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A note on compact graphs

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Abstract

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An undirected simple graph G is called *compact* iff its adjacency matrix A is such that the polytope $S(A)$ of doubly stochastic matrices X which commute with A has integral-valued extremal points only. We show that the isomorphism problem for compact graphs is polynomial. Furthermore, we prove that if a graph G is compact, then a certain naive polynomial heuristic applied to G and any partner G' decides correctly whether G and G' are isomorphic or not. In the last section we discuss some compactness preserving operations on graphs.

1. Introduction

We consider simple undirected graphs $G=(V_n,E)$ with vertex set $V_n=\{1,2,\dots,n\}$ and edge set E. The adjacency matrix $A=A(G)$ of such a graph is a symmetric square matrix of order n whose entries A_{ij} are equal to 1 or 0 depending on whether the edge $\langle i,j\rangle$ is in E or not. An automorphism π of G is a permutation $\pi:V_n\to V_n$ of the vertex set which preserves adjacency in G, i.e., $\langle i,j\rangle\in E$ iff $\langle\pi(i),\pi(j)\rangle\in E$. For a graph G with adjacency matrix A every automorphism is representable by some permutation matrix P satisfying

$$PA=AP. \tag{1}$$

Therefore, a permutation matrix P determines an automorphism of G iff P commutes with A. The set of permutation matrices P which commute with A forms a group under matrix multiplication. This group is a representation of the automorphism group of G, we denote it by $\mathrm{Aut}(A)$.

In the sequel we shall identify graphs with their adjacency matrices and automorphisms with their corresponding permutation matrices.

Any permutation matrix P is a doubly stochastic matrix having integral entries

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only. In general, a matrix X is called *doubly stochastic* iff it is a solution of the (continuous) linear programming problem

$$Xe = X^t e = e, \quad X \geq 0 \tag{2}$$

where e denotes the n-vector of 1's and where X^t is the transposed of X. The conditions (2) say that X should have nonnegative entries and that the sum of all entries in any row or column equals 1.

We may consider any real-valued matrix X as a point in $R^{n\times n}$. With this interpretation in mind the conditions (2) define a polytope $S_n \subset R^{n\times n}$, the well-known *assignment polytope*. Birkhoff [1] was the first to prove that the extremal points of this polytope correspond to integral-valued matrices P, in fact, the extremal points of S_n are exactly the permutation matrices of order n. This result is stated in the following well-known theorem.

Theorem 1 (Birkhoff). *Every doubly stochastic matrix X is a convex sum of permutation matrices.*

Now, given the graph A, add the constraints

$$XA = AX \tag{3}$$

to the linear programming problem (2). The solution set of (2) and (3) is a subpolytope of S_n which is canonically related to A. We denote this polytope by $S(A)$. The underlying set of matrices is closed under matrix multiplication. Hence, $S(A)$ may be considered as a semigroup, the *semigroup of the graph A*.

Any automorphism P of A satisfies (2) and (3) and is, therefore, an extremal point of $S(A)$. But in general, $S(A)$ will have more extremal points, consequently some of them must be nonintegral. Only in favorable cases each extremal point of $S(A)$ is integral. In such cases $S(A)$ is the convex hull of the automorphism group $\mathrm{Aut}(A)$. To study these cases we use the following definition.

Definition. A graph A will be called *compact* iff it satisfies the following condition:

> Every doubly stochastic matrix X which commutes with A is a convex sum of automorphisms of A.

Compact graphs have been discussed in [3,14–16]. In [16] they have been introduced as *Birkhoff graphs*. Proving that A is compact may be considered as a kind of extending Birkhoff's theorem (Theorem 1) to noncomplete graphs. (Note that Birkhoff's theorem is equivalent to the statement that complete graphs are compact.) Therefore, the notion *Birkhoff graph* is justified. The notion *compact* has been introduced by Brualdi in [3]. It seems to be more convenient (you can speak about *compactness* or use the term *supercompact* as in [3], and so on). For this reason, in this paper the class of graphs under consideration is called compact graphs.

