Towards the entropy of images

Pictures in less than a thousand words

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Outline

- What are we after?
- Information theory
- The maximum entropy method
- Toy models of image fluctuation
- The Wang-Landau algorithm
- Next steps

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$$\begin{aligned} S(X) &= \langle I_X \rangle_X \\ &= -\sum_{x \in \mathcal{X}} P(x) \log P(x) \end{aligned}$$

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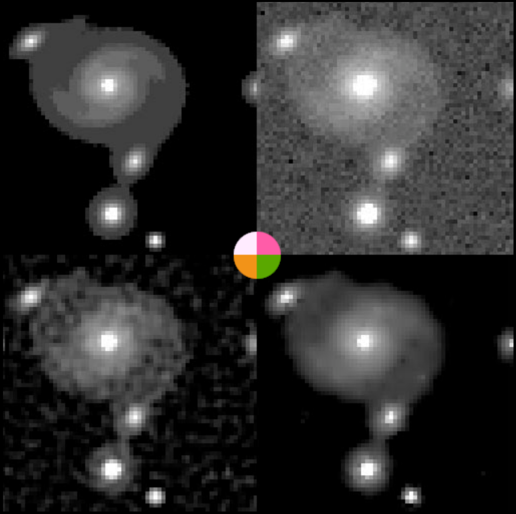
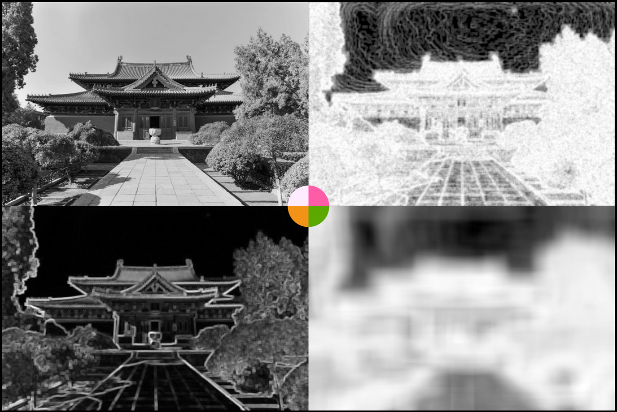
- Maximum entropy method: $P(I) = e^{\lambda S(I)}$



Original "Measured" MEM Multiscale MEM

E Pantin and J-L Starck. "Deconvolution of astronomical images using the multiscale maximum entropy method". In: *Astronomy and Astrophysics Supplement Series* 118.3 (1996), pp. 575–585

Intensity “Entropy”

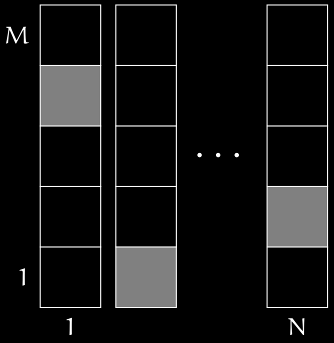


Original 5×5 IE 5×5 SD 41×41 IE

IE: Intensity entropy

SD: Standard deviation

Images as discrete-level systems



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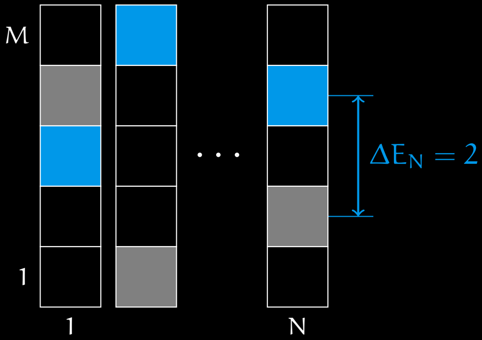

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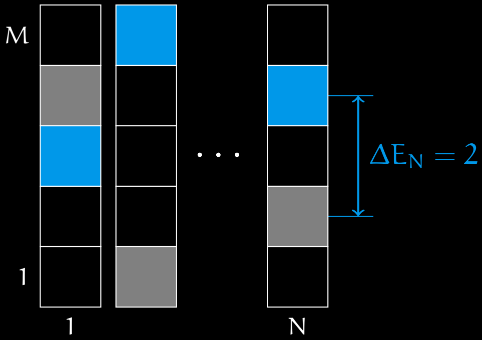

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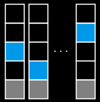

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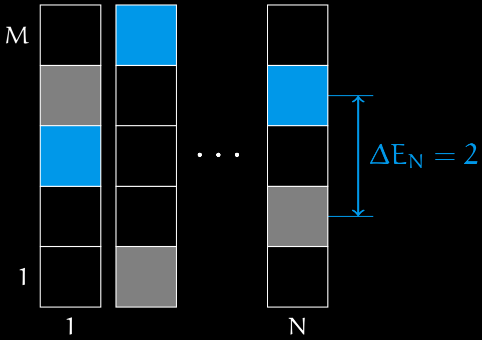

