Rensselaer 2020 REU Notebook

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Contents

1 Project description

May 27, 2020 The aim of this REU project is to quantify the information present in images by the principled application of methods from statistical physics. The approach is to find a suitable notion of entropy which captures the salient features of particular kinds of images. We will consider a variety of features motivated by intuition or domain knowledge, and then move to machine learning as a tool for discovering other features.

2 Getting started

May 27, 2020 The initial goal is to characterize the most naïve calculation, which I'll call the *intensity entropy*. This does *not* take into account the spatial correlation of pixels in an image.

3 Intensity-level entropy

Given a discrete random variable X on a probability space $(\Omega, \mathcal{F}, P)$ with image $\chi = \mathrm{im}(X)$, the *Shannon entropy* is

$$H = \sum_{x \in \chi} -P(x) \ln P(x).$$

The *intensity-level entropy* is the Shannon entropy of the empirical distribution of intensity values.

```
import numpy as np
```

```
def shannon_entropy(h):
    """The Shannon entropy in bits"""
    return -sum(p*np.log2(p) if p > 0 else 0 for p in h)

def intensity_entropy(data):
    """The intensity-level entropy of 8-bit image data"""
    hist, _ = np.histogram(data, bins=range(256+1), density=True)
    return shannon_entropy(hist)

def intensity_expected(f, data):
    """The intensity-distribution expected value of `f`."""
    hist, _ = np.histogram(data, bins=range(256+1), density=True)
    return sum(p*f(p) for p in hist)
```

4 Effect of smoothing on intensity-level entropy

```
import numpy as np
import numpy.linalg as linalg
import matplotlib.pyplot as plt
from PIL import Image, ImageFilter, ImageOps
from src.utilities import *
from src.intensity_entropy import *
```

4.1 Natural image

```
img = ImageOps.grayscale(Image.open('test.jpg'))
scale = max(np.shape(img))
data = np.array(img)
img
```

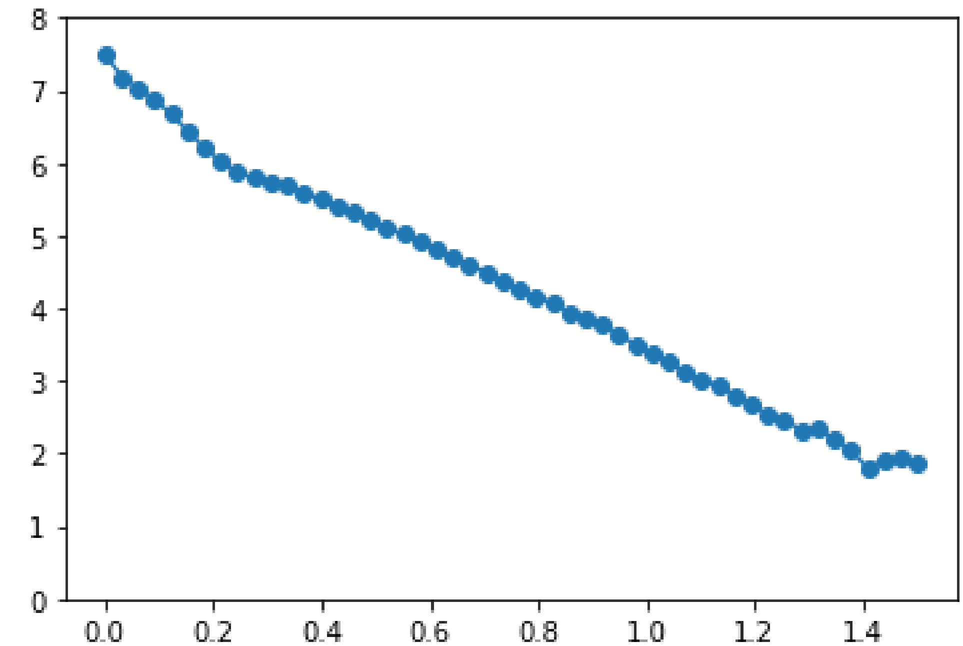
```
intensity_entropy(img)
```

```
7.51132356216608
```

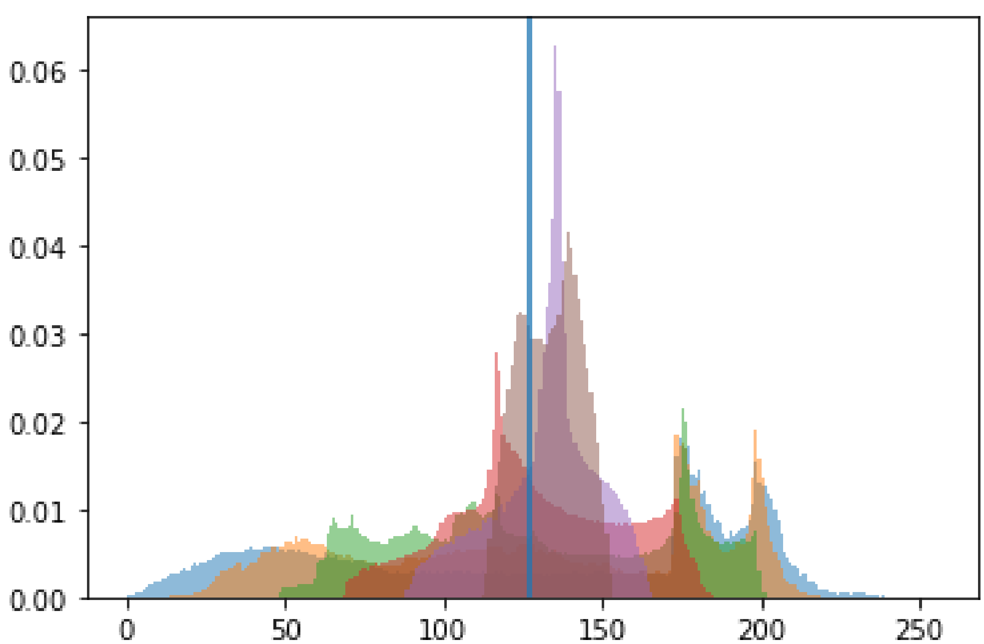
The problem with the intensity entropy is that it is usually near maximum (8 bits for these grayscale images).

```python
def intensity_blur(img, scales, display=True):
    scale = max(np.shape(img))

    results = []
    for k in scales:
        simg = img.filter(ImageFilter.GaussianBlur(k * scale))
        data = np.array(simg)
        ihist, ibins = np.histogram(data, bins=range(256+1), density=True)
        S = shannon_entropy(ihist)
        if display:
            hist = plt.hist(ibins[:-1], ibins, weights=ihist, alpha=0.5)
            results.append((k, simg, hist, S))
        else:
            results.append((k, S))

    if display:
        plt.axvline(x=np.mean(np.array(img)))

    return results
```

