Graphical Economics with Short Trading

Elliot Gorokhovsky

May 13, 2017

Abstract

In 2004, Kakade et. al. [2] introduced a graphical variant of the classic Arrow-Debreu model, in which agents are restricted to trade only with their neighbors in the network in which the economy is embedded. In this paper, we argue that their generalization does not accurately model strategic behavior, and that the addition of a form of *short trading* is necessary to allow agents to exploit their position in the network topology in a natural way.

1 Introduction

1.1 The Arrow-Debreu model

The Arrow-Debreu model of an economy [1] models strategic behavior of agents in a setting where

1. All agents can trade with all other agents without restriction, and
2. The endowments of individual agents are not large enough to allow agents to manipulate prices; i.e., individual deviation from an equilibrium strategy cannot affect prices.

The equilibrium is then defined as any price vector p and set of consumption plans $\{x^i\}$ satisfying, for all i,

$$x^i = \operatorname*{arg\,max}_{x \in [0,\infty)^c,\, p\cdot x \leq p\cdot e^i} u^i(x)$$

where

- c is the number of commodities being traded.
- u^i is agent i's utility function, $u^i : \mathbb{R}^c \to [0, \infty)$.
- e^i is agent i's endowment; e^i_k is the amount of commodity k that agent i is initially endowed with.

The predicate ensures that agents do not exceed the "cash value" of their endowments; it can be shown that any budget-constrained set of consumption plans corresponds to some sequence of cashless trades, so the use of "cash" is valid.

The objective is the "individual rationality" constraint: it ensures that, if the equilibrium prices were somehow imposed on the network, rational agents would not object to the consumption plans x^i, and would have no incentive to individually deviate (by the second assumption about the strategic setting). Thus, in a deep market, the equilibrium prices are in some sense a Nash.

1.2 Generalizing Arrow-Debreu to arbitrary graphs

As is fashionable nowadays, the classical model described above was generalized by Kakade et. al. to work in a network setting, where some pairs of agents are restricted from trading with each other. This can be seen as an attempt to consider economies in which assumption (1) of the Arrow-Debreu model is violated, but assumption (2) still holds. Presumably, this can be used to model settings where external restrictions are imposed on trade, especially in Internet settings where nodes may be "trading" resources like bandwidth or files or whatever, but not be connected to every other node in the network.

Mandatory cynicism aside, their model can be described by making the following modifications to Arrow-Debreu:

- Replace the consumption plans x^i with sub-plans $x^{j\to i}$, where $x^{j\to i}$ represents the vector of goods transferred from agent j to agent i.
- Replace the global price vector with agent-specific price vectors p^i.
- Replace the budget constraint with a sum that multiplies goods bought by specific agents by those agents' price vectors (where $N(i)$ is the neighbor set of i in the network, including i):

$$\sum_{j\in N(i)} p^j \cdot x^{j\to i} \leq p^i \cdot e^i$$

Note that the objective of the rationality constraint is not modified; we are merely restricting the set of feasible solutions to those in which trade respects the network topology. Indeed, we can use the rationality constraint as-is by simply setting

$$x^i = \sum_{j\in N(i)} x^{j\to i}$$

It's clear that if the network is a complete graph, then the graphical model is equivalent to the classical model, since the sum over $N(i)$ includes all agents.

1.3 Motivation: issues with the graphical model as-is

The graphical model as-is seems reasonably in line with intuition when commodities are evenly, or at least randomly, distributed throughout the network, and agents' utilities for commodities are similar. However, when agents have high utilities for commodities whose producers are not in their neighbor sets, the graphical model gives extremely counter-intuitive results, which was the inspiration for this paper.

For example, even if an agent has infinite utility for some commodity, any shortest-path distance greater than one from the producer of that commodity makes it entirely unobtainable. In contrast, in the real world, consumers frequently purchase commodities whose producers they do not have direct access to. Even a thousand years ago, wealthy Europeans, who had extremely high utilities for spices, paid middlemen huge sums to buy and then resell those spices from the Far East, consuming large quantities of commodities produced by agents they were not directly connected to in the "graph" of the world at the time.