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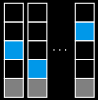
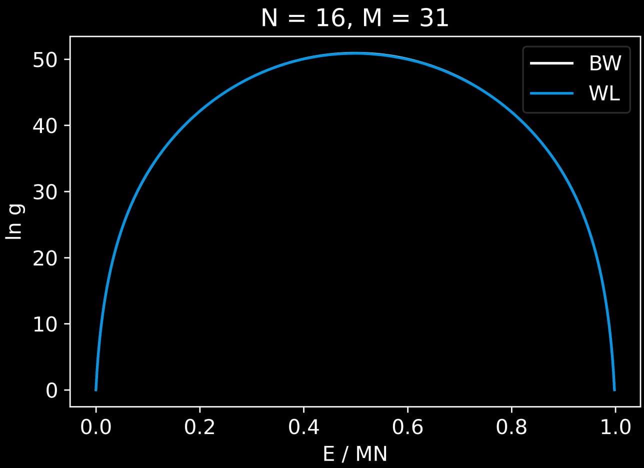


$$g(E) = \sum_k (-1)^k \binom{N}{k} \binom{N+E-kM-1}{E-kM}$$

The Wang-Landau Algorithm



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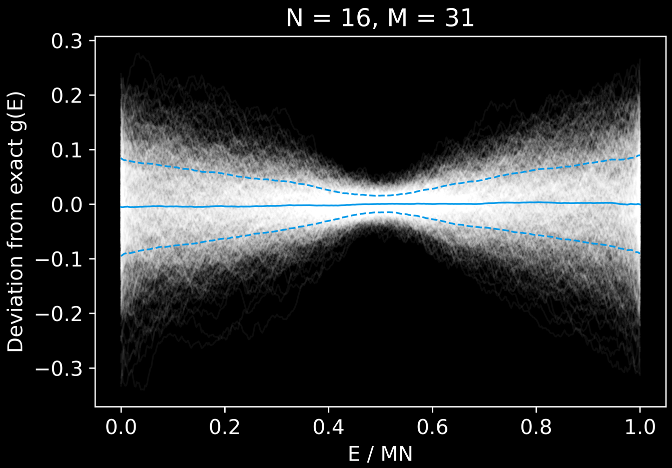

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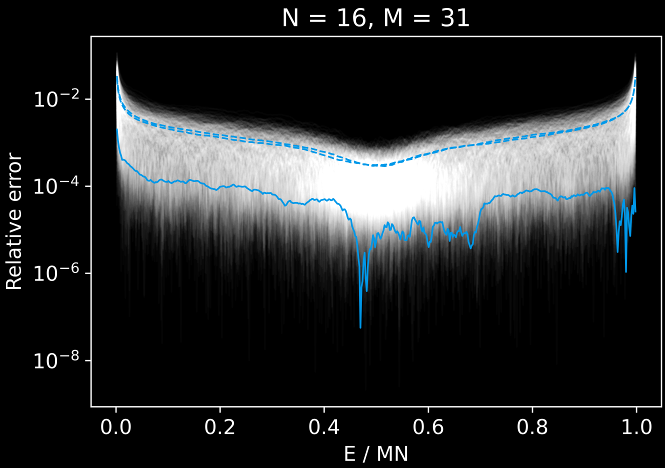

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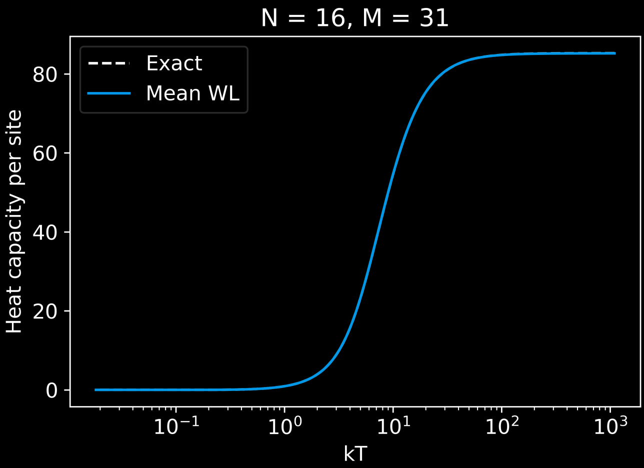

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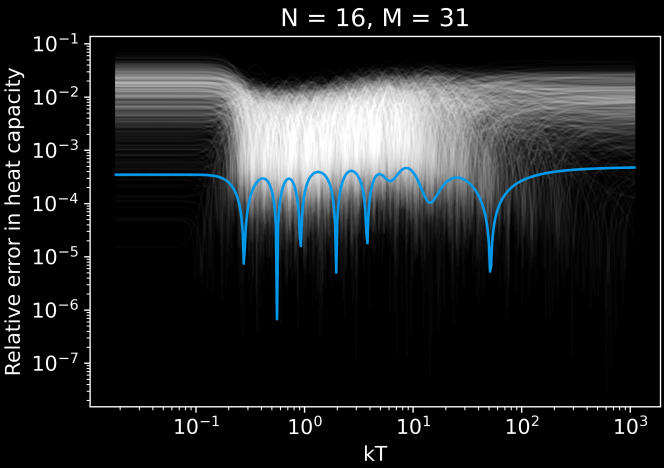

Next steps

- Connecting continuous and discrete
- Natural images
  - Spatial information
  - Color
  - Stereopsis
  - Motion
- High-level features like faces
- Generalization
- Inference

Acknowledgements

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- For a n-vector X with finite $H(\lfloor X \rfloor)$, $0 \leqslant d(X) \leqslant n$.

References

- Edwin T Jaynes. “Information theory and statistical mechanics”. In: *Physical review* 106.4 (1957), p. 620.
- E Pantin and J-L Starck. “Deconvolution of astronomical images using the multiscale maximum entropy method”. In: *Astronomy and Astrophysics Supplement Series* 118.3 (1996), pp. 575–585.
- Claude E Shannon. “A mathematical theory of communication”. In: *The Bell system technical journal* 27.3 (1948), pp. 379–423.