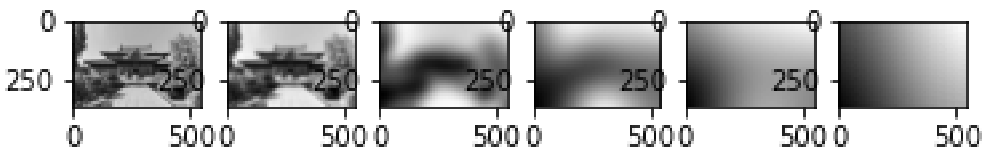
```python
results = intensity_blur(img, np.linspace(0, 1.5, num=50), False)

plt.plot(*np.transpose(results), 'o-')
plt.ylim((0, 8))
plt.xlabel = "Smoothing"
plt.ylabel = "Intensity Entropy (bits)"
```




```
rimgs = [img for _, img, _, _ in intensity_blur(img, [0, 0.01, 0.05, 0.125, 0.25, 0.5])]
plt.show()
```

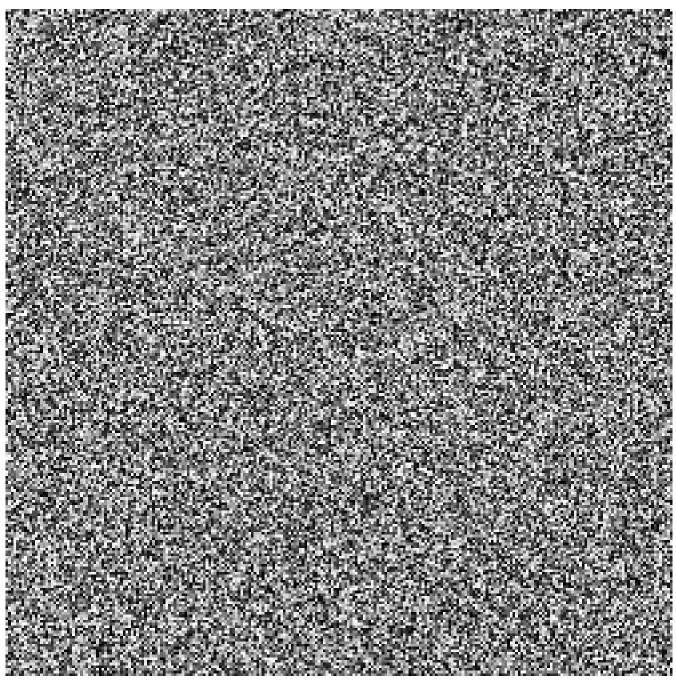


```python
_, axarr = plt.subplots(1, len(rimgs))
for i, subimg in enumerate(rimgs):
    axarr[i].imshow(subimg, cmap='gray')
plt.show()
```



4.2 Random pixel values

```python
rsize = 250
randimg = Image.fromarray((256*np.random.rand(*2*[rsize])).astype('uint8'))
randimg
```

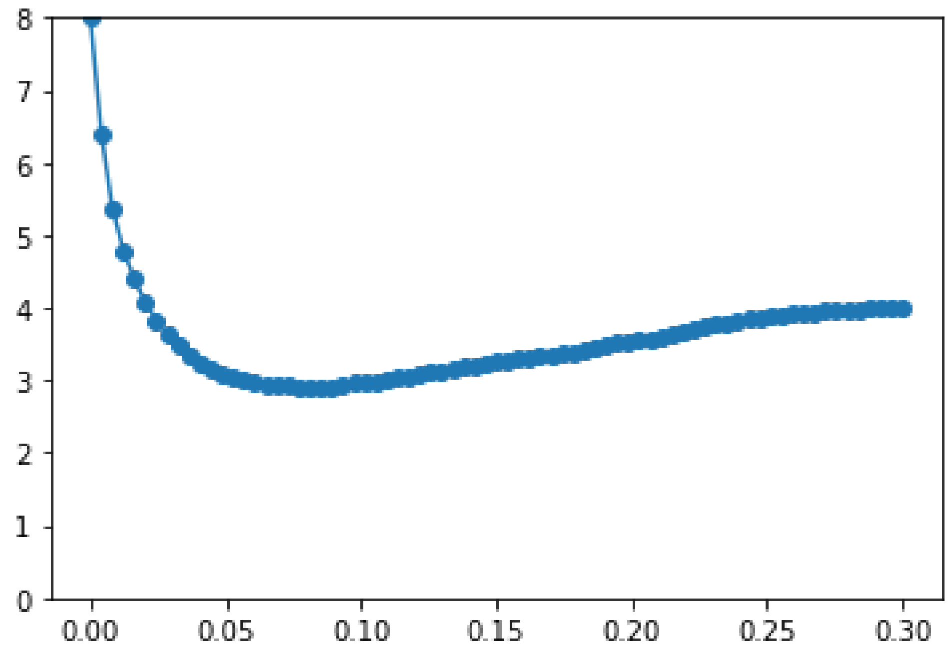


4.2.1 Beware: GIGO

The boundary effects and discrete kernel of `ImageFilter.GaussianBlur` renders the data unreliable after the "minimum" of the intensity entropy with smoothing.