Now, consider two graphs A and B of the same order n. A and B are called isomorphic iff there exists a permutation matrix P satisfying

$$PA = BP. \tag{4}$$

The isomorphism problem for graphs is the problem to decide whether two graphs A and B are isomorphic or not. The time-complexity of this problem class is unresolved. No polynomial-time algorithm for its solution is known nor is the problem known to be NP-complete. Many restrictions of the general graph isomorphism problem to particular graph classes are known to be polynomial, for instance the restrictions to planar graphs [7,9], or more general, to graphs of bounded genus [6,12], and the restriction to graphs with bounded vertex degrees [11]. Further examples are found with interval graphs and with permutation graphs [4,10]. On the other hand, there are numerous subclasses of graphs for which the isomorphism problem is known to be as hard as in the general case. For more details see [2].

The isomorphism relation is weakened if we replace (4) by

$$XA = BX \tag{5}$$

where X is a doubly stochastic matrix. The solution set of (5) and (2) is another (possibly empty) subpolytope of S_n which in general can have nonintegral extremal points. Denote this subpolytope by $S(A,B)$. A and B are isomorphic iff $S(A,B)$ has at least one integral extremal point. Therefore, the most favorable case appears when either *all* extremal points of $S(A,B)$ are integral or when *none* of them is integral. To observe this most favorable case it is necessary and sufficient that one of the two graphs A and B is compact, as the next theorem will show.

Theorem 2. *A graph A is compact iff the following statement is valid:*
For any graph B of the same order as A the polytope $S(A,B)$ either has integral extremal points only or none of its extremal points is integral.

Proof. The proof is obvious. Let A be compact and B an arbitrary partner for A. If B is not isomorphic to A, then $S(A,B)$ has no integral extremal points. However, if B is isomorphic to A, then $S(A,B)$ is simply $S(A)$ with coordinate order changed. This proves the only if part of the theorem. For the if part take $B=A$. □

Theorem 2 suggests how one could test a pair of graphs A and B for isomorphism if one of the graphs, say A, is known to be compact. The test procedure could be as follows:

Given A and B try to find a basic solution of

$$\begin{aligned} &XA = BX, \\ &Xe = X^{t}e = e, \\ &X \geq 0. \end{aligned} \tag{(2)$\wedge$(5)}$$

If this system is unsolvable or if the solution X is not integral, then A and B are not isomorphic.

Otherwise X is an isomorphism of A and B.

A basic solution to (2) and (5) (if there is any) is obtainable in polynomial time by an application of the ellipsoid method (see [8]) or Karmakar's algorithm [8]. From this it follows that the isomorphism problem for compact graphs is polynomial. However, the algorithms just mentioned are rather unattractive for graph-theoretical uses, and in fact, one can do much better. As it will be shown in the next section, if A is compact, then a simple try-and-error algorithm applied to A and an arbitrary partner B produces a correct isomorphism test and runs in polynomial time.

2. An algorithmic aspect of compact graphs

Let A be a graph on the vertex set V_n. An ordered partition $(W_1, \ldots, W_t)$ of V_n is called *feasible* for A iff

$$\pi(W_i) = W_i, \quad 1 \le i \le t \tag{6}$$

for every automorphism π of A.

Let $(W_1, \ldots, W_t)$ be feasible for A. For $v \in V_n$ denote the index i such that $v \in W_i$ by $L_0(v)$ and define

$$L_j(v) = |\{w \in W_j \mid A_{vw} = 1\}|, \quad 1 \le j \le t \tag{7}$$

$$(= \text{number of neighbours of } v \text{ in } W_j).$$

With every $v \in V_n$ associate the list

$$L(v) = (L_0(v), L_1(v), \ldots, L_t(v)). \tag{8}$$

Let t' be the number of different lists produced in this way and select vertices $w_1, w_2, \ldots, w_{t'}$ such that

$$L(w_1) <_l L(w_2) <_l \cdots <_l L(w_{t'})$$

where $<_l$ means lexicographically less. Define

$$W_i' = \{w \in V_n \mid L(w) = L(w_i)\}, \quad 1 \le i \le t'. \tag{9}$$

Evidently, $(W_1', \ldots, W_{t'}')$ is feasible for A and is at least as fine as $(W_1, \ldots, W_t)$. We use the notation

$$(W_1', \ldots, W_{t'}') = \mathrm{REFINE}_A(W_1, \ldots, W_t).$$

The result of a k-fold application of the operator REFINE_A is denoted by REFINE_A^k. There is a minimum index κ such that

$$\mathrm{REFINE}_A^{\kappa}(W_1, \ldots, W_t) = \mathrm{REFINE}_A^{\kappa+1}(W_1, \ldots, W_t).$$

$\mathrm{REFINE}_A^{\kappa}(W_1,\dots,W_t)$ is called the *closure* of $(W_1,\dots,W_t)$ with respect to A and is denoted by $\mathrm{CLOSURE}_A(W_1,\dots,W_t)$.

