The Computational Complexity of Immanants

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THE COMPUTATIONAL COMPLEXITY OF IMMANANTS*

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Abstract. Permanents and determinants are special cases of immanants. The latter are polynomial matrix functions defined in terms of characters of symmetric groups and corresponding to Young diagrams. Valiant has proved that the evaluation of permanents is a complete problem in both the Turing machine model (#P-completeness) as well as in his algebraic model (VNP-completeness). We show that the evaluation of immanants corresponding to hook diagrams or rectangular diagrams of polynomially growing width is both #P-complete and VNP-complete.

Key words. permanents, immanants, computational complexity, algebraic completeness

AMS subject classifications. 15A15, 68Q40, 68Q15

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1. Introduction.

The *permanent* $\mathrm{per}(A)$ of an n by n matrix $A = [a_{i,j}]$ is defined by

$$\mathrm{per}(A) := \sum_{\pi} \prod_{i=1}^{n} a_{i,\pi(i)},$$

where the summation is over all permutations π in S_n. Note that in contrast to the determinant, each term has a positive sign.

From the viewpoint of computational complexity, the determinant and permanent have, in spite of the similarity in their definitions, very little in common. While there are efficient polynomial time algorithms for the evaluation of the determinant, the best-known algorithm for the evaluation of the permanent of an n by n matrix needs $O(n2^n)$ arithmetic operations (Ryser [20]). A hypothesis due to Valiant in fact claims that the permanent cannot be computed with a polynomial number of arithmetic operations. This hypothesis is supported by Valiant's famous result [23] stating that the problem to evaluate the permanent of a matrix with entries in $\{0, 1\}$ is #P-complete, as well as his analogous VNP-completeness result [22] in a framework of algebraic computations.

Both permanents and determinants are special cases of immanants introduced by Littlewood [16]. To define these polynomial matrix functions, we have to rely on some basic facts about the characters of the symmetric groups, which can be found for instance in the books by Boerner [2], James and Kerber [13], or Fulton and Harris [9].

It is known that the irreducible characters of the symmetric group S_n can be labeled by *partitions* of n, i.e., by decreasing sequences $\lambda = (\lambda_1, \ldots, \lambda_s)$ of natural numbers adding up to n. A partition will be identified with its (Young) *diagram* $\{(i, j) \mid 1 \le j \le \lambda_i\}$, which can be visualized as a left-justified arrangement of λ_i boxes in the ith row. (Compare Figure 3.1.) We call $|\lambda| := \sum_i \lambda_i$ the *size* and λ_1 the *width* of λ and will use the notation $\lambda \vdash n$ to express that λ is a partition or a diagram of size n. A diagram is called *rectangular* iff $\lambda_1 = \cdots = \lambda_s$.

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Let $\chi_\lambda\colon S_n \to \mathbb{Z}$ denote the irreducible character of the symmetric group S_n corresponding to the diagram $\lambda \vdash n$. The *immanant* of an n by n matrix $A = [a_{i,j}]$ corresponding to λ is defined by

$$\mathrm{im}_\lambda(A) := \sum_{\pi \in S_n} \chi_\lambda(\pi) \prod_{i=1}^{n} a_{i,\pi(i)}.$$

For the "horizontal" diagrams given by $\lambda = (n)$ we have $\chi_\lambda = 1$, and the corresponding immanants specialize to the permanents. In the case of the "vertical" diagrams given by $\lambda = (1, \dots, 1)$ we get $\chi_\lambda = \mathrm{sgn}$, and the immanants specialize to the determinants. These diagrams are special cases of those described by $(k, 1, \dots, 1)$, which are called *hook diagrams* because of their shape. By *hook immanants* or *rectangular immanants* we will understand the immanant polynomials corresponding to diagrams of the corresponding shape. The reader may find some algebraic properties of immanants in Merris [17, 18].

Our interest for immanants stems from the fact that they constitute a natural parameterized set of polynomials, which allows us to study the change of computational complexity from easy (polynomial time computable) to difficult (complete) as the diagram λ of size n varies between the vertical and the horizontal diagram. To make this more specific, think for instance of the set of hook diagrams $(k, 1, \dots, 1) \vdash n$ as the parameter k varies between 1 and n.

In the previous paper [3] we have developed a fast algorithm to evaluate representations of general linear groups. As a byproduct, we obtained an algorithm to evaluate the immanant im_λ at a matrix A with nonscalar cost proportional to $n^2 s_\lambda d_\lambda$, where s_λ and d_λ denote the number of standard tableaus and semistandard tableaus on the diagram of λ, respectively. This upper bound improves previous bounds due to Hartmann [11] and Barvinok [1].

In the present article we complement the upper bounds in [3] by completeness results for certain families of immanants. The results will be formulated in Valiant's algebraic P-NP theory, centering around the notion of VNP-completeness. The main features of this theory are recalled in section 2. Note that all VNP-completeness statements in this article refer to a ground field of characteristic zero.

A sequence $(\lambda^{(n)})$ of diagrams such that the size of $\lambda^{(n)}$ is polynomially bounded in n will be called a *p-sequence of diagrams*. If, additionally, the width of $\lambda^{(n)}$ is growing at least polynomially in the size of $\lambda^{(n)}$, then we call such a sequence to be of *polynomially growing width*. The corresponding families of immanant polynomials will also be called so.

We have the following conjecture.

Conjecture 1.1. *Any family of immanant polynomials of polynomially growing width is VNP-complete.*

The achievement of this paper is the proof of this conjecture in two special situations: for hook and rectangular immanants. The statement for hook immanants generalizes a result by Hartmann [11], while the claim for rectangular immanants answers an open problem posed by Strassen [21, Problem 14.2].

Theorem 1.2. *Any family of hook immanants or rectangular immanants of polynomially growing width is VNP-complete.*

For sequences of diagrams of bounded width we have so far no clear idea of the complexity of the corresponding immanants. We raise the following question.

Problem 1.3. *Is the family of rectangular immanants corresponding to rectangles of width* 2 *VNP-complete?*

We remark that families of hook immanants of bounded width are p-computable. This is an immediate consequence of the upper complexity bound in [3] mentioned before.

Our proofs also yield #P-completeness results for the problem to evaluate immanants at matrices A with entries in $\{0,1\}$. However, note that $\mathrm{im}_\chi(A)$ may be negative, with absolute value bounded by $n!\,n^n \leq n^{2n}$ for $A \in \{0,1\}^{n\times n}$. We will therefore interpret the evaluation problem below as the modified problem to compute $\mathrm{im}_\chi(A) + n^{2n}$ from $A \in \{0,1\}^{n\times n}$.

COROLLARY 1.4. *Assume in Theorem* 1.2 *that the sequence of diagrams is polynomial time computable. Then the problem to evaluate the corresponding immanant at a given matrix with entries in* $\{0,1\}$ *is* #P*-complete.*

The paper is organized as follows. In section 2 we recall the main features of Valiant's algebraic P-NP theory. The goal of section 3 is the proof of an auxiliary result (Lemma 3.1) which is crucial for our completeness proofs. It expresses values of characters corresponding to rectangular diagrams by characters of hook diagrams. Section 4 is devoted to the proof that families of immanants are indeed p-definable.

In section 5, we provide the proof of Theorem 1.2 in several steps. The general strategy is to identify alternating sums of immanants corresponding to smaller partitions as a projection of a given immanant (Lemma 5.1), by applying the Murnaghan–Nakayama rule for the characters of the symmetric group. In combination with Lemma 3.1 we exhibit alternating sums of hook immanants as projections of rectangular immanants. This is then combined with technical results, which allows to obtain permanents or Hamilton cycle polynomials as projections of linear combinations of hook immanants.

2. Valiant's algebraic model of NP-completeness. We briefly recall the main features of Valiant's algebraic P-NP theory. For detailed expositions we refer to the survey by von zur Gathen [10] and the books by Bürgisser, Clausen, and Shokrollahi [7, Chapter 21] and Bürgisser [6].