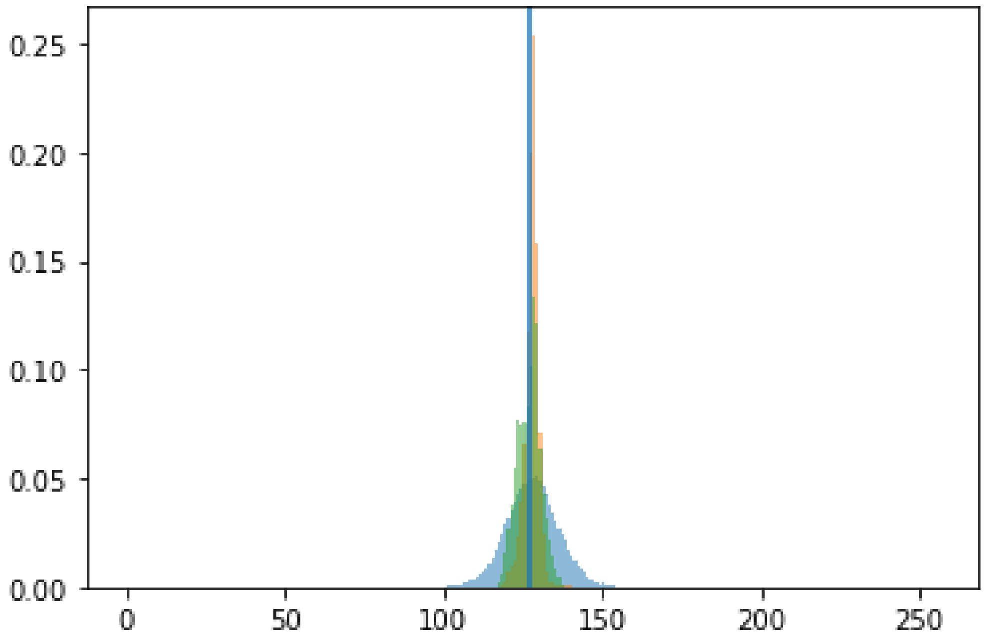
This is immediately clear after even small smoothing for random pixel values, since there are no spatial correlations.

```
results = intensity_blur(randimg, np.linspace(0, 0.3, num=75), False)

plt.plot(*np.transpose(results), 'o-')
plt.ylim((0, 8))
plt.xlabel = "Smoothing"
plt.ylabel = "Intensity Entropy (bits)"
```

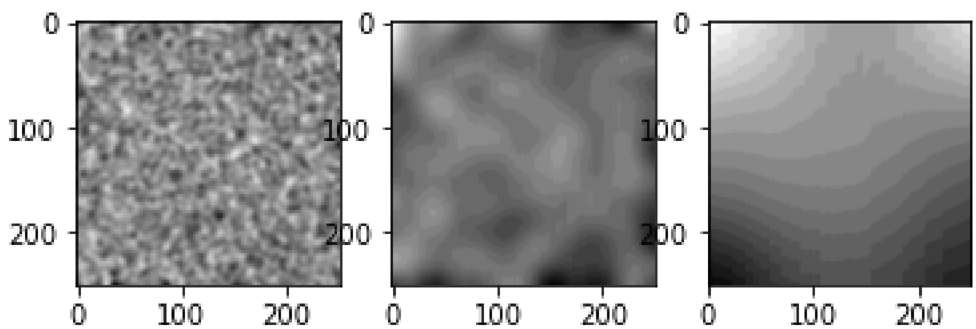



```
rimgs = [img for _, img, _, _ in intensity_blur(randimg, [0.01, 0.05, 0.25])]
```

```
plt.show()
```




```
_, axarr = plt.subplots(1, len(rimgs))
for i, subimg in enumerate(rimgs):
    axarr[i].imshow(subimg, cmap='gray')
plt.show()
```



The rightmost image should be uniform: the renormalization emphasizes incorrect deviations. These are what keep the intensity entropy from vanishing.

4.3 Comparing different levels of smoothing

Is composing n Gaussian blurs with variance σ^2 the same as doing one with variance $n\sigma^2$ (considering the boundary effects and discrete kernel)?

