# Adaptive and Array Signal Processing

## 1. Complex Analysis

### 1.1. Derivatives of Non-analytic functions

$$h : \mathbb{C} \ni z \mapsto h(z) \in \mathbb{C}$$
$$f : \mathbb{R}^2 \ni (x, y) \mapsto f(x, y) \in \mathbb{C},\ z = x + jy$$
$$g : \mathbb{C}^2 \ni (z_1, z_2) \mapsto g(z_1, z_2) \in \mathbb{C},\ x = \frac{z+z^*}{2},\ y = \frac{z-z^*}{2\mathrm{j}}$$

Derivative in direction $\varphi$:
$\frac{\mathrm{d}h}{\mathrm{d}z} = \left(\frac{\partial f}{\partial x}\cos(\varphi) + \frac{\partial f}{\partial y}\sin\varphi\right)\mathrm{e}^{-\mathrm{j}\varphi},\ \mathrm{d}z = \mathrm{e}^{\mathrm{j}\varphi}\mathrm{d}t, \quad \varphi, \mathrm{d}t \in \mathbb{R}$

$$\frac{\partial g}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - \mathrm{j}\frac{\partial f}{\partial y}\right), \quad \frac{\partial g}{\partial z^*} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + \mathrm{j}\frac{\partial f}{\partial y}\right)$$

### 1.2. Analytic functions

Definition:

$\frac{\mathrm{d}h}{\mathrm{d}z}$ independent of $\varphi$
$$\forall (x, y) \in \mathbb{R}^2 : \quad \frac{\partial f}{\partial x} + \mathrm{j}\frac{\partial f}{\partial y} = 0$$
$$\frac{\mathrm{d}h}{\mathrm{d}z} = \frac{\partial f}{\partial x} = -\mathrm{j}\frac{\partial f}{\partial y}$$
$$\frac{\partial g}{\partial z^*} \equiv 0$$

- Ansatz for obtaining the Lemma: compute derivative of $\frac{\mathrm{d}h}{\mathrm{d}z}$ wrt. to $\varphi$ and set it to 0.
- If $g()$ depends on $z^*$ it is not analytic

### 1.3. Minimization of $h(z) = g(z, z^*) \in \mathbb{R}$

Necessary condition for an extremum:
$$\frac{\partial g}{\partial z^*} = 0$$

Direction of steepest descent: $z \leftarrow z - \mu\frac{\partial g}{\partial z^*}, \quad \mu > 0$

Useful derivatives:
$$\frac{\partial\left(\boldsymbol{z}^{\mathrm{H}}\boldsymbol{p} + \boldsymbol{p}^{\mathrm{H}}\boldsymbol{z}\right)}{\partial \boldsymbol{z}^*} = \boldsymbol{p}$$
$$\frac{\partial\left(\boldsymbol{z}^{\mathrm{H}}\boldsymbol{R}\boldsymbol{z}\right)}{\partial \boldsymbol{z}^*} = \boldsymbol{R}\boldsymbol{z}$$
$$\frac{\partial \operatorname{tr}\left(\boldsymbol{S}^{\mathrm{H}}\boldsymbol{B}\right)}{\partial \boldsymbol{S}^*} = \boldsymbol{B}$$

### 1.4. Quadratic minimization with linear equality constraints

**Problem:**
$\min_{\boldsymbol{z}} \boldsymbol{z}^{\mathrm{H}}\boldsymbol{R}\boldsymbol{z}, \quad$ such that $\quad \boldsymbol{A}^{\mathrm{H}}\boldsymbol{z} = \boldsymbol{b}, \quad \boldsymbol{R} = \boldsymbol{R}^{\mathrm{H}} > \boldsymbol{0}$

**Corresponding Lagrange-ian function:**
$\mathcal{L} = \boldsymbol{z}^{\mathrm{H}}\boldsymbol{R}\boldsymbol{z} + \boldsymbol{\lambda}^{\mathrm{H}}\left(\boldsymbol{A}^{\mathrm{H}}\boldsymbol{z} - \boldsymbol{b}\right) + \left(\boldsymbol{z}^{\mathrm{H}}\boldsymbol{A} - \boldsymbol{b}^{\mathrm{H}}\right)\boldsymbol{\lambda}$

**Resulting dual optimization problem:** $\min_z \max_\lambda \mathcal{L}$

**Solution:**
$$\boldsymbol{z}_{\mathrm{opt}} = \boldsymbol{R}^{-1}\boldsymbol{A}\left(\boldsymbol{A}^{\mathrm{H}}\boldsymbol{R}^{-1}\boldsymbol{A}\right)^{-1}\boldsymbol{b}$$
$$\min \boldsymbol{z}^{H}\boldsymbol{R}\boldsymbol{z} = \boldsymbol{b}^{H}(\boldsymbol{A}^{H}\boldsymbol{R}^{-1}\boldsymbol{A})^{-1}\boldsymbol{b}$$

### 1.5. Real-valued quadratic minimization with linear inequality constraints

**Problem:**
$\min_{\boldsymbol{x}} \boldsymbol{x}^{\mathrm{T}}\boldsymbol{C}\boldsymbol{x}, \quad$ subject to $\quad \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}$

**Corresponding Lagrange-ian function:**
$\mathcal{L} = \boldsymbol{x}^{\mathrm{T}}\boldsymbol{C}\boldsymbol{x} + \boldsymbol{\lambda}^{\mathrm{T}}(\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b})$
**Resulting dual optimization problem:** $\min\max \mathcal{L}$ subject to $\lambda \geq 0$

**Algorithm for determining the solution:**

$$\min_{\boldsymbol{x}}\ \boldsymbol{x}^{\mathrm{T}}\boldsymbol{C}\boldsymbol{x}, \quad \text{subject to} \quad \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}, \quad \text{where } \boldsymbol{C} = \boldsymbol{C}^{\mathrm{T}} > \boldsymbol{0}.$$

1. $\boldsymbol{E} \leftarrow \frac{1}{4}\boldsymbol{A}\boldsymbol{C}^{-1}\boldsymbol{A}^{\mathrm{T}} \in \mathbb{R}^{M\times M}$
2. $\lambda_k \leftarrow 0, \quad$ for $k \in \{1, 2, \ldots, M\}$.
3. **repeat**
   **for** $k = 1 : M$ **do**
   $$\lambda_k \leftarrow \max\left(0\,,\ \frac{-1}{E_{k,k}}\left(\sum_{n=1}^{k-1} E_{k,n}\lambda_n + \frac{b_k}{2} + \sum_{n=k+1}^{M} E_{k,n}\lambda_n\right)\right)$$
   **end**
   **until** negligible change in all $\lambda_k$.
4. $\boldsymbol{x} \leftarrow -\frac{1}{2}\boldsymbol{C}^{-1}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{\lambda}$

## 2. Linear Algebra

### 2.1. Vector Space

A complex vector space $\mathcal{V}$ is a set with the following properties:
1. $\forall \boldsymbol{a}, \boldsymbol{b} \in \mathcal{V} : \quad \boldsymbol{a} + \boldsymbol{b} \in \mathcal{V}$
2. $\forall \boldsymbol{a}, \boldsymbol{b} \in \mathcal{V} : \quad \boldsymbol{a} + \boldsymbol{b} = \boldsymbol{b} + \boldsymbol{a}$
3. $\forall \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \mathcal{V} : \quad (\boldsymbol{a} + \boldsymbol{b}) + \boldsymbol{c} = \boldsymbol{a} + (\boldsymbol{b} + \boldsymbol{c})$
4. $\forall \boldsymbol{a} \in \mathcal{V} : \exists \boldsymbol{0} \in \mathcal{V} : \quad \boldsymbol{a} + \boldsymbol{0} = \boldsymbol{a}$
5. $\forall \boldsymbol{a} \in \mathcal{V} : \exists -\boldsymbol{a} \in \mathcal{V} : \quad \boldsymbol{a} + (-\boldsymbol{a}) = \boldsymbol{0}$
6. $\forall \boldsymbol{a} \in \mathcal{V} : \quad 1\boldsymbol{a} = \boldsymbol{a}$
7. $\forall \boldsymbol{a} \in \mathcal{V}, \forall \lambda, \mu \in \mathbb{C} : \quad \lambda(\mu\boldsymbol{a}) = (\lambda\mu)\boldsymbol{a}$
8. $\forall \boldsymbol{a}, \boldsymbol{b} \in \mathcal{V}, \forall \lambda \in \mathbb{C} : \quad \lambda(\boldsymbol{a} + \boldsymbol{b}) = \lambda\boldsymbol{a} + \lambda\boldsymbol{b}$
9. $\forall \boldsymbol{a} \in \mathcal{V}, \forall \lambda, \mu \in \mathbb{C} : \quad (\lambda + \mu)\boldsymbol{a} = \lambda\boldsymbol{a} + \mu\boldsymbol{a}$

### 2.2. Linear Subspace

A set $\mathcal{S}$ is called a subspace of a complex vector space $\mathcal{V}$ iff:
1. $\mathcal{S} \subseteq \mathcal{V}$
2. $\forall \boldsymbol{a}, \boldsymbol{b} \in \mathcal{V} : \quad \boldsymbol{a} + \boldsymbol{b} = \boldsymbol{b} + \boldsymbol{a}$
3. $\forall \boldsymbol{a}, \boldsymbol{b} \in \mathcal{S} : \quad \boldsymbol{a} + \boldsymbol{b} \in \mathcal{S}$
4. $\forall \boldsymbol{a} \in \mathcal{S}, \forall \lambda \in \mathbb{C} : \quad \lambda\boldsymbol{a} \in \mathcal{S}$

### 2.3. Linear (In)dependence

The vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n \in \mathcal{V}$ are said to be linearly independent iff:
$\sum_{k=1}^{n} a_k\boldsymbol{v}_k = \boldsymbol{0} \implies a_1 = \cdots = a_n = 0$

The vectors are linearly dependent iff:
$$\exists i : \exists b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_n \in \mathbb{C} : \quad \boldsymbol{v}_i = \sum_{k=1, k\neq i}^{n} b_k\boldsymbol{v}_k$$

- $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n \in \mathcal{V}$ are $LI$, and $\boldsymbol{s} \in \mathcal{V}$ cannot be expressed as a linear combination, then $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n, \boldsymbol{s}$ are LI
- $\dim(\mathcal{S})$ of a subspace is the maximum number of LI vectors that fit in it
- For every subspace $\mathcal{S}$, with $\dim(\mathcal{S}) = n$, and any LI vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n \in \mathcal{S}$ we have $\mathcal{S} = \mathrm{Sp}(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n)$
- Orthonormal vectors are LI

### 2.4. Gram-Schmidt

$$\boldsymbol{u}_2 = \boldsymbol{v}_2 - \boldsymbol{u}_1\frac{\boldsymbol{u}_1^{\mathrm{H}}\boldsymbol{v}_2}{\boldsymbol{u}_1^{\mathrm{H}}\boldsymbol{u}_1} \ldots$$

### 2.5. Matrix Cookbook

$\operatorname{tr}(\boldsymbol{A}\boldsymbol{B}) = \operatorname{tr}(\boldsymbol{B}\boldsymbol{A})$
$\operatorname{tr}(\boldsymbol{C}\boldsymbol{D}\boldsymbol{E}) = \operatorname{tr}(\boldsymbol{E}\boldsymbol{C}\boldsymbol{D}) = \operatorname{tr}(\boldsymbol{D}\boldsymbol{E}\boldsymbol{C})$
$(\boldsymbol{A}\boldsymbol{B})^{\mathrm{T}} = \boldsymbol{B}^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}}$
$(\boldsymbol{A}\boldsymbol{B})^{\mathrm{H}} = \boldsymbol{B}^{\mathrm{H}}\boldsymbol{A}^{\mathrm{H}}$
$(\boldsymbol{A}\boldsymbol{B})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$
$\left(\boldsymbol{A}^{-1}\right)^{\mathrm{H}} = \left(\boldsymbol{A}^{\mathrm{H}}\right)^{-1}$

### 2.6. Determinant

$det : \mathbb{C}^{m\times m} \ni \boldsymbol{A} \mapsto \det \boldsymbol{A} \in \mathbb{C}$ having the following properties:
1. $\det \mathbf{I}_m = 1$
2. If $\boldsymbol{A}$ has LD columns, then $\det \boldsymbol{A} = 0$.
3. $\det \boldsymbol{A}$ is linear in the columns of $\boldsymbol{A}$.