We will adopt the following useful convention: we denote matrix functions with small letters, but the corresponding function evaluated at a matrix with distinct indeterminate entries is written in capitals. For instance, $\mathrm{per}(A)$ is the permanent of the n by n matrix A, and

$$\mathrm{PER}_n = \sum_{\pi \in S_n} \prod_{i=1}^{n} X_{i,\pi(i)}$$

denotes the permanent of an n by n matrix with indeterminate entries $X_{i,j}$. Likewise, IM_λ denotes the immanant polynomial corresponding to the diagram $\lambda \vdash n$.

Throughout the paper, the discussion will refer to a fixed field k of characteristic zero. (The reader may assume $k = \mathbb{Q}$.) By a *p-family* we understand a sequence (f_n) of multivariate polynomials $f_n \in k[X_1, \ldots, X_{v(n)}]$ such that the number of variables $v(n)$ as well as the degree $\deg f_n$ are *p-bounded* functions of n, i.e., these functions are majorized by a polynomial in n. Interesting examples are the *permanent family* PER $= (\mathrm{PER}_n)$, the *determinant family* DET $= (\mathrm{DET}_n)$, and the family HC $= (\mathrm{HC}_n)$ of *Hamilton cycle polynomials* defined by

$$\mathrm{HC}_n = \sum_{\pi} \prod_{i=1}^{n} X_{i,\pi(i)},$$

where the sum is over all cycles $\pi \in S_n$ of length n. Note that the value of HC_n at the adjacency matrix of a digraph equals the number of its Hamilton cycles.

Let $L(f_n)$ denote the total *complexity* of $f_n \in k[X_1, \ldots, X_{v(n)}]$, that is, the minimum number of arithmetic operations $+, -, *$ to compute f_n from the variables X_i and constants in k by a straight-line program. We call a p-family *p-computable* iff $L(f_n)$ is p-bounded in n. The p-computable families constitute the complexity class VP.

A p-family (f_n) is called *p-definable* iff there exists a p-computable family (g_n), $g_n \in k[X_1, \ldots, X_{u(n)}]$, such that for all n

$$f_n(X_1, \ldots, X_{v(n)}) = \sum_{e \in \{0,1\}^{u(n)-v(n)}} g_n(X_1, \ldots, X_{v(n)}, e_{v(n)+1}, \ldots, e_{u(n)}).$$

The set of p-definable families form the complexity class VNP.

We will employ a very simple notion of reduction. A polynomial f_n is called a *projection* of a polynomial $g_m \in k[X_1, \ldots, X_u]$, written $f_n \leq g_m$, iff

$$f_n(X_1, \ldots, X_{v(n)}) = g_m(a_1, \ldots, a_u)$$

for some $a_i \in k \cup \{X_1, \ldots, X_{v(n)}\}$. That is, f_n can be derived from g_m through substitution by indeterminates and constants. A p-family (f_n) is called a *p-projection* of a family (g_m) iff there is a p-bounded function $t\colon \mathbb{N} \to \mathbb{N}$ such that f_n is a projection of $g_{t(n)}$ for all n. Finally, a p-definable family (g_m) is called *VNP-complete* iff any $(f_n) \in \mathrm{VNP}$ is a p-projection of (g_m).

In [22] Valiant proved that the p-families PER of permanents and HC of Hamilton cycles polynomials are VNP-complete (over fields k of characteristic different from two, which is a general assumption in this paper). Thus PER is not p-computable iff *Valiant's hypothesis* $\mathrm{VP} \neq \mathrm{VNP}$ is true.

One can prove that the "generating functions" corresponding to several NP-complete graph problems like cliques, graph factors, Hamilton cycles in planar graphs, etc. yield VNP-complete families as well (see [6, Chapter 3]). In fact, Valiant's hypothesis can be considered as a nonuniform algebraic counterpart of the well-known hypothesis $\mathrm{P} \neq \mathrm{NP}$ due to Cook [8]. For results relating these two hypotheses, see [4, 6]. We mention in passing that, by contrast with the classical P-NP theory, one knows specific p-definable families over finite fields, which are neither VNP-complete, nor p-computable, provided the polynomial hierarchy does not collapse (cf. [5, 6]).

For later use, we state some results going back to Valiant [24]; detailed proofs can be found in [6]. The first result shows that the complexity class VNP is closed under various natural operations.

PROPOSITION 2.1. *Let (f_n) and (g_n) be p-definable; say $f_n \in k[X_1, \ldots, X_{v(n)}]$. Then the following hold.*

(i) *Sum and product. $(f_n + g_n)$ and $(f_n \cdot g_n)$ are p-definable.*

(ii) *Substitution. $(f_n(g_1, \ldots, g_{v(n)}))$ is p-definable.*

(iii) *Coefficient. If $h_n \in k[X_{u(n)+1}, \ldots, X_{v(n)}]$ is the coefficient of some power product $X_1^{i_1} \cdots X_{u(n)}^{i_{u(n)}}$ in f_n for some $u(n) \leq v(n)$, then the family (h_n) is p-definable.*

The second result is a useful criterion for p-definability which connects the nonuniform counting complexity class #P/poly to the class VNP. Note that functions φ, which are computable in polynomial time on a Turing machine, are clearly contained in the class #P/poly. (For the definition of #P see [23]; a general definition of nonuniform complexity classes like #P/poly can be found in Karp and Lipton [15].)

PROPOSITION 2.2. *Suppose* $\varphi\colon \{0,1\}^* \to \mathbb{N}$ *is a function in the class* #P/poly. *Then the family* (f_n) *of polynomials defined by*

$$f_n = \sum_{e\in\{0,1\}^n} \varphi(e) X_1^{e_1}\cdots X_n^{e_n}$$

is p-definable.

3. Character formulas for the symmetric group. We first recall some character formulas for the symmetric group for later use. These formulas are then applied to derive Lemma 3.1, which is a crucial ingredient of our completeness proofs. More information on the characters of the symmetric groups can be found in the books by Boerner [2], James and Kerber [13], or Fulton and Harris [9].

We recall that $\lambda \vdash n$ means that $\lambda = (\lambda_1,\dots,\lambda_s)$ is a partition of n. The irreducible character of S_n corresponding to λ is denoted by χ_λ.

To a partition λ we may assign the strictly decreasing sequence

$$\ell = [\ell_1,\dots,\ell_s] := (\lambda_1,\dots,\lambda_s) + (s-1, s-2,\dots,1,0)$$

in $\mathbb{N}^s$ which satisfies $\sum \ell_i = n + \binom{s}{2}$. (We use square brackets to distinguish ℓ notationally from a partition λ.) It is useful to index the irreducible characters of S_n by such sequences, thus we set $\chi_\ell := \chi_\lambda$. We can extend this definition to any $\ell \in \mathbb{N}^s$ satisfying $\sum \ell_i = n + \binom{s}{2}$ by requiring the function $\ell \mapsto \chi_\ell$ to be alternating. In particular, χ_ℓ vanishes if two components of ℓ are equal. We also include the case $n = 0$ by setting $\chi_{[s-1,\dots,0]}(1) := 1$.

Conjugacy classes of permutations in S_n are described by their *cycle format* $(\rho_1,\dots,\rho_n)$, where ρ_i denotes the number of i-cycles. Clearly, $\sum_i i\rho_i = n$. It will be convenient to write cycle formats in frequency notation $\rho = 1^{\rho_1}\cdots n^{\rho_n}$, or shorter $\rho \models n$ in order to express that ρ is a cycle format of n. Moreover, we set $\chi_\ell(\rho) := \chi_\ell(\pi)$, where π is any permutation with cycle format ρ.