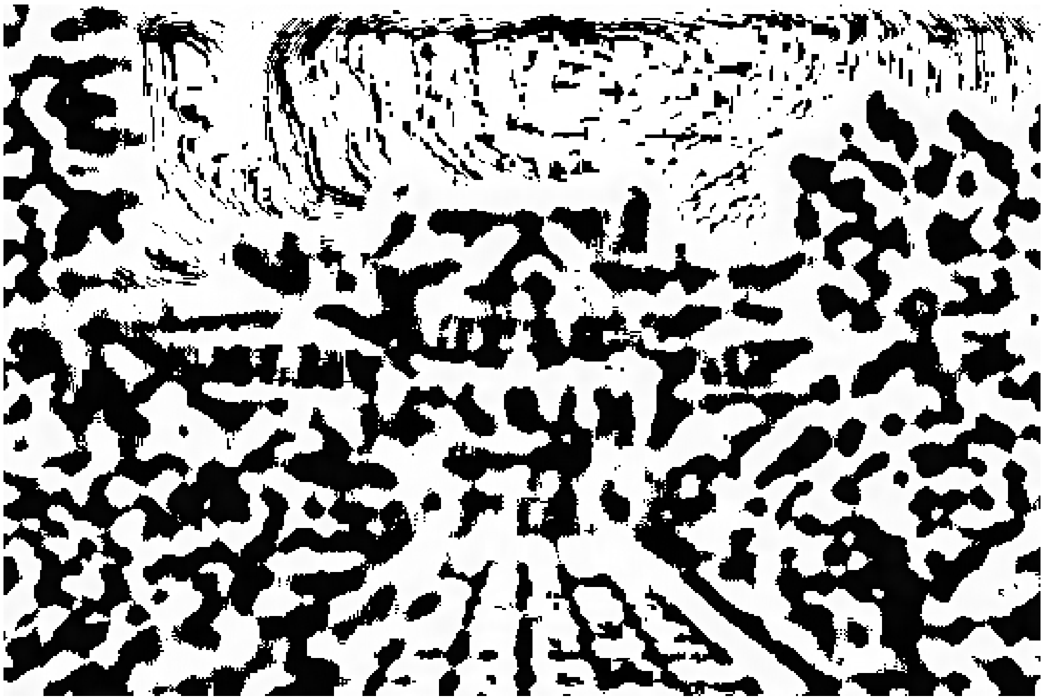
```
nsmooths = 10
cimg = img
oneimg = cimg.filter(ImageFilter.GaussianBlur(np.sqrt(nsmooths)*2))
oneimg
```

```
nimg = cimg
for _ in range(nsmooths):
    nimg = nimg.filter(ImageFilter.GaussianBlur(2))
nimg
```

Answer: **No**

The differences between results at different scales can be pretty wack.

```
Image.fromarray((255*rescale(np.array(nimg) - np.array(oneimg))).astype('uint8'))
```



```
smimg = img
smdiff = np.array(smimg.filter(ImageFilter.GaussianBlur(2))) - np.array(smimg.filter(ImageFilter.GaussianBlur(100)))
diffimg = Image.fromarray((255 * rescale(smdiff)).astype('uint8'))
diffimg
```

5 Local metrics

May 28, 2020 Given an image $I : X \times Y \to \mathbb{Z}_n$, we will now consider *local metrics* for the information it contains.

I want to be careful in understanding the statistical assumptions I am making, so I'll try to be explicit about distinguishing true distributions from empirical distributions, and how the assumptions behind postulating the existence of empirical distributions relate to the actual calculation being done. This should also aid in learning more solid probability theory.

5.1 Induced metrics

Definition 1 (Image distributions). An *image distribution* is a map D that takes an image I and produces a random variable $D(I) : \Omega \to E$.

We are constructing empirical distributions from image data according to some map $M : \mathsf{Img} \to \mathsf{List}(\Omega)$, which produces the list of values $V = M(I)$. Then the probability of $D(I)$ taking a value in a subset $S \subseteq E$ is

$$P(X \in S) = \frac{1}{|V|} \sum_{s \in S} \left| V^{-1}(\{s\}) \right|.$$

Example 1. The intensity-level entropy is a function of the *nonnegative* random variable from the image distribution of intensity values. That is, the map M takes an image and returns the list of its intensity values.

Definition 2 (Induced image distributions)**.** Given an image distribution D, and a subset $S \subseteq \operatorname{dom} I$, we construct the *induced image distribution* $D|_S$ by

$$D|_S(I) = D(I|_S).$$

Definition 3 (Induced random variable)**.** Given an image I, an image distribution D and collection of subsets $\{S_i\}$ of $\operatorname{dom} I$, a function H admits the random variables

$$H_i = (H \circ D|_{S_i})(I)$$

Definition 4. The *r-box* at (x, y) is $B_r(x, y) = [x - r, x + r] \times [y - r, y + r]$.

Given two real random variables A and B with joint PDF $f_{A,B}(a, b)$, the PDF of their sum is

$$f_{A+B}(c) = \int_{-\infty}^{\infty} \mathrm{d}a\, f_{A,B}(a, a - c) = \int_{-\infty}^{\infty} \mathrm{d}b\, f_{A,B}(b - c, b). \tag{1}$$

For independent A and B, EQ. 1 reduces to $f_{A+B} = f_A * f_B$ over the marginals.

6 Kernels

Generalized to arbitrary functions on subregions of images.

```
import numpy as np
```

```
def box(x, y, r):
    return np.s_[max(0, x-r) : x+r+1, max(0, y-r) : y+r+1]
def mapbox(r, f, a):
    return np.reshape([f(a[box(*i, r)]) for i in np.ndindex(np.shape(a))], np.shape(a))
def mapboxes(rs, f, a):
    return (mapbox(r, f, a) for r in rs)
```

