Interaction of the transverse-field Ising model with a bath

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Outline

Why care about the transverse-field Ising model?

What is an open quantum system?

What effect does the bath have?

How do we solve interacting spin systems?

How do we compute the jump operators?

What next?

The transverse-field Ising model

$$H = -\sum_{i \in \mathbb{Z}_N} \sigma_i^x \sigma_{i+1}^x - \lambda \sum_{i \in \mathbb{Z}_N} \sigma_i^z$$

- Cannot be described with classical statistical mechanics
- Show thermalization from microscopic interactions
- Undergoes a quantum phase transition at $\lambda = 1$
- Learn new techniques
- Relatively simple

Open quantum systems

A quantum system is described by a density operator ρ

Setup: Liouville space $\mathscr{L}$ of operators with finite norm

$$\langle A|B\rangle = \mathrm{tr}\left(A^\dagger B\right)$$

Expectations: Self-adjoint O has $\langle O\rangle = \langle \rho|O\rangle$

- $\langle \alpha I\rangle = \alpha$ for $\alpha \in \mathbb{C}$ requires $\mathrm{tr}\,\rho = 1$
- All $\langle O\rangle$ real requires $\rho = \rho^\dagger$
- All $\langle O\rangle$ positive for positive O requires ρ positive

Composite systems

Epistemically independent systems A and B form a composite system:

$$\mathscr{C}(\rho_A + \rho'_A, \rho_B) = \mathscr{C}(\rho_A, \rho_B) + \mathscr{C}(\rho'_A, \rho_B)$$

Equivalence classes are called tensors in $\mathscr{L}_A \otimes \mathscr{L}_B$
For $\rho_A \otimes \rho_B$ to be normalized,

$$\langle \rho_A \otimes \rho_B | \rho_A \otimes \rho_B \rangle = \langle \rho_A | \rho_A \rangle \langle \rho_B | \rho_B \rangle$$

This requires

$$(\rho_A \otimes \rho_B)(A \otimes B) = \rho_A A \otimes \rho_B B$$

Undo with $\mathrm{tr}_B\, \rho = \rho_A$.

Time evolution

A priori, anything can happen to ρ[1]

$$\rho \mapsto \sum_i B_i \rho B_i^\dagger, \quad \text{where} \quad \sum_i B_i B_i^\dagger \leqslant I$$

Usually $\rho(t)$ depends only on ρ now:

$$\rho(t) = \mathcal{V}_t \rho(0)$$
$$\mathcal{V}_{t+t'} = \mathcal{V}_t \mathcal{V}_{t'}$$

The Lindblad equation generates $\mathcal{V}_t$ that are CPTP

$$\dot{\rho} = -\mathrm{i}[H, \rho] + \sum_i \gamma_i \left(J_i \rho J_i^\dagger - \frac{1}{2}\left\{ J_i^\dagger J_i, \rho \right\} \right)$$

[1] ρ cannot start entangled if it follows a CPTP map

Weak-coupling approximations

System coupled to stationary bath:

$$H = H_S \otimes I + I \otimes H_B + H_I$$
$$H_I = \sum_i A_i \otimes B_i$$

Lindblad equation for $\rho_S(t) = \mathrm{tr}_B\, \rho(t)$ has

$$\gamma_i(\omega) = \int_{-\infty}^{\infty} \mathrm{d}s\, e^{\mathrm{i}\omega s} \left\langle B_i^\dagger(t) B_i(t-s) \right\rangle$$

Spins in a bosonic bath

For a single spin in a one-dimensional cavity:

$$H_B = \sum_{\omega} \omega\, a_{\omega}^{\dagger} a_{\omega}$$

$$H_I = \sigma_x \otimes \sum_{\omega} \sqrt{J(\omega)} \left(a_{\omega}^{\dagger} + a_{\omega} \right)$$

Only $J(\omega) \propto \omega$ for small ω produces interactions.
With an upper cutoff, this is the Ohmic spectral density

$$J(\omega) \propto \frac{\omega}{1 + (\omega/\Omega)^2}$$

Electric dipole interactions have the same coupling

Solving the transverse-field Ising model

$$H = -\sum_{i \in \mathbb{Z}_N} \sigma_i^x \sigma_{i+1}^x - \lambda \sum_{i \in \mathbb{Z}_N} \sigma_i^z$$

- A solution by E. Leib et. al. requires deriving and solving a N-eigenvalue problem.
- A series of transformations provide a simpler route.

$$\sigma_i \to c_i \to C_k \to \eta_k$$

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Fermions at each site with σ_i^+ and σ_i^-?

$$\left\{\sigma_i^+, \sigma_j^-\right\} \neq \delta_{ij} I$$

Actual fermions act like

$$c_i \left|\boldsymbol{n}\right\rangle = -(-1)^{\sum_{j<i} n_j} n_i \left|\boldsymbol{n}_{i \leftarrow 0}\right\rangle$$

$$\left|\boldsymbol{n}\right\rangle = \prod_i \left(c_i^\dagger\right)^{n_i} \left|\boldsymbol{0}\right\rangle$$

Same effect:

$$c_i = -\left(\prod_{j<i} -\sigma_j^z\right)\sigma_i^-$$

$$\left|\boldsymbol{n}\right\rangle = \prod_i \left(\sigma_i^+\right)^{n_i} \left|\boldsymbol{0}\right\rangle$$

$\sigma_i \to c_i \to C_k \to \eta_k$

After Jordan-Wigner transform:

$$H = \sum_i \left(c_i - c_i^\dagger\right)\left(c_{i+1}^\dagger + c_{i+1}\right) - \lambda \sum_i 2c_i^\dagger c_i + \lambda N I$$
$$- \left(I - \prod_{j<N-1} \left(2c_j^\dagger c_j - I\right)\right)\left(c_{N-1} - c_{N-1}^\dagger\right)\left(c_0^\dagger + c_0\right)$$

H is translation invariant without the boundary term.