Let $u_{s,i} := Z_1^i + Z_2^i + \cdots + Z_s^i$ denote the ith elementary power sum in the indeterminates $Z_1,\dots,Z_s$, and let

$$\Delta_s := \det(Z_i^{s-j})_{1\le i,j\le s} = \prod_{i<j}(Z_i - Z_j)$$

be the discriminant. The characters of S_n are determined by the remarkable *formula of Frobenius* (cf. [9, 4.10, p. 49]):

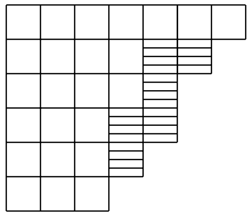
$$\Delta_s \cdot u_{s,1}^{\rho_1}\cdots u_{s,n}^{\rho_n} = \sum_\ell \chi_\ell(\rho) Z_1^{\ell_1}\cdots Z_s^{\ell_s}, \tag{3.1}$$

where the sum is over all $\ell \in \mathbb{N}^s$ satisfying $\sum \ell_i = n + \binom{s}{2}$. From this formula one easily deduces *Frobenius' recursion formula* for the characters of S_n (cf. [2, VI, section 3]): let $1 \le h \le n$ and $\rho \models n-h$. Then we have for all $\ell \in \mathbb{N}^s$ satisfying $\sum \ell_i = n + \binom{s}{2}$

$$\chi_\ell(\rho\cdot h) = \sum_i \chi_{[\ell_1,\dots,\ell_{i-1},\ell_i-h,\ell_{i+1},\dots,\ell_s]}(\rho), \tag{3.2}$$

where the sum is over all $1 \le i \le s$ such that $\ell_i \ge h$, and with the property that $\ell_1,\dots,\ell_{i-1},\ell_i - h,\ell_{i+1},\dots,\ell_s$ are pairwise distinct numbers.

Sometimes it is convenient to use a recursion formula related to (3.2), the so-called Murnaghan–Nakayama rule. We recall that a partition $\lambda \vdash n$ can be represented



$$\lambda = (7, 6, 5, 5, 4, 3)$$
$$\mu = (7, 4, 4, 3, 3, 3)$$
$$r(\lambda, \mu) = 3$$

FIG. 3.1. *A skew hook for the diagram of λ.*

by its *diagram* $\{(i, j) \mid 1 \leq j \leq \lambda_i\}$, which should be visualized as a left-justified arrangement of λ_i boxes in the ith row. A *skew hook* for λ is a connected region of boundary boxes for its diagram such that removing them leaves a diagram for another partition μ. We denote by $r(\lambda, \mu)$ the number of vertical steps in the skew hook, i.e., one less than the number of rows in the hook. For illustration see Figure 3.1.

The *Murnaghan–Nakayama* rule now reads as follows: For $\lambda \vdash n$, $1 \leq h \leq n$, and $\rho \models n - h$, we have

$$\chi_\lambda(\rho \cdot h) = \sum_\mu (-1)^{r(\lambda,\mu)} \chi_\mu(\rho), \tag{3.3}$$

where the sum is over all partitions $\mu \vdash n - h$ that can be obtained from λ by removing a skew hook containing h boxes (cf. [13, 2.4.7, p. 60] or [9, p. 59]).

As an example, let us compute the value of χ_λ on an n-cycle. The Murnaghan–Nakayama rule implies immediately that $\chi_\lambda(n^1) = (-1)^i$ if λ equals a hook partition $(n - i, 1, \ldots, 1)$, and that $\chi_\lambda(n^1) = 0$ otherwise.

For the rest of the paper, we will denote the characters corresponding to the hook partitions $(n - i, 1, \ldots, 1) \vdash n$ by $\chi_{n,i}$ and call them *hook characters*. The corresponding *hook immanant* polynomials will be denoted by $\mathrm{HI}_{n,i}$ for $0 \leq i < n$. Note that $\mathrm{HI}_{n,i}$ corresponds to a diagram of width $n - i$. For instance we have $\mathrm{HI}_{n,0} = \mathrm{PER}_n$ and $\mathrm{HI}_{n,n-1} = \mathrm{DET}_n$.

Now assume $\rho \models n$, $\rho \neq n^1$. The orthogonality relations (cf. [9, I, section 2.2]) and the above observation on the value of χ_λ on an n-cycle imply that

$$\sum_{\lambda \vdash n} \chi_\lambda(n^1)\chi_\lambda(\rho) = \sum_{i=0}^{n-1} (-1)^i \chi_{n,i}(\rho) = 0.$$

From this one easily concludes the following formula due to Merris [19]:

$$\sum_{i=0}^{n-1} (-1)^i \, \mathrm{HI}_{n,i} = n\mathrm{HC}_n. \tag{3.4}$$

We illustrate Frobenius' recursion formula (3.2) by computing the value $\chi_{(2,2,2,2)}$ for permutations in S_8 of cycle format 2^4. To the rectangular partition $(2, 2, 2, 2)$ there corresponds the sequence $\ell = [5, 4, 3, 2]$. The recursive application of (3.2) can be illustrated by the tree $T_{2,2}$ in Figure 3.2. This tree has depth 4 and its nodes carry a label $\ell \in \mathbb{N}^4$ having distinct components. The meaning of the tree $T_{2,2}$ is the following: Consider a node with label ℓ at level $1 \leq t \leq 4$, and let $\ell^{(1)}, \ldots, \ell^{(M)}$ be the labels of the sons of this node. Then we have by (3.2) that

$$\chi_\ell(2^t) = \sum_{i=1}^{M} \chi_{\ell^{(i)}}(2^{t-1}).$$

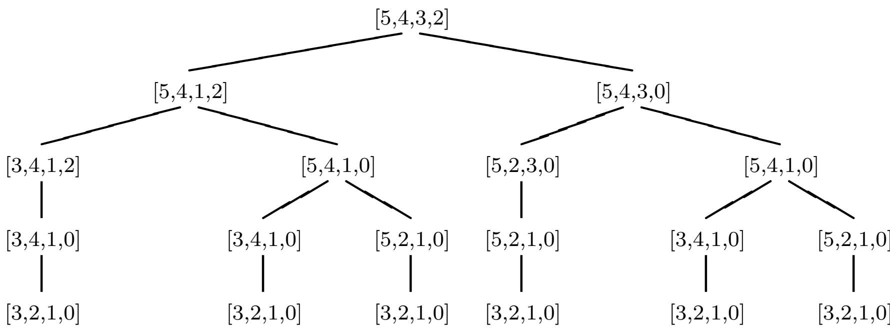


FIG. 3.2. *The tree $T_{2,2}$ illustrating the recursive application of Frobenius' recursion formula.*

From this it follows that $\chi_{[5,4,3,2]}(2^4)$ is the sum of $\chi_{[3,2,1,0]}(1)$ over all leaves of the tree. Hence $\chi_{[5,4,3,2]}(2^4) = 6\chi_{[3,2,1,0]}(1) = 6$.

To simplify notation, we will write χ_{m^s} for the character corresponding to a rectangular partition $(m, \ldots, m) \vdash sm$, and IM_{m^s} for the corresponding *rectangular immanant* polynomial.

The technical lemma below generalizes the above computational example and expresses certain values of χ_{m^s} by hook characters. The lemma will be crucial in our completeness proof in section 5.

LEMMA 3.1. *Let $s = qm + r$, $1 \leq r \leq m$, $q \geq 0$. Then we have for all cycle formats $\rho \models m$, $\rho \neq m^1$, that*

$$\chi_{m^s}(m^{s-1} \cdot \rho) = \gamma_{m,s} \sum_{i=0}^{r-1} (-1)^i \chi_{m,i}(\rho)$$

and

$$\chi_{m^s}(m^s) = r\gamma_{m,s} + m\beta_{m,s},$$

where

$$\gamma_{m,s} = \frac{(s-1)!}{q!^{m-r}(q+1)!^r}, \quad \beta_{m,s} = q\gamma_{m,s}.$$

Proof. As in the above example (Figure 3.2), we can describe the recursive application of Frobenius' recursion formula (3.2) for computing $\chi_{m^s}(m^s)$ by a labeled tree $T_{m,s}$. This tree is built up as follows. The root carries the label

$$\ell^{(0)} := [m+s-1, m+s-2, \ldots, m] = (m, m, \ldots, m) + (s-1, s-2, \ldots, 0).$$

To an already constructed node with label ℓ, we create a son for every $\ell - me_k$ which has nonnegative and distinct components. Here $e_k \in \{0,1\}^s$ denotes the canonical basis vector having a 1 at position $1 \leq k \leq s$.