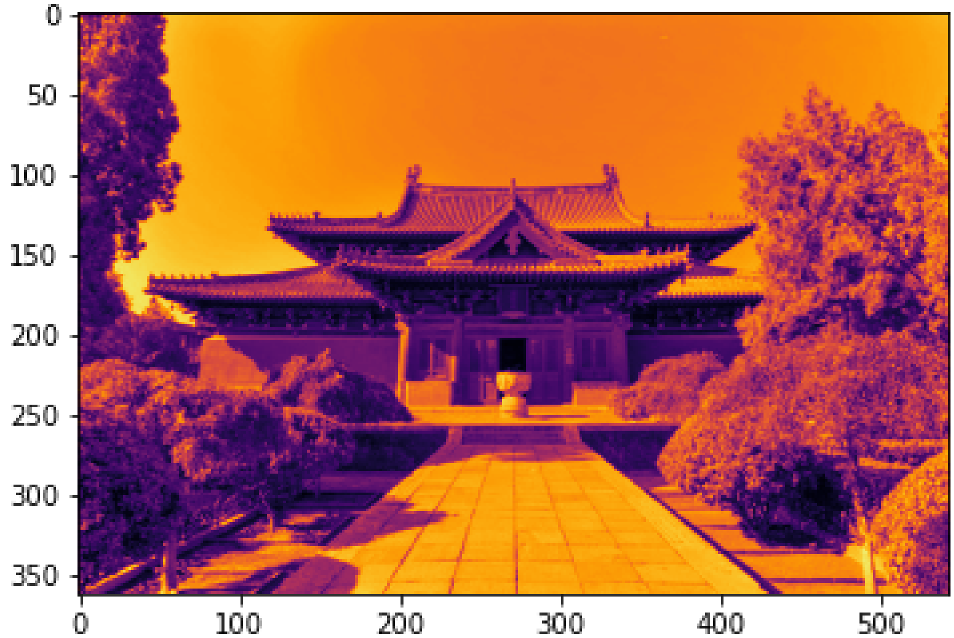
```python
def mapallboxes(f, a):
    return mapboxes(range(max(np.shape(a))), f, a)

def mapblocks(h, w, f, a):
    return np.array([[f(y) for y in np.array_split(x, w, axis=1)]
                     for x in np.array_split(a, h)])
```

7 Boxcar intensity-level entropy

```python
import numpy as np
import numpy.linalg as linalg
import matplotlib.pyplot as plt
from PIL import Image, ImageFilter, ImageOps
from src.utilities import *
from src.intensity_entropy import *
from src.kernels import *
plt.rcParams['image.cmap'] = 'inferno'
```

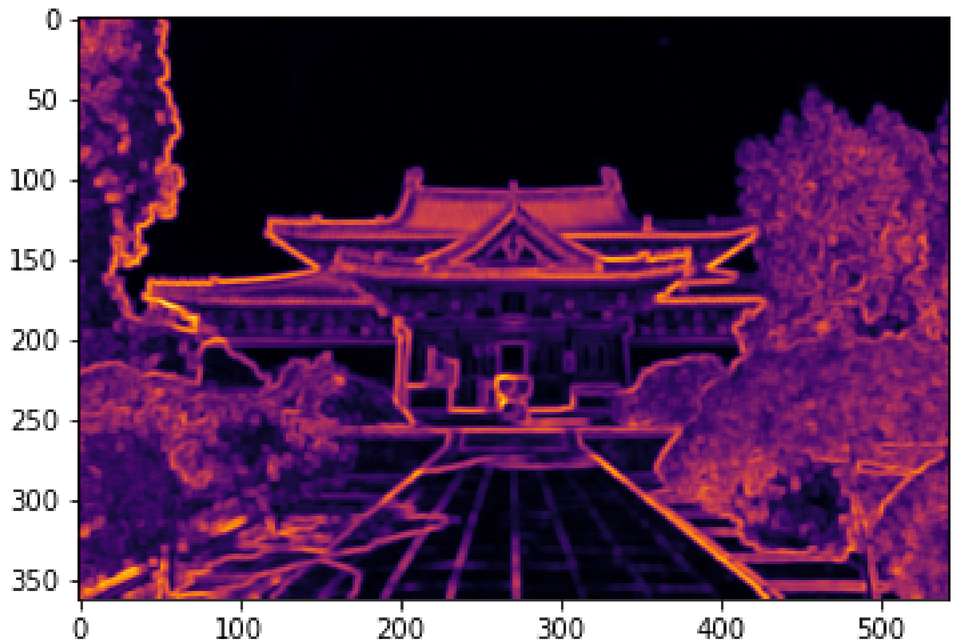
Let's compare the boxcar images for intensity entropy to those for a positive function on an image (the standard deviation) and for different functions of the induced intensity distribution.

```python
img = ImageOps.grayscale(Image.open('test.jpg'))
scale = max(np.shape(img))
data = np.array(img)
plt.imshow(img);
```



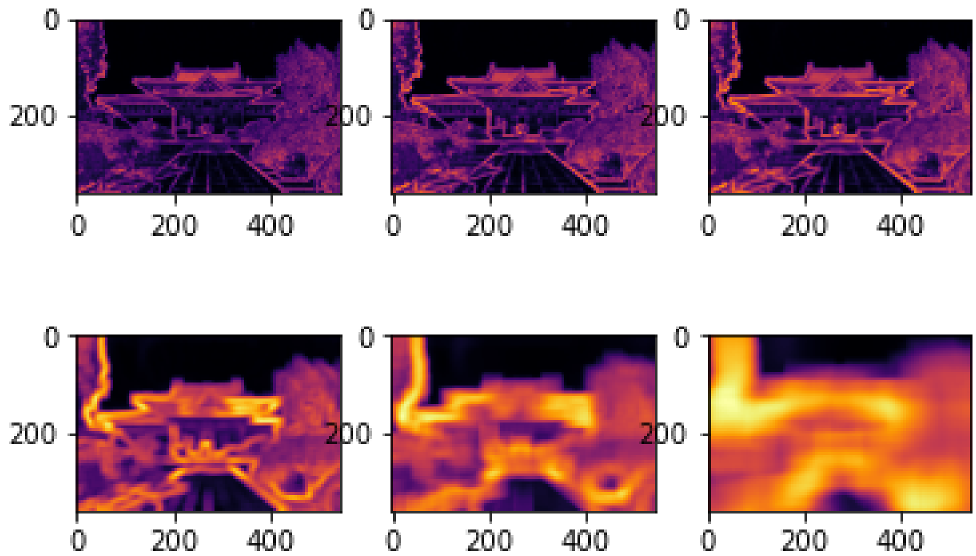
7.1 Standard deviation

```
plt.imshow(mapbox(2, np.std, np.array(img)));
```