For the determinant of MxM matrices the following is true:
1. For $m = 1 : \det A = A$.
2. $\det \boldsymbol{A} = \sum_{i=1}^{m}(-1)^{i+j}A_{i,j}\det \boldsymbol{A}^{[i,j]}$, for any $j \in \{1, \ldots m\}$, where $\boldsymbol{A}^{[i,j]}$ is the matrix that results from $\boldsymbol{A}$ if the $i$-th row and the $j$-th column are removed.
3. $\det(\boldsymbol{A}\boldsymbol{B}) = (\det \boldsymbol{A})(\det \boldsymbol{B})$
4. $\det\left(\boldsymbol{A}^{-1}\right) = (\det \boldsymbol{A})^{-1}$
5. $\det\left(\boldsymbol{A}^{\mathrm{T}}\right) = \det \boldsymbol{A}$

### 2.7. Eigenvalue Decomposition (EVD)

$$\boldsymbol{A} = \boldsymbol{B}\boldsymbol{\Lambda}\boldsymbol{B}^{-1}$$

$\operatorname{tr} \boldsymbol{A} = \sum_{i=1}^{m}\lambda_i$
$\det \boldsymbol{A} = \prod_{i=1}^{m}\lambda_i$
$\boldsymbol{A}^k = \boldsymbol{B}\boldsymbol{\Lambda}^k\boldsymbol{B}^{-1}$
Every scalar function which has a Taylor series expansion can be generalized to matrices:
$\boldsymbol{h}(\boldsymbol{A}) = \boldsymbol{B}\boldsymbol{h}(\boldsymbol{\Lambda})\boldsymbol{B}^{-1}$

### 2.8. Hermitian Matrices

$$\boldsymbol{A} = \boldsymbol{A}^{H} \in \mathbb{C}^{m\times m}$$

Properties:
- m orthogonal (LI) eigenvectors $\rightarrow$ EVD exists
- real eigenvalues
- EVD has the form $\boldsymbol{A} = \boldsymbol{B}\boldsymbol{\Lambda}\boldsymbol{B}^{\mathrm{H}}$

### 2.9. Gramian Matrices

$$\exists \boldsymbol{C} \in \mathbb{C}^{m\times m} : \quad \boldsymbol{A} = \boldsymbol{C}\boldsymbol{C}^{\mathrm{H}}$$

Properties:
- Hermitian matrix
- non-negative real eigenvalues
- EVD exists

### 2.10. Sherman-Morrison-Woodbury identity

$$(\boldsymbol{A}+\boldsymbol{B}\boldsymbol{C}\boldsymbol{D})^{-1} = \boldsymbol{A}^{-1} - \boldsymbol{A}^{-1}\boldsymbol{B}\left(\boldsymbol{C}^{-1} + \boldsymbol{D}\boldsymbol{A}^{-1}\boldsymbol{B}\right)^{-1}\boldsymbol{D}\boldsymbol{A}^{-1}$$

$$\boldsymbol{A} \in \mathbb{C}^{M\times M}, \boldsymbol{C} \in \mathbb{C}^{N\times N}, \boldsymbol{B} \in \mathbb{C}^{M\times N}, \boldsymbol{D} \in \mathbb{C}^{N\times M}$$
$$\operatorname{rank} \boldsymbol{A} = M, \operatorname{rank} \boldsymbol{C} = N$$

Tipps for Reformulation:
- insert $\boldsymbol{C}^{-1}\boldsymbol{C}$
- ausklammern wos geht

### 2.11. Singular Value Decomposition (SVD)

$$\boldsymbol{A} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^{\mathrm{H}}$$
$$\boldsymbol{U} \in \mathbb{C}^{m\times m}, \quad \text{with} \quad \boldsymbol{U}^{-1} = \boldsymbol{U}^{\mathrm{H}}$$
$$\boldsymbol{V} \in \mathbb{C}^{n\times n}, \quad \text{with} \quad \boldsymbol{V}^{-1} = \boldsymbol{V}^{\mathrm{H}}$$
$$\boldsymbol{\Sigma} \in \mathbb{R}^{m\times n}, \quad \text{with} \quad \boldsymbol{\Sigma}_{i,j\neq i} = 0, \quad \boldsymbol{\Sigma}_{i,i} \geq 0$$

- exists for every matrix

Relation to SVD:

- $\boldsymbol{V}$: SVD of $\boldsymbol{A}^{\mathrm{H}}\boldsymbol{A}$
- $\boldsymbol{U}$: SVD of $\boldsymbol{A}\boldsymbol{A}^{\mathrm{H}}$
- $\boldsymbol{\Sigma}\boldsymbol{\Sigma}^{\mathrm{T}} = \operatorname{diag}(\lambda_1 \ldots \lambda_m)$
- $s_i = \sqrt{\lambda_i}, \quad 1 \leq i \leq \min(m, n)$

$$\boldsymbol{A} = \boldsymbol{U}_1\boldsymbol{\Sigma}_1\boldsymbol{V}_1^{\mathrm{H}} = \begin{bmatrix} \boldsymbol{U}_1 & \boldsymbol{U}_2 \end{bmatrix}\begin{bmatrix} \boldsymbol{\Sigma}_1 & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{bmatrix}\begin{bmatrix} \boldsymbol{V}_1^{\mathrm{H}} \\ \boldsymbol{V}_2^{\mathrm{H}} \end{bmatrix}$$

Properties:
- $\boldsymbol{U}_1^{\mathrm{H}}\boldsymbol{U}_1 = \boldsymbol{V}_1^{\mathrm{H}}\boldsymbol{V}_1 = \mathbf{I}_r$
- $\boldsymbol{U}_2^{\mathrm{H}}\boldsymbol{U}_2 = \mathbf{I}_{m-r}$, $\mathbf{V}_2^{\mathrm{H}}\mathbf{V}_2 = \mathbf{I}_{n-r}$
- $\boldsymbol{U}_1^{\mathrm{H}}\boldsymbol{U}_2 = \mathbf{O}_{r,(m-r)}$, $\boldsymbol{U}_2^{\mathrm{H}}\boldsymbol{U}_1 = \mathbf{O}_{(m-r),r}$
- $\boldsymbol{V}_1^{\mathrm{H}}\boldsymbol{V}_2 = \mathbf{O}_{r,(n-r)}$, $\boldsymbol{V}_2^{\mathrm{H}}\boldsymbol{V}_1 = \mathbf{O}_{(n-r),r}$
- $\boldsymbol{U}_1\boldsymbol{U}_1^{\mathrm{H}} + \boldsymbol{U}_2\boldsymbol{U}_2^{\mathrm{H}} = \mathbf{I}_m$
- $\boldsymbol{V}_1\boldsymbol{V}_1^{\mathrm{H}} + \boldsymbol{V}_2\boldsymbol{V}_2^{\mathrm{H}} = \mathbf{I}_n$

- $\operatorname{im} \boldsymbol{V}_1$ and $\operatorname{im} \boldsymbol{V}_2$ are complementary subspaces
- $\operatorname{im} \boldsymbol{U}_1$ and $\operatorname{im} \boldsymbol{U}_2$ are complementary subspaces

**Four elementary subspaces of a matrix:**
Image/column space:
- $\operatorname{im} \boldsymbol{A} = \operatorname{im} \boldsymbol{A}\boldsymbol{A}^{\mathrm{H}} = \operatorname{im} \boldsymbol{U}_1$
- $\dim(\operatorname{im} \boldsymbol{A}) = \operatorname{rank}(\boldsymbol{A}) = r$
- $\operatorname{rank} \boldsymbol{A} = \operatorname{rank} \boldsymbol{A}^{\mathrm{H}} = \operatorname{rank} \boldsymbol{A}^{\mathrm{T}} = \operatorname{rank} \boldsymbol{A}\boldsymbol{A}^{\mathrm{H}} = \operatorname{rank} \boldsymbol{A}^{\mathrm{H}}\boldsymbol{A}$
- $\boldsymbol{P}_{\operatorname{im}\boldsymbol{A}} = \boldsymbol{U}_1\boldsymbol{U}_1^{\mathrm{H}}$

Null space:
- $\text{null } \boldsymbol{A} = \operatorname{im} V_2$
- $\dim(\text{ null } \boldsymbol{A}) = n - r$
- if $r = n$, the nullspace is $\{\boldsymbol{0}\}$
- $\boldsymbol{P}_{\text{null } \boldsymbol{A}} = \boldsymbol{V}_2\boldsymbol{V}_2^{\mathrm{H}}$

Left null space:
Right image space:

**Complementary subspaces:**
$\mathcal{S}_1 \bigcap \mathcal{S}_2 = \{\boldsymbol{0}\}$ and $\mathcal{S}_1 \bigcup \mathcal{S}_2 = \mathbb{C}^{m\times 1}$

### 2.12. Projectors

- $\operatorname{im} \boldsymbol{P}_{\mathcal{S}} = \mathcal{S}$
- $\boldsymbol{P}_{\mathcal{S}} = \boldsymbol{P}_{\mathcal{S}}^{\mathrm{H}}$
- $\boldsymbol{P}_{\mathcal{S}}\boldsymbol{P}_{\mathcal{S}} = \boldsymbol{P}_{\mathcal{S}}$

$$\boldsymbol{z} \in \mathcal{S} \iff \boldsymbol{P}_{\mathcal{S}}\boldsymbol{z} = \boldsymbol{z}$$

- Projectors are unique for their subspace

### 2.13. Eckart-Young

$\arg\min_{\boldsymbol{B}} \|\boldsymbol{A} - \boldsymbol{B}\|_{\mathrm{F}}^2, \quad$ s.t. $\quad \operatorname{rank} \boldsymbol{B} = k < r = \operatorname{rank} \boldsymbol{A}$

$$\boldsymbol{B} = \sum_{i=1}^{k}\boldsymbol{u}_i s_i\boldsymbol{v}_i^{\mathrm{H}}$$

$\|\boldsymbol{A} - \boldsymbol{B}\|_{\mathrm{F}}^2 = \|\boldsymbol{\Sigma} - \boldsymbol{M}\|_{\mathrm{F}}^2$, where $\boldsymbol{A} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^{\mathrm{H}}, \boldsymbol{M} = \boldsymbol{U}^{\mathrm{H}}\boldsymbol{B}\boldsymbol{V}$
$$= \sum_{i=1}|s_i - M_{i,i}|^2 + \sum_{i>r}|M_{i,i}|^2 + \sum_{i,j\neq i}|M_{i,j}|^2$$

Minimum: $M_{i,i} = 0, i > r,\ M_{i,j\neq i} = 0,\ M_{i,i} = s_i, i = 1 \ldots k$

### 2.14. Frobenius Norm

$\|\boldsymbol{C}\|_{\mathrm{F}}^2 = \operatorname{tr}\left(\boldsymbol{C}^{\mathrm{H}}\boldsymbol{C}\right) = \operatorname{tr}\left(\boldsymbol{C}\boldsymbol{C}^{\mathrm{H}}\right)$

$\|\boldsymbol{C}\|_{\mathrm{F}}^2 = \left\|\boldsymbol{U}^{\mathrm{H}}\boldsymbol{C}\boldsymbol{V}\right\|_{\mathrm{F}}^2$

With $\boldsymbol{M} = \boldsymbol{U}^{\mathrm{H}}\boldsymbol{B}\boldsymbol{V}$:
$\|\boldsymbol{A} - \boldsymbol{B}\|_{\mathrm{F}}^2 = \|\boldsymbol{\Sigma} - \boldsymbol{M}\|_{\mathrm{F}}^2 = \sum_{i=1}^r |s_i - \boldsymbol{M}_{i,i}|^2 + \sum_{i>r} |\boldsymbol{M}_{i,i}|^2 + \sum_{i,j\neq i} |\boldsymbol{M}_{i,j}|^2$

### 2.15. Linear System of Equations

**Exakt solution:**
If and only if $b \in \operatorname{im} \boldsymbol{A}$, the system $\boldsymbol{A}\boldsymbol{w} = \boldsymbol{b}$, with $\boldsymbol{A} \in \mathbb{C}^{m\times n}$, has the following exact solution(s):

$$\boldsymbol{w} = \boldsymbol{V}_1\boldsymbol{\Sigma}_1^{-1}\boldsymbol{U}_1^{\mathrm{H}}\boldsymbol{b} + \boldsymbol{V}_2\boldsymbol{z}, \quad \text{for any} \quad \boldsymbol{z} \in \mathbb{C}^{(n-r)\times 1}, \quad r = \operatorname{rank} \boldsymbol{A}$$

The solution is unique iff. $\boldsymbol{A}$ has full column rank.

