Closed form of the difference between the square of the sum and the sum of the squares of natural numbers

Alex Striff

May 14, 2017

1 Abstract

The goal is to find a closed form of the following expression, the difference between the square of the sum of natural numbers to n and the sum of the squares of the natural numbers to n.

$$\left(\sum_{i=1}^{n} i\right)^2 - \sum_{i=1}^{n} i^2$$

2 Triangle Numbers

The easiest subproblem here is to find the sum of the first n natural numbers:

$$T_n = \sum_{i=1}^{n} i$$

The sum of n numbers can be organized into $^n/_2$ pairs of numbers, pairing the first with the last, the second with the penultimate, and so on. The sum of a pair will always be $n+1$, so adding all $^n/_2$ pairs of $n+1$ gives $^{n(n+1)}/_2$.

$$\begin{aligned}
T_n &= 1 + 2 + 3 + \cdots + (n-2) + (n-1) + n \\
&= (1+n) + (2 + (n-1)) + (3 + (n-2)) + \cdots + (\frac{n}{2} + (n - \frac{n}{2})) \\
&= \frac{n(n+1)}{2}
\end{aligned}$$

A geometric approach would be to plot n rectangles in order with heights ranging from 1 to n and constant widths of 1. The area of these rectangles is the nth triangle number. Drawing a line from $(0,0)$ to (n,n) would make a right triangle with side length n on the bottom and n smaller right triangles of side length 1 on the top. The area of the larger triangle is $n^2/2$, and of any smaller triangle, $1^2/2$. Because there are n smaller triangles, the area of all of the smaller triangles is $n/2$. Adding together the two areas, the above $n(n+1)/2$ is obtained.

$$\begin{aligned}
T_n &= \frac{1}{2}(n)(n) + n \cdot \frac{1}{2}(1)(1) \\
&= \frac{n^2}{2} + \frac{n}{2} \\
&= \frac{n(n+1)}{2}
\end{aligned}$$

To find the nth square of the sum term, square T_n. The expression can now be simplified to:

$$\left(\frac{n(n+1)}{2}\right)^2 - \sum_{i=1}^{n} i^2$$

3 Square Numbers

The sum of the squares term is a bit more difficult. Upon writing down the first few squares, a pattern emerges.

$$\begin{aligned}
1^2 &= 1 &&= 1 \\
2^2 &= 4 &&= 1+3 \\
3^2 &= 9 &&= 1+3+5 \\
4^2 &= 16 &&= 1+3+5+7 \\
5^2 &= 25 &&= 1+3+5+7+9 \\
6^2 &= 36 &&= 1+3+5+7+9+11 \\
7^2 &= 49 &&= 1+3+5+7+9+11+13 \\
8^2 &= 64 &&= 1+3+5+7+9+11+13+15 \\
9^2 &= 81 &&= 1+3+5+7+9+11+13+15+17 \\
10^2 &= 100 &&= 1+3+5+7+9+11+13+15+17+19 \\
n^2 &= n^2 &&= \sum_{i=1}^{n} 2i-1
\end{aligned}$$

So the nth square appears to be the sum of the first n odd numbers. To use this relationship later, it must be proved that this is true for all $n \in \mathbb{N}$.

Proof.

$$\begin{aligned}
&\vdash 1^2 = 1 \\
&\vdash n^2 = \sum_{i=1}^{n} 2i - 1 \\
&= \left(\sum_{i=1}^{n-1} 2i - 1 \right) + (2i - 1) \\
&= (n-1)^2 + (2n - 1) \\
(n+1)^2 &= ((n+1) - 1)^2 + (2(n+1) - 1) \\
&= n^2 + 2n + 1 \\
&= (n+1)^2
\end{aligned}$$

Therefore $n^2 = \sum_{i=1}^{n} 2i - 1$ is true for all $\mathbb{N}$ by induction. $\square$

4 Odd Numbers

The task is now to sum squares by summing odd numbers. Expanding the sum:

$$\begin{aligned}
\sum_{i=1}^{n} i^2 &= \sum_{i=1}^{n} \sum_{k=1}^{i} 2k - 1 \\
&= \ 1 \\
&+ \ 1 + 3 \\
&+ \ 1 + 3 + 5 \\
&+ \ 1 + 3 + 5 + \cdots + (2n-5) \\
&+ \ 1 + 3 + 5 + \cdots + (2n-5) + (2n-3) \\
&+ \ 1 + 3 + 5 + \cdots + (2n-5) + (2n-3) + (2n-1) \\
&= (n)(1) + (n-1)(3) + (n-2)(5) \\
&+ \cdots + (3)(2n-5) + (2)(2n-3) + (1)(2n-1) \\
&= \sum_{i=1}^{n} (n - i + 1)(2i - 1)
\end{aligned}$$

Any odd number $O_n = 2n - 1$ can be written as $O_n = 2n - 1 + (1 - 1) = 2(n-1) + 1$, i.e. the sum of 1 and $(n-1)$ 2's.

$$
\begin{aligned}
\sum_{i=1}^{n} i^2 &= (n)(1) \\
&+ (n-1)(1+2) \\
&+ (n-2)(1+2+2) \\
&+ (n-3)(1+2+2+2) \\
&+ \cdots \\
&= n \\
&+ (n-1) + (n-1)(2) \\
&+ (n-2) + (n-2)(2+2) \\
&+ (n-3) + (n-3)(2+2+2) \\
&+ \cdots \\
&= T_n \\
&+ 2(n-1)(1) \\
&+ 2(n-2)(1+1) \\
&+ 2(n-3)(1+1+1) \\
&+ \cdots \\
&= T_n + 2\sum_{i=1}^{n} (n-i)i \\
&= T_n + 2\sum_{i=1}^{n} (ni - i^2) \\
&= T_n + 2n\left(\sum_{i=1}^{n} i\right) - 2\left(\sum_{i=1}^{n} i^2\right) \\
3\sum_{i=1}^{n} i^2 &= T_n(2n+1) \\
\sum_{i=1}^{n} i^2 &= T_n\left(\frac{2n+1}{3}\right)
\end{aligned}
$$

So a closed form for the sum of the squares is now known.

5 Conclusion

Now the closed form for the square of the sum minus the sum of the squares of the first n natural numbers may be written.

$$\begin{aligned}
\left(\sum_{i=1}^{n} i\right)^2 - \sum_{i=1}^{n} i^2 &= {T_n}^2 - T_n\left(\frac{2n+1}{3}\right) \\
&= T_n\left(\frac{n(n+1)}{2} - \frac{2n+1}{3}\right) \\
&= \frac{T_n}{6}(3n^2 - n - 2) \\
&= \frac{n^2 + 2n + 1}{12}(3n^2 - n - 2)
\end{aligned}$$

$$\boxed{\left(\sum_{i=1}^{n} i\right)^2 - \sum_{i=1}^{n} i^2 = \frac{n^4}{4} + \frac{n^3}{6} - \frac{n^2}{4} - \frac{n}{6}}$$