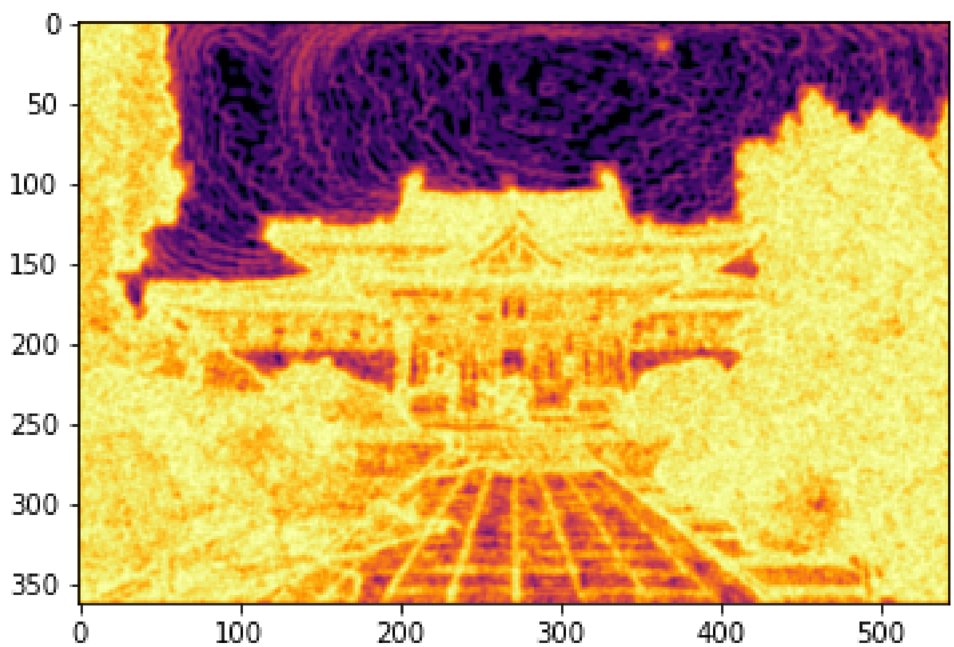

```
boxσs = list(mapboxes([1,2,3,10,20,50], np.std, np.array(img)))
```

```
_, axarr = plt.subplots(2, np.ceil(len(boxσs)/2).astype('int'))
for i, subimg in enumerate(boxσs[:3]):
    axarr[0,i].imshow(subimg)
for i, subimg in enumerate(boxσs[3:]):
    axarr[1,i].imshow(subimg)
plt.show()
```



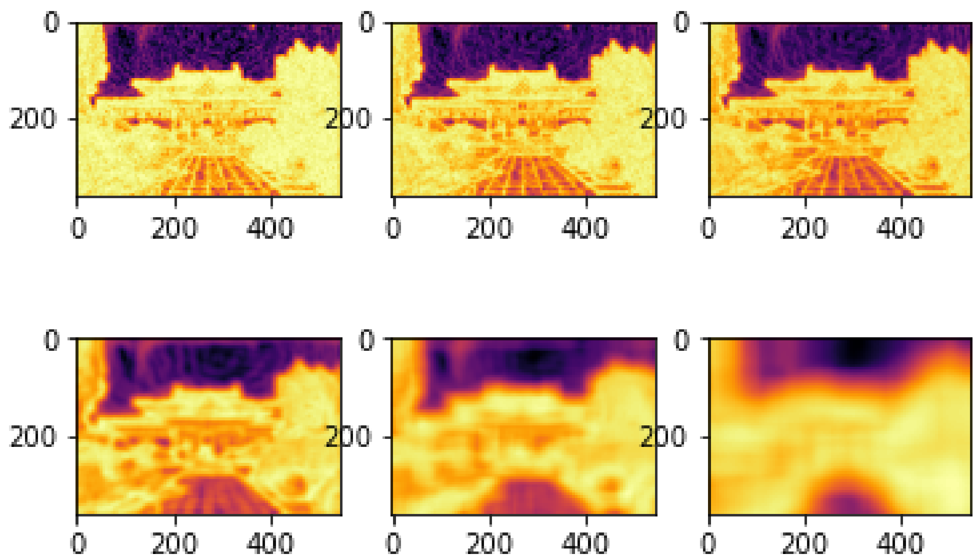
7.2 Intensity entropy

```
plt.imshow(mapbox(2, intensity_entropy, np.array(img)));
```



```python
boxSes = list(mapboxes([1,2,3,10,20,50], intensity_entropy, np.array(img)))
```

```python
_, axarr = plt.subplots(2, np.ceil(len(boxSes)/2).astype('int'))
for i, subimg in enumerate(boxSes[:3]):
    axarr[0,i].imshow(subimg)
for i, subimg in enumerate(boxSes[3:]):
    axarr[1,i].imshow(subimg)
plt.show()
```

