

While naivete and the exploitation of naivete can indeed be important in real-life contracting, a rigorous analysis falls beyond the scope of this dissertation. Henceforth we shall assume that all useful information other than that obtained through the agent's own investigation is automatically shared between principal and agent. The precision $rh=r^Ah$ of that information is known in advance, so that the agent can calculate how many measurements to take. If for some reason the mean W_o is not known in advance -- say, because the principal awaits reports from a third party -- the agent is assured his final estimate will incorporate W_o in best linear unbiased fashion. That is, the agent submits his n-measurement average W and his formula $(n+r)^{-1}(nW+rW_o)$ for calculating $\hat{W}_n$, which in turn is used once W_o and x are known to calculate agent's payoff $R(\hat{W}_n)-R'(\hat{W}_n)(\hat{W}_n-x)$.

We proceed to analyze the costs of inducing n measurements. Under the contract generated by (5.9), expected payoff given a truthful report $\hat{W}_n$ will be $b(r+n-1)(r+n)h(\hat{W}_n-W_o)^2$. Note that $\hat{W}_n$ is calculated on the basis of W_o and the average of n other independent measurements, where W_o has precision rh and the average has precision nh. Applying Lemma 5.2, we see that $\hat{W}_n$ has conditional mean W_o and conditional variance $nhr^{-1}h^{-1}(rh+nh)^{-1} = nr^{-1}(r+n)^{-1}h^{-1}$. Hence expected contract cost prior to investigation, given that n measurements are desired, equals

$$(5.11)\quad b(r+n-1)(r+n)hnr^{-1}(r+n)^{-1}h^{-1} = bn(r+n-1)r^{-1} = bn + bn(n-1)r^{-1}.$$

Thus expected contract cost can be separated into two components. The first, bn, is the cost of investigation proper. The second, $bn(n-1)r^{-1}$, is what we shall call the **expected cost of agency**. It equals the difference between what the principal expects to have to pay an investigating agent and what the principal would pay if she could carry out the same investigations herself with the same skill (same b) as the agent. For one investigation, expected agency cost is zero, but it is positive for two investigations or more. Intuitively, the contract is calibrated so that the agent's cost of undertaking the last investigation is just matched by his marginal expected benefits. From the standpoint of extra agency costs, this investigation is "free" to the principal. The preceding n-1 investigations, however, are "infra-marginal". The variance of reestimation induced by any of them is greater than that induced by the last investigation. Hence they are more lucrative to undertake.

For a large number of investigations, expected agency cost is approximately proportional to the square of n. It is directly proportional to the measurement cost and inversely proportional to the relative precision r of the prior. Alternatively, consider the ratio of agency cost to direct investigation cost: $\frac{n-1}{r}$. This is independent of measurement cost, inversely proportional to the prior estimate's relative precision, and directly proportional to the number of measurements less one. As explained above, no direct agency costs are imposed by the last investigation. However, the extra cost of the preceding investigations rises with the variance of the penultimate estimate around the prior. (Agency cost conditional on penultimate estimate W depends on $(W-W_o)^2$, so that unconditional agency cost depends on the variance of the penultimate estimate.). Since the penultimate estimate weights the average measurement relative to the prior in ratio n-1 to r, it is not surprising that relative agency costs should rise with n-1 and fall with r.

In analogy with total planning costs C(n) for n investigations without agency, let $C_a(n)$ denote total planning costs for n investigations with agency:

$$(5.12)\qquad C_A(n) = c(r+n)^{-1}h^{-1} + bn + bn(n-1)r^{-1} + cV^* \text{ for } n=0,1,2,\ldots$$

The planner chooses n to minimize $C_A(n)$. Ignoring the integer constraint, the optimal n^{**} will be the solution to

$$C_A'(n) = -c(r + n)^{-2}h^{-1} + b + b(2n - 1)r^{-1} = 0$$

$$\Rightarrow \quad -cr + b(r + n)^2rh + bh(r + n)^2(2n - 1) = 0$$

$$\Rightarrow \quad -cr + bh(r + n)^2(r + 2n - 1) = 0$$

$$\Rightarrow \quad -cr + 2bh(n + r)^2(n + \tfrac{r-1}{2}) = 0$$

$$\Rightarrow (5.13) \qquad (n + r)^2(n + \tfrac{r-1}{2}) = cr(2bh)^{-1}.$$

The exact solution for this cubic equation is rather messy. Fortunately, an approximation is adequate for our purposes. Letting Q denote the right-hand side of (5.13) and n^{**} the optimal number of measumrents, we see that $-r+Q^{1/3} < n^{**} < \frac{1-r}{2} + Q^{1/3}$. It follows that

$$(5.14) \qquad \text{for } n^{**} \gg r,\ \frac{n^{**}+r}{Q^{1/3}} \approx 1, \text{ where } Q = cr(2bh)^{-1}.$$

(5.14) says that the elasticities of n^{**} or $n^{**}+r$ with respect to various parameters are approximately one-third the corresponding elasticities for Q, provided the prior has little weight in the final estimate. In particular, the direction of change will be the same, unless the elasticities are close to zero. Using (5.14), final estimate precision $(n^{**}+r)h$ is seen to rise with the value of precision c and decline with the cost per unit precision $\frac{b}{h}$. This is the same direction of change as occurs without agency costs, and again accords with economic intuition. Unlike the no-agency-cost case, however, final estimate precision rises with the relative precision of the prior. As we have seen, having a more precise prior enables the principal to reduce expected agency costs $bn(n-1)r^{-1}$. Since the magnitude of reduction increases with the number of investigations, marginal conditions at the optimum shift to favor more investigations.