**Minimum Norm Solution:**
If $b \in \operatorname{im} \boldsymbol{A}$ and $\boldsymbol{A}$ has full row rank, the solution $w_{\mathrm{MN}}$, of $\boldsymbol{A}w_{\mathrm{MN}} = b$, which has the smallest euclidian norm is given by:

$$\boldsymbol{w}_{\mathrm{MN}} = \boldsymbol{A}^{\mathrm{H}}\left(\boldsymbol{A}\boldsymbol{A}^{\mathrm{H}}\right)^{-1}\boldsymbol{b} \text{ ( full row rank)}$$

**Least Squares Solution:**
While for $b \notin \operatorname{im} \boldsymbol{A}$, with $\boldsymbol{A} \in \mathbb{C}^{m\times n}$, there is no exact solution for the system $\boldsymbol{A}\boldsymbol{w} = \boldsymbol{b}$, an approximate solution, $\boldsymbol{w}_{\mathrm{LS}}$, can be defined as:

$$\boldsymbol{w}_{\mathrm{LS}} = \arg\min_{\boldsymbol{w}} \|\boldsymbol{A}\boldsymbol{w} - \boldsymbol{b}\|_2^2$$
$$\boldsymbol{w}_{\mathrm{LS}} = \left(\boldsymbol{A}^{\mathrm{H}}\boldsymbol{A}\right)^{-1}\boldsymbol{A}^{\mathrm{H}}\boldsymbol{b} \text{ (full column rank)}$$

### 2.16. Pseudoinverse

$$\boldsymbol{A}^+ = \begin{cases} \boldsymbol{V}_1\boldsymbol{\Sigma}_1^{-1}\boldsymbol{U}_1^{\mathrm{H}} & \text{for } \boldsymbol{A} \neq \boldsymbol{O} \\ \boldsymbol{A}^{\mathrm{T}} & \text{else.} \end{cases}$$

$$\boldsymbol{A}^+ = \begin{cases} \boldsymbol{A}^{\mathrm{H}}\left(\boldsymbol{A}\boldsymbol{A}^{\mathrm{H}}\right)^{-1} & \text{for full row-rank } \boldsymbol{A} \\ \left(\boldsymbol{A}^{\mathrm{H}}\boldsymbol{A}\right)^{-1}\boldsymbol{A}^{\mathrm{H}} & \text{for full column-rank } \boldsymbol{A} \\ \lim_{\epsilon\to 0} \boldsymbol{A}^{\mathrm{H}}\left(\boldsymbol{A}\boldsymbol{A}^{\mathrm{H}} + \epsilon\mathbf{I}\right)^{-1} & \\ \lim_{\epsilon\to 0} \left(\boldsymbol{A}^{\mathrm{H}}\boldsymbol{A} + \epsilon\mathbf{I}\right)^{-1}\boldsymbol{A}^{\mathrm{H}} & \end{cases}$$

Relation to the projectors:

- $\boldsymbol{P}_{\mathrm{im}\boldsymbol{A}} = \boldsymbol{A}\boldsymbol{A}^+ = \boldsymbol{U}_1\boldsymbol{U}_1^{\mathrm{H}}$
- $\boldsymbol{P}_{\text{null } \boldsymbol{A}} = \boldsymbol{I} - \boldsymbol{A}^+\boldsymbol{A} = \boldsymbol{V}_2\boldsymbol{V}_2^{\mathrm{H}}$

## 3. Random Processes

### 3.1. Discrete random processes

Definition: function $x[n]$ which is selected from an ensemble of possible functions by random.
Properties obtained by averaging over the ensemble:

Expectation function:
$$\mu_x[n] = \mathrm{E}[x[n]]$$
Autocorrelation function:
$$r_x[n,k] = \mathrm{E}[x[n]x[n-k]]$$
Autocovariance function:
$$c_x[n,k] = \mathrm{E}\left[(x[n] - \mu_x[n])\,(x[n-k] - \mu_x[n-k])\right]$$
$$= r_x[n,k] - \mu_x[n]\mu_x[n-k]$$
Cross-correlation function:
$$r_{x,y}[n,k] = \mathrm{E}[x[n]y[n-k]]$$
Cross-covariance function:
$$c_{x,y}[n,k] = \mathrm{E}\left[(x[n] - \mu_x[n])\,(y[n-k] - \mu_y[n-k])\right]$$
$$= r_{x,y}[n,k] - \mu_x[n]\mu_y[n-k]$$

### 3.2. Approximations obtained by averaging over time:

Expectation function:
$$\hat{\mu}_x^{[N]}[n] = \frac{1}{2N+1}\sum_{i=-N}^{N} x[n+i]$$
Cross-correlation function:
$$\hat{r}_{x,y}^{[N]}[n,k] = \frac{1}{2N+1}\sum_{i=-N}^{N} x[n+i]y[n+i-k]$$

The approximations are random processes themselves.

### 3.3. Wide-sense-stationary (WSS) processes

$\mu_x[n]$, $r_x[n,k]$, $c_x[n,k]$, $r_{x,y}[n,k]$ and $c_{x,y}[n,k]$ are independent of the time index $n$.

$$\mu_x = \mathrm{E}[x[n]]$$
$$r_x[k] = \mathrm{E}[x[n]x[n-k]],\ r_x[k] = r_x[-k]$$
$$r_{x,y}[k] = \mathrm{E}[x[n]y[n-k]],\ r_{x,y}[k] = r_{y,x}[-k]$$
$$\mathrm{E}\left[\hat{\mu}_x^{[N]}[n]\right] = \mu_x, \quad \mathrm{E}\left[\hat{r}_x^{[N]}[n,k]\right] = r_x[k]$$

For zero-mean processes: $\mu_x = 0$, $\mu_y = 0$, $\ldots$: correlation and covariance are identical

### 3.4. Ergodic processes

Averages over ensembles can be purely obtained from averages over time.
Ergodicity implies WSS.

$$\lim_{N\to\infty} \hat{\mu}_x^{[N]}[n] = \mu_x$$
$$\lim_{N\to\infty} \hat{r}_x^{[N]}[n,k] = r_x[k]$$

### 3.5. Complex processes

$u[n] = x[n] + \mathrm{j}y[n]$
Complex autocorrelation function:
$r_u[k] = \mathrm{E}\left[u[n]u^*[n-k]\right]$
$r_u[k] = r_x[k] + r_y[k] + \mathrm{j}\left(r_{x,y}[-k] - r_{x,y}[k]\right)$
Adjunct complex autocorrelation function:
$\tilde{r}_u[k] = \mathrm{E}[u[n]u[n-k]]$
$\tilde{r}_u[k] = r_x[k] - r_y[k] + \mathrm{j}\left(r_{x,y}[k] + r_{x,y}[-k]\right)$
Reformulation:
$r_x[k] = \frac{1}{2}\operatorname{Re}\{r_u[k] + \tilde{r}_u[k]\}$
$r_y[k] = \frac{1}{2}\operatorname{Re}\{r_u[k] - \tilde{r}_u[k]\}$
$r_{x,y}[k] = -\frac{1}{2}\operatorname{Im}\{r_u[k] - \tilde{r}_u[k]\}$

### 3.6. Proper WSS processes

Equivalent definitions (iff):
$\forall k: \quad \tilde{r}_u[k] = 0$
$\forall k: \quad r_x[k] = r_y[k], \quad \text{and} \quad \forall k: \quad r_{x,y}[k] = -r_{x,y}[-k]$

$$\mathrm{E}\left[\boldsymbol{u}[n]\boldsymbol{u}^{\mathrm{T}}[n]\right] = \begin{bmatrix} \tilde{r}_u[0] & \tilde{r}_u[1] & \tilde{r}_u[2] & \cdots \\ \tilde{r}_u[1] & \tilde{r}_u[0] & \tilde{r}_u[1] & \cdots \\ \vdots & \ddots & \ddots & \vdots \end{bmatrix} = \boldsymbol{O}$$

$\Rightarrow$ The autocorrelation function is completely described by the complex autocorrelation function.

## 4. Overview

### 4.1. Linear estimation for matrices

**Problem:** Find $\hat{\boldsymbol{S}}$ from $\boldsymbol{X}$

$$\boldsymbol{X} = \boldsymbol{A}\boldsymbol{S} + \boldsymbol{\Upsilon}$$

Four different cases:

1. nothing is known (except structure) $\rightarrow$ MUSIC
2. only $\boldsymbol{A}$ is known $\rightarrow$ Least Squares
3. $\boldsymbol{A}$ and $\mathrm{E}\left[\boldsymbol{\Upsilon}\boldsymbol{\Upsilon}^{\mathrm{H}}\right]$ are known $\rightarrow$ BLUE
4. $\boldsymbol{A}$, $E\left[\boldsymbol{\Upsilon}\boldsymbol{\Upsilon}^{\mathrm{H}}\right]$ and $\mathrm{E}\left[\boldsymbol{S}\boldsymbol{S}^{\mathrm{H}}\right]$ are known

## 5. Kolmogorov-Wiener Filters

### 5.1. Linear Filters

$$u = h * s$$

If $h$ has $K+1$ coefficients, its memory is $K$.
If $M$ output samples are required: $\boldsymbol{H} \in \mathbb{C}^{M\times(M+K)}$

### 5.2. Kolmogorov-Wiener filter for SISO/ time domain equalizer for the linear multipath channel

**Optimization problem:**

$$\mathbf{MSE} = \mathrm{E}[|\underbrace{y[n] - d[n]}_{e[n]}|^2]$$
$$(w[0], \ldots, w[M-1])_{\mathrm{opt}} = \arg\min_{(w[0],\ldots,w[M-1])} \mathrm{MSE}$$

$\boldsymbol{u}[n] \in \mathbb{C}^{M\times 1}$, $\boldsymbol{s}[n] \in \mathbb{C}^{N\times 1}$, $\boldsymbol{H} \in \mathbb{C}^{M\times N}$, $N = M + K$

$$y[n] = \boldsymbol{w}^{\mathrm{H}}\boldsymbol{u}[n] = \boldsymbol{w}^{\mathrm{H}}(\boldsymbol{H}\boldsymbol{s}[n] + \boldsymbol{v}[n])$$

$\boldsymbol{R} = \mathrm{E}\left[\boldsymbol{u}[n]\boldsymbol{u}^{\mathrm{H}}[n]\right] = \boldsymbol{H}\boldsymbol{R}_s\boldsymbol{H}^{\mathrm{H}} + \boldsymbol{H}\boldsymbol{R}_{s,v} + \boldsymbol{R}_{v,s}\boldsymbol{H}^{\mathrm{H}} + \boldsymbol{R}_v$
$\boldsymbol{p} = \mathrm{E}\left[\boldsymbol{u}[n]d^*[n]\right]$
$\sigma_s^2 = \mathrm{E}\left[d[n]d^*[n]\right]$

**Solutions in terms of u:**

$$\mathrm{E}\left[\boldsymbol{u}[n]e^*[n]\right] = 0 \text{ (principle of orthogonality)}$$
$$\boldsymbol{w}_{\mathrm{opt}} = \boldsymbol{R}^{-1}\boldsymbol{p}, \text{ or: } \boldsymbol{w}_{\mathrm{opt}}^{\mathrm{H}} = \boldsymbol{p}^{\mathrm{H}}\boldsymbol{R}^{-1}$$
$$\mathrm{MSE}_{\min} = \sigma_s^2 - \boldsymbol{p}^{\mathrm{H}}\boldsymbol{R}^{-1}\boldsymbol{p}$$

$\boldsymbol{w}_{\mathrm{opt}}$ minimizes MSE if: $\boldsymbol{R} > 0 \Leftrightarrow \boldsymbol{R}$ invertible ($\boldsymbol{R}$ is Gramian)
In general: $\mathbf{MSE} = \boldsymbol{w}^{\mathrm{H}}\boldsymbol{R}\boldsymbol{w} - \boldsymbol{w}^{\mathrm{H}}\boldsymbol{p} - \boldsymbol{p}^{\mathrm{H}}\boldsymbol{w} + \sigma_s^2$

Noise variance: $E[(\boldsymbol{w}_{\mathrm{opt}}^H\boldsymbol{v}[n])(\boldsymbol{w}_{\mathrm{opt}}^H\boldsymbol{v}[n])^H]$
Overall impulse response: coefficients given by $\boldsymbol{w}_{\mathrm{opt}}^H\boldsymbol{H}$

**Special case:** $d[n] = s[n-l]$, $s$ and $v$ are uncorrelated

$$\boldsymbol{w}_{\mathrm{opt}} = \left(\boldsymbol{H}\boldsymbol{R}_{\boldsymbol{s}}\boldsymbol{H}^{\mathrm{H}} + \boldsymbol{R}_{\boldsymbol{v}}\right)^{-1}\boldsymbol{H}\boldsymbol{R}_{\boldsymbol{s}}\mathbf{e}_{l+1}$$
$$= \boldsymbol{R}_{\boldsymbol{v}}^{-1}\boldsymbol{H}\left(\boldsymbol{R}_{\boldsymbol{s}}^{-1} + \boldsymbol{H}^{\mathrm{H}}\boldsymbol{R}_{\boldsymbol{v}}^{-1}\boldsymbol{H}\right)^{-1}\mathbf{e}_{l+1}$$

$\boldsymbol{R} = \boldsymbol{H}\boldsymbol{R}_s\boldsymbol{H}^{\mathrm{H}} + \boldsymbol{R}_v$
$\boldsymbol{p} = \boldsymbol{H}\boldsymbol{R}_s\mathbf{e}_{l+1}$
$\boldsymbol{R}_s = \mathrm{E}\left[\boldsymbol{s}[n]\boldsymbol{s}^{\mathrm{H}}[n]\right]$
$\boldsymbol{R}_{s,v} = \mathrm{E}\left[\boldsymbol{s}[n]\boldsymbol{v}^{\mathrm{H}}[n]\right] = \boldsymbol{O}$
$\boldsymbol{R}_v = \mathrm{E}\left[\boldsymbol{v}[\boldsymbol{n}]\boldsymbol{v}^{\mathrm{H}}[\boldsymbol{n}]\right]$

Optimization of $l_{\mathrm{opt}}$:
$$l_{\mathrm{opt}} = \arg\min_{l\in\{0,1,\ldots N-1\}} \sigma_s^2 - \boldsymbol{p}^{\mathrm{H}}\boldsymbol{R}^{-1}\boldsymbol{p}$$
$$= \arg\max_{l\in\{0,1,\ldots N-1\}} \boldsymbol{p}^{\mathrm{H}}\boldsymbol{R}^{-1}\boldsymbol{p}$$
$$= \text{argmin}\sigma_s^2 - \mathbf{e}_{l+1}^{\mathrm{T}}\boldsymbol{R}_{\boldsymbol{s}}\boldsymbol{H}^{\mathrm{H}}\left(\boldsymbol{H}\boldsymbol{R}_{\boldsymbol{s}}\boldsymbol{H}^{\mathrm{H}} + \boldsymbol{R}_v\right)^{-1}\boldsymbol{H}\boldsymbol{R}_{\boldsymbol{s}}\mathbf{e}_{l+1}$$
This is done by trying a few neighbors of $N/2 = (M+K)/2$.