Assume that $\mathrm{CLOSURE}_A(W_1,\dots,W_t)=(W_1^*,\dots,W_{t^*}^*)$. For any $v\in V_n$ let $L^*(v)$ be the list associated with v according to (8) but with respect to $(W_1^*,\dots,W_{t^*}^*)$. This partition is *closed* in the sense that the lists $L^*(v)$ do not vary on W_i^*, $1\le i\le t$, and, therefore, $\mathrm{REFINE}_A(W_1^*,\dots,W_{t^*}^*)=(W_1^*,\dots,W_{t^*}^*)$.

Now, start with the feasible partition (V_n) and let

$$(D_1^*,\dots,D_{d^*}^*)=\mathrm{CLOSURE}_A(V_n). \tag{10}$$

Again let $L^*(v)$ be the list associated with v with respect to (10). Consider the matrix $T(A)$ defined by

$$T(A)_{vj}=L_j^*(v),\quad 1\le v\le n,\quad 1\le j\le d^*. \tag{11}$$

It is a well-known fact in graph isomorphism theory that $T(A)=T(B)$ is necessary but not sufficient for A and B to be isomorphic [5]. However, in [15] it is proved that for two graphs A and B of the same order n, $T(A)=T(B)$ is necessary *and* sufficient for the existence of a doubly stochastic matrix X which satisfies

$$XA=BX.$$

This result can be generalized as follows. Given the graph A and a feasible partition $\mathcal{W}=(W_1,\dots,W_t)$ define a *test matrix* $T(A;\mathcal{W})$ according to (10) and (11) with $\mathrm{CLOSURE}_A(V_n)$ replaced by $\mathrm{CLOSURE}_A(\mathcal{W})$. The next theorem is the key for an algorithmic characterization of compact graphs although it does not refer to this restricted class of graphs explicitly.

Theorem 3. *Let A and B be graphs of the same order n. Let $\mathcal{W}=(W_1,\dots,W_t)$ and $\mathcal{U}=(U_1,\dots,U_t)$ be partitions of V_n which are feasible for A and B, respectively, and satisfy $\mathcal{W}=\mathrm{REFINE}_A(\mathcal{W})$ and $\mathcal{U}=\mathrm{REFINE}_B(\mathcal{U})$. Define*

$$\mathcal{U}\times\mathcal{W}=\bigcup_{i=1}^{t}U_i\times W_i.$$

There exits a doubly stochastic matrix X with

$$\begin{aligned} &XA=BX \\ \text{and}\quad & \\ &X_{ij}=0\quad \textit{for all } (i,j)\notin\mathcal{U}\times\mathcal{W}\end{aligned} \tag{12}$$

if and only if $T(A;\mathcal{W})=T(B;\mathcal{U})$.

Proof. Note that the second part of (12), namely $X_{ij}=0$ for all $(i,j)\notin\mathcal{U}\times\mathcal{W}$, defines a face of the polytope $S(A,B)$. Therefore, Theorem 3 gives a necessary and sufficient condition for the nonemptiness of this face.

Assume $T(A;\mathcal{W})=T(B;\mathcal{U})$. Define

$$X_{uw}=\begin{cases}|W_i|^{-1}, & \text{if } u\in U_i,\ w\in W_i,\\ 0, & \text{otherwise.}\end{cases}$$

It is easy to verify that X belongs to $S(A,B)$. Since X satisfies (12) the if part of Theorem 3 is proved.

Now assume there is an $X \in S(A,B)$ satisfying (12). For $1 \le i,j \le t$ consider the submatrices

$$X^{ii} = (X_{kl})_{k \in U_i, l \in W_i};$$
$$A^{ij} = (A_{kl})_{k \in W_i, l \in W_j};$$
$$B^{ij} = (B_{kl})_{k \in U_i, l \in U_j};$$

of X, A and B. Note that X^{ii} is a doubly stochastic matrix of order $|W_i|$. Since any doubly stochastic matrix is a square matrix we must have $|W_i| = |U_i|$, $1 \le i \le t$. (12) implies

$$X^{ii}A^{ij} = B^{ij}X^{jj} \tag{13}$$

for $1 \le i,j \le t$. Choose any pair (i,j) of indices. Summing up the entries in the rows of A^{ij} gives a vector r_A, the result of summing up along the columns is a vector c_A. Let r_B and c_B be the corresponding vectors with respect to B^{ij}.