Additional insights can be gleaned by comparing final estimate precisions with and without agency costs. With $n^{**} \gg r$, (5.14) can be used along with (5.1) to solve for the ratio:

$$(5.15) \qquad \frac{(n^{**}+r)h}{(n^{*}+r)h} \doteq [cr(2bh)^{-1}]^{1/3}[c^{1/2}(bh)^{-1/2}]^{-1}$$
$$= 2^{-1/3}b^{1/6}h^{1/6}c^{-1/6}r^{1/3}$$
$$= [2(\frac{n^{*}+r}{r})]^{-1/3} \doteq [2(\frac{n^{**}+r}{r})]^{-1/2}$$

The larger are b,h, and r, and the smaller is c, the greater is this ratio, that is, the smaller the ultimate precision penalty caused by agency. For an economic explanation, consider that the above-mentioned parameter changes increase costs or lower value per unit precision attained and hence reduce the net marginal expected benefits of measurement. Formerly beneficial measurements now return negative net benefits and have to be foregone. Since marginal costs rise much more steeply with agency than without (even more steeply when r is high), relatively fewer measurements have to be foregone in the former case than in the latter.

Regardless, the precision penalty imposed by agency will be high whenever $n^{**} \gg r$. If final estimate precision without agency is 15 times initial estimate precision, with agency it will be only 5 times initial estimate precision. If final estimate precision without agency is 100 times initial estimate precision, with agency it will be only 17 times initial estimate precision.

By analogy with EAC, let expected administrative costs under agency (EACA) denote $C_A(n^{**})$ less the unavoidable part cV^*. Substituting the solution to (5.13) into (5.12), EACA can be expressed as

$$c(r + n^{**})^{-1}h^{-1} + bn^{**} + bn^{**}(n^{**} - 1)r^{-1}$$
$$= b(n^{**} + r)(2n^{**} + r - 1)r^{-1} + bn^{**} + bn^{**}(n^{**} - 1)r^{-1}$$
$$= br^{-1}[2n^{**2} + 3n^{**}r + r^2 - n^{**} - r + n^{**}r + n^{**2} - n^{**}]$$
$$= br^{-1}[3(n^{**} + r)^2 - 2(n^{**} + r)(r + 1) - r].$$

For $n^{**} \gg r$, the first term inside the brackets will dominate the other two, so that the ratio of EACA to $3br^{-1}Q^{2/3}$ ($\doteq 1.890b^{1/3}c^{2/3}h^{-2/3}r^{-1/3}$) is close to one. Using this approximation, expected administrative costs with agency are seen to rise with b and c and fall with h and r, as was true of those costs without agency. But the rates of change are not the same. Comparing EACA with EAC, we have

$$(5.16)\quad \frac{EACA}{EAC} \doteq (1.89b^{1/3}c^{2/3}h^{-2/3}r^{-1/3})(2b^{1/2}c^{1/2}h^{-1/2})^{-1}$$
$$= 0.945c^{1/6}(bh)^{-1/6}r^{-1/3} \doteq 0.945\left(\frac{n^{*}+r}{r}\right)^{1/3} \doteq 1.06\left(\frac{n^{**}+r}{r}\right)^{1/2}$$

Relative agency-imposed administrative costs are expected to be least when b, h, and r are high and c is low, the same as we found above for agency-imposed precision penalties. Indeed, apart from sign, the corresponding elasticities in (5.15) and (5.16) are identical. Relative costs will be substantial whenever n^{**} is much larger than r. If final estimate precision without agency is 15 times initial estimate precision, agency serves to increase expected administrative costs by more than 130%. If final estimate precision without agency is 100 times initial estimate precision, the expected cost increase is over 330%.

4. Reducing Agency Costs Through In-House Investigation

The preceding discussion has shown that expected agency costs vary inversely with r, the ratio of the precision of the principal's prior estimate of the mean to the precision of a measurement by the agent. If the principal could somehow increase the precision of what she knows without the agent's help, she might drastically reduce agency costs and agency-induced inefficiencies. One way to attempt this would be for the principal to launch her own "in-house" investigations. Another would be to employ competing agents, and use each agent's report as a standard for rewarding or penalizing the other agent. These approaches are analyzed in the next two sections. This observation will form the cornerstone of our analysis for the next two sections.

Let us begin with the question of in-house investigations. Suppose the principal has no information about the true mean independent of investigation, but can take any number of independent measurements of precision h herself. The agent's direct measurement cost is b as before, while the principal's is β times that or βb. Let r be the number of measurements the principal takes and n be the number she hires the agent to take. The principal's problem is to choose the optimal r^o and n^o.

Clearly the principal should take all measurements herself if β is less than or equal to 1. The optimal number of investigations then corresponds to the optimal n^* of (5.1) with prior precision 0 and measurement cost βb, or

$$c^{1/2}(\beta bh)^{-1/2}, \tag{5.17}$$

at an expected administrative cost of

$$2(\beta bc)^{1/2}h^{-1/2}. \tag{5.18}$$

For β sufficiently greater than 1 to warrant at least one agent measurement, total planning costs amount to (5.12) plus an expenditure of βbr for r direct measurements, or

$$c(r + n)^{-1}h^{-1} + bn(n + r - 1)r^{-1} + cV^* + \beta br. \tag{5.19}$$

Once again, we ignore the integer constraints. Taking the partial derivative of (5.19) with respect to n yields the same first-order condition (5.13) as in the previous section:

$$(n + r)^2(n + \frac{r-1}{2}) = cr(2bh)^{-1}; \tag{5.13}$$

while taking the partial derivative with respect to r yields:

$$-c(r + n)^{-2}h^{-1} - bn(n - 1)r^{-2} + b\beta = 0. \tag{5.20}$$

Using (5.13), the left-hand term in (5.20) can be replaced by $-b(2n+r-1)r^{-1}$, so that (5.20) simplifies to:

$$(\beta - 1)r^2 - (2n - 1)r - n(n - 1) = 0; \tag{5.21}$$

$$\text{or} \quad r^o = \frac{2n^o - 1 + [4n^{o2}\beta - 4n^o\beta + 1]^{1/2}}{2(\beta - 1)}$$

