Note

Perfect elimination orderings of chordal powers of graphs [1]

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Abstract

Let $G = (V, E)$ be a finite undirected connected graph. We show that there is a common perfect elimination ordering of all powers of G which represent chordal graphs. Consequently, if G and all of its powers are chordal then all these graphs admit a common perfect elimination ordering. Such an ordering can be computed in $O(|V| \cdot |E|)$ time using a generalization of the Tarjan and Yannakakis' Maximum Cardinality Search.

Recently, powers of graphs from several classes have been investigated. One of the first results on powers of graphs is due to Duchet [5]: If G^k is chordal then also G^{k+2}. In particular, odd powers of chordal graphs are chordal, whereas even powers of chordal graphs are in general not. On the other hand, even powers of distance-hereditary graphs are chordal [1]. Powers of special chordal graphs have been studied in [4,8,2] and [7]. Chordal graphs are known as the class of graphs which admit a perfect elimination ordering [3]. Such orderings are algorithmically very useful and can be computed in linear time using the *Maximum Cardinality Search* [10].

In this note we prove that all chordal powers of an arbitrary graph have a common perfect elimination ordering. Such an ordering can be computed in $O(|V| \cdot |E|)$ time using a search procedure which generalizes the maximum cardinality search.

All graphs in this note are finite, undirected, simple (i.e. without loops and multiple edges), and connected. In a graph $G = (V, E)$ the *length* of a path between vertices $u, v \in V$ is the number of edges in the path. The *distance* $d(u, v)$ from vertex u to vertex v is the length of a shortest path from u to v and the *interval* $I(u, v)$ between

these vertices is the set

$$I(u,v) = \{w \in V : d(u,v) = d(u,w) + d(w,v)\}.$$

Let k be a positive integer. The *kth power* G^k of G has the same vertices as G and two vertices are joined by an edge in G^k if and only if their distance in G is at most k.

Let x be a vertex of G. By $N(x)$ we denote the *neighbourhood* of x consisting of all vertices adjacent to x. The *kth neighbourhood* of x, denoted by $N_k(x)$, is defined as the set of all vertices of distance k to x, that is $N_k(x) = \{y \in V : d(x,y) = k\}$.

The *disk* centred at x with radius k is the set of all vertices having distance at most k to x:

$$D(x,k) = \{v \in V : d(x,v) \leqslant k\}.$$

Let $G(S)$ denote the subgraph of G induced by $S \subset V$. A graph G is *distance-hereditary* iff for all induced connected subgraphs $G(S)$ the distances in $G(S)$ are the same as in G.

A vertex $v \in V$ is *simplicial* in G iff $N(v)$ is a clique in G. Let $G_i = G(\{v_i, v_{i+1}, \ldots, v_n\})$. A linear ordering $(v_1, \ldots, v_n)$ of V is a *perfect elimination ordering of G* iff for all $i \in \{1, \ldots, n\}$ the neighbourhood $N_i(v_i)$ restricted to G_i is a clique i.e. v_i is simplicial in G_i.

A graph G is *chordal* iff it does not contain any induced (chordless) cycles of length at least 4.

It is well-known [3] that a graph G is chordal iff it admits a perfect elimination ordering.

A *chordal power* of a graph G is a power G^k which is chordal.

For an arbitrary set of vertices $S \subseteq V$, let $c_k(x,S) = |D(x,k) \cap S|$. Consider an arbitrary increasing sequence of integers $0 < i_1 < i_2 < \cdots < i_p$. We define the p-tuples

$$c(x,S) = (c_{i_p}(x,S), \ldots, c_{i_2}(x,S), c_{i_1}(x,S)).$$

For two vertices $x, y \notin S$ let $x \leqslant y$ iff $c(x,S)$ is lexicographically not greater than $c(y,S)$, i.e. either $c(x,S) = c(y,S)$ or

$$c_{i_p}(x,S) = c_{i_p}(y,S), \ldots, c_{i_{k+1}}(x,S) = c_{i_{k+1}}(y,S),$$

while $c_{i_k}(x,S) < c_{i_k}(y,S)$. A vertex $x^* \notin S$ is called a *lexicographical maximum* with respect to the set S iff $x \leqslant x^*$ for any other vertex $x \notin S$.