From (13) it follows

$$c_A^{\mathrm{t}} = c_B^{\mathrm{t}} X^{jj}, \tag{14a}$$

$$r_B = X^{ii} r_A. \tag{14b}$$

Since $\mathcal{W} = \mathrm{REFINE}_A(\mathcal{W})$ and $\mathcal{U} = \mathrm{REFINE}_B(\mathcal{U})$, c_A, c_B, r_A and r_B are constant vectors. (Note that for a closed partition $\mathcal{W}$ and the corresponding lists $L^*(w)$ we have

$$\sum_{u \in W_i} A_{uw}^{ij} = L_i^*(w)$$

where these numbers are independent of $w \in W_j$.) This together with (14a) and (14b) implies

$$c_A = c_B \quad \text{and} \quad r_A = r_B.$$

Since (i,j) was arbitrary, we conclude that $T(A;\mathcal{W}) = T(B;\mathcal{U})$. This proves the only if part of the theorem. □

Next we consider a simple heuristic procedure for testing graph isomorphism which is based on Theorem 3. The procedure constructs a hierarchy of subfaces of $S(A,B)$ step-by-step, ending at the empty face if A and B are not isomorphic, and ending at an extremal point otherwise. At a general step of the procedure the subface under consideration is defined by a certain pair $(\mathcal{W}, \mathcal{U})$ of partitions of V_n satisfying $T(A;\mathcal{W}) = T(B;\mathcal{U})$. $\mathcal{W}$ and $\mathcal{U}$ are refined simultaneously by choosing an arbitrary pair (w,u) with $w \in W_i$, $u \in U_i$ for some i with $|U_i| > 1$ and putting w and u into different cells, respectively. This yields a new pair of partitions $\mathcal{W}'$ and $\mathcal{U}'$, and $\mathcal{W}$ and $\mathcal{U}$ are replaced by $\mathrm{CLOSURE}_A(\mathcal{W}')$ and $\mathrm{CLOSURE}_B(\mathcal{U}')$. This refinement step is a standard ingredient of classical graph isomorphism algorithms. It represents the basic "forward"-step in numerous much more elaborate algorithms for testing general graph isomorphism (see for example [5] or consult the bibliography [13]). In the case of general graphs, when an empty subface is reached ($T(A;\mathcal{W}) \neq T(B;\mathcal{U})$), then a "backward"-step is performed and a new "path" is tried such that the time-complexity of the resulting algorithm becomes exponential in the order n of the input graphs. However, in the case when one of the input graphs is compact "backward"-steps can be avoided and the time-complexity of the algorithm is polynomial.

A formal description of the algorithm is given below.

Algorithm GRAPHIS

Input: Two graphs A and B of the same order n;

(1) $\mathcal{W} \leftarrow \mathcal{U} \leftarrow (V_n)$; $k \leftarrow 0$;

(2) $k \leftarrow k+1$;

Define

$\mathcal{W}^k \leftarrow (W_1^k, \ldots, W_{w(k)}^k) \leftarrow \mathrm{CLOSURE}_A(\mathcal{W})$;

$\mathcal{U}^k \leftarrow (U_1^k, \ldots, U_{u(k)}^k) \leftarrow \mathrm{CLOSURE}_B(\mathcal{U})$;

and **compute** $T(A;\mathcal{W})$ and $T(B;\mathcal{U})$;

(3) **if** $T(A;\mathcal{W}) \neq T(B;\mathcal{U})$ **then goto** (6);

(4) **if** $w(k)=n$ **then goto** (7);

(5) **Take** an arbitrary triple (i, u, w) such that $|U_i|>1$ and $(u,w) \in U_i \times W_i$ and

define

$W_j \leftarrow W_j^k$; $U_j \leftarrow U_j^k$; $1 \leq j \leq w(k)$, $j \neq i$;

$W_i \leftarrow W_i^k - \{w\}$; $W_{w(k)+1} \leftarrow \{w\}$;

$U_i \leftarrow U_i^k - \{u\}$; $U_{w(k)+1} \leftarrow \{u\}$;

$\mathcal{W} \leftarrow (W_1, \ldots, W_{w(k)+1})$; $\mathcal{U} \leftarrow (U_1, \ldots, U_{w(k)+1})$; **goto** (2);

