Davis Lister and Jacob Fisher

Abstract

We will show that the Elliptic Curve discriminant $\Delta = -16(4a^3 + 27b^2) = 0$, if and only if the curve, S, is singular. For clarification, a singular curve is defined as having a point, the singular point, at which the derivative of the function is not well defined.

We wish to show that if S is singular, then $\Delta = 0$. Let $f(x, y) = y^2 - (x^3 + ax + b)$. Suppose S has a singular point at $P_0 = (x_0, y_0)$. Therefore,

$$\frac{\partial f}{\partial x} y_0^2 - (x_0^3 + ax_0 + b) = -3x_0^2 - a = 0 \Longrightarrow a = -3x_0^2,$$

$$\frac{\partial f}{\partial y} y_0^2 - (x_0^3 + ax_0 + b) = -2y_0 = 0 \Longrightarrow y_0 = 0.$$

Since $y_0 = 0$, all singular points will be roots of $y^2 = x^3 + ax + b$. Observe,

$$0 = x_0^3 - 3x_0^3 + b \Longrightarrow b = 2x_0^3.$$

Thus,

$$\Delta = -16(4(-3x_0^2)^3 + 27(2x_0^3)^2) = 0.$$

We wish to show that if $\Delta = 0$, then S is singular. First we will prove a lemma: if $\Delta = 0$, $y = x^3 + ax + b$ has a double root x_0. Note that

$$-16(4a^3 + 27b^2) = 0 \Longrightarrow b = \sqrt{\frac{-4a^3}{27}}.$$

Furthermore, observe that given the above our equation becomes

$$y = x^3 + ax + \sqrt{\frac{-4a^3}{27}}.$$

The roots of the above equation are

$$x_1 = \frac{a}{\sqrt{3}\sqrt[6]{-a^3}} - \frac{\sqrt[6]{-a^3}}{\sqrt{3}}, x_2 = \frac{i\sqrt{3}\sqrt[3]{-a^3} + \sqrt[3]{-a^3} + i\sqrt{3}a - a}{2\sqrt{3}\sqrt[6]{-a^3}}, x_3 = \frac{-i\sqrt{3}\sqrt[3]{-a^3} + \sqrt[3]{-a^3} - i\sqrt{3}a - a}{2\sqrt{3}\sqrt[6]{-a^3}}.$$

Note that

$$b = \sqrt{\frac{-4a^3}{27}} \Longrightarrow a < 0 \,\text{for}\, b \in \mathbb{R}.$$

Therefore, our second two solutions become identical

$$x_0 = \frac{\sqrt[3]{-a^3} - a}{2\sqrt{3}\sqrt[6]{-a^3}}.$$

Thus $y = x^3 + ax + b$ has a double root x_0. By the lemma, we can deduce that one of the roots of $y = x^3 + ax + b$ is a root of its derivative, $y' = 3x^2 + a$. Recall that $f(x, y) = y^2 - (x^3 + ax + b)$. Therefore,

$$f(x_0, 0) = 0^2 - (x_0^3 + ax_0 + b) = 0,$$

$$\frac{\partial f}{\partial x}(x_0, 0) = -3x_0^2 - a = 0,$$

$$\frac{\partial f}{\partial y}(x_0, 0) = 2(0) = 0.$$

Thus S is singular. Furthermore, we can conclude that S is singular if and only if $\Delta = 0$.