$\sigma_i \to c_i \to C_k \to \eta_k$

Fourier transform:

$$c_i = \frac{1}{\sqrt{N}} \sum_k e^{\mathrm{i}ki} C_k$$

Now the Hamiltonian is

$$\begin{aligned} H &= -\sum_k (\lambda + \cos k)\left(C_k^\dagger C_k - C_{-k} C_{-k}^\dagger\right) + \sum_k \mathrm{i} \sin k \left(C_{-k} C_k - C_k^\dagger C_{-k}^\dagger\right) \\ &= \sum_k \boldsymbol{v}_k^\dagger \boldsymbol{H}_k \boldsymbol{v}_k, \end{aligned}$$

where

$$\boldsymbol{H}_k = \begin{bmatrix} -(\lambda + \cos k) & -\mathrm{i} \sin k \\ \mathrm{i} \sin k & \lambda + \cos k \end{bmatrix}$$

$$\boldsymbol{v}_k = \begin{bmatrix} C_k \\ C_{-k}^\dagger \end{bmatrix}$$

$\sigma_i \to c_i \to C_k \to \eta_k$

Diagonalize the quadratic form:

$$E_k = \sqrt{\lambda^2 + 2\lambda \cos k + 1}$$

$$\eta_k = \frac{\mathrm{i} \sin k}{\sqrt{2E_k(E_k + \cos k + \lambda)}} C_k + \sqrt{\frac{E_k + \cos k + \lambda}{2E_k}} C_{-k}^\dagger$$

Now the Hamiltonian is

$$\begin{aligned} H &= \sum_k \begin{bmatrix} \eta_k^\dagger & \eta_{-k} \end{bmatrix} \begin{bmatrix} E_k & 0 \\ 0 & -E_k \end{bmatrix} \begin{bmatrix} \eta_k \\ \eta_{-k}^\dagger \end{bmatrix} \\ &= \sum_k 2E_k \eta_k^\dagger \eta_k - I \sum_k E_k \end{aligned}$$

$c_i \to \eta_k$ is unitary, so we have independent fermions

Computing jump operators

Jump operators are superoperators of the Hamiltonian:

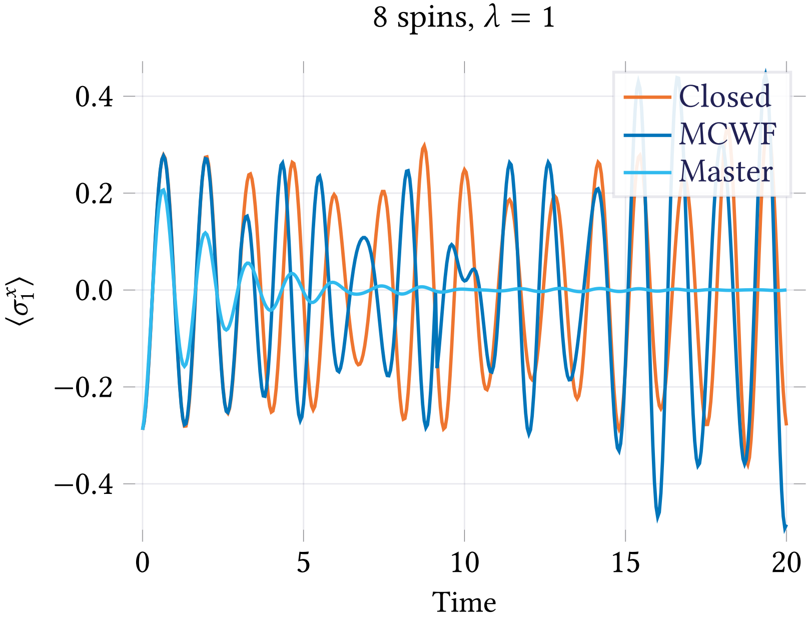
$$[H, J_i(\omega)] = \omega J_i(\omega),$$

where

$$J_i(\omega) = \sum_{E_2 - E_1 = \omega} P(E_1)\sigma_i^x P(E_2)$$

$$P(E) = \sum_{\boldsymbol{n} : E = \sum_k E_k n_k} |\boldsymbol{n}\rangle\langle\boldsymbol{n}|$$

Do inverse transformations $\eta_k \to \sigma_i$ for σ_i^x


8 spins, $\lambda = 1$
0.4
0.2
0.0
−0.2
−0.4
$\langle \sigma_1^x \rangle$
0
5
10
15
20
Time
Closed
MCWF
Master

References

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