Noise variance: $\sigma_n^2 = E[(\boldsymbol{w}_{\mathrm{opt}}^H\boldsymbol{v}[n])(\boldsymbol{w}_{\mathrm{opt}}^H\boldsymbol{v}[n])^H]$
Overall impulse response: coefficients given by $\boldsymbol{h} \leftarrow \boldsymbol{w}_{\mathrm{opt}}^H\boldsymbol{H}$
Signal-to-interference ratio:

$$\mathrm{SINR} = \frac{\sigma_s'^2|h[l]|^2}{\sigma_s'^2\sum_{i,i\neq l}|h[i]|^2 + \sigma_n^2}, \quad \text{assuming } \boldsymbol{R}_s = \sigma_s^2\boldsymbol{I}$$

.

### 5.3. Kolmogorov-Wiener filter for diversity reception

L multipath channels with just **one transmit signal**:
$\boldsymbol{u}_i[n] = \boldsymbol{H}_i\boldsymbol{s}[n] + \boldsymbol{v}_i[n], i \in \{1,2,\ldots L\}$, $\boldsymbol{H}_i \in \mathbb{C}^{M\times(M+K)}$

$$\underbrace{\begin{bmatrix} u_1[n] \\ u_2[n] \\ \vdots \\ u_L[n] \end{bmatrix}}_{\boldsymbol{u}[n]} = \underbrace{\begin{bmatrix} H_1 \\ H_2 \\ \vdots \\ H_L \end{bmatrix}}_{\boldsymbol{H}\in\mathbb{C}^{LM\times(M+K)}} s[n] + \underbrace{\begin{bmatrix} v_1[n] \\ v_2[n] \\ \vdots \\ v_L[n] \end{bmatrix}}_{\boldsymbol{v}[n]}$$

Same formula as for SISO but with different matrices:
$\boldsymbol{w}_{\mathrm{opt}} = \left(\boldsymbol{H}\boldsymbol{R}_{\boldsymbol{s}}\boldsymbol{H}^{\mathrm{H}} + \boldsymbol{R}_{\boldsymbol{v}}\right)^{-1}\boldsymbol{H}\boldsymbol{R}_{\boldsymbol{s}}\mathbf{e}_{l+1}$
$\boldsymbol{R}_{\boldsymbol{s}} = \mathrm{E}\left[\boldsymbol{s}[n]\boldsymbol{s}[n]^{\mathrm{H}}\right] \in \mathbb{C}^{(M+K)\times(M+K)}$
$\boldsymbol{R}_{\boldsymbol{v}} = \mathrm{E}\left[\boldsymbol{v}[n]\boldsymbol{v}[n]^{\mathrm{H}}\right] \in \mathbb{C}^{LM\times LM}$
$y[n] = \boldsymbol{w}_{\mathrm{opt}}^{\mathrm{H}}\boldsymbol{u}[n]$
usually $l_{\mathrm{opt}} = \lfloor(M+K)/2\rfloor$

Diversity reception

- one signal is transmitted over multiple channels
- others can still be used if one breaks (redundancy)
- space-diversity, frequency-diversity, time-diversity, or combinations

### 5.4. Kolmogorov-Wiener filter for multi-streaming

L channels with **different transmit signals**:

$$\underbrace{\begin{bmatrix} \boldsymbol{u}_1[n] \\ \boldsymbol{u}_2[n] \\ \vdots \\ \boldsymbol{u}_{m_{\mathrm{Rx}}}[n] \end{bmatrix}}_{\boldsymbol{u}[n]} = \underbrace{\begin{bmatrix} \boldsymbol{H}_{1,1} & \boldsymbol{H}_{1,2} & \cdots & \boldsymbol{H}_{1,m_{\mathrm{Tx}}} \\ \boldsymbol{H}_{2,1} & \boldsymbol{H}_{2,2} & \cdots & \boldsymbol{H}_{2,m_{\mathrm{Tx}}} \\ \vdots & \vdots & \vdots & \vdots \\ \boldsymbol{H}_{m_{\mathrm{Rx}},1} & \boldsymbol{H}_{m_{\mathrm{Rx}},2} & \cdots & \boldsymbol{H}_{m_{\mathrm{Rx}},m_{\mathrm{Tx}}} \end{bmatrix}}_{\boldsymbol{H}} \underbrace{\begin{bmatrix} \boldsymbol{s}_1[n] \\ \boldsymbol{s}_2[n] \\ \vdots \\ \boldsymbol{s}_{m_{\mathrm{Tx}}}[n] \end{bmatrix}}_{\boldsymbol{s}[n]} + \underbrace{\begin{bmatrix} \boldsymbol{v}_1[n] \\ \boldsymbol{v}_2[n] \\ \vdots \\ \boldsymbol{v}_{m_{\mathrm{Rx}}}[n] \end{bmatrix}}_{\boldsymbol{v}[n]}$$

$\boldsymbol{H}_{i,j} \in \mathbb{C}^{M\times(M+K)}$ connects j-th transmitter to i-th receiver
$\mathbf{H} \in \mathbb{C}^{Mm_{\mathrm{Rx}}\times(M+K)m_{\mathrm{Tx}}}$
$\boldsymbol{s}[n] \in \mathbb{C}^{(M+K)m_{\mathrm{Tx}}}$
$\boldsymbol{v}[n] \in \mathbb{C}^{Mm_{\mathrm{Rx}}}$
$K$ is the maximum channel memory of all channels

Main difference to before: Ordering of the elements in $\boldsymbol{s}[n]$ has changed to groups of time series.

**Solution:**

$$\boldsymbol{w}_{j,\mathrm{opt}} = \underbrace{\left(\boldsymbol{H}\boldsymbol{R}_{\boldsymbol{s}}\boldsymbol{H}^{\mathrm{H}} + \boldsymbol{R}_v\right)^{-1}}_{\boldsymbol{R}^{-1}}\underbrace{\boldsymbol{H}\boldsymbol{R}_{\boldsymbol{s}}\mathbf{e}_{(M+K)(j-1)+l_{j+1}}}_{\boldsymbol{p}}$$

$$y_j[n] = \boldsymbol{w}_{j,\mathrm{opt}}^{\mathrm{H}}\boldsymbol{u}[n], \quad j \in \{1,2,\ldots,m_{\mathrm{Tx}}\}$$

$\boldsymbol{w}_{j,\mathrm{opt}}$ is the optimum filter for recovering the signal of the j-th transmitter with time delay $l_j \in \{0,\ldots,M+K-1\}$

### 5.5. Kolmogorov-Wiener Filter for SISO

**Setup:**

$$\boldsymbol{u} = \boldsymbol{h}s + \boldsymbol{v}, \quad s = \boldsymbol{w}^H\boldsymbol{u}$$
$$\boldsymbol{R}_{\boldsymbol{s}} = 1, \quad \boldsymbol{R}_v = \mathbf{I}, \quad s, v \text{ uncorrelated}$$

**MSE solution:**

$$\boldsymbol{w} = (\boldsymbol{I} + \boldsymbol{h}\boldsymbol{h}^H)^{-1}\boldsymbol{h} = \frac{\boldsymbol{h}}{1 + \boldsymbol{h}^H\boldsymbol{h}}$$

### 5.6. General Properties of Kolmogorov-Wiener

- Linear filter: can only use first order correlations of $d$ and $u$, $\rightarrow p$ must be $\neq 0$
- Filtering without noise: MSE drops exponentially when increasing reconstruction filter length (if $\boldsymbol{H}$ has full column rank)
- Filtering with noise: MSE saturates and cannot drop exponentially

## 5.7. Kolmogorov-Wiener Filter for SNR $\to \infty$

**Setup:**

$$\boldsymbol{u} = \boldsymbol{H}\boldsymbol{s} + \boldsymbol{v}, \quad \hat{\boldsymbol{s}} = \boldsymbol{w}^H \boldsymbol{u}$$

$$\boldsymbol{R_s} = \sigma_s^2 \mathbf{I}, \quad \boldsymbol{R}_v = \sigma_v^2 \mathbf{I}, \quad s, v \text{ uncorrelated}$$

**MSE solution:**

$$\lim_{\sigma_s^2/\sigma_v^2 \to \infty} \boldsymbol{w}_{\text{opt}}^{\mathrm{H}} = \begin{cases} \mathbf{e}_{l+1}^{\mathrm{T}} \boldsymbol{H}^{\mathrm{H}} \left(\boldsymbol{H}\boldsymbol{H}^{\mathrm{H}}\right)^{-1}, & \text{full row rank } \boldsymbol{H} \\ \mathbf{e}_{l+1}^{\mathrm{T}} \left(\boldsymbol{H}^{\mathrm{H}}\boldsymbol{H}\right)^{-1} \boldsymbol{H}^{\mathrm{H}}, & \text{full column rank} \boldsymbol{H} \end{cases}$$

$$= \mathbf{e}_{l+1}^{\mathrm{T}} \boldsymbol{H}^{+}$$

Perfect reconstruction only if $H$ has full column rank:

$$\hat{\boldsymbol{s}}_{l+1} = \mathbf{e}_{l+1}^{\mathrm{T}} \boldsymbol{H}^{+} \boldsymbol{H} \boldsymbol{s} = \mathbf{e}_{l+1}^{\mathrm{T}} \underbrace{\left(\boldsymbol{H}^{\mathrm{H}}\boldsymbol{H}\right)^{-1} \boldsymbol{H}^{\mathrm{H}}\boldsymbol{H}}_{\mathbf{I}} \boldsymbol{s} = \boldsymbol{s}_{l+1}$$

Imperfect reconstruction if $H$ has not full column rank:

$$\hat{s}_{l+1} = \mathbf{e}_{l+1}^{\mathrm{T}} \underbrace{\boldsymbol{H}^{+}\boldsymbol{H}}_{\neq \mathbf{I}} \boldsymbol{s} \neq \boldsymbol{s}_{l+}$$

Because with $\operatorname{rank} \boldsymbol{H} < N$:

$$\boldsymbol{H}^{+}\boldsymbol{H} = \mathbf{I} - \boldsymbol{P}_{\text{null } \boldsymbol{H}}$$

$$\dim \text{ null } \boldsymbol{H} = N - \operatorname{rank} \boldsymbol{H} > 0$$

$$\boldsymbol{P}_{\text{null } \boldsymbol{H}} \neq \mathbf{O}$$

## 5.8. Steepest Descent Algorithm (SDA)

Kolmogorov-Wiener filters need to solve $\boldsymbol{R}\boldsymbol{w}_{\text{opt}} = \boldsymbol{p}$, where:

$\boldsymbol{R} = \mathrm{E}\left[\boldsymbol{u}[n]\boldsymbol{u}^{\mathrm{H}}[n]\right]$ and $\boldsymbol{p} = \mathrm{E}\left[\boldsymbol{u}[n] d^*[n]\right]$

Problems with solving explicitly

- Accuracy: computation of $m^2$ complex numbers for $R^{-1}$ and matrix vector multiplication add errors when $m$ is large
- Computational load: Recomputing $R^{-1}$ at every time step is costly
- Gaussian elimination needs to restart the entire computation every time
- Triangular schemes bring no benefit as $R$ also changes

**Gradient of MSE**:

$$\mathrm{MSE}(w) = F\left(\tfrac{1}{2}\left(w + w^{\star}\right), \tfrac{1}{2}\left(w - w^{\star}\right)/\mathrm{j}\right) = G\left(w, w^{*}\right)$$

$$\mathrm{dMSE} = \left(\frac{\partial G}{\partial \boldsymbol{w}}\right)^{\mathrm{T}} \mathrm{d}\boldsymbol{w} + \left(\frac{\partial G}{\partial \boldsymbol{w}^*}\right)^{\mathrm{T}} \mathrm{d}\boldsymbol{w}^*$$

$$= \left(\frac{\partial G^*}{\partial \boldsymbol{w}^*}\right)^{\mathrm{H}} \mathrm{d}\boldsymbol{w} + \left(\left(\frac{\partial G}{\partial \boldsymbol{w}^*}\right)^{\mathrm{H}} \mathrm{d}\boldsymbol{w}\right)^*$$

$$= 2\operatorname{Re}\left\{\left(\frac{\partial G}{\partial w^{\star}}\right)^{\mathrm{H}} \mathrm{d}w\right\} = 2\operatorname{Re}\left\{(\boldsymbol{R}w - \boldsymbol{p})^{\mathrm{H}} \mathrm{d}\boldsymbol{w}\right\}$$

$$\leq 2\left|(\boldsymbol{R}w - \boldsymbol{p})^{\mathrm{H}} \mathrm{d}\boldsymbol{w}\right|, \text{ with equality for } \mathrm{d}\boldsymbol{w} = (\boldsymbol{R}w - \boldsymbol{p})\mathrm{d}t$$

