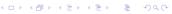
Elliptic Curves

Jake Fisher and Davis Lister

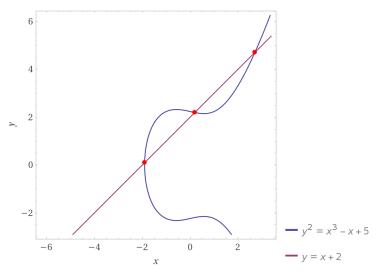
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Group Law for Elliptic Curves

- ▶ Elliptic Curves are a group under the + operation with the set $\{K \times K : y^2 = x^3 + ax + b\} \cup \{\mathcal{O}\}$ where \mathcal{O} is the point at infinity and K is a field. We notate the set represented by an elliptic curve E over a field K, E(K).
- Let us define the discriminant $\Delta = -16(4a^3 + 27b^2)$. If $\Delta = 0$ the group law does not hold.
- ▶ The + operation is defined geometrically on two points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ thus: draw the secant line and find the third point where it intersects the curve (x', y'), which can include \mathcal{O} , finally find (x', -y'), the resulting point.
- ▶ Under the + operation \mathcal{O} is the identity and $(x,y)^{-1} = (x,-y)$.
- ▶ The + operation is closed since each secant line will intersect the curve at exactly one other point. The definition of the operation does not discriminate between P_1 and P_2 . Therefore, (E(K), +) is an abelian group.



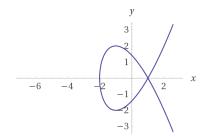
A Visualization of the + Operation over E(K)



Computed by Wolfram | Alpha

More on the Discriminant

- ▶ The condition we impose on the discriminant is the is motivated by the need to have a tangent line be well defined at all points on the elliptic curve S. If the tangent line is not defined at a point $P_0 = (x_0, y_0)$, then S is singular.
- ▶ If S is singular, then $P_0 + P_0$ is not well-defined, which breaks the group law.
- ▶ Geometrically, a singularity is a cusp or self-intersection.





Definition of *B*-power smooth

- ▶ Let *B* be a positive integer. The prime factorization of an integer $n = \prod p_i^{e_i}$. If $\forall i \ p_i^{e_i} \leq B$, *n* is *B*-power smooth.
- ► For instance, 60 is 5-power smooth but 150 is 25-power smooth.

Pollard's p-1 Factoization

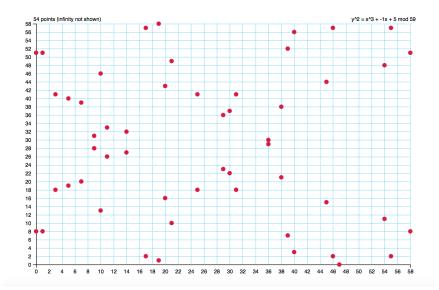
- We wish to find a nontrivial factor of a large positive integer N using the Pollard p-1 method.
- Let us choose a positive integer B. Suppose that there is a prime factor p of N such that p-1 is B-power smooth.
- ▶ Let us choose *a* > 1 such that p does not divide a. Often we will choose *a* = 2 for convenience.
- ▶ By Fermat's Little Theorem $a^{p-1} \equiv 1 \mod p$.
- Let m = lcm(1, 2, 3, ..., B). Since p 1 is B-power smooth, $p 1 \mid m \Longrightarrow p \mid \text{gcd}(a^m 1, N) > 1$.
- ▶ If $gcd(a^m 1, N) < N$, then $gcd(a^m 1, N)$ is a nontrivial factor of N.
- ▶ The algorithm becomes more transparent if we consider m = k(p-1), where $k \in \mathbb{Z}$.



Factoring Magic!

- An example of integer factorization using Pollard's p-1 method.
- Let N = 5917 and let B = 5. m = lcm(1, 2, 3, 4, 5) = 60.
- Note that $2^{60} 1 = 3416 \mod 5917$, and $gcd(2^{60} 1, 5917) = gcd(3416, 5917) = 61$.
- ▶ 61 is a factor of 5917!
- ▶ But if p-1 and q-1 (where pq=N) are not B-power smooth, Pollard p-1 does not work.
- ▶ The issue is that $(\mathbb{Z}/p\mathbb{Z})^*$ has order p-1.
- Additionally, if p-1 is the product of many small primes, then the algorithm will return N.

Elliptic Curves over Finite Fields



Lenstra's Elliptic Curve Factorization

- ▶ Choose $N, B \in \mathbb{Z}^+$.
- ▶ Compute m = lcm(1, 2, ..., B).
- ▶ Choose a random $a \in \mathbb{Z}/N\mathbb{Z}$ such that $4a^3 + 27 \in \mathbb{Z}/N\mathbb{Z}^*$. Thus, P = (0,1) is a point on the elliptic curve $y^2 = x^3 + ax + 14$ over $\mathbb{Z}/N\mathbb{Z}$.
- Attempt to compute mP using the + operation for the group (E(K),+). If at some point we cannot compute a sum of points because $\gcd(x_1-x_2,N)\neq 1$ (where x_1-x_2 is the denominator of the slope expression we compute in order to execute the + operation), compute and return $\gcd(x_1-x_2,N)$ if $\gcd(x_1-x_2,N)\neq N$. If some point $kP=\mathcal{O}$ for $k\leq m$, terminate and output, "Fail." Additionally, if mP can be computed using the + operation, output, "Fail."