For even moderate-sized n^o (or β close to 1), $[4n^{o2}\beta-4n^o\beta+1]^{1/2} = [(2n^o-1)^2\beta+1-\beta]^{1/2} \doteq (2n^o-1)\beta^{1/2}$ (The error is less than 6% for

n^o=2, 2% for n^o=3, and 1% for n^o=4 or more.). Hence $r^o \doteq (1+\beta^{1/2})(2n^o-1)(\beta-1)^{-1} = (n^o-0.5)(\beta^{1/2}-1)^{-1}$, so that for n^o large

$$(5.22) \qquad \frac{r^o}{n^o+r^o} = (1 + \frac{n^o}{r^o})^{-1} \doteq (1 + \beta^{1/2} - 1)^{-1} = \beta^{-1/2}.$$

Equation (5.22) indicates that when many investigations are required, the fraction of direct investigations in the total essentially depends on the ratio of direct and indirect investigation costs per measurement and nothing else. This is a striking result. Even more striking is the degree to which direct measurements are "favored" over indirect ones. If the principal is only one-quarter as efficient as the agent in taking measurements (β=4), half of all measurements should be direct ones. At one-ninth efficiency, half as many direct as indirect investigations are warranted. Even at one-hundredth efficiency, one measurement out of ten should be taken by the principal.

This has a very strong policy implication for government R&D contracting. Many such contracts, and especially military ones, are "sole-source" contracts. That is, only one firm is considered to have the expertise to carry out the project successfully, so that no competitive bids are taken. Often such claims of monopoly expertise are exaggerated. Leaving aside that question, however, let us say that the practice of providing only cursory review of sole-sourcers' cost forecasts is highly inadvisable from a budget planning standpoint. When project award is assured, sole-sourcers may indeed have no incentive to lie. But neither may they have much incentive to take cost estimation seriously. Independent investigations, be they costly and imprecise, can drastically reduce the costs of encouraging good forecasting.

To solve for the optimal n^o+r^o when n^o is large, approximation (5.19) can no longer be used since r^o is large relative to n^o. Instead, substitute (5.22) into (5.13) to establish that

$$(5.23) \qquad n^o + r^o \approx c^{1/2}(bh)^{-1/2}(2\beta^{1/2} - 1)^{-1/2}.$$

Using (5.13), (5.22), and (5.23), expected administrative costs amount to

$$(5.24) \qquad c(r^o + n^o)^{-1}h^{-1} + bn^o(n^o + r^o - 1)r^{o-1} + b\beta r^o$$
$$= b(n^o + r^o)r^{o-1}(2n^o + r^o - 1) + bn^o r^{o-1}(n^o + r^o - 1) + b\beta r^o$$
$$= b\beta^{1/2}[(2\beta^{1/2} - 1)r^o - 1] + b(\beta^{1/2} - 1)(\beta^{1/2}r^o - 1) + b\beta r^o$$
$$= br^o(4\beta - 2\beta^{1/2}) - 2b\beta^{1/2} + b = b(2\beta^{1/2} - 1)(2\beta^{1/2}r^o - 1)$$
$$\approx b(2\beta^{1/2} - 1)2\beta^{1/2}r^o \quad \text{if } 2\beta^{1/2}r^o >> 1$$
$$= 2\beta^{1/2}b(2\beta^{1/2} - 1)\beta^{-1/2}c^{1/2}(bh)^{-1/2}(2\beta^{1/2} - 1)^{-1/2}$$
$$= 2[cbh^{-1}(2\beta^{1/2} - 1)]^{1/2}$$

Comparing (5.23) and (5.24) with (5.17) and (5.18), we see that final estimate precision is about $\beta^{1/2}(2B^{1/2}-1)^{-1/2}$ times as great when the principal has access to a β-times-more-efficient agent as when she does not, while the corresponding ratio of expected administrative costs is exactly the inverse of this. For β=4, final estimate precision improves by 15% and costs decrease by 13% when agency is allowed. For β=9, final estimate precision improves by 34% and costs decrease by 25%. For β=100, final estimate precision improves by 129% and costs decrease by 54%.

For another perspective on (5.23) and (5.24), suppose the principal were confined to one direct investigation (r=1). Comparing

(5.19) with (5.23), the ratio of final estimate precision with and without the restriction would for large n^{**} be approximately

$$(5.25)\qquad c^{1/2}(bh)^{-1/2}(2\beta^{1/2} - 1)^{-1/2}(2bhc^{-1})^{-1/3}$$
$$= 2^{-1/3}c^{2/3}(bh)^{-2/3}(2\beta^{1/2} - 1)^{-1/2} \doteq 1.26(n^{**}+1)^{2}(2\beta^{1/2} - 1)^{-1/2},$$

while from (5.21) and (5.24) the corresponding ratio of expected administrative costs would be

$$2(bc)^{1/2}h^{-1/2}(2\beta^{1/2} - 1)^{1/2}(1.89b^{1/3}c^{2/3}h^{-2/3})^{-1}$$
$$= 1.06(bh)^{1/4}c^{-1/4}(2\beta^{1/2} - 1)^{1/2} \doteq 0.943(n^{**} + 1)^{-1/2}(2\beta^{1/2} - 1)^{1/2}.$$

If under the restriction 5 total measurements are taken and β=4, then 18 measurements will be taken without the restriction for an administrative cost savings of 27%. If under the restriction 10 total measurements are taken and β=9, then 56 measurements will be taken without the restriction for an administrative cost savings of 33%. At 20 total measurements with the restriction and β=100, 26 measurements will be taken without the restriction for an administrative costs savings of 8%.