**Gradient descent**:

$$\boldsymbol{w}_{n+1} = \boldsymbol{w}_n - (\boldsymbol{R}\boldsymbol{w}_n - \boldsymbol{p})\mu, \quad \mu > 0$$

$$\boldsymbol{w}_{n+1} = (\mathbf{I} - \mu \boldsymbol{R})\boldsymbol{w}_n + \mu \boldsymbol{p}$$

**Convergence**:

$$\boldsymbol{c}_n = \boldsymbol{w}_n - \boldsymbol{w}_{\text{opt}}, \boldsymbol{c}_{n+1} = (\mathbf{I} - \mu \mathbf{R})\boldsymbol{c}_n$$

$$\boldsymbol{R} = \boldsymbol{Q}\boldsymbol{\Lambda}\boldsymbol{Q}^H, \boldsymbol{z}_n = \mathbf{Q}^{\mathrm{H}}\boldsymbol{c}_n, \boldsymbol{z}_{n+1} = (\mathbf{I} - \boldsymbol{\mu}\boldsymbol{\Lambda})\boldsymbol{z}_n$$

$$\Rightarrow \forall i \in \{1, 2, \ldots, m\} : |1 - \mu\lambda_i| < 1 \Leftrightarrow 0 < \mu < 2/\lambda_{\text{max}}$$

sufficient (not necessary): $0 < \mu < \frac{2}{\operatorname{tr} R}$

**Steepest Descent Algorithm**:

$\mu_{\text{opt}}$ is obtained by minimizing $\mathrm{MSE}_{n+1}$ wrt. $\mu$, i.e. $\frac{\partial \mathrm{MSE}_{n+1}}{\partial \mu^*} = 0$

$$\mathbf{MSE}_n = \boldsymbol{w}_n^{\mathrm{H}}\boldsymbol{R}\boldsymbol{w}_n - \boldsymbol{w}_n^{\mathrm{H}}\boldsymbol{p} - \boldsymbol{p}^{\mathrm{H}}\boldsymbol{w}_n + \sigma_d^2$$

$$\boldsymbol{w}_{n+1} = \boldsymbol{w}_n + \mu \boldsymbol{r}_n, \quad \boldsymbol{r}_n = \boldsymbol{p} - \boldsymbol{R}\boldsymbol{w}_n$$

**Steepest Descent Algorithm (SDA):**

*Input*: $\mathbb{C}^{m \times m} \ni \boldsymbol{R} = \boldsymbol{R}^{\mathrm{H}} > \boldsymbol{0}$, and $\boldsymbol{p} \in \mathbb{C}^{m\times 1}$, some $\epsilon > 0$ and initial value $\boldsymbol{w}_0$
*Output*: An approximate solution of $\boldsymbol{R}\boldsymbol{w} = \boldsymbol{p}$ for $\boldsymbol{w}$, such that $\|\boldsymbol{R}\boldsymbol{w} - \boldsymbol{p}\|_2^2 \leq \epsilon$.

1. Init: $\boldsymbol{w} \leftarrow \boldsymbol{w}_0$
2. Search direction: $\boldsymbol{r} \leftarrow \boldsymbol{p} - \boldsymbol{R}\boldsymbol{w}$
3. Test: if $\boldsymbol{r}^{\mathrm{H}}\boldsymbol{r} \leq \epsilon$ terminate and return $\boldsymbol{w}$.
4. Step-size: $\mu \leftarrow \boldsymbol{r}^{\mathrm{H}}\boldsymbol{r}/(\boldsymbol{r}^{\mathrm{H}}\boldsymbol{R}\boldsymbol{r})$
5. Update: $\boldsymbol{w} \leftarrow \boldsymbol{w} + \mu \boldsymbol{r}$
6. Continue: goto step 2.

Complexity: $M^2 \log M$ scalar arithmetic operations/series time-steps
Complexity with full parallelization: $(\log M)^2$ time-steps
Steps 2-5 require $4m^2 + 5n - 1$ scalar arithmetic operations per iteration
Number of iterations: $\approx 15(\log(M) - 1), M > 15$

## 5.9. SDA for constrained optimization

**Problem:**

$$\min_{w} \boldsymbol{w}^{\mathrm{H}}\boldsymbol{A}\boldsymbol{w}, \quad \text{s.t.} \quad \boldsymbol{B}^{\mathrm{H}}\boldsymbol{w} = \boldsymbol{c}$$

$\mathbb{C}^{M \times M} \ni \boldsymbol{A} = \boldsymbol{A}^{\mathrm{H}} > \boldsymbol{0}, \boldsymbol{B} \in \mathbb{C}^{M \times L}, \boldsymbol{c} \in \mathbb{C}^{L \times 1}$, rank $\boldsymbol{B} = L$

**Ansatz:**

Decompose solution into fixed term $\boldsymbol{w}_q$ and variable term $\boldsymbol{z}$

$$\boldsymbol{B}^{\mathrm{H}}\boldsymbol{w}_{\mathrm{q}} = \boldsymbol{c}, \quad \boldsymbol{z} \in \text{ null } \boldsymbol{B}^{\mathrm{H}}$$

$$\boldsymbol{w} = \boldsymbol{w}_q - \boldsymbol{z}$$

Parameterize $\boldsymbol{z}$ by $\boldsymbol{w}_a$:

$$\boldsymbol{B} = \underbrace{\begin{bmatrix} \boldsymbol{U}_1 & \boldsymbol{U}_2 \end{bmatrix}}_{\boldsymbol{U}} \begin{bmatrix} \boldsymbol{\Sigma}_1 & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{bmatrix} \begin{bmatrix} \boldsymbol{V}_1^{\mathrm{H}} \\ \boldsymbol{V}_2^{\mathrm{H}} \end{bmatrix}$$

$$\boldsymbol{z} = \boldsymbol{U}_2\boldsymbol{w}_{\mathrm{a}}, \quad \boldsymbol{w}_{\mathrm{a}} \in \mathbb{C}(M - L) \times 1 \Rightarrow \boldsymbol{z} \in \text{null}\boldsymbol{B}^H \forall \boldsymbol{w}_a$$

Obtaining $\boldsymbol{U}_2$ without SVD:

1. Init: $\boldsymbol{U} \leftarrow \begin{bmatrix} \boldsymbol{B} & \boldsymbol{F} \end{bmatrix}$, where $\boldsymbol{F} \in \mathbb{C}^{M \times (M-L)}$ has i.i.d. random components.
2. Orthogonalize with all yet orthogonalized columns.
   For $i = 0$ to $M - 1$:
   $$\boldsymbol{u}_i \leftarrow \boldsymbol{u}_i - \sum_{j=1}^{i-1} \boldsymbol{u}_j \left(\boldsymbol{u}_j^{\mathrm{H}}\boldsymbol{u}_i\right) / \left(\boldsymbol{u}_j^{\mathrm{H}}\boldsymbol{u}_j\right)$$
3. Normalize: For $i \in \{L{+}1, L{+}2, \ldots M\}$ do $\boldsymbol{u}_i \leftarrow \boldsymbol{u}_i / \sqrt{\boldsymbol{u}_i^{\mathrm{H}}\boldsymbol{u}_i}$
4. Output: $\boldsymbol{U}_2 \leftarrow \left[\boldsymbol{u}_{L+1}\boldsymbol{u}_{L+2} \cdots \boldsymbol{u}_M\right] \in \mathbb{C}^{M \times (M-L)}$

Finding $w_q$ by SDA:

$$w_q = \boldsymbol{B} \underbrace{\left(\boldsymbol{B}^{\mathrm{H}}\boldsymbol{B}\right)^{-1} c}_{q}$$

$$\left(\boldsymbol{B}^{\mathrm{H}}\boldsymbol{B}\right) q = c$$

$$w_q = \boldsymbol{B}q$$

Reformulate:

$$\min_{\boldsymbol{w}_{\mathrm{a}}} \left(\boldsymbol{w}_{\mathrm{q}}^{\mathrm{H}} - \boldsymbol{w}_{\mathrm{a}}^{\mathrm{H}}\boldsymbol{U}_2^{\mathrm{H}}\right) \boldsymbol{A} \left(\boldsymbol{w}_{\mathrm{q}} - \boldsymbol{U}_2\boldsymbol{w}_{\mathrm{a}}\right)$$

**Solution:** run SDA with:

$$\boldsymbol{R} = \boldsymbol{U}_2^{\mathrm{H}}\boldsymbol{A}\boldsymbol{U}_2 \in \mathbb{C}^{(M-L)\times(M-L)}, \quad \text{and} \quad \boldsymbol{p} = \boldsymbol{U}_2^{\mathrm{H}}\boldsymbol{A}\boldsymbol{w}_{\mathrm{q}}$$

## 5.10. Steepest Descent Procedure (SDP)

Problem: $R$ and $p$ are unknown and need to be estimated.
Drop assumption that $u[n]$ is WSS (e.g. channel changes) $\to \boldsymbol{R}[n]$
Estimation with exponential weighting:

$$\widehat{\boldsymbol{R}}[n] = \frac{\sum_{k=0}^{\infty} \boldsymbol{u}[n-k]\boldsymbol{u}^{\mathrm{H}}[n-k]\alpha[k]}{\sum_{k=0}^{\infty} \alpha[k]}$$

$$\alpha[k] = \begin{cases} 1 & \text{for } k = 0 \\ \eta^k & \text{for } k > 0 \\ 0 & \text{else} \end{cases}, \quad \sum_{k=0}^{\infty} \eta^k = \frac{1}{1 - \eta}$$

$$\widehat{\boldsymbol{R}}[n] = \eta\widehat{\boldsymbol{R}}[n-1] + (1-\eta)\boldsymbol{u}[n]\boldsymbol{u}^{\mathrm{H}}[n]$$

$$\widehat{\boldsymbol{p}}[n] = \eta\widehat{\boldsymbol{p}}[n-1] + (1-\eta)\boldsymbol{u}[n]d^*[n]$$

**Steepest Descent Procedure (SDP)**

1. Init: $\boldsymbol{w} \leftarrow \boldsymbol{0}_M$, $\widehat{\boldsymbol{R}} \leftarrow \mathbf{O}_{M\times M}$, $\widehat{\boldsymbol{p}} \leftarrow \boldsymbol{0}_M$, $n \leftarrow 0$
2. Update auto-correlation: $\widehat{\boldsymbol{R}} \leftarrow \eta\widehat{\boldsymbol{R}} + (1-\eta)\boldsymbol{u}[n]\boldsymbol{u}^{\mathrm{H}}[n]$
3. Update cross-correlation: If $d[n]$ available: $\widehat{\boldsymbol{p}} \leftarrow \eta\widehat{\boldsymbol{p}} + (1-\eta)\boldsymbol{u}[n]d^*[n]$
4. Update weight: $\boldsymbol{r} \leftarrow \widehat{\boldsymbol{p}} - \widehat{\boldsymbol{R}}\boldsymbol{w}$, $\boldsymbol{w} \leftarrow \boldsymbol{w} + \boldsymbol{r}\left(\boldsymbol{r}^{\mathrm{H}}\boldsymbol{r}\right) / \left(\boldsymbol{r}^{\mathrm{H}}\widehat{\boldsymbol{R}}\boldsymbol{r} + 2^{-52}\right)$
5. Output current weight: $\boldsymbol{w}[n] \leftarrow \boldsymbol{w}$
6. Next time index: $n \leftarrow n + 1$
7. Round ribbon loop: goto step 2.