The classical Arrow-Debreu model does not need to account for this behavior, because it is set on a complete graph: it wouldn't make sense to buy commodities through a middleman, because one can buy them directly from the producers (we will formalize this intuition later). But once completeness is removed, it becomes necessary to address this problem: supply and demand are no longer the only forces at play. As will be shown, in a more reasonable model, even agents with zero endowment can make huge profits by exploiting strategic positions in the network!

More generally, one would intuitively assume that *when trade is constrained to respect a network topology, control of trade routes should be as important, or even more important, than supply and demand.* The goal of this paper is to demonstrate that a simple modification of the Kakade et. al. model is sufficient to make it respect that intuition.

2 Extending the Kakade et. al. model

2.1 Graphical economics as multi-commodity flow

What we want, essentially, is to model the movement of continuous quantities of commodities through a network. One natural way to model the movement of a continuous quantity through a network is using a *network flow.* Specifically, we don't need a mechanism enabling agents to establish arbitrary trade routes, or trade with arbitrary agents. All we need is a mechanism allowing agents to sell some of what they buy from their neighbors to their other neighbors. Any flow of commodities can be decomposed into a superposition of paths, which are sequences of such transactions, so the model is sufficiently general.

Luckily, it turns our there is a simple, elegant way to extend the model to allow this. Namely, we simply allow agents to decide how much of their consumption plan to resell, r, and how much to keep, k. We allow agents to sell not only their entire endowment, but also the part of their purchase plan they allocate for resale. The utility is taken not over the entire purchase plan, but over k only, ensuring agents will only resell if it is more profitable than keeping. First, we redefine the budget constraint, using a partition function p to split the consumption plans between neighbor nodes:

$$\beta^i(r,k,\lambda) = \left(\sum_{j \in N(i)} p^j \cdot \lambda(r+k,j) \leq p^i \cdot (e^i + r) \right) \bigwedge \\ \left(\sum_{j \in N(i)} \lambda(r+k,j) = r+k \right) \quad (1)$$

Note the addition of r on the right side of the inequality: this is the crucial difference between our model and the model as-is. The right side of the equation represents the "cash value" of everything agent i gives away. By adding in r^i, we effectively allow agent i to sell things he is not endowed with: things he bought from other agents, which he is now transporting through the network. Because if the trades were actually to be executed, the agent would be temporarily in debt (in between buying the r vector and reselling it), we call this "short trading". (Note further that here we use the partition λ to formalize our notion of "subplan", which we formerly denoted $x^{j \to i}$.)

The equilibrium is simply defined as any set of price vectors p^i such that the values of r^i, k^i, and λ^i maximize $u^i(k^i)$ while satisfying $\beta^i(r^i, k^i, \lambda^i)$, for all i.

2.2 Example

The use of the Lagrangian (whatever that means) "cash" values in this cashless setting can be confusing, so it is instructive to work through an example to see how the r vector allows agents to move goods throughout the network. Consider once again our "spice trade" example:

👸 $\rightleftharpoons$ ⛵ $\rightleftharpoons$ 👳

Here the noblewoman 👸 wants spice, and has gold, the merchant ⛵ wants everything, and has nothing, and the spice grower 👳 wants gold, and has spice.

Formally, we can specify the utilities by

i	u^i(🌶️)	u^i(💰)
👸	10	1
⛵	10	10
👳	1	10

and the endowments by[1]

i	$e^i_{🌶️}$	$e^i_{💰}$
👸	0	1
⛵	0	0
👳	1	0

In the model as-is, no goods would change hands, because the parties that have goods do not have anyone to trade with: they are connected to the merchant, but the merchant has nothing to bring to the table.