Table 5.1: COSTS OF DELEGATION

β	B f.e.p.	C % investigations by principal	B to C change in f.e.p.	B to C change in EAC	A to C change in f.e.p.	A to C change in EAC
1	5	100%	+200%	-57%	--	--
1	17	100%	+500%	-77%	--	--
4	5	50%	+260%	-27%	+ 15%	-13%
9	10	33%	+460%	-33%	+ 34%	-25%
100	20	10%	+ 30%	- 8%	+129%	-54%

Notation:

A = Principal performs all investigations
B = Agent performs all but one investigation (r=1)
C = Investigations divided to minimize EAC
f.e.p. = final estimate precision
β = agent's efficiency relative to principal
EAC = expected adminstrative cost

5. Reducing Agency Costs Through Competition

If in-house investigation is not feasible, it may still be possible to reduce agency costs by hiring a competing agent to furnish estimates. Of course, in this case the principal must worry about contractual incentives for the second agent as well as for the first. Provided the agents do not collaborate, however, and leaving aside possible fixed costs of arranging competition, employing two agents can always be expected to be cheaper than employing one agent. Indeed, given sufficiently many agents, expected contract costs can be reduced to the direct measurement costs themselves, so that all agency-imposed inefficiencies disappear.

The key to this approach is to use each agent's report to refine the principal's base estimate vis-a-vis every other agent. W_0 is taken as the estimate based on the reports of all the agents except j, and W as the estimate based on all the reports including j's. Each agent will then want to report truthfully as long as every other agent does.

Let us try to formalize this argument. Suppose it is desired that each agent j take m_j measurements of precision h, where measurements cost the agent b_j apiece. Let us assume further that every measurement of every agent is independent of every other and of the principal's prior. Now let us consider various BLU estimators of the true mean W^* and the reports of those estimators. Based solely on agent j's measurements, the average of these measurements $\hat{W}^j$ is BLUE with precision $m_j h$. Let W^j be agent j's report of $\hat{W}^j$. Based on all the information **except** agent j's, the BLUE $\hat{W}^{\setminus j}$ is $(\sum_{i \neq j} m_i \hat{W}^i)(N_{\setminus j})^{-1}$ with precision $N_{\setminus j}h$, where $N_{\setminus j} = \sum_{i \neq j} m_i$, and where to simplify the notation the principal is taken as agent 0 with estimate $W^0 = W_0$ based on $m_0 = r$ measurements. Let $W^{\setminus j}$ represent the calculation of $\hat{W}^{\setminus j}$ based on the reports W^i instead of the true $\hat{W}^i$'s. Based on all the information, the BLUE $\hat{W}$ is $(\sum m_j \hat{W}^j)N^{-1}$ with precision Nh, where $N = \sum m_j$. Equivalently, $\hat{W} = (m_j \hat{W}^j + N_{\setminus j}\hat{W}^{\setminus j})(m_j + N_{\setminus j})^{-1}$.

Now consider the compensation function $H^j(g^j(\cdot;W^{\setminus j}),\cdot;W^{\setminus j})$ defined by

$$(5.26) \qquad R^j(Y;W^{\setminus j}) = b_j hN(N-1)(Y - W^{\setminus j})^2$$

$$(5.27)\qquad H^j(Y,x;W^{\setminus j}) = R^j(Y;W^{\setminus j}) - R^{j\prime}(Y;W^{\setminus j})(Y - x)$$

$$(5.28)\qquad g^j(W^j;W^{\setminus j}) = (m_jW^j + N_{\setminus j}W^{\setminus j})(m_j + N_{\setminus j})^{-1}$$

In words, the function $g_j(\cdot)$ constructs a final estimate out of separate reports; that estimate becomes the argument in a surplus function $R_j(\cdot)$, which in turn generates a scoring rule $H_j(\cdot)$. From Theorem 5.2, $H^j(\cdot,\cdot;\hat{W}^{\setminus j})$ is the optimal contract for inducing the agent to take m_j measurements of precision h and cost b_j and reporting the final estimate $Y=\hat{W}$ truthfully, given a prior $\hat{W}^{\setminus j}$ of precision $(N-m_j)h = N_{\setminus j}h$. Since $g^j(\hat{W}^j;\hat{W}^{\setminus j}) = \hat{W}$, an agent offered the contract $H^j(g^j(\cdot;\hat{W}^{\setminus j}),\cdot;\hat{W}^{\setminus j})$ would truthfully report $W^j=\hat{W}^j$. Hence if every agent j $(j\neq 0)$ is offered the contract $H^j(g^j(\cdot;W^{\setminus j}),\cdot;W^{\setminus j})$, with $W^{\setminus j}$ constructed on the basis of every other agent's reports as described above, each agent will tell the truth as long as everyone else does. Formally, when $\{H^j(g^j(W^j;W^{\setminus j}),x;W^{\setminus j})\}$ is the set of payoff functions for a multi-person game with simultaneous reporting, truth-telling is a Nash equilibrium.