## 5.11. SDP: Determining $\eta$

If $R[n]$ and $p[n]$ are changing slowly $\to$ choose $\eta$ close to 1
If $R[n]$ and $p[n]$ remain const for $N_1$ time slots, but have completely changed after $N_2 > N_1$:
$\eta^{N_2} = \gamma \ll 1$ and $\eta^{N_1} = 1 - \gamma$
$\to \eta = x^{1/N_1}, \quad$ where $\quad x^{N_2/N_1} + x - 1 = 0, \quad 0 < x < 1$
$N_2/N_1 = 30$ is a reasonable choice in practice $\to \eta = \exp\left(-\frac{2.5}{N_2}\right)$

**Radio Communications:**

$N_2$ is the number of samples in which the receiver has moved by $\lambda$. For sample rate (bandwidth) $B$, wavelength $\lambda$, and speed $v$:

$$\eta = \exp\left(-2.5\frac{v}{B\lambda}\right)$$

## 5.12. Least Mean Square (LMS)

Special Case of SDP for $\eta = 0$:

$$\widehat{\boldsymbol{R}}[n] = \boldsymbol{u}[n]\boldsymbol{u}^{\mathrm{H}}[n], \quad \widehat{\boldsymbol{p}}[n] = \boldsymbol{u}[n]d^*[n]$$

$$\mu = \frac{\boldsymbol{r}^{\mathrm{H}}\boldsymbol{r}}{\boldsymbol{r}^{\mathrm{H}}\widehat{\boldsymbol{R}}\boldsymbol{r}} = \frac{1}{\|\boldsymbol{u}[n]\|_2^2}$$

$$\boldsymbol{r} = \widehat{\boldsymbol{p}} - \widehat{\boldsymbol{R}}\boldsymbol{w} = \boldsymbol{u}[n]\underbrace{(d^*[n] - \overbrace{\boldsymbol{u}^{\mathrm{H}}[n]\boldsymbol{w}}^{y^*[n]})}_{e^*[n]} = \boldsymbol{u}[n]e^*[n]$$

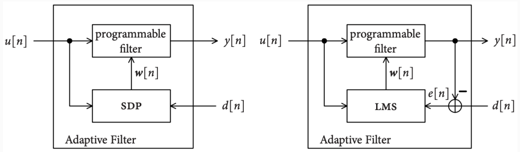
**Least Mean Square (LMS)**

1. Init: $\boldsymbol{w} \leftarrow \boldsymbol{0}_M$, $n \leftarrow 0$
2. Update weight: If $e[n]$ available: $\boldsymbol{w} \leftarrow \boldsymbol{w} + \boldsymbol{u}[n]\frac{e^*[n]}{\alpha + \|\boldsymbol{u}[n]\|^2}\beta$
3. Output current weight: $\boldsymbol{w}[n] \leftarrow \boldsymbol{w}$
4. Next time index: $n \leftarrow n + 1$
5. Round ribbon loop: goto step 2.

$\alpha > 0$: avoids very large step sizes, if $\boldsymbol{u}[n]$ is small
$\beta > 0$: is for fine-tuning (found experimentally as well as $\alpha$)

## 5.13. Block diagrams

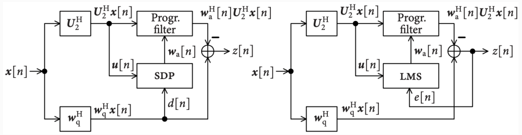
Standard SDP and LMS:




Constrained SDP and LMS:

$$z[n] = \boldsymbol{w}^{\mathrm{H}}[n]\boldsymbol{x}[n]$$

$$\min_{w[n]} \mathrm{E}\left[|z[n]|^2\right], \quad \text{s.t.} \quad \boldsymbol{B}^{\mathrm{H}}\boldsymbol{w}[n] = \boldsymbol{c}$$




$$\boldsymbol{x}[n] = s_i[n]\boldsymbol{b}_i \quad \Longrightarrow \quad z[n] = s_i[n]c_i^*$$

$$\boldsymbol{x}[n] \notin \operatorname{im} \boldsymbol{B} \quad \Longrightarrow \quad z[n] = 0$$

# 6. Least Squares

## 6.1. Least Squares

Only $\boldsymbol{A}$ is known, no information about $\boldsymbol{\Upsilon}$ → ignore it all together

**Least squares problem:**

$$\boldsymbol{X} \approx \boldsymbol{A}\boldsymbol{S}$$
$$\hat{\boldsymbol{S}}_{\text{LS}} = \arg\min_{\boldsymbol{S}} \|\boldsymbol{X} - \boldsymbol{A}\boldsymbol{S}\|_{\text{F}}^2$$

**Derivation:**

$$\mathcal{E} = \|\boldsymbol{X} - \boldsymbol{A}\boldsymbol{S}\|_{\text{F}}^2 = \operatorname{tr}\left((\boldsymbol{X} - \boldsymbol{A}\boldsymbol{S})^{\text{H}}(\boldsymbol{X} - \boldsymbol{A}\boldsymbol{S})\right)$$

$$\frac{\partial \mathcal{E}}{\partial \boldsymbol{S}^*} = -\boldsymbol{A}^{\text{H}}\boldsymbol{X} + \boldsymbol{A}^{\text{H}}\boldsymbol{A}\boldsymbol{S} \leftarrow \frac{\partial \operatorname{tr}\left(\boldsymbol{S}^{\text{H}}\boldsymbol{B}\right)}{\partial \boldsymbol{S}^*} = \boldsymbol{B}$$

$$\boxed{\begin{aligned}\hat{\boldsymbol{S}}_{\text{LS}} &= \left(\boldsymbol{A}^{\text{H}}\boldsymbol{A}\right)^{-1}\boldsymbol{A}^{\text{H}}\boldsymbol{X} \text{ (full column rank)} \\ &= \boldsymbol{A}^{+}\boldsymbol{X} = \boldsymbol{A}^{+}\boldsymbol{A}\boldsymbol{S} + \boldsymbol{A}^{+}\boldsymbol{\Upsilon} \text{ (general)}\end{aligned}}$$

$$\boldsymbol{A} \text{ has full column rank } \implies \hat{\boldsymbol{S}}_{\text{LS}} = \boldsymbol{S} + \boldsymbol{A}^{+}\boldsymbol{\Upsilon}$$
$$\mathrm{E}[\boldsymbol{\Upsilon}] = \boldsymbol{0} \implies \text{ unbiased}$$

Estimation noise:

$$\mathrm{E}\left[\left\|\boldsymbol{A}^{+}\boldsymbol{\Upsilon}\right\|_{\text{F}}^2\right] = \operatorname{tr}\left(\boldsymbol{A}^{+}\mathrm{E}\left[\boldsymbol{\Upsilon}\boldsymbol{\Upsilon}^{\text{H}}\right]\boldsymbol{A}^{+\text{H}}\right)$$

For white noise $\mathrm{E}\left[\boldsymbol{\Upsilon}\boldsymbol{\Upsilon}^{\text{H}}\right] = \mathbf{I}$ and full column rank $\boldsymbol{A}$:

$$\mathrm{E}\left[\left\|\boldsymbol{A}^{+}\boldsymbol{\Upsilon}\right\|_{\text{F}}^2\right] = \operatorname{tr}\left(\left(\boldsymbol{A}^{\text{H}}\boldsymbol{A}\right)^{-1}\right) = \sum_i \frac{1}{\lambda_i},$$

where $\lambda_i$ are the eigenvalues of $\boldsymbol{A}^H\boldsymbol{A}$ → minimal if all are the same

For white observation noise the smallest possible variance is achieved iff:

$$\boldsymbol{A}^{\text{H}}\boldsymbol{A} = c\mathbf{I}, \quad \text{for any } c > 0$$

among all matrices $\boldsymbol{A} \in \mathbb{C}^{M \times d}$ with $\|\boldsymbol{A}\|_{\text{F}}^2 = c$.

- $\boldsymbol{A}^{\text{H}}\boldsymbol{A}$ has $\lambda_i = c/d$
- $\boldsymbol{A}$ must have orthogonal columns with the same euclidean norm
- use orthogonal pilot sequences
- purely deterministic approach, no need to know statistical properties

## 6.2. Pilot Sequence

**Setup:**

$$\boldsymbol{u} = \boldsymbol{H}\boldsymbol{p} + \boldsymbol{v} \iff \boldsymbol{u} = \boldsymbol{A}\boldsymbol{h} + \boldsymbol{v}$$

$$\boldsymbol{A} = \begin{bmatrix} P_0 & P_1 & \cdots & P_K \\ P_1 & P_2 & \cdots & P_{K+1} \\ \vdots & \vdots & \vdots & \vdots \\ P_{q-K-1} & P_{q-K} & \cdots & P_{q-1} \end{bmatrix}$$

$$\boldsymbol{h} = \begin{bmatrix} h_0 & h_1 & \cdots & h_K \end{bmatrix}^{\text{T}}$$

Necessary for full column rank: $q \geq 2K + 1$

## 6.3. LS curve fitting

$$\hat{y}(x) = \frac{a}{x} + b + cx + dx^2 + ex^3$$

$$\underbrace{\begin{bmatrix} 1/x_1 & 1 & x_1 & x_1^2 & x_1^3 \\ 1/x_2 & 1 & x_2 & x_2^2 & x_2^3 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1/x_N & 1 & x_N & x_N^2 & x_N^3 \end{bmatrix}}_{\boldsymbol{A}} \underbrace{\begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix}}_{\boldsymbol{w}} \approx \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix}}_{\boldsymbol{y}}$$

$$\boldsymbol{w}_{\text{LS}} = \boldsymbol{A}^{+}\boldsymbol{y}$$

## 6.4. Numerical integration with LS

Within every 3x3 window:

$$\hat{f}(x, y) = a + bx^2 + cy^2 + dx^2y^2$$

$$\begin{bmatrix} 1 & h_x^2 & h_y^2 & h_x^2h_y^2 \\ 1 & 0 & h_y^2 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & h_x^2 & h_y^2 & h_x^2h_y^2 \end{bmatrix} \underbrace{\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}}_{w} \approx \begin{bmatrix} f(-h_x, h_y) \\ f(0, h_y) \\ \vdots \\ f(h_x, -h_y) \end{bmatrix}$$

$$\hat{F} = \frac{4}{9}h_xh_y\left(9a + 3bh_x^2 + \left(3c + dh_x^2\right)h_y^2\right)$$

And without solving $w_{\text{LS}} = \boldsymbol{A}^{+}\boldsymbol{f}$:

$$\begin{aligned}\hat{F} = &(f(-h_x, h_y) + f(h_x, h_y) + f(-h_x, -h_y) + f(h_x, -h_y)) \\ &+ 4(f(0, h_y) + f(-h_x, 0) + f(h_x, 0) + f(0, -h_y)) \\ &+ 16f(0, 0))\frac{h_xh_y}{9}\end{aligned}$$

3x3 window: quadratic convergence when increasing $M$

5x5 window: cubic convergence when increasing $M$

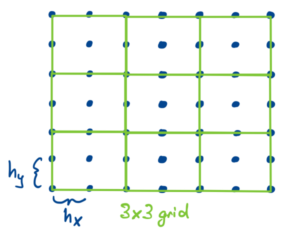
Procedure:

$$F = \int_{y=y_{\min}}^{y_{\max}} \int_{x=x_{\min}}^{x_{\max}} f(x, y)\mathrm{d}x\,\mathrm{d}y$$

$$h_x = \frac{x_{\max} - x_{\min}}{M - 1}, \quad h_y = \frac{y_{\max} - y_{\min}}{M - 1}$$

3x3 window: $M \in \{3, 5, 7, \ldots\}$, 5x5 window: $M \in \{5, 9, 13, \ldots\}$

Divide the integration region into $M$x$M$ parts. Move the local approximation by $3 - 1$, or $5 - 1$ grid points and sum up all.




## 6.5. Least squares as a projection

**Setup:**

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \quad \text{where} \quad \boldsymbol{b} \notin \operatorname{im}\boldsymbol{A}$$
$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} + \Delta\boldsymbol{b}, \quad \text{where} \quad \boldsymbol{b} + \Delta\boldsymbol{b} \in \operatorname{im}\boldsymbol{A}$$
$$\Delta\boldsymbol{b}_{\text{opt}} = \arg\min_{\Delta\boldsymbol{b}} \|\Delta\boldsymbol{b}\|_2^2, \quad \text{s.t.} \quad \boldsymbol{b} + \Delta\boldsymbol{b} \in \operatorname{im}\boldsymbol{A}$$

Derivation:

$$\Delta\boldsymbol{b}_{\text{opt}} = \arg\min_{\Delta\boldsymbol{b}} \|\Delta\boldsymbol{b}\|_2^2, \quad \text{s.t.} \quad \boldsymbol{P}(\boldsymbol{b} + \boldsymbol{\Delta}\boldsymbol{b}) = \boldsymbol{b} + \boldsymbol{\Delta}\boldsymbol{b}$$

$$\Delta\boldsymbol{b}_{\text{opt}} = \arg\min_{\Delta\boldsymbol{b}} \|\Delta\boldsymbol{b}\|_2^2, \quad \text{s.t.} \quad (\mathbf{I} - \boldsymbol{P})(\boldsymbol{b} + \Delta\boldsymbol{b}) = \boldsymbol{0}$$

Lagrangian optimization yields:

$$\Delta\boldsymbol{b} = -(\mathbf{I} - \boldsymbol{P})\boldsymbol{b}$$

The least squares solution $\boldsymbol{x} = \boldsymbol{A}^{+}\boldsymbol{b}$ is the exact solution of $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{P}\boldsymbol{b}$. I.e. the least squares estimation projects the measurements on $\operatorname{im}\boldsymbol{A}$

$$\boldsymbol{P} = \boldsymbol{A}\boldsymbol{A}^{+}$$

## 6.6. Total Least Squares

**Setup:**

$$\min \left\|\begin{bmatrix} \Delta\boldsymbol{A} & \Delta\boldsymbol{b} \end{bmatrix}\right\|_F^2 \quad \text{s.t.} \quad (\boldsymbol{A} + \Delta\boldsymbol{A})\boldsymbol{x} = \boldsymbol{b} + \Delta\boldsymbol{b}$$