We will soon prove that all leaves of $T_{m,s}$ carry the label $[s-1, \ldots, 1, 0]$. Using this already, we see that the nodes one level above the leaves carry a label of the form $L_k := [s-1, \ldots, 1, 0] + me_k$ for $1 \leq k \leq m$. Let α_k denote the number of these nodes. By a repeated application of the recursion formula (3.2), we obtain for all $\rho \models m$

$$\chi_{m^s}(m^{s-1} \cdot \rho) = \sum_{k=1}^{m} \alpha_k \chi_{L_k}(\rho).$$

A k-cycle transforms the sequence $(m-k+1,1,\dots,1,0,\dots,0)+(s-1,\dots,1,0)$ into L_k, and thus χ_{L_k} equals up to a sign a hook character: $\chi_{L_k}=(-1)^{k-1}\chi_{m,k-1}$. Now the point is that

$$\alpha_1=\cdots=\alpha_r,\ \alpha_{r+1}=\cdots=\alpha_m. \tag{3.5}$$

Assuming this for the moment, we may conclude for $\rho\models m$, $\rho\neq m^1$, that

$$\begin{aligned}\chi_{m^s}(m^{s-1}\cdot\rho)&=\alpha_1\sum_{i=0}^{r-1}(-1)^i\chi_{m,i}(\rho)+\alpha_{r+1}\sum_{i=r}^{m-1}(-1)^i\chi_{m,i}(\rho)\\&=(\alpha_1-\alpha_{r+1})\sum_{i=0}^{r-1}(-1)^i\chi_{m,i}(\rho),\end{aligned}$$

where we have used formula (3.4) in the last equality. We also get (recall that $\chi_{m,i}(m^1)=(-1)^i$)

$$\chi_{m^s}(m^s)=r\alpha_1+(m-r)\alpha_{r+1}=r(\alpha_1-\alpha_{r+1})+m\alpha_{r+1}.$$

So it remains to show (3.5) and to check that indeed $\alpha_1-\alpha_{r+1}=\gamma_{m,s}$ and $\alpha_{r+1}=q\gamma_{m,s}$.

To a leaf of $T_{m,s}$ there corresponds bijectively the path from the root to this leaf, which can be uniquely described by the sequence of labels $(\ell^{(0)},\dots,\ell^{(s)})$ of the nodes along this path. By assigning to $\ell^{(i)}$ the set $A_i\subseteq B:=\{0,1,\dots,m+s-1\}$ of its components, we get a sequence $(A_0,\dots,A_s)$ of subsets of B all having cardinality s, and which satisfy

$$A_{i+1}=(A_i\setminus\{a_i\})\cup\{a_i-m\} \tag{3.6}$$

with some $a_i\in A_i$ such that $a_i\geq m$ and $a_i-m\notin A_i$. It is easy to see that this correspondence is in fact a bijection between the paths of $T_{m,s}$ from the root to a leaf and the sequences $(A_0,\dots,A_s)$ of subsets of B satisfying (3.6) and such that $A_0=\{m,m+1,\dots,m+s-1\}$.

Now consider the complements $\overline{A_i}:=B\setminus A_i$. They are all of cardinality m. By induction on i one shows that the remainders modulo m of the elements of $\overline{A_i}$ are pairwise distinct. Hence we may write

$$\overline{A_i}=\{p_1^{(i)}m,p_2^{(i)}m+1,\dots,p_m^{(i)}m+m-1\} \tag{3.7}$$

with a uniquely determined vector $p^{(i)}=(p_1^{(i)},\dots,p_m^{(i)})$ contained in

$$W:=\{0,1,\dots,q+1\}^r\times\{0,1,\dots,q\}^{m-r}.$$

Let $a_i\in A_i$ be as in (3.6). As $a_i-m\notin A_i$, we have $a_i-m=p_{\mu_i}^{(i)}m+\mu_i-1$ for some $1\leq\mu_i\leq m$. On the other hand, $a_i=(p_{\mu_i}^{(i)}+1)m+\mu_i-1$ is by construction not contained in A_{i+1}. This implies that

$$p^{(i+1)}=p^{(i)}+e'_{\mu_i}, \tag{3.8}$$

where $e'_{\mu_i}\in\{0,1\}^m$ is the canonical basis vector having a 1 at position μ_i. We can thus regard $(p^{(0)},\dots,p^{(s)})$ as a walk in W which starts in $p^{(0)}=\underline{0}:=(0,\dots,0)$ and must end in the opposite corner $p^{(s)} = w := (q+1,\dots,q+1,q,\dots,q)$. In this walk, a successor of a point is obtained by incrementing exactly one coordinate by 1.

It is now straightforward to check that we have found a bijection between the leaves of $T_{m,s}$ and the above-described walks in W from $\underline{0}$ to the opposite corner w. In particular, all leaves of $T_{m,s}$ carry the same label $[s-1,\dots,1,0]$ corresponding to w, as was claimed at the beginning of the proof.

In what follows we will assume that $q \geq 1$. (The case $q = 0$ can be checked separately.) A node N of $T_{m,s}$ one level above the leaves corresponds to a walk in W ending in one of the points $w - e'_M$, where $1 \leq M \leq m$. To such a walk in turn there corresponds a sequence of sets $(A_0,\dots,A_{s-1})$. Assume first that $1 \leq M \leq r$. Then it is easily checked that

$$A_{s-1} = (\{0,1,\dots,s-1\} \setminus \{qm+M-1\}) \cup \{(q+1)m+M-1\}.$$

By comparing this with the set of components of L_k, we obtain $qm+M-1 = s-k$, and hence $k = r-M+1$. We conclude that the node N carries the label L_{r-M+1}. The number of nodes of level $s-1$ carrying the label L_{r-M+1} equals the number of walks in W from $\underline{0}$ to $w-e'_M$. Such a walk is uniquely described (cf. (3.8)) by a sequence $(M_0, M_1, \dots, M_{s-2})$ in $\{1,2,\dots,m\}^{s-1}$, in which $\mu \in \{M\} \cup \{r+1,\dots,m\}$ occurs with frequency q, and $\mu \in \{1,\dots,r\} \setminus \{M\}$ occurs with frequency $q+1$. The number of these sequences equals the multinomial coefficient

$$\alpha := \frac{(s-1)!}{q!^{m-r+1}(q+1)!^{r-1}}.$$

This proves that $\alpha_1 = \cdots = \alpha_r = \alpha$.

In the case $r < M \leq m$ one can show similarly that N carries the label $L_{r-M+m+1}$ and that the number of nodes carrying this label equals

$$\beta := \frac{(s-1)!}{(q-1)!\,q!^{m-r-1}(q+1)!^{r}},$$

which yields $\alpha_{r+1} = \cdots = \alpha_m = \beta$. A straightforward calculation shows that indeed $\alpha - \beta = \gamma_{m,s}$ and $\beta = q\gamma_{m,s}$, where $\gamma_{m,s} = (s-1)!/(q!^{m-r}(q+1)!^r)$. ∎

4. P-definability of immanants. Frobenius' formula (3.1) for the generating function of S_n-characters allows us to express immanants as coefficients of p-computable families of polynomials. We will use this observation to prove the following proposition.

PROPOSITION 4.1. *Any family of immanant polynomials corresponding to a p-sequence of diagrams is p-definable.*

Proof. Let the *cycle format polynomial* of ρ be defined as $\mathrm{CF}_\rho := \sum_\pi \prod_{i=1}^n X_{i,\pi(i)}$, where the sum is over all permutations π having cycle format $\rho \models n$.