In our model, however, the merchant has a great deal to bring to the table: he can set r as large as he wants both in spice and in gold, sell his now-large "endowment" accordingly (paying off the loan), and use any remaining "cash" to buy goods for himself. Specifically, suppose the merchant set

$$e^{⛵}_{🌶️} = 1/10$$
$$e^{⛵}_{💰} = 1/10$$

If the merchant uses prices

$$p^{⛵}_{🌶️} = 10$$
$$p^{⛵}_{💰} = 10$$

he can sell the r vector to obtain a profit of 2 units of "cash", since both the noblewoman and the spice grower have a rate of substitution of 10 between their preferred and non-preferred commodities, and thus would not object to paying 1 unit of "cash" for 1/10 units of commodity. Using his 2 units of "cash", and supposing the endpoints price similarly, the merchant can then buy the remaining 9/10 of both commodities, ending with a much larger endowment, and utility, than he started with. The noblewoman and spice grower end with the same utility that they started with, but replace their non-preferred good with their preferred good. (Any price lower than 10, however, would still be an equilibrium, leaving the middle node with a smaller final utility, and allowing the endpoints to increase their utilities. Prices higher than 10 would also be equilibria, but no trade would occur).

This example is a nice illustration of how the r vector facilitates the flow of commodities. It also raises an important issue: many equilibria can exist

[1] Note that since e^i is a vector, we can subscript it, but since the utilities can be nonlinear, we must represent them as functions; inputting scalars to those functions is technically an abuse of notation, but when one uses emojis in TEX, one mustn't be too pedantic about abusing notation, or one might risk collapsing on one's own inconsistencies. (A more formal alternative would be to think of $\{🌶️, 💰\}$ as a basis for the vector space of possible endowment vectors.)

in the model, leading to very different final allocations. For example, if the endpoints price differently, the merchant will put all his "cash" into whichever endpoint prices lower, since he has equal utility for both commodities. So there are at least three possible final allocations: merchant ends up with all 🌶️ or all 💰, merchant ends up with equal amounts of 🌶️ and 💰, or merchant ends up with nothing. All are equilibria, and only the first two are really "strategic," so further work is clearly needed to bring the model more in line with real-life play.

2.3 Arguing for the r vector

One can make two strong arguments for adding the r vector to the model. First, if agents are *not* allowed to resell their purchases, one wonders in what sense their "trades" are at all a model for what actually happens when a commodity is traded: one exchanges control of one piece of property for control over another, and one of the things one is allowed to do with property one controls is trade it again. If agent's can't resell property they bought, they do not have the same rights to that property that the original owner had, which is not consistent with the meaning of "trade" in the real world. Thus, reselling should be allowed.

The second argument focuses more on the power that reselling gives agents, rather than the reselling itself. Namely, it's important to understand that the unique graphical setting of the Kakade et. al. model introduces extreme incentives to resell, by giving some agents access to vastly larger markets than others. In fact, the inspiration for this project (IIRC) was the realization that in the Kakade et. al. model, even an agent with full control of all trade routes (as in a hub-and-spoke graph) would not be able to benefit at all from its extremely strategic position in the network topology. In fact, the fundamental difference between the graphical and non-graphical setting is that the graphical setting gives some agents control over other agents' access to markets: any model that ignores that is not accurately modelling strategic behavior. As the example shows, even agents with zero endowment can make huge profits in our model if they are in a strategic position in the network topology; this is precisely what one wants in a graphical economics model.

3 Towards demonstrating economic soundness: equivalence of the model with Arrow-Debreu on a complete graph

One important property *any* network-generalization of Arrow-Debreu should satisfy is equivalence on a complete graph: if there is no restriction of trade, then an Arrow-Debreu equilibrium should also be an equilibrium in the generalized model. This trivially holds for Kakade et. al. model: the only change made is the set being summed over in the budget constraint, and on a complete graph, the neighbor set *is* the agent set, so the sums equal.