(6) STOP; "A and B are not isomorphic";

(7) **for** $1 \leq i \leq n$ map the unique element $w \in W_i^k$ onto the unique element $u \in U_i^k$ and STOP;

"This mapping is an isomorphism of A and B";

Algorithm GRAPHIS is a heuristic procedure which works correctly for any pair of nonisomorphic graphs A and B, even if none of them is compact. For isomorphic input graphs there are in general two possible outcomes depending on which sequence of pairs (u, w) is chosen in the sequence of steps (5). The probability for a correct outcome depends on the structure of A. However, if one of the graphs is compact, then each run of Algorithm GRAPHIS leads to the correct result, no matter what the second graph is like. This behaviour of compact graphs is an interesting feature of this class of graphs.

Theorem 4. *If A is a compact graph, then each run of Algorithm GRAPHIS ap-*

plied to A and an arbitrary graph B of the same order as A decides correctly whether A is isomorphic to B or not.

Proof. Let A be compact and B any graph of the same order as A. We have to deal with the case where Algorithm GRAPHIS stops under label (6).

Let κ be the actual value of the variable k when Algorithm GRAPHIS turns to label (6). If $\kappa = 1$, then

$$\mathscr{W}^1 = \mathrm{CLOSURE}_A(V_n), \qquad \mathscr{U}^1 = \mathrm{CLOSURE}_B(V_n)$$

and

$$T(A;\mathscr{W}) \neq T(B;\mathscr{U}).$$

Therefore, by the only if part of Theorem 3, $S(A,B)=\emptyset$, A and B are not isomorphic and the algorithm stops giving the correct answer.

Thus, assume $\kappa > 1$. We must have $\kappa < n$, since if $\kappa = n$, then Algorithm GRAPHIS turns to label (7) and stops with an isomorphism of A and B.

Any pair of partitions

$$\mathscr{W} = (W_1, \dots, W_t), \qquad \mathscr{U} = (U_1, \dots, U_t)$$

implicitly defines a face $\mathscr{F}_{A,B}(\mathscr{U}\times\mathscr{W})$ of $S(A,B)$ consisting of all matrices $X \in S(A,B)$ such that $(u,w) \notin \mathscr{U}\times\mathscr{W}$ implies $X_{uw}=0$. Denote the face corresponding to $\mathscr{U}^k \times \mathscr{W}^k$ by $\mathscr{F}_k$, $1 \geq k \geq \kappa$. By assumption we have

$$T(A;\mathscr{W}^{\kappa-1}) = T(B;\mathscr{U}^{\kappa-1})$$

but

$$T(A;\mathscr{W}^{\kappa}) \neq T(B;\mathscr{U}^{\kappa}).$$

Thus, the if part of Theorem 3 implies $\mathscr{F}_{\kappa-1} \neq \emptyset$, the only if part implies $\mathscr{F}_\kappa = \emptyset$. Furthermore, in the proof of Theorem 3 it was shown that $T(A;\mathscr{W}) = T(B;\mathscr{U})$ implies the existence of a matrix $X \in \mathscr{F}_{A,B}(\mathscr{U}\times\mathscr{W})$ with $X_{uw} > 0$ for all $(u,w) \in \mathscr{U}\times\mathscr{W}$. Assume that $\bar{u}$ and $\bar{w}$ are the actual values of u and w during the last execution of step (5) by Algorithm GRAPHIS. We have

$$(\bar{u},\bar{w}) \in \mathscr{U}^{\kappa-1} \times \mathscr{W}^{\kappa-1},$$

hence there exists a matrix $X \in \mathscr{F}_{\kappa-1}$ with $X_{\bar{u}\bar{w}} > 0$. Since $\mathscr{F}_\kappa = \emptyset$, there is no $Y \in \mathscr{F}_{\kappa-1}$ with $Y_{\bar{u}\bar{w}} = 1$. From this we conclude that $\mathscr{F}_{\kappa-1}$, and therefore $S(A,B)$, must have nonintegral extremal points. Since A is compact, by Theorem 2, A and B cannot be isomorphic, and Algorithm GRAPHIS stops with the correct answer. □

One could be tempted to conjecture that also the converse of Theorem 4 is valid: *If a graph A is such that each run of Algorithm GRAPHIS applied to A and any partner B is a correct isomorphism test, then A is compact.*

As yet there is no proof for this conjecture.