From (5.27), $g^j(W^j;W^{\setminus j}) - W^{\setminus j} = \frac{m}{N}^j(W^j-W^{\setminus j})$, so that expected payoff for truthful reporting is:

$$(5.29)\qquad R^j(g^j(W^j;W^{\setminus j});W^{\setminus j}) = b_jhm_j{}^2N^{-1}(N-1)(W^j-W^{\setminus j})^2.$$

Indeed, as long as the precision-weighted average of the W^j's (given by $g^j(W^j;W^{\setminus j})$) equals the BLU estimate $\hat{W}$, agent j can secure expected payoff (5.29). This shows that our scheme as described is vulnerable to collusion, since widening the spread between the W^i's while preserving their precision-weighted average will always serve to increase expected payoffs. Moreover, collusion would be extremely difficult to prevent, for the offenders need not communicate directly during or after investigation. All they need is an agreement before contracting begins over who will raise and who will lower their estimates, and by how much. This is a problem common to Groves mechanisms, to which class this scheme belongs.

Having noted this potentially major drawback, let us proceed on the assumption that collusion does not occur -- say, that the identities of the competing agents are completely concealed from each other. The next question we will address is how the principal can best divide up measurements between agents, assuming that final estimate precision Nh is known. From (5.29), the expected cost of contract j is

$$
\begin{aligned}
(5.30)\qquad & b_j h m_j^2 N^{-1}(N-1)(\mathrm{Var}[W^j] + \mathrm{Var}[W^{\setminus j}]) \\
&= b_j h m_j^2 N^{-1}(N-1)[(m_j h)^{-1} + (N_{\setminus j} h)^{-1}] \\
&= b_j h m_j^2 N^{-1}(N-1)(m_j + N_{\setminus j})(m_j N_{\setminus j} h)^{-1} \\
&= b_j m_j (N-1)(N-m_j)^{-1},
\end{aligned}
$$

where the first line follows from the fact that W^j and $W^{\setminus j}$ are independent and have identical means. To minimize total expected contract cost subject to the constraint that the total number of agent measurements equals N-r, form the Lagrangean

$$(N-1)\sum_1^T [b_j m_j (N-m_j)^{-1}] + \lambda(N - r - \sum_1^T m_j),$$

where T is the number of agents. First-order conditions require (ignoring the integer constraint on the m_j's) that $b_j(N-m_j)^{-2}$ be identical across agents: i.e., there is some ξ such that $m_j=N-\xi b_j^{1/2}$ for all j. Hence if two agents have different b_j's, the agent with the lower b_j should be induced to take more investigations than the other, which is intuitively reasonable. Plugging these equalities into the constraint to determine ξ, we find that

$$(5.30) \qquad m_j = N - (\sum_{i}^{T} b_i^{1/2})^{-1}(TN - N + r)b_j^{1/2} \quad \forall j=1,\ldots,T.$$

For (5.30) to be a reasonable solution, each m_j must be positive. This requires that

$$(5.31) \qquad b_j^{1/2} < N(\sum_{i \neq j} b_i^{1/2})[(T-2)N + r]^{-1}.$$

If agents are numbered so that $b_1 \leq b_2 \leq \cdots \leq b_T$, agent 2 should be hired if and only if b_2 satisfies (5.31) for agent 1 the only other agent, agent 3 should be hired if and only if b_3 satisfies (5.31) for agents 1 and 2 the only other agents, and so on.

When all the b_j's are the same ($\equiv b$), each agent should perform $\frac{N-r}{T}$ measurements. If there are as many agents as there are measurements to take, total expected contract cost is just $(N-r)b$, which is just the direct cost of N-r investigations without any additional agency costs. All inefficiency disappears, so that the optimal N-r corresponds to the n^* given by (5.1). But even a relative handful of competitors can significantly reduce agent costs. Consider the case when r is close to zero. Investigation by only one agent is infeasible, as expected agency costs are extremely high. For T>1 agents, expected total contract cost is $(N-1)bT(T-1)^{-1}$, to which must be added expected planning losses of $c(Nh)^{-1}+cV^*$. This is minimized when $N = [c(T-1)(bhT)^{-1}]^{1/2}$, so that the fractional loss in final estimate precision compared to the efficient solution equals $1-(1-T^{-1})^{1/2}$, or 29% for T=2, 18% for T=3, 13% for T=4, and -- using a Taylor's series expansion -- $50T^{-1}$ percent when T is 5 or more. The fractional increase in costs is about the same (precisely, $(1-T^{-1})^{-1/2}-1-[2N-2NT^{-1}]^{-1}$).

6. A Note on Agent Payoff Constraints

Crucial to our findings of substantial inefficiencies in (non-competitive) agent investigation has been assumption (5.6): that expected payoff from reporting the mean truthfully is at least zero regardless of the underlying distribution. This was justified earlier on the grounds that an agent expecting a negative marginal return would withhold the report. This argument is not wholly convincing, however. Given that investigation offers prospects of a substantial profit, an aspiring forecaster might gladly pay an entry fee T to the principal to assure hiring. Indeed, two agents might compete so fiercely for the opportunity to be the principal's sole forecaster that the principal could extract the whole surplus as rent.

Let us work out the optimal contract $R(W)-R'(W)(W-x)-T$ when agents are Nash competitors. For n investigations, R(W) is given by (5.9) as $b(r+n-1)(r+n)h(W-W_0)^2$. T is the expected value of R(W), which from (5.11) equals $bn(r+n-1)r^{-1}$. The optimal n, however, is no longer given by (5.13). Rather, it is the same as the optimal n for the no-agent case, which is $-r+c^{1/2}(bh)^{-1/2}$ from (5.1). Combining equations, the optimal contract is

$$(5.32)\qquad H(W,x) = [c - (bhc)^{1/2}][W_o^2 - W^2 + 2(W - W_o)x] - c(rh)^{-1} + (1 + r)r^{-1}(bc)^{1/2}h^{-1/2} - b$$

In practice, there are limits to competition and the agent's ability to pay. But surely the agent must be willing to bear some losses or would not accept the contract under any conditions, since payoffs $R(W)-R'(W)(W-x)$ may be negative even when the expected surplus $R(W)$ is not.

