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2 I want to thank a few people.



# Abstract

The preface pretty much says it all.





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# 1 Introduction

2   Introductory content.



# Chapter 1

## Density operator theory

IF QUANTUM MECHANICS is so weird, then why aren't we? The answer lies in the phenomenon of *decoherence*, which was first considered in depth in the 1970's by Zeh [1]. Despite being a late comer to the history of quantum mechanics, the theory of decoherence is crucial to understanding how classical results are obtained from many interacting quantum systems. While there are other routes to decoherence, the most common is through interaction with a memoryless (Markovian) environment [2]. This leads to the theory of quantum Markovian master equations, which describe transformations of a system in the presence of such an environment. The most general form of these transformations was first extended to quantum mechanics by Gorini, Kossakowski, Lindblad, and Sudarshan to give the GKLS or *Lindblad equation* [3, 4].

This chapter will explain the relevant theoretical background in Secs. 1.1 and 1.2, before presenting the general theory which leads to the Lindblad equation in Sec. 1.3, following the text of Breuer and Petruccione [5] and to a lesser extent [6] (which has some flaws). This leads us to consider the weak-coupling limit in Sec. 1.4 and an application to atomic physics in Sec. 1.6. Issues with the method are discussed briefly in Sec. 1.7.

### 1.1 Different perspectives on density operators

For a statistical perspective on quantum mechanics, we will make two postulates. The mathematical background is the *Liouville space*  $\mathcal{L}(\mathcal{H})$  for the Hilbert space  $\mathcal{H}$ .

**Definition 1** (Liouville space). The space  $\mathcal{L}(\mathcal{H})$  is the complex Hilbert space of operators  $A$  on  $\mathcal{H}$  for which the norm induced by the inner product  $\langle A|B \rangle \equiv \text{tr}(A^\dagger B)$  is finite.

**Postulate 1.** A quantum system may be understood as a statistical ensemble  $\rho$  with observables  $O$  which are both described by elements of  $\mathcal{L}(\mathcal{H})$ , where the ensemble average of  $O$  is  $\langle O \rangle_\rho \equiv \langle O|\rho \rangle$  and  $O$  is Hermitian.

The usual properties of the *density operator*  $\rho$  follow from considering various averages. The only way for  $\langle \alpha I \rangle_\rho = \alpha$  for all physical constants  $\alpha \in \mathbb{C}$  is if  $\text{tr } \rho = 1$ . For  $\langle O \rangle_\rho$  to be real,  $\rho$  must be *self-adjoint*, and if  $O$  is also positive, then  $\rho$  must be *positive* for  $\langle O \rangle_\rho$  to be positive [7].

**Postulate 2.** The density operator for an isolated system with Hamiltonian  $H$  evolves unitarily in time according to the *Liouville-von Neumann equation*

$$\dot{\rho} = [H, \rho]/i\hbar. \quad (1.1)$$

While we usually consider the density operator to change in time, the time dependence may be shifted onto the observables. Consider a quantum system with unitary time-evolution operator  $U(t)$ , so that we may express the density operator for the system as  $\rho(t) = U(t)\rho U^\dagger(t)$ , where  $\rho = \rho(0)$ . If we compute the ensemble average of an observable  $O(t)$  and cycle the trace, we find

$$\begin{aligned} \langle O(t) \rangle_{\rho(t)} &= \text{tr}(O(t)U(t)\rho U^\dagger(t)) \\ &= \text{tr}(U^\dagger(t)O(t)U(t)\rho) \equiv \langle O_H(t) \rangle_\rho \end{aligned} \quad (1.2)$$

where  $O_H$  is the observable in the *Heisenberg picture*, as opposed to the usual *Schrödinger picture*. If we can split the Hamiltonian into the form  $H = H_0 + H_I(t)$ , then  $U(t)$  splits into the product of  $U_0(t) = e^{H_0 t/i\hbar}$  and  $U_I(t) = U_0^\dagger(t)U(t)$ . Cycling over only  $U_I(t)$  in (1.2) gives the *interaction picture* operators

$$O_I(t) = U_0^\dagger(t)O(t)U_0(t) \quad (1.3a)$$

$$\rho_I(t) = U_I(t)\rho U_I^\dagger(t). \quad (1.3b)$$

Without  $H_I(t)$ , (1.3) reduces to the Schrödinger picture, and without  $H_0$ , (1.3) reduces to the Heisenberg picture. The time-dependence of the interaction picture density operator from differentiating (1.3b) is (suppressing time dependences)