We multiply Frobenius' formula (3.1) for $s = n$ with CF_ρ and take the sum over all cycle formats $\rho \models n$. This yields

$$F_n := \Delta_n \sum_{\rho \models n} u_{n,1}^{\rho_1} \cdots u_{n,n}^{\rho_n}\, \mathrm{CF}_\rho = \sum_{\ell} \left(\sum_{\rho \models n} \chi_\ell(\rho)\, \mathrm{CF}_\rho \right) Z_1^{\ell_1} \cdots Z_n^{\ell_n}.$$

The immanent $\mathrm{IM}_\lambda = \sum_\rho \chi_\ell(\rho)\mathrm{CF}_\rho$ appears in F_n as the coefficient of the power product $\prod_j Z_j^{\ell_j} = \prod_j Z_j^{\lambda_j+n-j}$. By Proposition 2.1(iii) it is therefore sufficient to prove that (F_n) is p-definable. Let $T_{i,j}$ be further indeterminates for $1 \le i,j \le n$, and define

$$G_n := \sum_{\rho \models n} \mathrm{CF}_\rho \prod_{(i,j): j \le \rho_i} T_{i,j}.$$

By substituting $T_{i,j}$ by $u_{n,i} = Z_1^i + \cdots + Z_n^i$ in G_n and multiplying the resulting polynomial with $\Delta_n = \prod_{i<j}(Z_i - Z_j)$, we obtain F_n. By Proposition 2.1(i)–(ii) it is therefore enough to show that the family (G_n) is p-definable. If we denote by $\rho_i(\pi)$ the number of i-cycles of π, we may write

$$\begin{aligned} G_n &= \sum_{\pi \in S_n} \left(\prod_{\alpha \le n} X_{\alpha,\pi(\alpha)} \right) \left(\prod_{(i,j): j \le \rho_i(\pi)} T_{i,j} \right) \\ &= \sum_{e,\epsilon \in \{0,1\}^{n\times n}} \varphi_n(e,\epsilon) \prod_{1 \le \alpha,\beta \le n} X_{\alpha,\beta}^{e_{\alpha,\beta}} \prod_{1 \le i,j \le n} T_{i,j}^{\epsilon_{i,j}} \end{aligned}$$

with a uniquely determined function

$$\varphi_n \colon \{0,1\}^{n\times n} \times \{0,1\}^{n\times n} \to \{0,1\}.$$

It is obvious that the extension of all φ_n to a function $\varphi\colon \{0,1\}^* \to \{0,1\}$ is computable by a polynomial time Turing machine. Therefore, Proposition 2.2 implies that (G_n) is p-definable. $\square$

We can alternatively deduce Proposition 4.1 from the following interesting result due to Hepler [12], which shows that computing characters of the symmetric groups is a #P-complete problem. This result should be contrasted with Proposition 7.4 of [3], which shows that the evaluation of characters of GL_n is possible in polynomial time.

THEOREM 4.2. *The function which assigns to $\lambda \vdash n$, $\rho \models n$ the value $\chi_\lambda(\rho) + n^n$ is* #P*-complete (n given in unary).*

In fact, to deduce Proposition 4.1, we need only the easy part of this theorem stating that the above function is contained in #P. Let a p-sequence of diagrams $(\lambda^{(n)})$ be given. We can interpret $\lambda^{(n)}$ as a "polynomial advice" for n in the sense of Karp and Lipton [15]. By Theorem 4.2, the function which assigns to the description π of the power product $\prod_{i=1}^n X_{i,\pi(i)}$ its coefficient in the polynomial $\mathrm{IM}_{\lambda^{(n)}} + n^n \mathrm{PER}_n$ is in the class #P/poly. Therefore, by Proposition 2.2, the family $(\mathrm{IM}_{\lambda^{(n)}} + n^n\mathrm{PER}_n)_n$ is p-definable. This again implies Proposition 4.1.

5. Completeness proofs. This section is divided into five parts. We first describe the general strategy of our completeness proofs. Using this strategy and applying Lemma 3.1, we then provide a short proof of a special case of Theorem 1.2, namely for rectangular diagrams satisfying a certain divisibility condition.

In order to settle the general case of Theorem 1.2, we develop in subsection 5.3 technical results which allow us to obtain permanents or Hamilton cycle polynomials as projections of linear combinations of hook immanants. These technical results in combination with the general strategy then allow us to prove Theorem 1.2 for hook immanants. Finally, we are able to complete the proof of Theorem 1.2 for rectangular immanants.

5.1. The general proof strategy. We introduce some notation and work out the relationship between matrices and weighted digraphs.

There is a canonical bijective correspondence between square matrices and (edge) weighted digraphs. Namely, we can regard an n by n matrix $A = [a_{i,j}]$ over some field as the adjacency matrix of a weighted digraph G on n nodes, where $a_{i,j} \neq 0$ gives the weight of the edge from node i to node j, and there is no such edge if $a_{i,j} = 0$. We define the weight $\mathrm{wt}(\mathcal{C})$ of a subgraph $\mathcal{C}$ of G as the product of the weights of all its edges. The permutations $\pi \in S_n$ with $\prod_{i=1}^{n} a_{i,\pi(i)} \neq 0$ correspond bijectively to the *cycle covers* $\mathcal{C}$ of G, i.e., n-node subgraphs of G consisting of node-disjoint cycles. Note that $\mathrm{wt}(\mathcal{C}) = \prod_i a_{i,\pi(i)}$ if $\mathcal{C}$ corresponds to π. A cycle cover $\mathcal{C}$ has a (cycle) format $\rho \models n$, where ρ_i counts the number of i-cycles of $\mathcal{C}$.

We define now the *cycle format value* $\mathrm{cf}_\rho(G)$ of a weighted digraph G as the sum of the weights of all cycle covers of G having the format ρ. A special case of this is the *Hamilton cycle value* $\mathrm{hc}(G) := \mathrm{cf}_n(G)$. (Recall our convention on notations at the beginning of section 2.) The *immanant* $\mathrm{im}_\lambda(G)$ of the weighted digraph G is defined as

$$\mathrm{im}_\lambda(G) := \sum_{\rho \models n} \chi_\lambda(\rho) \mathrm{cf}_\rho(G).$$

If all the weights of G are either rationals, constants, or indeterminates, then $\mathrm{im}_\lambda(G)$ is obviously a projection of IM_λ. Moreover, if DK_n denotes the complete weighted digraph DK_n on n nodes with edge (i,j) carrying the indeterminate weight $X_{i,j}$, then $\mathrm{im}_\lambda(\mathrm{DK}_n) = \mathrm{IM}_\lambda$. Similarly, we define the *cycle format polynomial* by setting $\mathrm{CF}_\rho := \mathrm{cf}_\rho(\mathrm{DK}_n)$. Finally, we will denote the value of the *hook immanant* $\mathrm{HI}_{n,i}$ on G by $\mathrm{hi}_{n,i}(G)$.

We explain now our general strategy for proving completeness for a family of immanants. Let $1 \leq h \leq n$, and let G be the disjoint union of the complete weighted digraph DK_{n-h} on $n-h$ nodes with a directed cycle Z of length h, all of whose edges carry the weight 1. Every cycle cover $\mathcal{C}$ of G consists of a cycle cover $\mathcal{C}'$ of DK_{n-h} and the h-cycle Z, and we have $\mathrm{wt}(\mathcal{C}) = \mathrm{wt}(\mathcal{C}')$. Therefore, we obtain for $\rho \models n-h$

$$\mathrm{cf}_{\rho \cdot h^1}(G) = \mathrm{cf}_\rho(DK_{n-h}) = \mathrm{CF}_\rho,$$

and all other cycle format values of G vanish. From this and the Murnaghan–Nakayama rule (3.3), we obtain

$$\begin{aligned} \mathrm{im}_\lambda(G) &= \sum_{\rho \models n-h} \chi_\lambda(\rho \cdot h^1)\, \mathrm{CF}_\rho = \sum_{\rho \models n-h} \sum_{\mu} (-1)^{r(\lambda,\mu)} \chi_\mu(\rho)\, \mathrm{CF}_\rho \\ &= \sum_{\mu} (-1)^{r(\lambda,\mu)} \sum_{\rho \models n-h} \chi_\mu(\rho)\, \mathrm{CF}_\rho = \sum_{\mu} (-1)^{r(\lambda,\mu)} \mathrm{IM}_\mu. \end{aligned}$$

Let us summarize this important insight in slightly more general form.

LEMMA 5.1. *Let $\lambda^{(1)}, \ldots, \lambda^{(t)} \vdash n$, $\alpha_1, \ldots, \alpha_t \in \mathbb{Q}$, and $1 \leq h \leq n$. Then the linear combination of immanants $\sum_{i=1}^{t} \alpha_i \mathrm{IM}_{\lambda^{(i)}}$ has as a projection the linear combination of immanants*

$$\sum_{i=1}^{t} \alpha_i \sum_{\mu^{(i)}} (-1)^{r(\lambda^{(i)}, \mu^{(i)})} \mathrm{IM}_{\mu^{(i)}},$$

where $\mu^{(i)}$ runs over all partitions $\mu^{(i)} \vdash n-h$ which can be obtained from $\lambda^{(i)}$ by removing a skew hook containing h boxes.