Advantages of the Lenstra Method

- The advantage of the Lenstra method is that if the algorithm fails, we may choose a different elliptic curve and repeat the algorithm.
- ▶ We have more flexibility with our groups since we work with many different groups $E(\mathbb{Z}/N\mathbb{Z})$, which will have order $p+1\pm s$.
- ▶ In Pollard p-1 we work with $\mathbb{Z}/N\mathbb{Z}^*$, which always has order p-1.

An Example of the Lenstra Method

- ▶ Let us choose N = 5959 and B = 8.
- ► Therefore, m = 840 and our random a = 6, where $a \in \mathbb{Z} \mid 4a^3 + 27 \neq 0$.
- Note that P = (0,1) is on the elliptic curve $y^2 = x^3 + ax + 1$ over $\mathbb{Z}/N\mathbb{Z}$. We now attempt to compute mP.
- ▶ Observe that 60P = (649, 2654). Therefore $\lambda \equiv \frac{2653}{649}$ mod 5959. Since $649 \not\equiv 0 \mod 5959$, which would indicate that the next point were \mathcal{O} , we encounter a contradiction to the group law.
- Note that $(649, 2654)^{-1} = (649, -2654)$.
- ▶ Therefore, $p = \gcd(649, 5959)$ is a nontrivial factor of N, if 1 .
- ▶ $gcd(649, 5959) = 59 \Longrightarrow 59$ is a nontrivial factor of 5959.

Elliptic Curve Cryptography

- ► The Diffie-Hellman key exchange can be implemented on an Elliptic Curve.
- ▶ Alice and Bob publicly agree on a prime p and an elliptic curve S over $\mathbb{Z}/p\mathbb{Z}$. They then agree on a point $P \in S(\mathbb{Z}/p\mathbb{Z})$.
- ▶ Alice chooses a private key *m* and sends Bob *mP*.
- ▶ Bob chooses a private key *n* and sends Alice *nP*.
- ▶ Alice and Bob both compute *mnP*, their shared secret key.

A Numerical Example of Diffie-Hellman Key Exchange with Elliptic Curves

- Alice and Bob publicly agree on p = 571 and the elliptic curve S given by $y^2 = x^3 12x + 5$. They then agree on P = (16, 324).
- Alice privately computes and sends Bob 39P = (148, 387). Bob privately computes and sends 121P = (465, 556).
- ► Alice privately computes 39(465, 556) = (202, 445), and Bob privately computes 121(148, 387) = (202, 445).

Elliptic Curve Discrete Logarithm Problem

- ► The security of an elliptic-curve cryptosystem is dependent on the solution to the elliptic curve discrete logarithm problem.
- ▶ Let S be an elliptic curve over $\mathbb{Z}/p\mathbb{Z}$ and $P \in S(\mathbb{Z}/p\mathbb{Z})$.
- ▶ Given Q a multiple of P, we aim to find $n \in \mathbb{Z}$ such that nP = Q.
- The naive approach is to simply check each possible value of n until one arrives at the solution Q. However, the naive approach becomes computationally infeasible as p is sufficiently large.
- ▶ Currently, it appears that the discrete logarithm problem on $E(\mathbb{Z}/p\mathbb{Z})$ is more difficult than the discrete logarithm problem on $\mathbb{Z}/p\mathbb{Z}^*$.
- ► An elliptic curve cryptosystem can offer equivalent security to many cryptosystems currently in use with much smaller numbers, which allows for great gains in efficiency.

Why Should You Care?

- You probably shouldn't.
- But, if you want to...
- Elliptic curves are much more space-efficient than other methods – a 1024-bit RSA key is equivalent to a 163-bit ECC key in security.
- However, because it's not as well-established, it has other potential issues.

Dual_EC_DRBG

- Published first in 2006.
- ▶ It was pushed by NSA to be standardized, even going so far as to pay a security company \$10 million to use it.
- Suspected to contain a backdoor.
- Removed by NIST in 2014.

Applications of Elliptic Curve Cryptography

- Presently, Bitcoin uses an elliptic curve digital signature algorithm to verify coin ownership.
- Google also utilizes elliptic curve public keys for encryption and verification.

```
Public Key Info
Algorithm
Parameters Elliptic Curve Public Key ( 1.2.840.10045.2.1 )
Public Key
Key Size
Key Usage Encrypt, Verify, Derive
Signature 256 bytes : 21 BF D2 FF A4 E4 D4 02 ...
```