$$\begin{aligned} i\hbar \dot{\rho}_I(t) &= i\hbar \frac{d}{dt} (U_0^\dagger \rho(t) U_0) \\ &= -U_0^\dagger H_0^\dagger \rho(t) U_0 + U_0^\dagger \rho(t) H_0 U_0 \\ &\quad + U_0^\dagger [H_0, \rho(t)] U_0 + U_0^\dagger [H_I, \rho(t)] U_0 \\ &= U_0^\dagger [U_0 H_I' U_0^\dagger, \rho(t)] U_0 \\ &= [H_I', \rho_I(t)], \end{aligned} \quad (1.4)$$

where  $H_I'$  denotes the interaction Hamiltonian  $H_I(t)$  in the interaction picture. This is just (1.1) with the interaction Hamiltonian.

With this understanding of the behavior of isolated systems, it may be surprising that postulates 1 and 2 are actually insufficient to describe common systems. For example, the allowed energies for the harmonic oscillator are unbounded, so the Hamiltonian is not an element of the Liouville space. We will see later how this issue is related to the dynamics of a composite quantum system in Sec. 1.7, but will now move on to considering the more general dynamics of interacting quantum systems.

## 1.2 Composite quantum systems

Consider two quantum systems in Hilbert spaces  $\mathcal{H}_A$  and  $\mathcal{H}_B$ . The Hilbert space of the composite system is the *tensor product*  $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$  (Def. 2). How might a density

operator  $\rho$  for the composite system admit a *reduced density operator*  $\rho_A$  for system  $A$ ? Consider an observable  $O$  of system  $A$ , for which the corresponding composite observable is  $O \otimes I$ . Regardless of the representation (composite or subsystem), the ensemble average of  $O$  should be the same:

$$\langle O_A \otimes I \rangle_\rho = \langle O_A \rangle_{\rho_A}. \quad (1.5)$$

Suppose that there is a map  $\mathcal{P}$  that sends  $\rho$  to  $\rho_A$ . To see what this is, take an orthonormal complete basis  $\{A_j\}$  of Hermitian operators for  $\mathcal{L}(\mathcal{H}_A)$  and  $\{B_k\}$  for  $\mathcal{L}(\mathcal{H}_B)$ , so the density operator may be expressed as

$$\rho = \sum_{jk} A_j \langle A_j | \rho \rangle \otimes B_k \langle B_k | \rho \rangle.$$

We may then compute that (summing over repeated indices)

$$\begin{aligned} \mathcal{P}\rho &= A_i \text{tr}(A_i \mathcal{P}\rho) \\ &= A_i \text{tr}[(A_i \otimes I)(A_j \langle A_j | \rho \rangle \otimes B_k \langle B_k | \rho \rangle)] \quad (\text{by (1.5)}) \\ &= A_i \text{tr}(A_i A_j \langle A_j | \rho \rangle) \text{tr}(B_k \langle B_k | \rho \rangle) \\ &= A_i \langle A_i | A_j \langle A_j | \rho \rangle \rangle \text{tr}(B_k \langle B_k | \rho \rangle) \\ &= A_j \langle A_j | \rho \rangle \text{tr}(B_k \langle B_k | \rho \rangle) \\ &\equiv \text{tr}_B (A_j \langle A_j | \rho \rangle \otimes B_k \langle B_k | \rho \rangle) = \text{tr}_B \rho = \rho_A, \end{aligned}$$

which is the *partial trace* defined by  $\text{tr}_B(A_j \otimes B_k) \equiv A_j \text{tr} B_k$  and extending linearly.

However, the reduction of the density operator by “tracing over”  $B$  comes at the cost of losing information about the correlation between  $A$  and  $B$ . Quantitatively, the *relative entropy* between the correlated and uncorrelated density operators is

$$\begin{aligned} S(\rho \| \rho_A \otimes \rho_B) &\equiv \text{tr} \rho (\ln \rho - \ln(\rho_A \otimes \rho_B)) \\ &= \text{tr}(\rho \ln \rho) - \text{tr}_A \text{tr}_B (\rho \ln(\rho_A \otimes I)) - \text{tr}_B \text{tr}_A (\rho \ln(I \otimes \rho_B)) \\ &= S(\rho_A) + S(\rho_B) - S(\rho). \end{aligned}$$

Together with the Klein inequality which states that relative entropies are non-negative (Thm. 1.9.1), we have that

$$S(\rho) \leq S(\rho_A) + S(\rho_B),$$

with equality when  $\rho = \rho_A \otimes \rho_B$ . Other expected properties hold, such as that the relative entropy is invariant under unitary transformations (Thm. 1.9.2), or that the relative entropy between subsystems is less than that between combined systems (Thm. 1.9.3).