3. Compactness preserving operations

In this section some operations will be discussed which applied to compact graphs lead to new compact graphs. We use the following notations. J_n denotes the square matrix of 1's of order n, $J_{n,m}$ the rectangular matrix of 1's of size $n \times m$. For any matrix X of order n and the n-vector e_n of 1's define

$$r_X = Xe_n,$$
$$c_X = X^{\mathrm{t}}e_n.$$

Let A be any graph of order n. The graph $\bar{A} = J_n - A - I_n$ (I_n the unit matrix) is called the complement of A. Clearly $S(\bar{A}) = S(A)$. Therefore, $\bar{A}$ is compact iff A is compact. This statement is trivial. However, for bipartite graphs A we have the possibility of constructing the complement with respect to the complete bipartite graph having A as a partial subgraph. Assume

$$A = \begin{pmatrix} 0 & B \\ B^{\mathrm{t}} & 0 \end{pmatrix} \tag{15}$$

where B is any $m \times n$-matrix of 0's and 1's, and consider the graph

$$\tilde{A} = \begin{pmatrix} 0 & J_{m,n} - B \\ J_{n,m} - B^{\mathrm{t}} & 0 \end{pmatrix}. \tag{16}$$

Call $\tilde{A}$ the *b-complement* of A. We shall discuss under which conditions compactness of A implies compactness of $\tilde{A}$. The main result is given in the following theorem.

Theorem 5. *If A and $\tilde{A}$ are both connected, then $\tilde{A}$ is compact iff A is compact.*

Proof. We give a proof of the theorem by showing that under the hypothesis we even have $S(A) = S(\tilde{A})$. Let A be as in (15) and $X \in S(A)$. Write

$$X = \begin{pmatrix} Y & Z \\ U & V \end{pmatrix}$$

according to the block decomposition of A. We have $X \in S(A)$ iff

$$ZB^{\mathrm{t}} = BU, \quad VB^{\mathrm{t}} = B^{\mathrm{t}}Y, \quad YB = BV, \quad UB = B^{\mathrm{t}}Z. \tag{17}$$

Due to [15, Lemma 1] $X_{ij} > 0$ implies that i and j have the same degree in A. From this we find after multiplying $YB = BV$ by e_m^{t} from the left

$$\sum_i c_{Y,i} B_{ij} = c_{B,j} c_{V,j}, \quad 1 \leq j \leq n.$$

Using the notation $F_{ij} = c_{B,j}^{-1} B_{ij}$ this reads

$$c_Y^{\mathrm{t}} F = c_V^{\mathrm{t}}. \tag{18}$$

Analogously, using $H_{ij} = c_{B^t,j}^{-1} B_{ij}^t$ we find from $VB^t = B^tY$,

$$c_V^t H = c_Y^t. \tag{19}$$

Combining (18) and (19) yields

$$c_Y^t F H = c_Y^t.$$

Now, H^tF^t is stochastic and the graph associated with this matrix is connected if A is connected. Therefore, if A is connected, then any eigenvector of H^tF^t which belongs to the eigenvalue 1 must be a scalar multiple of e_m. This proves that Y has constant column sums. Analogously, we find that r_Y, r_Z, r_U, r_V, c_Z, c_U, c_V are constant vectors. Furthermore,

$$r_Y + r_Z = e_m, \qquad r_U + r_V = e_n,$$

$$c_Y + c_U = e_m, \qquad c_Z + c_V = e_n.$$

But this implies that X commutes with

$$\begin{pmatrix} 0 & J_{m,n} \\ J_{n,m} & 0 \end{pmatrix}.$$

Thus, we have $S(A) \subset S(\tilde{A})$. Starting with $\tilde{A}$ we find $S(\tilde{A}) \subset S(A)$. This proves Theorem 5. $\square$

Next, let us turn to the operation of disjoint union of connected graphs. Let

$$A = \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix}$$

where B and C are connected graphs of order m and n, respectively. Assume $X \in S(A)$ and decompose X into blocks

$$X = \begin{pmatrix} Y & Z \\ U & V \end{pmatrix}$$

according to the blocks of A. In a way completely analogous to the proof of Theorem 5 we find that all the vectors r_L and c_L, $L \in \{Y, Z, U, V\}$ are constant vectors.