Next we present a special numbering of vertices of an arbitrary graph G under the assumption that an increasing sequence of integers $0 < i_1 < i_2 \cdots < i_p$ is given. The algorithm is a generalization of the maximum cardinality search (MCS) algorithm of [10] for finding a perfect elimination ordering of chordal graphs. According to MCS the vertices of a graph are numbered from n to 1 in decreasing order. As the next vertex to number select a vertex adjacent with the largest number of previously numbered vertices, breaking ties arbitrarily [10]. In our algorithm in the next step we number

the vertex which is a lexicographical maximum with respect to the set of previously numbered vertices. We call this procedure the *lexicographic maximum cardinality search* (LMCS). If $i_p = 1$ and $p = 1$ then we obtain the maximum cardinality search. Subsequently, we prove that LMCS gives a common perfect elimination ordering of all chordal powers $G^{i_1}, G^{i_2}, \ldots, G^{i_p}$ of a graph G.

Procedure (LMCS)

   Input: A connected graph $G = (V, E)$ with $|V| = n > 1$ and integers
      $0 < i_1 < \cdots < i_p$ such that $G^{i_1}, \ldots, G^{i_p}$ are chordal.

   Output: A perfect elimination ordering $(v_1, \ldots, v_n)$ of the graphs $G^{i_1}, \ldots, G^{i_p}$.

   (0) Initially, all $v \in V$ are unnumbered;

   (1) choose an arbitrary vertex $v \in V$, number v with n, i.e. $v_n = v$ and let $S := \{v_n\}$;

repeat

   (2) among all unnumbered vertices select a vertex v which is a lexicographic maximum with respect to S and adjacent to some vertex of S;

   (3) number the vertex v with maximal possible number between 1 and $n - 1$ which is still free;

   (4) let $S := S \cup \{v\}$

until all vertices are numbered.

Theorem. *Let $G = (V, E)$ be a graph whose powers $G^{i_1}, \ldots, G^{i_p}$ are chordal. Then the algorithm LMCS finds a common perfect elimination ordering of the graphs $G^{i_1}, \ldots, G^{i_p}$.*

The proof of this theorem requires some additional notions and auxiliary results. First, recall that the subset S of G is called *induced-path convex* (or *m-convex* [6]) iff for any two vertices of S any induced path connecting them is contained in S. For a vertex $x \notin S$ let $d(x, S) = \min\{d(x, v): v \in S\}$ be the *distance* from x to the set S. By $Pr(x, S) = \{v \in S: d(x, v) = d(x, S)\}$ we denote the *metric projection* of x on S.

Lemma 1. *Let S be an induced-path convex set of a graph G. For any vertex $x \notin S$ the set $P_r(x, S)$ induces a clique. Moreover, for any vertex $v \in S$ there is a vertex $y \in Pr(x, S)$ such that*

$$d(x, v) = d(x, y) + d(y, v).$$

Proof. Let $y', y'' \in Pr(x, S)$ and consider two shortest paths P' and P'' between vertices x, y' and x, y'', respectively. Evidently, $P' \cap S = \{y'\}$ and $P'' \cap S = \{y''\}$. If the vertices y' and y'' are assumed to be non-adjacent then any induced path between y' and y'' contained in $P' \cup P''$ has at least one vertex outside S. This contradicts to the assumption that S is induced-path convex. So $Pr(x, S)$ is a clique.

   For an arbitrary vertex $v \in S$, let y be a vertex of $I(v, x) \cap S$ which is furthest from v. Then any shortest path between x and y does not contain other vertices of S except y. As in the preceding case we can show that y is adjacent to all vertices of $Pr(x, S)$.

From the choice of y we conclude that $d(y,x) = d(y',x)$ for any vertex $y' \in Pr(x,S)$ i.e. $y \in Pr(x,S)$. $\square$

Lemma 2. *Let S be an induced-path convex set of a graph. Then there exists a neighbour of some vertex of S which is a lexicographically maximum vertex with respect to S.*

Proof. Let x^* be a lexicographical maximum and suppose that $d(x^*,S) > 1$. Pick any vertex $y \in Pr(x^*,S)$ and let x be a neighbour of y on a shortest path between x^* and y. By Lemma 1 we get $d(x^*,v) \geq d(x,v)$ for any vertex $v \in S$. Therefore,

$$c_{i_1}(x,S) \geq c_{i_1}(x^*,S), \ldots, c_{i_p}(x,S) \geq c_{i_p}(x^*,S)$$

and thus $x \geq x^*$. $\square$

Lemma 3. *Let $S \subseteq V$ be an induced-path convex set of all chordal powers $G^{i_1}, \ldots, G^{i_p}$ of a graph G and let x^* be a lexicographical maximum with respect to the set S and adjacent to some vertex of S. Then the set $S \cup \{x^*\}$ is induced-path convex in all graphs $G^{i_1}, \ldots, G^{i_p}$.*

Proof. Assume the contrary and let i_k be the largest index such that the set $S \cup \{x^*\}$ is not induced-path convex in the chordal graph G^{i_k}. This means that there is an induced path $P = (x^*, \ldots, x, y)$ of G^{i_k} whose vertices, except y, are outside S. Then necessarily $d(x,y) \leq i_k$, while $d(x^*,y) > i_k$.