We would like to know the *reduced dynamics* of the quantum system  $S$  when in contact with an *environment* system  $B$ . We suppose that the composite system has a Hamiltonian of the form

$$H_{SB}(t) = H_S \otimes I + I \otimes H_B + H_I(t) \quad (1.6)$$

and that the environment is in equilibrium, so the composite density operator is  $\rho(t) = \rho_S(t) \otimes \rho_B$ . In terms of the unitary time-evolution operator  $U(t)$  for the system, (1.1)

becomes  $\rho(t) = U(t)\rho(0)U^\dagger(t)$ . Taking the partial trace over the environment gives the time-evolved system density operator

$$\rho_S(t) = \text{tr}_B \left( U(t)(\rho_S(0) \otimes \rho_B)U^\dagger(t) \right). \quad (1.7)$$

Whatever (1.7) evaluates to, it will be an example of a *dynamical map*  $\mathcal{V}(t)$  that time-evolves the system according to  $\rho_S(t) = \mathcal{V}(t)\rho_S(0)$ . While  $\mathcal{V}(t)$  seems abstract, we know it should output a density operator. Then for  $\mathcal{V}(t)$  to be a valid map on system density operators, it should *preserve the trace* of the input density operator. In fact, as a valid map  $\mathcal{V}(t) \otimes \mathbb{1}$  on the composite system, the composite density operator should remain positive. This property of  $\mathcal{V}(t)$  is called *complete positivity*. Thus the valid maps on system density operators are *completely positive and trace-preserving* (CPTP).

### 1.3 The Lindblad equation

With the idea of random interactions with an environment in mind, we will assume that the maps  $\{\mathcal{V}(t) : t \geq 0\}$  are also *memoryless* or *Markovian*, so that they form a *quantum dynamical semigroup* satisfying

$$\mathcal{V}(t_1)\mathcal{V}(t_2) = \mathcal{V}(t_1 + t_2) \quad \text{for } t_1, t_2 \geq 0.$$

The action of the dynamical semigroup on the system describes an irreversible process. As such, the relative entropy between an arbitrary system ensemble  $\rho(t)$  and an equilibrium ensemble  $\rho_0$  cannot decrease (by (1.7) and theorems 1.9.2 and 1.9.3):

$$\begin{aligned} S(\mathcal{V}(t)\rho \| \mathcal{V}(t)\rho_0) &= S(\text{tr}_B [U(t)(\rho \otimes \rho_B)U^\dagger(t)] \| \rho_0) \\ &\leq S(U(t)(\rho \otimes \rho_B)U^\dagger(t) \| \rho_0 \otimes \rho_B) \\ &= S(\rho \otimes \rho_B \| \rho_0 \otimes \rho_B) \\ &= S(\rho \| \rho_0). \end{aligned}$$

We would like to determine the *infinitesimal generator*  $\mathcal{L}$  for the quantum dynamical semigroup which allows the dynamical maps to be expressed as  $\mathcal{V}(t) = e^{\mathcal{L}t}$ , analogously to how a time-independent Hamiltonian is a generator for the unitary time-evolution operator  $e^{Ht/\hbar}$ . Following this analogy, the Schrödinger equation is replaced by the *Markovian quantum master equation*  $\dot{\rho}_S = \mathcal{L}\rho_S$ , which generalizes (1.1) to typically non-unitary CPTP maps of density operators, provided that they are Markovian.