5.2. Completeness of particular rectangular immanants. We present here the proof of a special case of Theorem 1.2, namely for rectangular immanants satisfying a certain divisibility condition.

A rectangular diagram $(m,\dots,m)\vdash sm$ is said to be of *height* s.

PROPOSITION 5.2. *Take a sequence of rectangular diagrams $(\lambda^{(m)})$ of polynomially growing width such that the width of $\lambda^{(m)}$ is a divisor of the height of $\lambda^{(m)}$ for all m. Then the corresponding family of rectangular immanants is VNP-complete.*

Proof. By Proposition 4.1 we know already that the given family is p-definable, so it suffices to show that the family of Hamilton cycle polynomials is a p-projection of the given family.

Let us write $\lambda^{(m)}=(m,\dots,m)\vdash s_m m$ and $s_m=q_m m+r_m$ with $1\le r_m\le m$, $q_m\ge 0$. The height s_m is p-bounded in m since $(\lambda^{(m)})$ is a p-sequence of diagrams. We will use the divisibility assumption $r_m=m$ only at the end of the proof, in order to make the following reasonings reusable.

Let G_m be the disjoint union of the complete weighted digraph DK_m with s_m-1 directed cycles of length m, all of whose edges having weight 1. By applying our general strategy explained in section 5.1, we obtain that

$$f_m:=\mathrm{im}_{m^{s_m}}(G)=\sum_{\rho\models m}\chi_{m^{s_m}}(m^{s_m-1}\cdot\rho)\,\mathrm{CF}_\rho$$

is a projection of $\mathrm{IM}_{m^{s_m}}$. Lemma 3.1 implies that

$$\begin{aligned}f_m&=(r_m\gamma_{m,s_m}+m\beta_{m,s_m})\mathrm{CF}_m+\sum_{\substack{\rho\models m\\ \rho\neq m}}\gamma_{m,s_m}\sum_{i=0}^{r_m-1}(-1)^i\chi_{m,i}(\rho)\,\mathrm{CF}_\rho\\ &=m\beta_{m,s_m}\mathrm{CF}_m+\sum_{\rho\models m}\gamma_{m,s_m}\sum_{i=0}^{r_m-1}(-1)^i\chi_{m,i}(\rho)\,\mathrm{CF}_\rho.\end{aligned}$$

Therefore, we get

$$f_m=m\beta_{m,s_m}\mathrm{HC}_m+\gamma_{m,s_m}\sum_{i=0}^{r_m-1}(-1)^i\,\mathrm{HI}_{m,i}. \tag{5.1}$$

Since we assume that $r_m=m$, we conclude with formula (3.4) that f_m is a nonzero scalar multiple of HC_m. This shows that HC is a p-projection of $\mathrm{IM}_{m^{s_m}}$ and proves our proposition. ∎

5.3. Projections of linear combinations of hook immanants. In order to settle the general case of Theorem 1.2, we have to develop some technical results which allow us to obtain permanents or Hamilton cycle polynomials as projections of linear combinations of hook immanants.

The following lemma is proved similarly to Theorem 2 in Hartmann [11].

LEMMA 5.3.

(i) *Let $0\le r<n$ and $\alpha_0,\dots,\alpha_r,\beta\in\mathbb{Q}$. Put $p:=\lfloor n/(r+1)\rfloor$ and write $\gamma:=\sum_{i=0}^r(-1)^i\alpha_i$. Then $\beta\mathrm{HC}_p+\gamma\mathrm{PER}_p$ is a projection of $\beta\mathrm{HC}_n+\sum_{i=0}^r\alpha_i\mathrm{HI}_{n,i}$.*

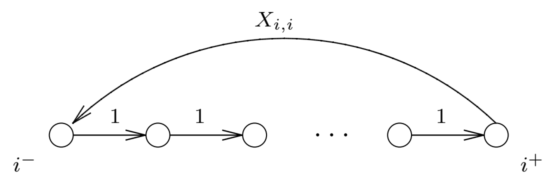
(ii) *Let $1\le r<n$ and $\alpha_0,\alpha_r,\dots,\alpha_{n-1},\beta\in\mathbb{Q}$. Put*

$$p:=\left\lfloor\frac{n}{n-r+\delta}\right\rfloor,\ R:=n-p(n-r+\delta),\ \gamma:=(-1)^{n+R-1}\sum_{i=r}^{n-1}(-1)^i\alpha_i,$$

where $\delta \in \{0,1\}$, $\delta \equiv n-r+1 \bmod 2$. *Then* $\alpha_0\mathrm{PER}_p+\beta\mathrm{HC}_p+\gamma\mathrm{DET}_p$ *is a projection of* $\alpha_0\mathrm{PER}_n+\beta\mathrm{HC}_n+\sum_{i=r}^{n-1}\alpha_i\mathrm{HI}_{n,i}$.

Proof. We note first that by the Murnaghan–Nakayama rule we have for the hook characters $\chi_{n,i}$ that $\chi_{n,i}(\rho) = (-1)^i$ if the cycle format ρ does not contain cycles of length at most i. This observation is crucial for the proof.

(i) For each $i \in \{2,\dots,p\}$ we introduce a cycle of length $r+1$ with edge weights as indicated in the following figure.



Analogously, we introduce a cycle of length $n-(p-1)(r+1)$ for $i=1$. Moreover, we connect the node i^+ with the node j^- by a (directed) edge of weight $X_{i,j}$ for all distinct $i,j \in \{1,2,\dots,p\}$. The resulting weighted digraph G has n nodes and girth $\geq r+1$; that is, all cycles of G have length at least $r+1$. A cycle cover $\tilde{\mathcal{C}}$ of G corresponds bijectively to a cycle cover $\mathcal{C}$ of the complete weighted digraph DK_p. Moreover, $\mathrm{wt}(\tilde{\mathcal{C}}) = \mathrm{wt}(\mathcal{C})$, and $\tilde{\mathcal{C}}$ is a Hamilton cycle iff $\mathcal{C}$ is so.

By the observation at the beginning of the proof we have $\chi_{n,i}(\tilde{\mathcal{C}}) = (-1)^i$ for all $0 \leq i \leq r$ and cycle covers $\tilde{\mathcal{C}}$ of G (which we identify with permutations in S_n). We get therefore for $0 \leq i \leq r$

$$\mathrm{hi}_{n,i}(G) = \sum_{\mathcal{C}} \chi_{n,i}(\tilde{\mathcal{C}})\mathrm{wt}(\tilde{\mathcal{C}}) = (-1)^i \sum_{\mathcal{C}} \mathrm{wt}(\mathcal{C}) = (-1)^i\,\mathrm{PER}_p,$$

and $\mathrm{hc}(G) = \mathrm{HC}_p$; hence

$$\left(\beta\,\mathrm{hc} + \sum_{i=0}^{r} \alpha_i \mathrm{hi}_{n,i}\right)(G) = \beta\,\mathrm{HC}_p + \gamma\mathrm{PER}_p.$$

This proves statement (i).

(ii) We assume that $n-r$ is odd; thus $\delta = 0$. (In the case where $n-r$ is even one can argue analogously.) As in the proof of (i) we construct a weighted digraph G by introducing for each $i \in \{2,\dots,p\}$ a cycle of length $n-r$, and for $i=1$ a cycle of length $n-(p-1)(n-r) = n-r+R$. Again, a cycle cover $\tilde{\mathcal{C}}$ of G corresponds bijectively to a cycle cover $\mathcal{C}$ of DK_p. Note that an ℓ-cycle of DK_p which does not pass through 1 is assigned to a cycle of G having length $\ell(n-r)$, and that $\ell \equiv \ell(n-r) \bmod 2$. To an ℓ-cycle of DK_p passing through 1 there corresponds a cycle of DK_p having length $\ell(n-r)+R$, which is congruent to $\ell+R$ modulo 2. From this we see that for any cycle cover $\mathcal{C}$ of DK_p

$$\mathrm{sgn}(\tilde{\mathcal{C}}) = (-1)^R\mathrm{sgn}(\mathcal{C}).$$

By interchanging rows and columns in the hook diagram $(n-i,1,\dots,1)$ we get the diagram of $(i+1,1,\dots,1)$. Therefore, the corresponding characters are conjugated:

$$\chi_{n,i}(\tilde{\mathcal{C}}) = \mathrm{sgn}(\tilde{\mathcal{C}})\chi_{n,n-i-1}(\tilde{\mathcal{C}})$$