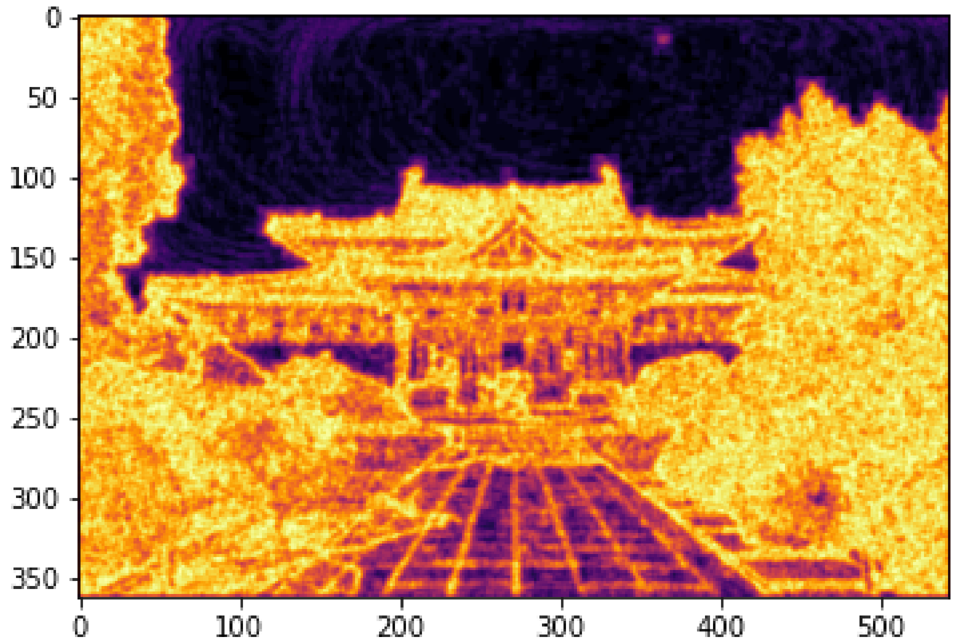


7.3 Replace surprisal with other functions

To what extent do the surprisal-related results depend upon the specific form of the *surprisal* $x \mapsto -\log(x)$ in the expected value of the intensity distribution? We will replace the expected surprisal with the expected f, for different functions f on the empirical probabilities of a pixel taking some intensity.

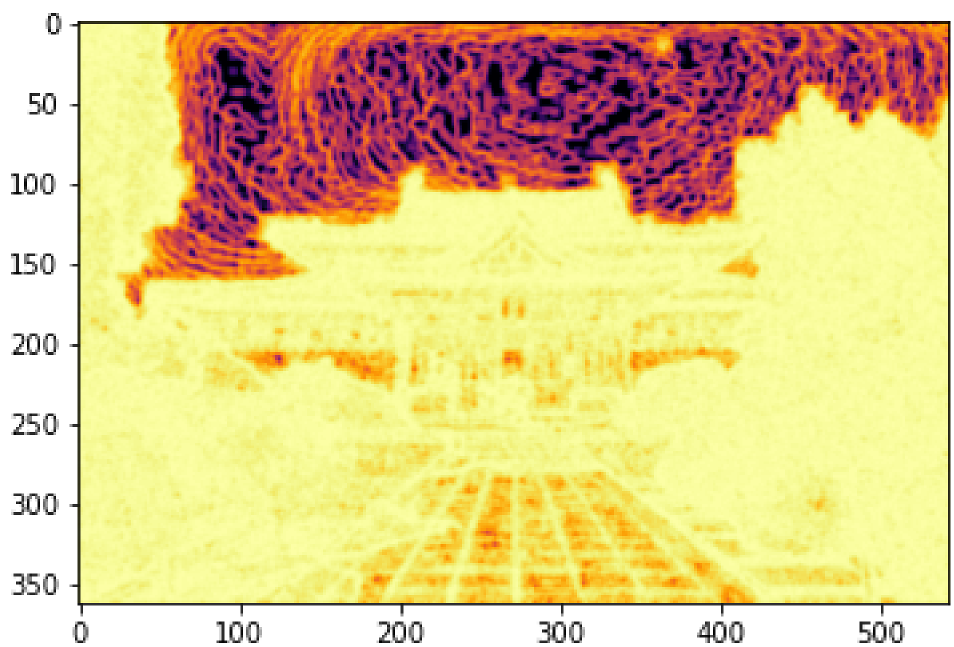
Laurent: $p \mapsto -1 + 1/p$.

```python
plt.imshow(mapbox(2, lambda I: intensity_expected(lambda p: -1 + 1/p if p > 0 else 0, I),
↪ np.array(img)));
```