We will find that the most general form of  $\mathcal{L}$  is given by the *Lindblad equation* (1.15). To obtain this result, first consider diagonalizing  $\rho_B$  as  $\rho_B = \sum_j \lambda_j |\phi_j\rangle\langle\phi_j|$  with orthonormal vectors  $\phi_j \in \mathcal{H}_B$ , where  $\sum_j \lambda_j = 1$ . Then (1.7) becomes (writing  $\rho_S$  as  $\rho$ )

$$\rho(t) = \sum_{ij} \langle\phi_i|U(t)(\rho(0) \otimes \lambda_j |\phi_j\rangle\langle\phi_j|)U^\dagger(t)|\phi_i\rangle \quad (1.8)$$

$$= \sum_{ij} \lambda_j \langle\phi_i|U(t)|\phi_j\rangle \rho(0) \langle\phi_j|U^\dagger(t)|\phi_i\rangle \quad (1.9)$$

$$= \sum_{ij} M_{ij}(t) \rho(0) M_{ij}^\dagger(t), \quad (1.10)$$



where  $M_{ij}(t) \equiv \sqrt{\lambda_j} \langle \phi_i | U(t) | \phi_j \rangle$ . This decomposition in terms of the  $M_{ij}$  is an instance of the Choi-Kraus representation theorem (Thm. 1.9.4). We can express the  $M_{ij}$  in terms of an orthonormal complete basis  $\{F_n\}$  for  $\mathcal{L}(\mathcal{H}_S)$  as  $M_{ij} = \sum_k F_k \langle F_k | M_{ij} \rangle$ . Then (1.10) becomes

$$\rho(t) = \sum_{mn} c_{mn}(t) F_m \rho F_n^\dagger,$$

where

$$c_{mn}(t) \equiv \sum_{ij} \langle F_m | M_{ij}(t) \rangle \langle M_{ij}(t) | F_n \rangle. \quad (1.11)$$

For convenience, we may choose  $F_{d^2} = 1/\sqrt{d}$ , where  $d = \dim(\mathcal{H}_S)$ . With an eye towards simplifying (1.13), we eliminate the explicit time dependence of (1.11) by defining

$$a_{mn} \equiv \lim_{t \rightarrow 0^+} \frac{c_{mn}(t) - d\delta_{d^2} d^2}{t} \quad (1.12)$$

and introduce the sum of Kraus operators

$$\begin{aligned} F &= \frac{1}{\sqrt{d}} \sum_{n=1}^{d^2-1} a_{nd^2} F_n \\ &= \frac{F + F^\dagger}{2} + i \frac{F - F^\dagger}{2i} \equiv G + H/i\hbar, \end{aligned}$$

where we have decomposed the sum  $F$  into Hermitian and anti-Hermitian parts and included  $\hbar$  so that  $H$  will have dimensions of energy. Now we may write the master equation  $\mathcal{L}\rho = \dot{\rho}$  as

$$\begin{aligned} \dot{\rho} &= \lim_{\Delta t \rightarrow 0^+} \frac{\mathcal{V}(\Delta t)\rho - \rho}{\Delta t} \\ &= \lim_{\Delta t \rightarrow 0^+} \left( \frac{c_{d^2 d^2}(\Delta t) - d}{d\Delta t} \rho + \sum_{m,n=1}^{d^2-1} \frac{c_{mn}(\Delta t)}{\Delta t} F_m \rho F_n^\dagger \right. \\ &\quad \left. + \frac{1}{\sqrt{d}} \sum_{n=1}^{d^2-1} \left( \frac{c_{nd^2}(\Delta t)}{\Delta t} F_n \rho + \frac{c_{d^2 n}(\Delta t)}{\Delta t} \rho F_n^\dagger \right) \right) \quad (1.13) \end{aligned}$$

$$\begin{aligned} &= \frac{a_{d^2 d^2}}{d} \rho + F \rho + \rho F^\dagger + \sum_{m,n=1}^{d^2-1} a_{mn} F_m \rho F_n^\dagger \\ &= \frac{a_{d^2 d^2}}{d} \rho + \{G, \rho\} + \frac{[H, \rho]}{i\hbar} + \sum_{m,n=1}^{d^2-1} a_{mn} F_m \rho F_n^\dagger \\ &= \{G', \rho\} + \frac{[H, \rho]}{i\hbar} + \sum_{m,n=1}^{d^2-1} a_{mn} F_m \rho F_n^\dagger, \quad (1.14) \end{aligned}$$

where  $G' = G + a_{d^2 d^2} I / d$ . Since  $\mathcal{V}(t)$  is trace-preserving,  $\text{tr } \dot{\rho} = 0$ . Applying this condition to (1.14) and cycling the trace gives

$$0 = \text{tr} \left( 2G' \rho + \sum_{m,n=1}^{d^2-1} a_{mn} F_n^\dagger F_m \rho \right),$$

so  $G' = -\sum_{m,n=1}^{d^2-1} a_{mn} F_n^\dagger F_m / 2$ . This allows us to write (1.14) as