Rewrite:

$$\begin{bmatrix} \boldsymbol{A} & \boldsymbol{b} \end{bmatrix}\begin{bmatrix} x \\ -1 \end{bmatrix} = 0, \quad \boldsymbol{b} \notin \operatorname{im}\boldsymbol{A}$$

$\boldsymbol{b} \notin \operatorname{im}\boldsymbol{A}$, thus: $\operatorname{rank}[\boldsymbol{A}\boldsymbol{b}] = N + 1$ and $\operatorname{null}[\boldsymbol{A}\boldsymbol{b}] = \{0\}$

$$\begin{bmatrix} \boldsymbol{A} & \boldsymbol{b} \end{bmatrix} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^{\text{H}} = \sum_{k=1}^{N+1} s_k\boldsymbol{u}_k\boldsymbol{v}_k^{\text{H}}$$

Allow for solutions by increasing the dimensionality of the nullspace to 1:

$$\begin{bmatrix} \boldsymbol{A} + \Delta\boldsymbol{A} & \boldsymbol{b} + \Delta\boldsymbol{b} \end{bmatrix} = \sum_{k=1}^{N} s_k\boldsymbol{u}_k\boldsymbol{v}_k^{\text{H}}$$

$$\begin{bmatrix} \boldsymbol{A} + \Delta\boldsymbol{A} & \boldsymbol{b} + \Delta\boldsymbol{b} \end{bmatrix}\boldsymbol{v}_{N+1}\alpha = \boldsymbol{0}$$

The solution is optimal as we used the best approximation by Eckart-Young

$$\boxed{\boldsymbol{x} = \frac{-\begin{bmatrix} \mathbf{I}_N & \mathbf{0}_N \end{bmatrix}\boldsymbol{v}_{N+1}}{\begin{bmatrix} \mathbf{0}_N^{\text{T}} & 1 \end{bmatrix}\boldsymbol{v}_{N+1}}}$$

# 7. BLUE

## 7.1. Best Linear Unbiased Estimator (BLUE)

$\boldsymbol{A}$ and $\mathrm{E}\left[\boldsymbol{\Upsilon}\boldsymbol{\Upsilon}^{\text{H}}\right]$ are known

**Setup:**

$$\boldsymbol{X} = \boldsymbol{A}\boldsymbol{S} + \boldsymbol{\Upsilon}$$

$$\boldsymbol{W}_{\text{BLUE}} = \arg\min_{\boldsymbol{W}} \mathrm{E}\left[\left\|\boldsymbol{W}^{\text{H}}\boldsymbol{\Upsilon}\right\|_{\text{F}}^2\right] \quad \text{s.t.} \quad \boldsymbol{W}^{\text{H}}\boldsymbol{A} = \mathbf{I}$$

→ minimize estimation variance while being unbiased

Derivation:

$$\mathrm{E}\left[\left\|\boldsymbol{W}^{\text{H}}\boldsymbol{\Upsilon}\right\|_{\text{F}}^2\right] = \operatorname{tr}\left(W^{\text{H}}\mathrm{E}\left[\boldsymbol{\Upsilon}\boldsymbol{\Upsilon}^{\text{H}}\right]\boldsymbol{W}\right)$$

$$\boldsymbol{W}^{\text{H}}\boldsymbol{A} = \mathbf{I} \iff \forall i : \boldsymbol{w}_i^{\text{H}}\boldsymbol{A} = \mathbf{e}_i^{\text{T}}$$

Lagrangian optimization yields:

optimal weights:

$$\boldsymbol{W}_{\text{BLUE}} = \left(\mathrm{E}\left[\boldsymbol{\Upsilon}\boldsymbol{\Upsilon}^{\text{H}}\right]\right)^{-1}\boldsymbol{A}\left(\boldsymbol{A}^{\text{H}}\left(\mathrm{E}\left[\boldsymbol{\Upsilon}\boldsymbol{\Upsilon}^{\text{H}}\right]\right)^{-1}\boldsymbol{A}\right)^{-1}$$

minimal noise variance:

$$\mathrm{E}\left[\left\|\boldsymbol{W}_{\text{BLUE}}^{\text{H}}\boldsymbol{\Upsilon}\right\|_{\text{F}}^2\right] = \operatorname{tr}\left(\left(\boldsymbol{A}^{\text{H}}\left(\mathrm{E}\left[\boldsymbol{\Upsilon}\boldsymbol{\Upsilon}^{\text{H}}\right]\right)^{-1}\boldsymbol{A}\right)^{-1}\right)$$

**Special case:** $\mathrm{E}\left[\boldsymbol{\Upsilon}\boldsymbol{\Upsilon}^{\text{H}}\right] = \mathbf{I}\sigma_v^2$ white observation noise:

$$\boldsymbol{W}_{\text{BLUE}}^{\text{H}} = \boldsymbol{A}^{+} = \boldsymbol{W}_{\text{LS}}^{\text{H}}$$

- noise must be colored in order for BLUE to improve the result
- $\boldsymbol{S} = 0$ must be transmitted to estimate $\mathrm{E}\left[\boldsymbol{\Upsilon}\boldsymbol{\Upsilon}^{\text{H}}\right]$ → extra payload
- if noise is WSS, $\mathrm{E}\left[\boldsymbol{\Upsilon}\boldsymbol{\Upsilon}^{\text{H}}\right]$ can be estimated once
- if noise is not WSS: tradeoff between tracking performance and estimation error

## 7.2. BLUE for uncorrelated signal and noise

**Special case:** noise and signal are uncorrelated:

For $\mathrm{E}\left[\boldsymbol{S}\boldsymbol{\Upsilon}^{\text{H}}\right] = 0$:

$$\boxed{\boldsymbol{W}_{\text{BLUE}} = \left(\mathrm{E}\left[\boldsymbol{X}\boldsymbol{X}^{\text{H}}\right]\right)^{-1}\boldsymbol{A}\left(\boldsymbol{A}^{\text{H}}\left(\mathrm{E}\left[\boldsymbol{X}\boldsymbol{X}^{\text{H}}\right]\right)^{-1}\boldsymbol{A}\right)^{-1}}$$

If the statistics are unknown, they can be estimated using the observation matrix $\boldsymbol{X}$:

$$\widehat{\boldsymbol{W}}_{\text{BLUE}} = \left(\boldsymbol{X}\boldsymbol{X}^{\text{H}}\right)^{-1}\boldsymbol{A}\left(\boldsymbol{A}^{\text{H}}\left(\boldsymbol{X}\boldsymbol{X}^{\text{H}}\right)^{-1}\boldsymbol{A}\right)^{-1}$$

- no need to switch of signal for estimating the noise variance
- if noise is WSS: estimation is easy (once)
- if noise is not WSS: tradeoff between estimation error and tracking performance
- $XX^H$ must be invertible → needs enough samples
- loss of generality
- latency: samples $\boldsymbol{X}$ have to be gathered before estimation can start (LS can start immideately)

## 7.3. BLUE for multi-user detection

**General Setup:**

Q users with signals $\tilde{\boldsymbol{S}}_i$ and full column rank channels $\tilde{H}_i$, additive white noise, signals $S_i$ and $S_j$ are uncorrelated for $i \neq j$

$$\boldsymbol{X} = \sum_{i=1}^{Q} \tilde{\boldsymbol{H}}_i\tilde{\boldsymbol{S}}_i + \boldsymbol{\Theta}, \quad \mathrm{E}\left[\boldsymbol{\Theta}\boldsymbol{\Theta}^{\text{H}}\right] = \mathbf{I}, \quad \mathrm{E}\left[\tilde{\boldsymbol{S}}_i\tilde{\boldsymbol{S}}_i^{\text{H}}\right] = \sigma_i^2\mathbf{I}$$

**Normalized Setup:** $\boldsymbol{H}_i = \tilde{\boldsymbol{H}}_i\sigma_i$, and $\boldsymbol{S}_i = \tilde{\boldsymbol{S}}_i/\sigma_i$

$$\boldsymbol{X} = \sum_{i=1}^{Q} \boldsymbol{H}_i\boldsymbol{S}_i + \boldsymbol{\Theta}, \quad \mathrm{E}\left[\boldsymbol{\Theta}\boldsymbol{\Theta}^{\text{H}}\right] = \mathbf{I}, \quad \mathrm{E}\left[\boldsymbol{S}_i\boldsymbol{S}_i^{\text{H}}\right] = \mathrm{I}$$

For reconstructing the signal of the k-th user:

$$\boldsymbol{X} = \boldsymbol{H}_k\boldsymbol{S}_k + \underbrace{\sum_{i=1, i\neq k}^{Q} \boldsymbol{H}_i\boldsymbol{S}_i + \boldsymbol{\Theta}}_{\boldsymbol{\Upsilon}_k} = \boldsymbol{H}_k\boldsymbol{S}_k + \boldsymbol{\Upsilon}_k$$

Solution for the BLUE:

$$\boldsymbol{W}_k = \left(\mathrm{E}\left[\boldsymbol{X}\boldsymbol{X}^{\text{H}}\right]\right)^{-1}\boldsymbol{H}_k\left(\boldsymbol{H}_k^{\text{H}}\left(\mathrm{E}\left[\boldsymbol{X}\boldsymbol{X}^{\text{H}}\right]\right)^{-1}\boldsymbol{H}_k\right)^{-1}$$

$$\mathrm{E}\left[\boldsymbol{X}\boldsymbol{X}^{\text{H}}\right] = \mathbf{I} + \sum_{i=1}^{Q} \boldsymbol{H}_i\boldsymbol{H}_i^{\text{H}}$$

Procedure for estimating all signals:

1. Find the signal with the lowest estimation noise, by computing all noises and finding the minimum

$$\begin{aligned}\xi_k &= \mathrm{E}\left[\left\|\boldsymbol{W}_k^{\text{H}}\boldsymbol{\Upsilon}_k\right\|_{\text{F}}^2\right] \\ &= \operatorname{tr}\left(\boldsymbol{W}_k^{\text{H}}\left(\mathbf{I} + \sum_{i=1, i\neq k}^{Q} \boldsymbol{H}_i\boldsymbol{H}_i^{\text{H}}\right)\boldsymbol{W}_k\right)\end{aligned}$$

$$k_* = \arg\min_{k\in\{1,2,\ldots Q\}} \xi_k$$

2. compute the estimated signal and subtract it from the observation

$$\hat{\boldsymbol{S}}_{k_*} = \boldsymbol{W}_{k_*}^{\text{H}}\boldsymbol{X}$$
$$\boldsymbol{X} \leftarrow \boldsymbol{X} - \boldsymbol{H}_{k_*}\boldsymbol{S}_{k_*}$$

3. repeat the procedure with the remaining measurement

- BLUE usually does not yield the minimum MSE (also unbiased)
- optimum filter wrt. MSE is given by: $(\mathrm{E}[\|\boldsymbol{W}^H\boldsymbol{X} - \boldsymbol{S}\|_{\text{F}}^2])$

$$\boldsymbol{W}_{\text{opt}} = \left(\mathrm{E}\left[\boldsymbol{X}\boldsymbol{X}^{\text{H}}\right]\right)^{-1}\mathrm{E}\left[\boldsymbol{X}\boldsymbol{S}^{\text{H}}\right]$$

# 8. MUSIC

## 8.1. MUSIC: Multiple Signal Classification

Nothing is known except for algebraic structure

**Setup and restrictions:**

$$\boldsymbol{X} = \boldsymbol{A}\boldsymbol{S} + \boldsymbol{\Upsilon}$$

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{a}(\theta_1) & \boldsymbol{a}(\theta_2) & \cdots & \boldsymbol{a}(\theta_d) \end{bmatrix} \in \mathbb{C}^{M\times d}$$

$$\boldsymbol{a}(\theta) = \begin{bmatrix} a_1(\theta) \\ a_2(\theta) \\ \vdots \\ a_M(\theta) \end{bmatrix}$$

$$\operatorname{rank}\left(\boldsymbol{S} \in \mathbb{C}^{d\times N}\right) = d, \quad \operatorname{rank}(\boldsymbol{A}) = d, d < M$$

$$\theta_1, \theta_2, \ldots, \theta_M \text{ pairwise different}$$

$$\Downarrow$$

$$\begin{pmatrix} \boldsymbol{a}(\theta_1) & \boldsymbol{a}(\theta_2) & \cdots & \boldsymbol{a}(\theta_M) \end{pmatrix} \text{ are linearly independent}$$

**Goal:** find $d$, $\theta_i$, and $\boldsymbol{S}$

**Derivation without noise:**

$$\operatorname{im}(\boldsymbol{A}) = \operatorname{im}\left(\boldsymbol{A}\boldsymbol{S}\boldsymbol{S}^{\mathrm{H}}\boldsymbol{A}^{\mathrm{H}}\right)$$

$$\boldsymbol{X}\boldsymbol{X}^{H} = \boldsymbol{A}\boldsymbol{S}\boldsymbol{S}^{\mathrm{H}}\boldsymbol{A}^{\mathrm{H}} = [\underbrace{\boldsymbol{u}_1 \cdots \boldsymbol{u}_d}_{\boldsymbol{U}_1}\underbrace{\boldsymbol{u}_{d+1} \cdots \boldsymbol{u}_M}_{\boldsymbol{U}_2}]\boldsymbol{\Lambda}\boldsymbol{U}^{\mathrm{H}}$$