(cf. [9, Ex. 4.4, p. 47]). On the other hand, we have for all $r \leq i < n$ and cycle covers $\tilde{\mathcal{C}}$ that $\chi_{n,n-i-1}(\tilde{\mathcal{C}}) = (-1)^{n-i-1}$, since $\mathrm{girth}(G) \geq n-r$. From these observations we conclude that for all $r \le i < n$

$$
\begin{aligned}
\mathrm{hi}_{n,i}(G) &= \sum_{\mathcal{C}} \chi_{n,i}(\tilde{\mathcal{C}})\mathrm{wt}(\tilde{\mathcal{C}}) \\
&= \sum_{\mathcal{C}} (-1)^R \mathrm{sgn}(\mathcal{C}) \chi_{n,n-i-1}(\tilde{\mathcal{C}})\mathrm{wt}(\mathcal{C}) \\
&= (-1)^{n+R-1}(-1)^i \sum_{\mathcal{C}} \mathrm{sgn}(\mathcal{C})\mathrm{wt}(\mathcal{C}) \\
&= (-1)^{n+R-1}(-1)^i\, \mathrm{DET}_p.
\end{aligned}
$$

Taking into account that $\mathrm{hc}(G) = \mathrm{HC}_p$ and $\mathrm{per}(G) = \mathrm{PER}_p$, the claim follows now readily. ∎

LEMMA 5.4. *Let $\alpha, \beta \in \mathbb{Q}$, $\alpha \ne 0$. Then the following hold.*

(i) PER_{n-1} *is a projection of* $\alpha\mathrm{PER}_n + \beta\mathrm{DET}_n$.

(ii) HC_{n-2} *is a projection of* $\alpha\mathrm{HC}_n + \beta\mathrm{DET}_n$.

Proof. (i) We concatenate the matrix $[X_{i,j}]_{1\le i,j<n}$ of indeterminates with the column $[0,\dots,0,(2\alpha)^{-1}]^T$ and repeat in the resulting matrix the last row. In this way we get an n by n matrix M having determinant zero. By expanding along the last column we get $\mathrm{per}(M) = 2(2\alpha)^{-1}\mathrm{per}([X_{i,j}]_{i,j<n}) = \alpha^{-1}\mathrm{PER}_{n-1}$. Hence $(\alpha\,\mathrm{per} + \beta\,\det)(M) = \mathrm{PER}_{n-1}$.

(ii) Consider the n by n matrix

$$
M := \begin{bmatrix}
0 & 0 & 1 & X_{1,2} & \dots & X_{1,n} \\
\alpha^{-1} & 0 & 0 & 0 & \dots & 0 \\
0 & 0 & 1 & X_{1,2} & \dots & X_{1,n} \\
0 & X_{2,1} & 0 & X_{2,2} & \dots & X_{2,n} \\
\vdots & \vdots & \vdots & \vdots & & \vdots \\
0 & X_{n,1} & 0 & X_{n,2} & \dots & X_{n,n}
\end{bmatrix}.
$$

As the first and third row of M coincide, we have $\det(M) = 0$. We claim that $\mathrm{hc}(M) = \alpha^{-1}\mathrm{HC}_{n-2}$. This is best seen by considering the weighted digraph with adjacency matrix M which is built up as follows from DK_{n-2}. Assume that $\{1,2,\dots,n-2\}$ is the set of nodes of DK_{n-2}. We delete the loop $(1,1)$ and split the node 1 into nodes 1^- and 1^+. The edges $(i,1)$ of DK_{n-2} are thus replaced by edges $(i,1^-)$, and the edges $(1,i)$ of DK_{n-2} are replaced by edges $(1^+,i)$ for $2 \le i \le n$. Now we introduce a new node O, an edge $(1^-,O)$ of weight α^{-1}, an edge $(O,1^+)$ of weight 1, a loop $(1^+,1^+)$ of weight 1, and edges (O,i) of weight $X_{1,i}$ for $2 \le i \le n$. The reader may easily verify that the resulting weighted digraph G has indeed the adjacency matrix M. On the other hand, it is clear from the construction that every Hamilton cycle of G passes through the edges $(1^-,O)$ and $(O,1^+)$. This shows that indeed $\mathrm{hc}(G) = \alpha^{-1}\mathrm{hc}(\mathrm{DK}_{n-2}) = \alpha^{-1}\mathrm{HC}_{n-2}$. Altogether, we have $(\alpha\,\mathrm{hc}+\beta\,\det)(G) = \mathrm{HC}_{n-2}$, which proves the claim. ∎

We call a p-family (f_n) *monotone* iff f_n is a projection of f_{n+1} for all n. The above lemma in particular shows that PER is monotone. Also, HC is monotone, which can be demonstrated similarly as part (ii) of the above lemma.

5.4. Completeness of hook immanants. The technical results of the previous section in combination with our general strategy allow us to provide the proof of Theorem 1.2 for hook immanants, stated explicitly below.

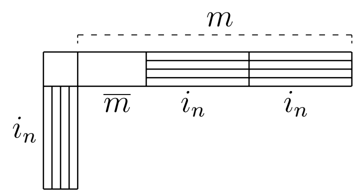


FIG. 5.1. *The hook diagram* $(n - i_n, 1, \ldots, 1)$ *when* $q = 2$.

PROPOSITION 5.5. *Any family of hook immanants of polynomially growing width is VNP-complete.*

Proof. The following notation will be useful: we write $\varphi(n) \gtrsim \psi(n)$ for functions $\varphi, \psi \colon \mathbb{N} \to (0, \infty)$ iff $\liminf_{n\to\infty} \varphi(n)/\psi(n) \geq 1$.

Let $\epsilon > 0$, and let (i_n) be a sequence of natural numbers satisfying $n - i_n \geq n^\epsilon$. We have to prove that (HI_{n,i_n}) is VNP-complete. As we already know from Proposition 4.1 that (HI_{n,i_n}) is p-definable, it is sufficient to prove that PER is a p-projection of this p-family. We distinguish several cases.

Case 1. $i_n \leq \sqrt{n}$.

Let $p_n := \lfloor \frac{n}{i_n+1} \rfloor$. Then $p_n \gtrsim \sqrt{n}$ and Lemma 5.3(i) implies that $(-1)^{i_n}\mathrm{PER}_{p_n}$ is a projection of HI_{n,i_n}.

Case 2. $i_n > \sqrt{n}$.

Put $m := n - i_n - 1$ and let $m = qi_n + \overline{m}$, where $0 \leq \overline{m} < i_n$. By applying Lemma 5.1 with $h = i_n$ we obtain that $f_1 := \mathrm{HI}_{n-i_n,i_n} + (-1)^{i_n-1}\mathrm{HI}_{n-i_n,0}$ is a projection of HI_{n,i_n} (cf. Figure 5.1). Again applying this lemma with $h = i_n$ yields that

$$f_2 := \mathrm{HI}_{n-2i_n,i_n} + (-1)^{i_n-1}\mathrm{HI}_{n-2i_n,0} + (-1)^{i_n-1}\mathrm{HI}_{n-2i_n,0}$$

is a projection of f_1. If we continue in this way, we see that with $\overline{n} := n - qi_n = \overline{m} + i_n + 1$, the polynomial

$$f_q := \mathrm{HI}_{\overline{n},i_n} + q(-1)^{i_n-1}\mathrm{PER}_{\overline{n}}$$

is a projection of HI_{n,i_n}.

Subcase 2.1. $\overline{m} \geq n^{1/4}$.

We apply Lemma 5.1 with $h = i_n$ to f_q and get that $(q+1)(-1)^{i_n-1}\mathrm{PER}_{\overline{m}+1}$ is a projection of HI_{n,i_n}. (Recall that $\overline{m} < i_n$.) Note that the coefficient $(q+1)(-1)^{i_n-1}$ is nonzero!

Subcase 2.2. $\overline{m} < n^{1/4}$ and $q \geq 1$.