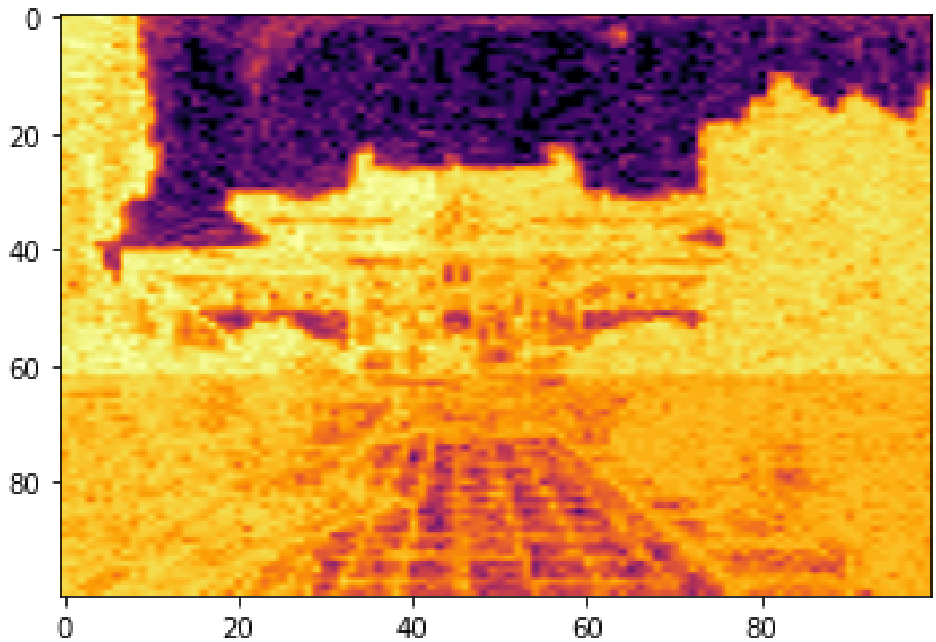
Taylor: $p \mapsto -(1+p)$.

```
plt.imshow(mapbox(2, lambda I: intensity_expected(lambda p: -(1+p), I), np.array(img)));
```

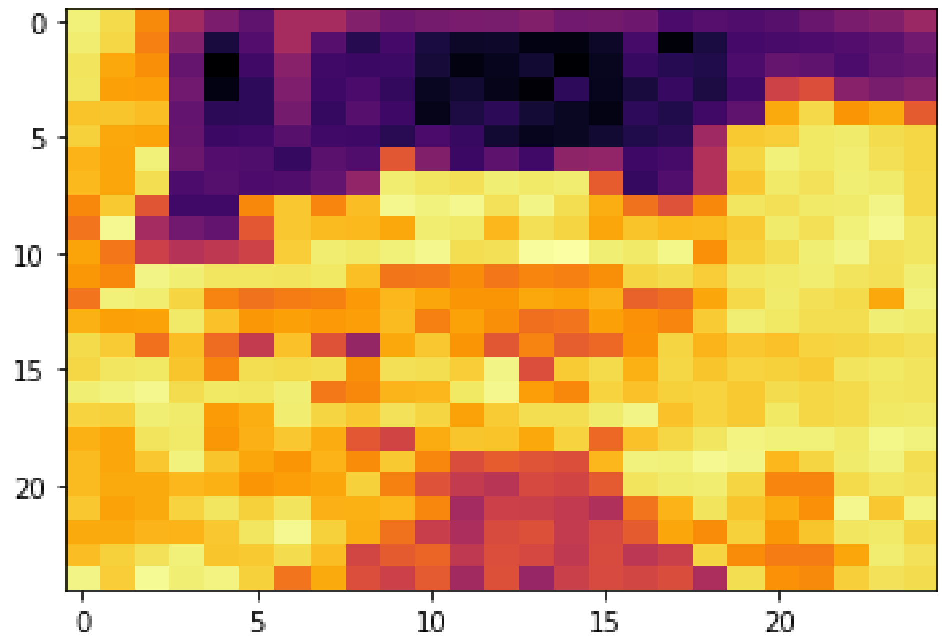


7.4 Intensity entropy on disjoint blocks

```
plt.imshow(mapblocks(100, 100, intensity_entropy, np.array(img)),
           aspect=np.divide(*np.shape(img)));
```




```
plt.imshow(mapblocks(25, 25, intensity_entropy, np.array(img)),
           aspect=np.divide(*np.shape(img)));
```



8 Fractal dimensions

May 29, 2020 The previous results hint at characterizing the growth of the intensity entropy with different discretizations.

Definition 5. The *Rényi entropy of order $\alpha \geq 0$* of a discrete random variable X with support χ is

$$H_\alpha(X) = \frac{1}{1-\alpha} \log \sum_{x \in \chi} P(x)^\alpha = \frac{\alpha}{1-\alpha} \log \|P\|_\alpha,$$

where $\|P\|_\alpha$ denotes the α-norm of the vector of probability values. The limit $\alpha \to 1$ reproduces the Shannon entropy.

Definition 6. Given a real random variable X, define a discretized random variable

$$\langle X \rangle_\varepsilon = \frac{\lfloor \varepsilon X \rfloor}{\varepsilon}.$$

Then the *generalized dimension* of X is

$$d_\alpha(X) = \lim_{\varepsilon \to 0} \frac{H_\alpha(\langle X \rangle_\varepsilon)}{\log \varepsilon} = \lim_{\varepsilon \to 0} \frac{\alpha}{1-\alpha} \log \left(\|\langle X \rangle_\varepsilon\|_\alpha - \varepsilon \right).$$

The case $\alpha \longrightarrow 1$ is the *information dimension* of X. The generalized dimension may be estimated from linear regression with dependent ε on the line

$$\|\langle X\rangle_\varepsilon\|_\alpha = \exp\left(\frac{\alpha}{\alpha-1}\right)e^d + \varepsilon \equiv \eta + \varepsilon.$$

Given the intercept η, the generalized dimension is estimated as

$$\hat{d}_\alpha(X) = \ln \eta + \frac{\alpha}{1-\alpha}.$$

A Mathematical details

Definition 7 (Lists). Given a set S, the collection of lists of elements from S is

$$\mathsf{List}(S) = \bigcup_{n\in\mathbb{Z}_{\geq 0}} S^n,$$

where a list (tuple) $s \in S^n$ is a map $s : \mathbb{Z}_n \longrightarrow S$ and $|s| = n$.