$$\dot{\rho} = \frac{[H, \rho]}{i\hbar} + \sum_{m,n=1}^{d^2-1} a_{mn} \left( F_m \rho F_n^\dagger - \frac{1}{2} \{ F_n^\dagger F_m, \rho \} \right),$$

which is the first form of the *Lindblad equation*. This may be simplified further if we diagonalize the coefficient matrix  $a$  by applying a unitary transformation  $u$  to give  $a = u \gamma u^\dagger$ , where the  $\{\gamma_k\}_{k=1}^{d^2-1}$  are the non-negative eigenvalues of  $a$ . This is possible since the coefficient matrix  $c$  is seen from (1.11) to be Hermitian, and (1.12) then gives that  $a$  is Hermitian. We may then express  $F_{n \neq d^2} = \sum_{k=1}^{d^2-1} L_n u_{nk}$  in terms of the *Lindblad operators*  $L_n$  to find

$$\dot{\rho} = \frac{[H, \rho]}{i\hbar} + \sum_{k=1}^{d^2-1} \gamma_k \left( L_k \rho L_k^\dagger - \frac{1}{2} \{ L_k^\dagger L_k, \rho \} \right) \equiv \mathcal{L} \rho, \quad (1.15)$$

which is the *diagonal form* of the Lindblad equation. The eigenvalues  $\gamma_k$  have dimensions of inverse time and the Lindblad operators may be taken to be traceless. The second term is often called the *dissipator*  $\mathcal{D}$  (see Sec. 1.4), so the Lindbladian may be separated into unitary and non-unitary parts.

## 1.4 The weak-coupling limit

Now that we have found the general form for a stochastic CTCP generator, we must now determine the conditions for interaction Hamiltonian in (1.6) to give rise to Markovian dynamics. While there are several different regimes where this is true, we will consider the *weak-coupling* limit which we justify by supposing that the environment is similar to a *harmonic bath* of many harmonic oscillators.

We start by expressing the interaction Hamiltonian in terms of Hermitian operators as

$$H_I = \sum_{\alpha} A_{\alpha} \otimes B_{\alpha}.$$

We suppose that the system in isolation would have *discrete* energy levels, so the eigenoperators of the superoperator  $\mathcal{S} = [H_S, -]$  form a complete basis for  $\mathcal{L}(\mathcal{H}_S)$ . We then may write  $A_{\alpha} = \sum_{\omega} A_{\alpha\omega}$ , where

$$[H_S, A_{\alpha\omega}] = -\omega A_{\alpha\omega}. \quad (1.16)$$

Using (1.16) to commute past the exponential in (1.3a) gives  $A'_{\alpha\omega} = e^{-i\omega t} A_{\alpha\omega}$  in the interaction picture. Thus the interaction Hamiltonian in the interaction picture is

$$H'_I = \sum_{\alpha\omega} e^{-i\omega t} A_{\alpha\omega} \otimes B'_{\alpha}, \quad (1.17)$$

where  $B'_\alpha(t) = e^{-H_B t/\hbar} B_\alpha e^{H_B t/\hbar}$  per (1.3a).

Since we are interested in how fluctuations in different environment modes are related, we will consider the *reservoir correlation functions*

$$\langle B_\alpha^\dagger(t) B_\beta(t-s) \rangle_{\rho_B} \quad (1.18)$$

and their one-sided Fourier transform

$$\Gamma_{\alpha\beta}(\omega) \equiv \int_0^\infty ds e^{i\omega s} \langle B_\alpha^\dagger(t) B_\beta(t-s) \rangle_{\rho_B} \quad (1.19)$$

$$\equiv iS_{\alpha\beta}(\omega) + \gamma_{\alpha\beta}(\omega)/2, \quad (1.20)$$

where the corresponding matrix  $S = (\Gamma - \Gamma^\dagger)/2i$  is Hermitian and the matrix corresponding to the full Fourier transform

$$\gamma_{\alpha\beta}(\omega) \equiv \int_{-\infty}^\infty ds e^{i\omega s} \langle B_\alpha^\dagger(t) B_\beta(t-s) \rangle_{\rho_B}$$

is positive.