We apply Lemma 5.3 (ii) to f_q and obtain

$$g_n := \gamma_n\mathrm{DET}_{p_n} + q(-1)^{i_n-1}\mathrm{PER}_{p_n}$$

as a projection of f_q for some $\gamma_n \in \{-1, 1\}$ and some

$$p_n \geq \left\lfloor \frac{\overline{n}}{\overline{n} - i_n + 1} \right\rfloor \geq \left\lfloor \frac{i_n}{\overline{m} + 2} \right\rfloor \gtrsim n^{1/4}.$$

By invoking Lemma 5.4(i), we see that PER_{p_n-1} is a projection of g_n and thus of HI_{n,i_n}.

Subcase 2.3. $q = 0$.

As in subcase 2.1, we apply Lemma 5.1 with $h = i_n$ to $f_q = \mathrm{HI}_{n,i_n}$ and get that $(-1)^{i_n-1}\mathrm{PER}_{\overline{m}+1}$ is a projection of HI_{n,i_n}. But $\overline{m}+1 = n - i_n \geq n^\epsilon$ by assumption.

To summarize, let $\delta := \min\{\epsilon, 1/4\}$. We have shown the existence of a sequence $N_n \gtrsim n^\delta$ of natural numbers such that PER_{N_n} is a projection of HI_{n,i_n} for all sufficiently large n. Taking into account that PER is monotone, this implies that PER is a p-projection of $(\mathrm{HI}_{n,i_n})_n$ and finishes the proof. □

5.5. Completeness of rectangular immanants. The proof of Theorem 1.2 will be achieved by identifying either a hook immanant or a Hamilton cycle polynomial as a projection of a rectangular immanant. This will be sufficient to establish completeness according to the following little observation.

LEMMA 5.6. *Let $f = (f_n)$ and $g = (g_n)$ be VNP-complete families and assume f to be monotone. Moreover, let $I \subseteq \mathbb{N}$ and define the corresponding* mixture $h = (h_n)$ *of f and g as*

$$h_n := \begin{cases} f_n & \text{if } n \in I, \\ g_n & \text{otherwise.} \end{cases}$$

Then h is VNP-complete as well.

Proof. The p-family $(f_1, g_1, f_2, g_2, f_3, g_3, \dots)$ is obviously p-definable and h is a p-projection of it; hence h is also p-definable.

Let $\varphi = (\varphi_n)$ be any p-definable family. Since f is complete, there is a p-bounded function $n \mapsto t_1(n)$ such that φ_n is a projection of $f_{t_1(n)}$: $\varphi_n \leq f_{t_1(n)}$ for all n. We put $D(n) := \max\{\deg g_i \mid 1 \leq i \leq n\}$ and consider the p-definable family

$$\Phi_n := \varphi_n + Z^{1+D(t_1(n))},$$

where Z is a new variable. As g is complete, there is a p-bounded $n \mapsto t_2(n)$ such that $\Phi_n \leq g_{t_2(n)}$ for all n. In particular,

$$D(t_1(n)) < \deg \Phi_n \leq \deg g_{t_2(n)},$$

which implies $t_2(n) > t_1(n)$. By substituting $Z \mapsto 0$ we see that $\varphi_n \leq g_{t_2(n)}$. On the other hand, we also have $\varphi_n \leq f_{t_2(n)}$, as f is monotone. From this it immediately follows that $\varphi_n \leq h_{t_2(n)}$ for all n. This shows that h is a complete family. □

We remark that the monotonicity assumption is necessary. Namely, if (f_n) is complete, then the families $(f_1, 0, f_2, 0, \dots)$ and $(0, f_1, 0, f_2, \dots)$ are both complete as well, but their mixture with respect to $I = \{2, 4, 6, 8, \dots\}$ equals the zero sequence.

Finally, we present the proof of Theorem 1.2 in the general situation.

Proof. Suppose that (s_m) is a p-bounded sequence of natural numbers. We wish to show that the sequence of rectangular immanants $\mathrm{IM}_{m^{s_m}}$ is VNP-complete. We write $s_m = q_m m + r_m$ with $1 \leq r_m \leq m$, $q_m \geq 0$.

As in the proof of Proposition 5.2, (5.1), we find that

$$f_m = m\beta_{m,s_m}\mathrm{HC}_m + \gamma_{m,s_m} \sum_{i=0}^{r_m-1} (-1)^i\, \mathrm{HI}_{m,i}$$

is a projection of $\mathrm{IM}_{m^{s_m}}$.

We distinguish now two cases.

Case 1. $1 \leq r_m \leq m - \sqrt{m}$.

We apply Lemma 5.1 with $h = 1$ and obtain that

$$g_m := \gamma_{m,s_m}\left(\mathrm{HI}_{m-1,0} + \sum_{i=1}^{r_m-1} (-1)^i \left[\mathrm{HI}_{m-1,i} + \mathrm{HI}_{m-1,i-1}\right]\right)$$

is a projection of $\gamma_{m,s_m} \sum_{i=0}^{r_m-1} (-1)^i \, \mathrm{HI}_{m,i}$. Recall that this is shown by adding to DK_{m-1} an isolated vertex with a loop (of weight 1). The resulting digraph does not have a Hamilton cycle. From this one easily sees that g_m is also a projection of f_m. The formula for g_m simplifies to (telescoping sum)

$$g_m = \gamma_{m,s_m}(-1)^{r_m-1}\mathrm{HI}_{m-1,r_m-1}.$$

If $r_m > m - \sqrt{m}$, then we define $g_m := \mathrm{HI}_{m-1,0}$. The family (g_m) of hook immanants is complete by Proposition 5.5.

Case 2. $m - \sqrt{m} < r_m \leq m$.

By using relation (3.4) we can rewrite f_m as follows:

$$\begin{aligned} f_m &= m\beta_{m,s_m}\mathrm{HC}_m + \gamma_{m,s_m}\left(m\mathrm{HC}_m - \sum_{i=r_m}^{m-1} (-1)^i \, \mathrm{HI}_{m,i}\right) \\ &= \kappa_m \mathrm{HC}_m + \gamma_{m,s_m} \sum_{i=r_m}^{m-1} (-1)^{i+1}\mathrm{HI}_{m,i}, \end{aligned}$$

where $\kappa_m := m(\beta_{m,s_m} + \gamma_{m,s_m}) > 0$. Lemma 5.3(ii) shows that

$$\varphi_m := \kappa_m \mathrm{HC}_{p_m} + c_m \mathrm{DET}_{p_m}$$

is a projection of f_m for some $c_m \in \mathbb{Q}$ and some

$$p_m \geq \left\lfloor \frac{m}{m - r_m + 1} \right\rfloor \geq \left\lfloor \frac{m}{\sqrt{m} + 1} \right\rfloor \geq \lfloor \sqrt{m} \rfloor - 1.$$

Lemma 5.4(ii) together with the fact that HC is monotone implies that the Hamilton cycle polynomial $h_m := \mathrm{HC}_{\lfloor \sqrt{m} \rfloor - 3}$ is a projection of φ_m and thus of f_m. The family (h_m) is monotone and complete, as HC has these properties.

To summarize, we have proved that some mixture of the families (g_m) of hook immanants and (h_m) of Hamilton cycle polynomials is a p-projection of (f_m), and thus of $(\mathrm{IM}_{m^{s_m}})$. Hence the family of rectangular immanants $(\mathrm{IM}_{m^{s_m}})$ is complete by Lemma 5.6. □

Finally, we remark that in order to obtain Corollary 1.4 from Theorem 1.2, one can check that the projections $f_n \leq g_m$ occurring in the proof of this theorem actually yield relations $N_n f_n(X) = g_m(a)$, where the components of a are either indeterminates, 0, or 1, and the factor N_n is a nonzero integer. Moreover, m, N_n, and a are computable in polynomial time from n. Thus we get "weakly parsimonious" reductions (cf. [14, p. 107]) between the corresponding problems to evaluate f_n, g_m at 0, 1-values. Moreover, one can obtain from Theorem 4.2 that the problem to compute $\mathrm{im}_{\lambda^{(n)}}(A) + n^{2n}$ from $A \in \{0,1\}^{n\times n}$ is in contained in #P for any polynomial time computable map $n \mapsto \lambda^{(n)}$.

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