One way to model the fixing of an "intermediate" entry fee is as follows. Let us relax the assumption, which was not very compelling anyway, that the principal knows everything the agent does before investigation begins. Let rh and $(r+r_a)h$ be the precision of the prior information of principal and agent respectively, and let the corresponding means be W_o and W_a. To simplify matters, we assume the agent knows everything the principal does, so that r_a measures the agent's "extra" information. Leaving aside entry fees for the moment, the optimal contract for inducing n investigations is generated, as we know, by $R(W) = b(r+r_a+n-1)(r+r_a+n)h(W-W_o)^2$. From the agent's point of view, the final estimate $\hat{W}_n$ will be calculated as on the basis of W_a and the average of n independent measurements, where W_a has precision $(r+r_a)h$ and the average has precision nh. Applying Lemma 5.2, $\hat{W}_n$ has conditional mean W_a and conditional variance $n(r+r_a)^{-1}(r+r_a+n)^{-1}h^{-1}$. Thus the conditional mean of $R(\cdot)$ of W, given the prior estimate W_a, is

$$(5.33)\qquad bn(r+r_a+n-1)(r+r_a)^{-1} + b(r+r_a+n-1)(r+r_a+n)h(W_a-W_o)^2.$$

The agent would willing to pay up to the value of (5.33), less the cost of investigations bn, to be awarded the contract. Since the second term is at least zero, an entry fee of at least

(5.34) $$bn(n - 1)(r + r_a)^{-1}$$

can be levied. If W_a were known to differ from W_o, more could be charged. However, the principal does not know the true value, and when questioned the agent will always claim that is W_o in order to keep the entry fee down. Therefore, if it is required that all agents be willing to take the contract, the optimal entry fee is given by (5.34).

Now let us calculate expected contract cost from the principal's point of view. Applying (5.11) with n replaced by r_a+n, the expected value of R(·) of W equals $b(r_a+n)(r+r_a+n-1)r^{-1}$. Expected contract cost is then the difference between this and (5.34), which can be simplified to

(5.35) $$br_ar^{-1}(r_a + 2n - 1) + br_a(n^2 - n)(r + r_a)^{-1}$$

If r_a equals 0, the agent has no private information and we are back to the previously discussed case, when expected contract cost is zero. In the general case total planning costs $C_a(n)$ for n agent measurements equals (5.35) plus direct planning losses $c(r+r_a+n)^{-1}h^{-1}+cV^*$. These are minimized when

(5.36) $$C_a'(n) = -c(r+r_a+n)^{-2}h^{-1} + 2br_ar^{-1} + br_a(2n-1)(r+r_a)^{-1} = 0.$$

If the optimal n^{***} is much larger than r and r_a, (5.36) has approximate solution

$$(5.37) \qquad n^{***} \approx c^{1/3}(2bh)^{-1/3}[1 + \frac{r}{r_a}]^{1/3}$$

Comparing (5.37) and the approximate solution (5.14) generated by assumption (5.6), we see that the only difference is that now $1+\frac{r}{r_a}$ replaces r. Hence the qualitative features of the previously-discussed comparative statics results would remain the same, although the quantitative dimensions might not.

7. Discussion

We see that for a proper contract of the form $R(W)-R'(W)(W-x)+T(x)$, incentives for careful estimation of the mean depend on how convex $R(\cdot)$ is. If the variance of estimate correction from additional investigation is known but other characteristics of the investigation are not, a quadratic $R(\cdot)$ is the best way for the principal to ensure that the measurement is taken. Even with the optimal $R(\cdot)$, however, agency costs can be quite substantial relative to the direct costs of investigation. Relative agency costs tend to rise with the number of measurements desired and fall with the relative precision of the principal's prior estimate.

Because of these costs, a planner forced to rely on agent reports will settle for less precise estimates than she would if she could carry out the same investigation herself at the same direct cost. The same factors that raise expected planning costs without agency problems -- high measurement costs, high measurement variances, high cost of estimate imprecision, and low initial estimate precision --

also raise expected planning costs. But the increase is even more rapid with agency than without, making for increased relative inefficiencies.

To reduce relative agency costs with a given measurement technology, planners can try to decrease their relative dependence on any one agent for information. The two basic ways to do this are (a) undertake more direct investigations, and (b) hire competing agents. Direct investigations by the principal can trim expected administrative costs substantially, even when the principal's measurement abilities are very poor. Competition is potentially even more effective. If there are enough skilled agents to have every measurement taken by a different agent, and if there are no fixed costs of arranging competition, agency-imposed inefficiencies can be completely eliminated. The optimal number of investigations and expected administrative cost would be the same as in the no-agency case discussed at the beginning of the chapter.

Unfortunately, the competitive contract schemes described are extremely vulnerable to collusion between agents. By agreeing to alter their forecasts in compensating fashion, colluding agents could raise their expected payoffs to any specified level. in order Moreover, the agreement would be self-enforcing and not require any side communication between agents once contracting begins. For example, if two agents are performing equally many investigations and if agent 1's report is thought to be inflated by x units, then it is in agent 2's interest to deflate his report by x units in order to generate a correct overall estimate. Conversely, it is in agent 1's interest to inflate his report by x as long as agent 2's report is expected to be deflated by x. Hence collusion would be difficult to prevent, short of concealing agents' identities from each other or changing the nature of the contracts.