With this setup, we may now move to the main derivation. It is helpful to consider the interaction picture time evolution (1.4) in the integral form

$$\rho(t) = \rho(0) - i \int_0^t ds [H_I(s), \rho(s)].$$

Applying (1.4) again and tracing out the environment gives the closed equation

$$\dot{\rho}_S(t) = - \int_0^t ds \text{tr}_B [H_I(t), [H_I(s), \rho_S(s) \otimes \rho_B]]$$

for the system density operator. In doing so we have made two assumptions: that

$$\text{tr}_B [H_I(t), \rho(0)] = 0,$$

which is the *weak-coupling approximation*, and that

$$\rho(t) = \rho_S(t) \otimes \rho_B,$$

which is the *Born approximation*. It should be noted that weak-coupling follows if the reservoir averages of the interactions vanish:  $\langle B_\alpha(t) \rangle_{\rho_B} = 0$ .

We now make the *Markov approximation* that  $\rho_S(s) = \rho_S(t)$ , so that the time-evolution only depends on the present time, to obtain the *Redfield equation*. To simplify further, we make the substitution  $s \mapsto t - s$  and set the upper limit of the integral to infinity:

$$\dot{\rho}_S = - \int_0^\infty ds \text{tr}_B [H_I(t), [H_I(t-s), \rho_S(t) \otimes \rho_B]]. \quad (1.21)$$

This is justified when the reservoir correlation functions in (1.19) vanish quickly over a time  $\tau_B$  that is smaller than the relaxation time  $\tau_R$  (see Sec. 1.7). Substituting (1.17) into (1.21) and using (1.19) gives

$$\dot{\rho}_S = 2 \text{He} \sum_{\alpha\beta\omega\omega'} e^{i(\omega'-\omega)t} \Gamma_{\alpha\beta}(\omega) \left( A_{\beta\omega} \rho_S A_{\alpha\omega'}^\dagger - A_{\alpha\omega'}^\dagger A_{\beta\omega} \rho_S \right), \quad (1.22)$$

where  $\text{He} \Gamma \equiv (\Gamma + \Gamma^\dagger)/2$ . If the typical times

$$\tau_S = |\omega' - \omega|^{-1} \quad \text{for} \quad \omega' \neq \omega$$

for system evolution are large compared to the relaxation time  $\tau_R$ , then the contribution from the fast-oscillating terms of (1.22) where  $\omega' \neq \omega$  may be neglected. This *rotating wave* or *secular* approximation is analogous to how we consider the high-energy position distribution in the infinite square well to be uniform, even though it is actually a fast-oscillating function. By coarse-graining in this sense, we obtain

$$\dot{\rho}_S = 2 \text{He} \sum_{\alpha\beta\omega} \Gamma_{\alpha\beta}(\omega) \left( A_{\beta\omega} \rho_S A_{\alpha\omega}^\dagger - A_{\alpha\omega}^\dagger A_{\beta\omega} \rho_S \right).$$

Now applying the decomposition (1.20) gives the interaction picture Lindblad equation

$$\dot{\rho}_S = i[H_{LS}, \rho_S] + \mathcal{D}\rho_S, \quad (1.23)$$

where the *Lamb shift Hamiltonian* is

$$H_{LS} = \sum_{\alpha\beta\omega} S_{\alpha\beta}(\omega) A_{\alpha\omega}^\dagger A_{\beta\omega},$$

and the *dissipator* is

$$\mathcal{D}\rho_S = \sum_{\alpha\beta\omega} \gamma_{\alpha\beta} \left( A_{\beta\omega} \rho_S A_{\alpha\omega}^\dagger - \frac{1}{2} [A_{\alpha\omega}^\dagger A_{\beta\omega}, \rho_S] \right).$$

The Lamb shift (or environment renormalization) Hamiltonian commutes with the system Hamiltonian since (1.16) implies that  $[H_S, A_{\alpha\omega}^\dagger A_{\beta\omega}] = 0$ . Adding the system's Hamiltonian  $H_S$  to  $H_{LS}$  and diagonalizing gives the Schrödinger picture Lindblad equation (1.15).

## 1.5 Relaxation to thermal equilibrium

The system will generally relax from its initial configuration to a stationary solution of (1.15) (see Sec. 1.7). We expect that the thermal state

$$\rho_S = \frac{e^{-\beta H_S}}{Z} \quad \text{where} \quad Z = \text{tr}(e^{-\beta H_S})$$

would be the equilibrium state. This is true when the reservoir correlation functions obey the KMS condition [8, 9]

$$\langle B_\alpha^\dagger(t) B_\beta(0) \rangle_{\rho_B} = \langle B_\beta(0) B_\alpha^\dagger(t + i\beta) \rangle_{\rho_B},$$

which is true when the environment is in the thermal state  $\rho_B = e^{-\beta H_B} / \text{tr}(e^{-\beta H_B})$ .