Assume for the moment that B and C are isomorphic and that $C = PBP^t$ where P is an appropriate permutation matrix. Since $m = n$ in this case, all blocks of X are quadratic and there are reals σ and ϱ such that each of the matrices σY, ϱZP, $\varrho P^t U$ and $\varrho P^t VP$ is in $S(B)$. Assuming $\sigma \neq 0$, if B is compact, we have

$$Y = \sum_i \sigma_i P_i \quad \text{and} \quad V = \sum_i \sigma_i' P P_j P^t$$

where $0 \leq \sigma_i, \sigma_i' \leq 1$, $\sum \sigma_i = \sum \sigma_i' = 1/\sigma$, and the summation is over all automorphisms $P_i \in \mathrm{Aut}(B)$. Thus, the matrix

$$\begin{pmatrix} \sigma Y & 0 \\ 0 & \sigma V \end{pmatrix}$$

is in the convex hull of

$$\left\{\begin{pmatrix} P_i & 0 \\ 0 & P_k \end{pmatrix} \mid P_i \in \mathrm{Aut}(B),\ P_k \in \mathrm{Aut}(C)\right\} \subset \mathrm{Aut}(A).$$

(Proof by induction on the number of positive entries in Y and V as in one of the standard proofs of Birkhoff's theorem.)

Analogously, we find that

$$\begin{pmatrix} 0 & \varrho Z \\ \varrho U & 0 \end{pmatrix} \in \mathrm{conv}(\mathrm{Aut}(A)).$$

Since $1/\sigma + 1/\varrho = 1$,

$$X = \frac{1}{\sigma}\begin{pmatrix} \sigma Y & 0 \\ 0 & \sigma V \end{pmatrix} + \frac{1}{\varrho}\begin{pmatrix} 0 & \varrho Z \\ \varrho U & 0 \end{pmatrix}$$

is a convex sum of elements of $\mathrm{Aut}(A)$. This proves the following theorem.

Theorem 6. *The disjoint union of isomorphic connected compact graphs is compact.*

Theorem 6 is a generalization of a result of Brualdi [3, Theorem 3.5].

We return now to the case where B and C are nonisomorphic. We have $X \in S(A)$ iff

$$YB = BY, \qquad ZC = BZ,$$

$$VC = CV, \qquad UB = CU.$$

Assume that Z^* is a solution of $ZC = BZ$ with $e_m^t Z^* = \sigma e_n \neq 0$ and $Z^* e_n = \varrho e_m$. Then $U^* = Z^{*t}$ is a nontrivial solution of $UB = CU$. Take any $Y^* \in S(B)$ and any $V^* \in S(C)$ and define

$$X^* = \begin{pmatrix} (1-\varrho)Y^* & Z^* \\ U^* & (1-\sigma)V^* \end{pmatrix}.$$

We have $X^* \in S(A)$, but since $Z^* \neq 0$ and since there is no automorphism of A sending a vertex of B onto a vertex of C, X^* is never a convex sum of automorphisms of A.

For any matrix T and any natural number n write n^*T for the matrix derived from T by replacing each row t by a block of n copies of t. Given B and C as above consider the matrices $T(B)$ and $T(C)$ as defined by (11). In [16] it has been proved that $n^*T(B) = m^*T(C)$ is necessary and sufficient for the existence of a solution Z^* to $ZB = CZ$ having constant row sums ϱ and constant column sums $\sigma \neq 0$. This proves the following theorem.

Theorem 7. *The disjoint union of nonisomorphic compact connected graphs $A_1, \ldots, A_k$ is compact iff for no two graphs A_i, A_j of them $n_j^* T(A_i) = n_i^* T(A_j)$ where n_i and n_j are the orders of A_i and A_j, respectively.*

4. Final remarks

The recognition problem for compact graphs turns out to be extremely difficult. There is a similarity to the recognition problem for perfect graphs. Recognizing compact graphs means, at least implicitly, proving that a certain polytope, here $S(A)$, is integral. General problem classes of this kind are even not known to belong to NP, however, they belong definitively to co-NP.

Regular graphs offer an interesting special case. A compact regular graph A must be vertex-transitive, that means, its group $\mathrm{Aut}(A)$ must act transitively on the vertex set V_n. In the case of a prime number n the only vertex-transitive graphs are *circulant* graphs (see [14] for definition and references). From the results of [14] it follows that in the case of regular graphs of prime order n the compactness recognition problem is time-polynomially reducable to the recognition problem for circulant graphs.

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