These results carry some broader implications for the theory of organizations. The gathering, transfer, and processing of information is central to the functioning of bureaucracies. But bureaucracies are generally not perceived to handle this task very well. The very term "bureaucracy", in its pejorative connotations, calls up images of costly, duplicated efforts and haphazard transmission of messages.

Our analysis indicates that bureaucracies can indeed generally be expected to be inefficient. This inefficiency stems from the need to provide agents with expected rents to acquire and transmit information. Reducing the rents tends to worsen the quality of information, and vice-versa, improving the quality of information tends to raise relative rents. The tradeoff can be "softened" by procuring information from multiple sources. In this case, however, care has to be taken to ensure that the sources do not collude with each other. In other words, to streamline bureaucracy, it may be helpful to arrange more competition in information-gathering.

If this is so, there may be some potential efficiency advantages to what is sometimes considered a quintessentially bureaucratic curse: the lack of cooperation across bureaucratic lines. Obviously, this is not to say that vertical information flows (between lower and higher hierarchical levels) are always to be preferred to horizontal ones (between agents at the same levels). On the one hand, not all horizontal competition is channeled productively. On the other hand, teamwork has its own creative advantages and can ease strains on the information-processing capabilities of the center. In deciding which organizational forms to use, one factor to be considered is the external environment of the hierarchy. When a small hierarchy is imbedded in a large competitive environment, the prices and output decisions of other firms may provide enough information to warrant relatively unstructured internal communication. As the hierarchy grows or external competition recedes, relatively more internal competition is favored.

Appendix: Some Basic Results on Estimation

Lemma 5.1: Let $W_1,\ldots,W_n$ be n independent and random measurements of W^*, with expectation $E[W_i]=W^*$ and variance $Var[W_i]=\sigma_i^2=1/h_i$ for each $i=0,\ldots,n$. The best linear unbiased estimate (BLUE) of W^* based on these measurements is given by

$$(5.38)\qquad (\Sigma h_i W_i)(\Sigma h_j)^{-1} = \frac{h_1 W_1+\ldots+h_n W_n}{h_1+\ldots+h_n},$$

with variance

$$(5.39)\qquad (\Sigma h_j)^{-1} = \frac{1}{h_1+\ldots+h_n},$$

where each summation is understood to run from 1 to n.

Proof: The BLUE is defined as $\Sigma p_i W_i$, where the p_i's solve:

$$\text{minimize } Var[\Sigma p_i W_i] \text{ subject to } E[\Sigma p_i W_i] = W^*.$$

Since $E[\Sigma p_i W_i] = \Sigma p_i E[W_i] = (\Sigma p_i)W^*$, the constraint amounts to $\Sigma p_i = 1$. Furthermore, since the W_i's are independent, $Var[\Sigma p_i W_i] = \Sigma p_i Var[W_i] = \Sigma p_i \sigma_i^2$. So the p_i's must minimize the following Lagrangean:

$$\Sigma p_i \sigma_i^2 + \lambda(1-\Sigma p_i).$$

First-order conditions require $p_i\sigma_i^2=\lambda$ or $p_i=h_i\lambda$. Plugging into the constraint, $\Sigma h_i\lambda = 1$, or $\lambda = (\Sigma h_i)^{-1}$. Substitution establishes (5.38). Since $Var[aY+bZ] = a^2Var[Y]+b^2Var[Z]$ for constants a and b and independent Y and Z, we have

$$Var[BLUE] = (\Sigma h_i^2 Var[W_i])(\Sigma h_j)^{-2} = (\Sigma h_i)(\Sigma h_j)^{-2} = (\Sigma h_j)^{-1}.\blacksquare$$

Lemma 5.2: Let Y be an unbiased estimate of W^* with precision g. Suppose an additional random measurement Z of W^* can be taken, where

Z is independent of Y with mean W^* and precision h. From Lemma 5.1, Y and Z together will yield a BLUE of $(gY+hZ)(g+h)^{-1}$. Then the conditional mean of BLUE, given $Y=Y^\circ$, is Y°, and its conditional variance is $hg^{-1}(g+h)^{-1}$.

Proof: Define $\epsilon_z=Z-W^*$ and $\epsilon_y=Y-W^*$. ϵ_y and ϵ_z can be viewed as random variables with mean 0 and precisions g and h respectively. Combining, $Z = Y-\epsilon_y+\epsilon_z$, so that conditional on $Y=Y^\circ$, Z's expected value is Y° and its variance is $E[(\epsilon_y-\epsilon_z)^2] = g^{-1}+h^{-1} = (g+h)g^{-1}h^{-1}$. Hence $E[\text{BLUE}|Y=Y^\circ] = (gY^\circ+hE[Z|Y=Y^\circ])(g+h)^{-1}) = Y^\circ(g+h)(g+h)^{-1} = Y^\circ$. $\text{Var}[\text{BLUE}|Y=Y^\circ] = (g^2\text{Var}[Y|Y=Y^\circ]+h^2\text{Var}[Z|Y=Y^\circ])(g+h)^{-2} = h^2(g+h)g^{-1}h^{-1}(g+h)^{-2} = hg^{-1}(g+h)^{-1}$.∎

CHAPTER 6: CONCLUSION

We have tried in this work to develop a general theory of the elicitation of information about uncertainty and apply the results to analysis of various "practical" contracting problems. In the basic model, a project's anticipated outcome is random or perceived as if it were random, so that planning must be based on forecasts. To encourage the agent to make these forecasts sincerely and reasonably accurately, the principal ties the agent's compensation to the forecast and to the mutually observed outcome. The compensation function is known as a scoring rule. The agent is presumed to choose the forecast so as to maximize expected score or expected utility of the score. Scoring rules which always provide nonnegative [positive] incentives for sincere reporting, no matter what the truth is, are called proper [strictly proper].