## 1.6 A two-level atom

To demonstrate the use of the Lindblad equation, we will study a model for the decay of a two-level atom. Our aim is to glimpse why electrons in atoms undergo optical decay, even though excited states are stable atomic states. Suppose that the atom has Hamiltonian  $H_S = \hbar\omega\sigma_3/2$ , where  $\sigma_3 = |1\rangle\langle 1| - |0\rangle\langle 0|$ . The operators  $\sigma_- = |0\rangle\langle 1|$  and  $\sigma_+ = |1\rangle\langle 0|$  are Lindblad operators, since they are eigenoperators of the superoperator  $[\mathcal{H}_S, -]$ , like in (1.16). These correspond to lowering and raising the energy by  $\hbar\omega$ , and will be our analogues of the emission and absorption processes. The derivation of Sec. 1.3 is similar for a bath of photons in equilibrium, and our assumptions are justified because typical atomic relaxation times of about 20 ns are much slower than the periods of electromagnetic waves [10]. Ignoring the Lamb shift (which only offsets) and considering only the effects at  $\omega$ , (1.23) becomes

$$\begin{aligned}\dot{\rho} = & \gamma_0(N+1)\left(\sigma_-\rho\sigma_+ - \frac{1}{2}\{\sigma_+\sigma_-, \rho\}\right) \\ & + \gamma_0N\left(\sigma_+\rho\sigma_- - \frac{1}{2}\{\sigma_-\sigma_+, \rho\}\right) \equiv \mathcal{D}\rho,\end{aligned}$$

where  $N = 1/(e^{\beta\hbar\omega} - 1)$ . This is straightforward to solve given the properties of the Pauli matrices. From the initial density operator  $\rho(0) = |1\rangle\langle 1|$ , we find that the population of upper level is

$$\rho_{11}(t) = \frac{N}{2N+1}(1 - e^{-\gamma t}), \quad \text{where } \gamma = \gamma_0(2N+1).$$

This is consistent with what we observe in atomic spectra: an exponential decay to an equilibrium level which gives Lorentzian peaks. At low temperatures ( $N \rightarrow 0$ ) the system approaches the ground state in accordance with the third law of thermodynamics, and at high temperatures ( $N \gg 1$ ), the level is half-occupied and the absorption is saturated [11] (Cf. <sup>1</sup>).

## 1.7 Limitations

Though the Lindblad equation is widely applicable, there are some situations in which key assumptions in its derivation break down. For one, we have glossed over the issue of *ergodicity* in considering a harmonic bath. Since we needed to assume a discrete spectrum, the correlation functions (1.18) will be quasi-periodic and will not decay as we required. It is only in the limit of a reservoir with infinitely many degrees of freedom that we expect non-periodic behavior like decay to emerge, but then there may be issues with unbounded operators. This can lead to the interesting behavior of spontaneous symmetry breaking and phase transitions.

There are also many systems with dynamics that occur on time scales comparable to the relaxation time. For example, a paper published less than a month ago (!) demonstrates how fast pulsed laser experiments can probe the relaxation of temporarily polarized gas molecules due to collisions [12, 13].

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<sup>1</sup>It's cool how this provides a fundamental explanation of the *Doppler-free saturated absorption spectroscopy of Rubidium vapor* JLAB experiment that I did.

## 1.8 Conclusion

We have seen how the general consideration of Markovian CTCP maps on density operators leads to the Lindblad equation, and considered the weak-coupling limit as an example of a physical regime where the assumption of stochastic dynamics is valid. However, we have only scratched the surface of what can be done with the Lindblad equation, especially with respect to solving it. Since the two-level atom is a small system, it is simple to diagonalize, but larger systems provide more difficulty as the dimension of the Hilbert space grows. Numerical solutions are complicated by the additional requirement of trace preservation, but they can still be done in many situations [6, 12]. The theory of open quantum systems gives some fundamental justifications for the assumptions of equilibrium statistical mechanics, as was briefly noted in Sec. 1.5, and has made the picture of decoherence a bit more clear.