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d > 0$$

$$\operatorname{im}(\boldsymbol{A}) = \operatorname{im}(\boldsymbol{U}_1) \to y \in \operatorname{im}(\boldsymbol{A}) \iff \boldsymbol{U}_2^{\mathrm{H}}\boldsymbol{y} = 0$$

$$\forall i \in \{1, 2, \ldots, d\}: \quad \boldsymbol{U}_2^{\mathrm{H}}\boldsymbol{a}(\theta_i) = \boldsymbol{0}$$

In summary:

$$\boxed{\left\|\boldsymbol{U}_2^{\mathrm{H}}\boldsymbol{a}(\theta)\right\|_2^2 \begin{cases} = 0 & \text{for } \theta \in \{\theta_1, \theta_2, \ldots, \theta_d\} \\ > 0 & \text{else.} \end{cases}}$$

**Modification with noise:**

$$\mathrm{E}\left[\boldsymbol{\Upsilon}\boldsymbol{\Upsilon}^{\mathrm{H}}\right] = \sigma_\Upsilon^2\mathbf{I}, \quad \text{and} \quad \mathrm{E}\left[\boldsymbol{S}\boldsymbol{\Upsilon}^{\mathrm{H}}\right] = \mathbf{O}$$

$$\mathrm{E}\left[\boldsymbol{X}\boldsymbol{X}^{\mathrm{H}}\right]$$

$$= [\boldsymbol{U}_1\boldsymbol{U}_2]\operatorname{diag}(\lambda_1 + \sigma_\Upsilon^2, \ldots, \lambda_d + \sigma_\Upsilon^2, \sigma_\Upsilon^2, \ldots, \sigma_\Upsilon^2)[\boldsymbol{U}_1\boldsymbol{U}_2]^{H}$$

Approximation:

$$\boldsymbol{X}\boldsymbol{X}^{H} = [\hat{\boldsymbol{U}}_1\hat{\boldsymbol{U}}_2]\operatorname{diag}(\hat{\lambda}_1, \ldots, \hat{\lambda}_M)[\hat{\boldsymbol{U}}_1\hat{\boldsymbol{U}}_2]^{H}$$

For determining $d$, find the last significant drop in $\hat{\lambda}_i$.

**MUSIC spectrum:** has maxima close to $\theta_i$

$$\boxed{F(\theta) = \frac{\|\boldsymbol{a}(\theta)\|_2^2}{\left\|\widehat{\boldsymbol{U}}_2^{\mathrm{H}}\boldsymbol{a}(\theta)\right\|_2^2}}$$

Procedure:

1. Compute eigenvalue decomposition of $XX^H$
2. Scan eigenvalues for jumps and determine $\hat{d}$
3. Find locations $\hat{\theta}_i$ at $\hat{d}$ strongest peaks of the MUSIC spectrum $F(\theta)$
4. Form the matrix $\hat{\boldsymbol{A}} = \begin{bmatrix} \boldsymbol{a}\left(\widehat{\theta}_1\right) & \boldsymbol{a}\left(\widehat{\theta}_2\right) & \cdots & \boldsymbol{a}\left(\widehat{\theta}_{\hat{d}}\right) \end{bmatrix}$
5. Solve for $\hat{\boldsymbol{S}}$ with pseudo-inverse $\hat{\boldsymbol{S}} = \hat{\boldsymbol{A}}^{+}\boldsymbol{X}$

Signal-to-noise ratio:

$$\mathrm{SNR} = \frac{\mathrm{E}\left[\|\boldsymbol{A}\boldsymbol{S}\|_{\mathrm{F}}^2\right]}{\mathrm{E}\left[\|\boldsymbol{\Upsilon}\|_{\mathrm{F}}^2\right]}$$

Maximum number of signals:
At most $d = M - 1$ complex components can be resolved $\to \boldsymbol{U}_2$ does not exist for larger $d$. Otherwise, $M$ has to be increased by either lowering $N$ or increasing the number if samples.

Sharpness of the peaks:
The peaks get sharper when $N$ is increased $\to \hat{\boldsymbol{U}}_2$ more accurate. The higher $N$, the closer targets can be resolved.

**MUSIC with colored noise:**

$$\begin{aligned} \boldsymbol{X}' &= \left(\mathrm{E}\left[\boldsymbol{\Upsilon}\boldsymbol{\Upsilon}^{\mathrm{H}}\right]\right)^{-1/2}\boldsymbol{X} \\ &= \underbrace{\left(\mathrm{E}\left[\boldsymbol{\Upsilon}\boldsymbol{\Upsilon}^{\mathrm{H}}\right]\right)^{-1/2}\boldsymbol{A}}_{\boldsymbol{A}'}\boldsymbol{S} + \underbrace{\left(\mathrm{E}\left[\boldsymbol{\Upsilon}\boldsymbol{\Upsilon}^{\mathrm{H}}\right]\right)^{-1/2}\boldsymbol{\Upsilon}}_{\boldsymbol{\Upsilon}'} \\ &= \boldsymbol{A}'\boldsymbol{S} + \boldsymbol{\Upsilon}'. \end{aligned}$$

## 8.2. MUSIC for multiple frequency estimation

**Setup:**

$$\boldsymbol{a}(\omega_i) = \begin{bmatrix} 1 \\ \mathrm{e}^{\mathrm{j}\omega_i T} \\ \mathrm{e}^{\mathrm{j}2\omega_i T} \\ \vdots \\ \mathrm{e}^{\mathrm{j}(M-1)\omega_i T} \end{bmatrix}$$

$$x[n] = \sum_{i=1}^{d} c_i \mathrm{e}^{\mathrm{j}\omega_i T n}$$

$$\boldsymbol{x}[n] = \begin{bmatrix} x[n] \\ x[n+1] \\ x[n+2] \\ \vdots \\ x[n+M-1] \end{bmatrix} = \sum_{i=1}^{d} \begin{bmatrix} c_i \mathrm{e}^{\mathrm{j}\omega_i T n} \\ c_i \mathrm{e}^{\mathrm{j}\omega_i T(n+1)} \\ c_i \mathrm{e}^{\mathrm{j}\omega_i T(n+2)} \\ \vdots \\ c_i \mathrm{e}^{\mathrm{j}\omega_i T(n+M-1)} \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} \boldsymbol{a}(\omega_1) & \cdots & \boldsymbol{a}(\omega_d) \end{bmatrix}}_{\boldsymbol{A}} \underbrace{\begin{bmatrix} c_1 \mathrm{e}^{\mathrm{j}\omega_1 T n} \\ c_2 \mathrm{e}^{\mathrm{j}\omega_2 T n} \\ \vdots \\ c_d \mathrm{e}^{\mathrm{j}\omega_d T n} \end{bmatrix}}_{\boldsymbol{s}[n]} = \boldsymbol{A}\boldsymbol{s}[n]$$

$$\boldsymbol{X} = \begin{bmatrix} \boldsymbol{x}[n] & \boldsymbol{x}[n+1] & \cdots & \boldsymbol{x}[n+N-1] \end{bmatrix}$$

$\boldsymbol{A}$ is Vandermode $\to$ for pairwise different $\theta_i$, the columns of $\boldsymbol{A}$ are LI

## 8.3. Sampling of plane waves

$$p(t, \vec{r}) = P(t - \vec{\kappa}\cdot\vec{r}/c)$$

$\vec{\kappa}$: direction unit vector, $c$: propagation speed
Harmonic planar wave:

$$p(t, \underline{r}) = A\cos\left(\omega_0 t - k_0\vec{\kappa}\cdot\vec{r} + \varphi\right), k_0 = \omega_0/c = 2\pi f_0/c$$

Modulated harmonic planar wave:

$$p(t, \vec{r}) = A(t - \vec{\kappa}\cdot\vec{r}/c)\cos\left(\omega_0 t - k_0\vec{\kappa}\cdot\vec{r} + \varphi(t - \vec{\kappa}\cdot\vec{r}/c)\right)$$

If changes occur only slowly in time (approx. constant withing one period):
$A(t - \lambda_0/c) = A(t - 1/f_0) \approx A(t)$
$\varphi(t - \lambda_0/c) = \varphi(t - 1/f_0) \approx \varphi(t)$

$$p\left(t, \vec{r} + \vec{\kappa}\lambda_0\right) = p(t, \vec{r})$$

Short notation with complex numbers:

$$s(t, \vec{r}) = A(t - \vec{\kappa}\cdot\vec{r}/c)\mathrm{e}^{-\mathrm{j}k_0\vec{\kappa}\cdot\vec{r} + \mathrm{j}\varphi(t - \vec{\kappa}\cdot\vec{r}/c)}$$

$$p(t, \vec{r}) = \operatorname{Re}\left\{s(t, \vec{r})\mathrm{e}^{\mathrm{j}\omega_0 t}\right\}$$

After modulating to the baseband and filtering with a LP:

$$\boldsymbol{x}[n] = \begin{bmatrix} s\left(nT, \vec{r}_0\right) \\ s\left(nT, \vec{r}_1\right) \\ \vdots \\ s\left(nT, \vec{r}_{M-1}\right) \end{bmatrix}$$

If sampling points are close together: $\forall i: \quad |\vec{r}_i| \ll cT, \ (cT/30)$

$$\begin{aligned} \boldsymbol{x}[n] &= \underbrace{A(nT)\mathrm{e}^{j\varphi(nT)}\mathrm{e}^{j\psi}}_{s[n]} \overbrace{\begin{bmatrix} \mathrm{e}^{-j(k_0\vec{\kappa}\cdot\vec{r}_0 + \psi)} \\ \mathrm{e}^{-j(k_0\vec{\kappa}\cdot\vec{r}_1 + \psi)} \\ \vdots \\ \mathrm{e}^{-j\left(k_0\vec{\kappa}\cdot\vec{r}_{M-1} + \psi\right)} \end{bmatrix}}^{\boldsymbol{a}(\vec{\kappa})} \\ &= s[n]\boldsymbol{a}(\vec{\kappa}) \end{aligned}$$

The angle $\psi$ is arbitrary as it cancels out $\to$ used for simplifying

## 8.4. MUSIC for DOA estimation using ULAs

M sensors on a line with constant distance $\delta$ along the z-axis:

$$\vec{r}_i = \left(\frac{1}{2}(M-1) - i\right)\delta\vec{e}_z, \quad i \in \{0, 1, \ldots, M-1\}$$

$$\vec{\kappa} = -\cos(\phi)\sin(\theta)\mathbf{e}_x - \sin(\phi)\sin(\theta)\mathbf{e}_y - \cos(\theta)\mathbf{e}_z$$

The angle is measured from the side of the ULA. The steering vector becomes:

$$\boldsymbol{a}(\theta) = \begin{bmatrix} 1 \\ \mathrm{e}^{-\mathrm{j}k_0\delta\cos\theta} \\ \mathrm{e}^{-2\mathrm{j}k_0\delta\cos\theta} \\ \vdots \\ \mathrm{e}^{-(M-1)\mathrm{j}k_0\delta\cos\theta} \end{bmatrix}$$

with

$$\psi = \frac{1}{2}(M-1)k_0\delta\cos\theta$$

spacial normalized angular frequency:

$$\mu = -k_0\delta\cos\theta = -2\pi\frac{\delta}{\lambda_0}\cos\theta$$

$$\boldsymbol{a}(\mu) = \begin{bmatrix} 1 \\ \mathrm{e}^{\mathrm{j}\mu} \\ \mathrm{e}^{\mathrm{j}2\mu} \\ \vdots \\ \mathrm{e}^{\mathrm{j}(M-1)\mu} \end{bmatrix}$$

aliasing:

$$\boldsymbol{a}(\mu) = \boldsymbol{a}(\mu + 2\pi n), \quad n \in \{\pm 1, \pm 2, \cdots\}$$

$$\text{no directional aliasing} \iff k_0\delta < \pi \iff \delta < \lambda_0/2$$

If $0^\circ$ is chosen to be orthogonal to the ULA and measured from the center to the right:

$$\boldsymbol{a}(\theta) = \begin{bmatrix} 1 \\ \mathrm{e}^{-\mathrm{j}k_0\delta\sin\theta} \\ \mathrm{e}^{-2\mathrm{j}k_0\delta\sin\theta} \\ \vdots \\ \mathrm{e}^{-(M-1)jk_0\delta\sin\theta} \end{bmatrix}$$

## 8.5. Useful identities

$$1 + z + z^2 + \cdots + z^{M-1} = \frac{z^M - 1}{z - 1}$$

$$z^M - 1 = 0 \Leftrightarrow z = e^{j2\pi n/M}, \quad n \in \{\pm 1, \pm 2, \ldots\}$$

$$\boldsymbol{A}^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$