In Chapter 2 a general characterization of scoring rules was presented. For virtually any message function, we were able (a) to determine whether the messages could be elicited by means of strictly proper scoring rules and (b) if so, to specify the functional form scoring rules would have to take. Complete classifications were provided for a number of message functions. Proper scoring rules were derived as well for restricted families of distributions.

In Chapter 3 we narrowed our focus to elicitation of the mean but relaxed a number of other assumptions. We began with an examination of conditional contracting, in which a project is only awarded (and thus the outcome only observed) as long as forecasts fall below a threshold value. Proper scoring rules for this problem were derived as subsets of the general class. A similar approach was taken towards elucidation of scoring rules that deter cost-padding. We saw how scoring rules could be applied within a competitive framework, and how competition could help lower elicitation costs. The impact of agent risk-aversion on the functioning of scoring rules was studied, and contracts were modified to take risk-aversion into account. However, we were unable to derive proper rules which simultaneously elicit the probability of an unknown outcome and reveal the agent's true utility function. Indeed, in threshold level contracting we were able to show that such schemes do not exist, and conjectured that the same is true more generally. Finally, we expanded scoring rules to encompass multiple planning criteria.

Chapter 4 addressed issues of multi-period contracting. We began with the case of two independent outcomes, deriving the entire class of proper scoring rules for a large family of message functions. Results were applied to analysis of various threshold value contracting arrangements. In general, although the unrestricted class is somewhat more extensive than the corresponding class of single-variable rules, it shrinks rapidly once other criteria are imposed. When both pre-project rejection and mid-project cancellation are allowed, for example, there may not be any smooth strictly proper scoring rules. It is not clear, however, to what extent these results can be generalized to non-smooth rules and to rules with more than two messages elicited. At least some non-smooth counterexamples exist. As for the apparently more "interesting" families of proper scoring rules for dependent variables, these are shown to be subsets of proper scoring rules for independent variables.

Missing from these chapters was a thorough examination of optimality: the choice of rules within a proper class. Chapter 5 partly remedied this problem, by incorporating agent learning costs. The agent, who was asked to report the mean of an uncertain or random outcome, would begin with a rough estimate. By making cost investigations, the agent could make this estimate more precise, but effort would not be directly observed by the principal. Contracts for inducing a desired degree of learning were derived. By comparing their expected administrative costs to their benefits, the optimal degree of learning could be determined. When agents are assured nonnegative expected payoffs regardless of the underlying distribution, agency costs turn out to be substantial, prompting cutbacks in investigation.

Possibilities were explored of reducing administrative costs though direct investigation and through employment of competing agents. Competition in particular seemed to offer tremendous possibilities for cost reduction. In a method recalling that of a second-price auction, every agent's report could be used to help evaluate the new information content of every other agent's reports and to help target rewards accordingly. Unfortunately, such schemes were shown to be extremely vulnerable to collusion.

In closing, let us note two major limitations to this work. The first is that we made only scarce provision for moral hazard, i.e., an agent's ability to influence the probability distribution through adding or withholding effort. To be sure, cost-padding is a form of moral hazard. But our modeling of it was quite superficial: agents are presumed to receive a constant fraction of audited costs in the form of project perquisites, or they are allowed to effortlessly reduce costs to the "efficient" level and no further. Some preliminary work with less restrictive specifications suggests that scoring rule selection can be posed as an optimal control problem. Unfortunately, the control variables tend to be defined implicitly rather than explicitly, making the general problem difficult to solve.

Connected with this issue is the question of why information should be elicited at all. Apart from any bureaucratic penchants for record-keeping, there are basically two reasons. The first is to help motivate agents, by setting standards for judging performance. The problem with this approach is that if the agent anticipates a report being used "against" him later, it may be hard to secure truthful reports. As mentioned before, this is a persistent difficulty in Soviet-type planning. In the USSR, an enterprise manager's estimate of production capacity is used to set output targets for the enterprise, with bonuses offered for target fulfillment and overfulfillment. Eager to secure the bonuses, managers tend to understate production capabilities. Consequently, Soviet planners have to check managers' information against that culled from other sources, such as records from similar plants or from direct Party investigations. The same types of problems help explain why very large corporations in Western economies often have their own divisions audited by outsiders, or set up internal accounting divisions outside the direct chain of command. In general, standard-setting is most effective when each agent's report is used to evaluate <u>other</u> agent's performances but not his own. However, there are cases involving moral hazard in which agents may be usefully called on to set standards for themselves. Two recent papers by Melamud and Reichelstein (1985a, 1985b) begin to explore this topic.

The second reason for eliciting information is to assist in budget planning. Even if the project is certain to be awarded, an estimate may still be useful in helping gauge the funds remaining for other purposes. To be sure, some budget flexibility must be maintained, in case actual project cost deviates greatly from its estimate. But allowing for every contingency is itself expensive. A baseline is needed; cost estimates help provide one.

Still, it is not at all clear what sort or sorts of estimates are needed. Underlying much of the discussion was the notion of a planner's loss function. The requested forecast was intended to minimize the expected loss, so that any negative multiple (or a constant plus a negative multiple) of the loss function could serve as a proper scoring rule. But why does the loss function take the particular form it does and not some other? It is hoped that future research will address this question.

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