## 1.9 Mathematical details

**Definition 2** (Tensor product). Consider vector spaces  $V(k)$ ,  $W(k)$ , and  $Z$ . For any bilinear map  $h : V \times W \rightarrow Z$ , the *tensor product*  $V \otimes W$  and associated bilinear map  $\phi : V \times W \rightarrow V \otimes W$  map have the property that there is a unique bilinear map  $g : V \otimes W \rightarrow Z$  such that  $h = g \circ \phi$ . For tensor products of Hilbert spaces, the inner product is defined on each element of a product and then the space is completed. There is then a natural correspondence between the element  $v \otimes f$  of the tensor product  $V \otimes V^*$  and the linear map  $T : V \rightarrow V$  defined by  $Tx = f(x)v$ .

This induces an extension of Dirac notation where all pairs  $f \otimes x$  of dual and usual vectors from the same space are evaluated as  $\langle f|x \rangle = f(x)$  and extended linearly. For example, given a linear operator  $U : V \otimes W \rightarrow V \otimes W$  and a basis  $|\phi_i\rangle$  for  $W$ , the partial trace over  $W$  may be expressed as  $\text{tr}_W U = \langle \phi_i|U|\phi_i \rangle$ . This forms the justification of the step from (1.8) to (1.9) and of the manipulations in Thm. 1.9.4.

**Theorem 1.9.1** (Klein inequality). *For density operators  $\rho$  and  $\rho'$ ,  $S(\rho\|\rho') \geq 0$ , with equality if and only if  $\rho = \rho'$ .*

*Proof.* The case for equality is trivial, so we will consider  $\rho \neq \rho'$ . Let  $\mathcal{F}\rho = \rho \ln \rho$ , so that we may express the relative entropy as

$$S(\rho\|\rho') = \text{tr}(\mathcal{F}\rho - \mathcal{F}\rho' - \delta\mathcal{F}'\rho'),$$

where  $\delta = \rho - \rho'$ . We then have for  $0 < t < 1$  that

$$\rho' + t\delta = t\rho + (1-t)\rho'.$$

Now let  $f(t) = \text{tr}(\mathcal{F}(\rho' + t\delta))$ . Since the trace is monotonic and convex,  $f$  is convex and  $f(t) \leq f(0) + t(f(1) - f(0))$ . Rearranging and taking the limit as  $t \rightarrow 0^+$  gives

$$f'(0) \leq f(1) - f(0),$$

which evaluates to

$$\text{tr}(\delta\mathcal{F}'\rho') \leq \text{tr}\mathcal{F}\rho - \text{tr}\mathcal{F}\rho'.$$

**Theorem 1.9.2.** For a unitary operator  $U$  and density operators  $\rho$  and  $\rho'$ ,

$$S(U\rho U^\dagger \| U\rho' U^\dagger) = S(\rho \| \rho').$$

*Proof.* Since we may cycle the traces, it suffices to show that

$$\ln(U\rho U^\dagger) = \ln \rho.$$

This follows from Jacobi's formula for invertible matrices when applied to the logarithm that takes us from a Lie group to its corresponding Lie algebra, giving  $\text{tr} \circ \det = \text{tr} \circ \log$ .  $\square$

**Theorem 1.9.3.** For density operators  $\rho$  and  $\rho'$ ,

$$S(\text{tr}_B \rho \| \text{tr}_B \rho') \leq S(\rho \| \rho'),$$

with equality if and only if  $\rho$  or  $\rho'$  is uncorrelated.

**Theorem 1.9.4** (Choi-Kraus representation [6]). A superoperator  $\mathcal{S}$  on a density operator  $\rho$  is completely positive and trace-preserving if and only if it may be represented as

$$\rho = \sum_{k=1}^K M_k \rho M_k^\dagger, \quad \text{where} \quad \sum_{k=1}^K M_k M_k^\dagger = I.$$





# Conclusion

<sup>2</sup> HERE'S a conclusion, demonstrating the use of all that manual incrementing and table of  
<sup>3</sup> contents adding that has to happen if you use the starred form of the chapter command.  
<sup>4</sup> The deal is, the chapter command in  $\LaTeX$  does a lot of things: it increments the chapter  
<sup>5</sup> counter, it resets the section counter to zero, it puts the name of the chapter into the table  
<sup>6</sup> of contents and the running headers, and probably some other stuff.



<sup>1</sup> Appendix A

<sup>2</sup> The First Appendix

<sup>3</sup> Appendix!



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