We aim to show that any Arrow-Debreu equilibrium on a complete graph is also an equilibrium in the short-trading model. Formally, the necessary theorem is the following:

Theorem 1. *Suppose (V, E) is a complete graph, and the agents V have endowments $\{e^i\}$ and utilities $\{u^i\}$. Let $(p, \{x^i\})$ be an Arrow-Debreu equilibrium in the non-graphical economy $(\{e^i\}, \{u^i\})$. Finally set, for all i,*

$$\begin{aligned} p^i &= p \\ k^i &= x^i \\ r^i &= 0 \end{aligned}$$

Then $(\{p^i\}, \{k^i\}, \{r^i\})$ is an equilibrium in the short-trading graphical model.

Proof. If having $r^i = 0$ for all i is compatible with individual rationality, then $(\{p^i\}, \{k^i\}, \{r^i\})$ is necessarily an equilibrium, because $r^i = 0$ collapses the short-trading model down to the Kakade et. al. model, which in turn collapses to Arrow-Debreu (on a complete graph), as desired. So it is sufficient to show that, for all i, and with r and k ranging over all possible consumption vectors and λ ranging over all possible partition functions,

$$\max_{\beta^i(0,k,\lambda)} u^i(k) = \max_{\beta^i(r,k,\lambda)} u^i(k)$$

Note that, by assumption, all the price vectors are the same! So we can distribute out all the $p^j = p$ and get rid of the dependence on λ:

$$\beta^i(r, k, \lambda) = \left[p \cdot (r + k) \leq p \cdot (e^i + r)\right]$$

But this means that, for all i,

$$p \cdot k^i \leq p \cdot e^i$$

So constraining $r^i = 0$ cannot affect the objective of the max, and in fact one rational allocation for each agent is simply to set

$$k^i = \arg\max_{p \cdot k^i \leq p \cdot e^i} u^i(k^i)$$

Which, again, is independent of r^i. □

4 Proof of existence of an equilibrium

The proof of existence of an equilibrium in the case where agents' utilities are both semi-Lesbegue-differentiable and semi-Fourier-convex, but neither Lesbegue-Fourier-convex nor Fourier-Lesbegue-differentiable, is so trivial that we leave it as an exercise to the reader.

5 Future work

The immediate shortcomings of this work are that we have neither a proof of existence of an equilibrium (except for when the network is a complete graph), nor an algorithm for computing an equilibrium. We have hope that the algorithm described in [2] can be ported to our model without major modification, but do not believe that their proof of equilibrium existence can be ported at all, because it relies on a special injective mapping from graphical economies to Arrow-Debreu economies that is not possible in our model. Those two problems are the subject of our current work.

However, a more general issue is determining to what extent the model agrees with real-world behavior. Specifically, as we showed in the example, multiple equilibria, with different final resource allocations, can be possible in our model, and only some of those allocations correspond to intuitively strategic behavior. Determining in what circumstances, if any, our equilibrium corresponds to, say, a Nash or coordinated equilibrium may be a useful step in determining the practical utility of our model.

6 Conclusion

In this paper, we showed that an extremely simple modification to the Kakade et. al. model of a graphical economy allows agents to trade across their entire connected component in an incentive-compatible way, using only local flow constraints, with a natural economic interpretation ("short trading"). We argued that this may be a much more accurate model of strategic, or at least real-world, behavior, even giving an example where strategic play corresponds to an equilibrium in our model, but not in the Kakade et. al. model. Further, we showed some promising results in the direction of proving our model is economically sound; specifically, we showed that on a complete graph, our model admits Arrow-Debreu equilibria, which is a necessary, but not sufficient, condition for being economically sound. As discussed above, we hope to show that the model is fully economically sound, which would allow us to decisively claim our model a much better way to understand trade in a restricted setting than the Kakade et. al. model.

7 References

References

[1] Kenneth J Arrow and Gerard Debreu. "Existence of an equilibrium for a competitive economy". In: *Econometrica: Journal of the Econometric Society* (1954), pp. 265–290.

[2] Sham M Kakade, Michael Kearns, and Luis E Ortiz. “Graphical economics”. In: *International Conference on Computational Learning Theory*. Springer. 2004, pp. 17–32.