Consider an arbitrary vertex $v \in S$ such that $d(x^*,v) \leq i_k$, i.e. vertices x^* and v are adjacent in G^{i_k}. Since S is an induced-path convex set of G^{i_k} the path $(v,x^*) \cup P$ between v and y must be not induced. This is possible only if v and y are adjacent in G^{i_k}. Therefore we obtain a cycle $(v,x^*, \ldots, x, y, v)$ in the graph G^{i_k}. Since G^{i_k} is chordal and the path P is induced the vertex v in G^{i_k} is adjacent to all vertices of P. In particular, we obtain that $d(x,v) \leq i_k$ and thus $D(x^*,i_k) \cap S \subseteq D(x,i_k) \cap S$. However, since $d(x,y) \leq i_k$ while $d(x^*,y) > i_k$ this inclusion must be strict and $c_{i_k}(x,S) > c_{i_k}(x^*,S)$.

We claim that for any other $i_j > i_k$ we have

$$D(x^*,i_j) \cap S \subseteq D(x,i_j) \cap S.$$

Assume that such an inclusion holds for all $j = k+1, \ldots, l-1 < p$ and consider a vertex $v \in S$ with $d(x^*,v) = i_l > i_k$. By Lemma 1 there is a shortest path between v and x^* which passes through a vertex v^* of the set $Pr(x^*,S)$. Since x^* and v^* are adjacent $d(v^*,v) \geq i_l - i_1$ holds. Let v^+ be a vertex on a shortest path between v^* and v at distance $i_l - i_{l-1}$ from v. Then $v^+ \in I(v^*,v) \subseteq I(x^*,v)$ and therefore $d(x^*,v^+) = i_{l-1}$. Since $D(x^*,i_{l-1}) \cap S \subseteq D(x,i_{l-1}) \cap S$ was assumed also $d(x,v^+) \leq i_{l-1}$ holds. Hence, we obtain

$$d(x,v) \leq d(x,v^+) + d(v^+,v) \leq i_{l-1} + i_l - i_{l-1} = i_l,$$

i.e. $D(x^*,i_l) \cap S \subseteq D(x,i_j) \cap S.$

Summarizing, we obtain that if $S \cup \{x^*\}$ is not induced-path convex in some chordal power G^{i_k} of G then there exists a vertex $x \notin S$ such that

$$c_{i_j}(x, S) \geqslant c_{i_j}(x^*, S) \quad \text{for all } j = p, \ldots, k-1$$

while $c_{i_k}(x, S) > c_{i_k}(x^*, S)$. This contradicts to our assumption that x^* is a lexicographical maximum with respect to S. $\square$

Proof of the theorem. Let $(v_1, \ldots, v_n)$ be the ordering obtained by the algorithm LMCS. Let $S_i = \{v_i, v_{i+1}, \ldots, v_n\}$. Proceeding by induction on n by Lemma 3 we get that the sets $S_n, S_{n-1}, \ldots, S_1$ are induced-path convex in all graphs $G^{i_1}, \ldots, G^{i_p}$. Then necessarily any vertex v_i is simplicial in all subgraphs

$$G^{i_1}(S_i), \ldots, G^{i_p}(S_i),$$

i.e. $(v_1, \ldots, v_n)$ is a common perfect elimination ordering of the graphs $G^{i_1}, \ldots, G^{i_p}$. $\square$

Corollary 1. *If G is chordal then there exists a common perfect elimination ordering of the graphs $G, G^3, G^5, \ldots$*

Corollary 2. *If G^2 of a graph G is chordal then there exists a common perfect elimination ordering of the graphs $G^2, G^4, G^6, \ldots$*

Corollary 3. *If G and G^2 are chordal then there exists a common perfect elimination ordering of the graphs $G, G^2, G^3, \ldots$*

In this note we proved the fact that all chordal powers of an arbitrary graph have a common perfect elimination ordering. For the proof we used a generalization of Maximum Cardinality Search which leads to a $O(|V| \cdot |E|)$ time procedure for finding such an ordering. We believe that this procedure is not time-optimal. It could well be that by using *Doubly Lexical Search* a better time bound may be obtained for the same purpose. We state this as an open problem for future research.

References

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