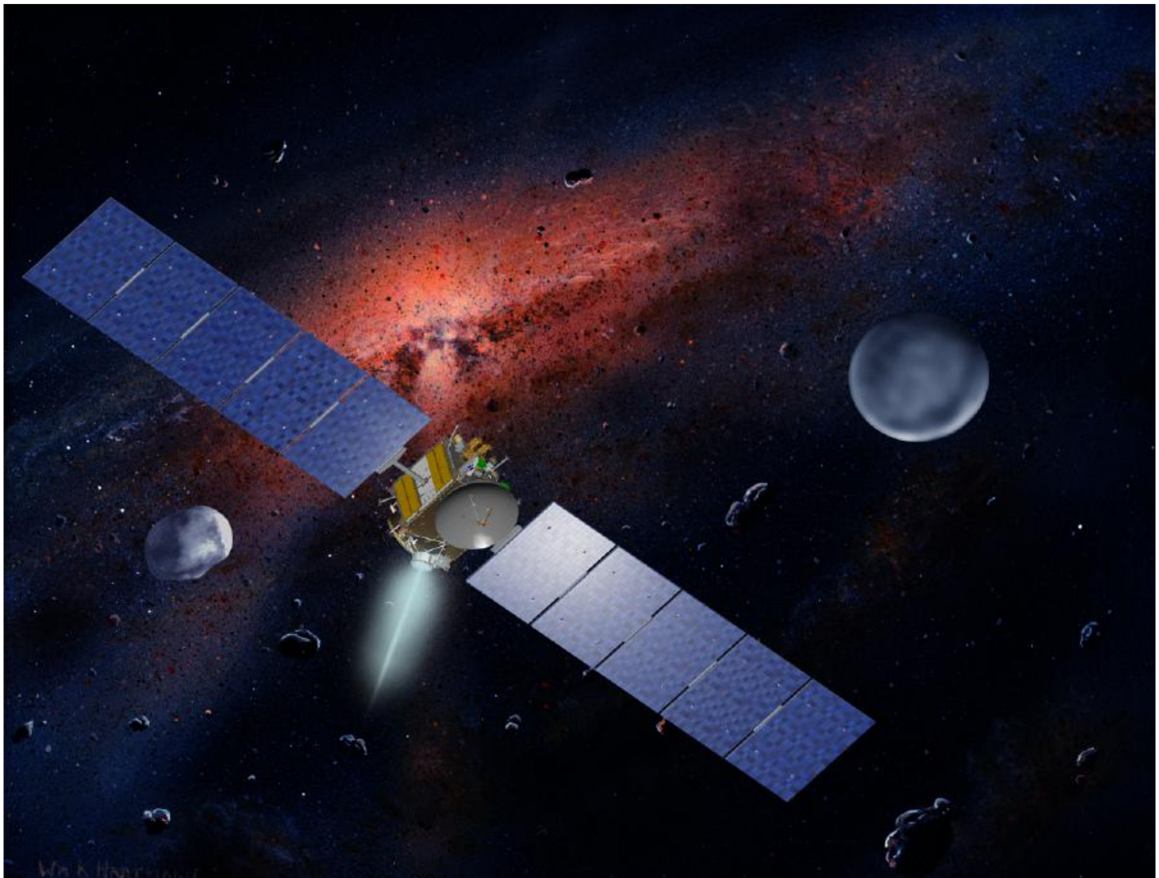

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SIMULATING AND PERFORMING AN INTERPLANETARY JOURNEY

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Abstract

This article looks into many aspects of sending a rocket on an interplanetary journey. Our satellite travels from our designated home planet, Akkli, to our chosen destination planet, Driddu. We decided to travel to Driddu, because of its proximity to Akkli, and because it was the only real option we had. First of all we had to create a vessel. A rocket, which we simulated using a couple of assumptions. These assumptions lead us to an approximation of a rocket engine, and thereby the output force of the rocket. Our next step was to simulate the n-body problem for the star system. We needed the positions of our planets for our orbit transfer maneuver. Once on its way to Driddu our satellite needed software to orient itself in space. Utilizing data of Doppler shift relative to our rest frame and relative to our satellite, the distance to the planets and the star, the satellite will be able to do properly orient itself. Once we had completed our injection burn into orbit around Driddu, we analyzed its atmosphere to calculate the dimensions of our landers parachute. In the end we touched down at Driddu, and proceeded to perform a huge number of experiments regarding general and special relativity. The results from these calculations and experiments can be found in the appendix, sections A and B. Lastly we took a closer look at our system's star, which is the last part of the appendix(C).

Contents

1. Introduction

We have been chosen to be the test subjects of an experiment performed by an extraterrestrial race of beings. They want to see how far we can come scientifically given a fresh start in a new star system with all the equipment and resources we need for space exploration.

An interplanetary journey consists of several highly precise maneuvers which require careful calculation and a solid grasp of the laws of physics. Everything from the planets' atmospheres to the temperature of the fuel in the rocket has to be taken into account, and some of these calculations are potentially catastrophic should you get them wrong. Luckily, we have the power of computer simulation at our hands, and are able to use trial and error in a safe environment before actually embarking on such a dangerous journey. Our goal is ultimately just to impress the alien species by successfully completing an interplanetary mission and performing as many scientific experiments as possible. Since we are actually exploring this star system for the first time we also want to make sure we choose the most interesting target planet so that we make the most out of our mission.

Due to the superior intellect and scientific knowledge of the alien race, we are able to create a rocket engine with the ability of instantaneous boosts, which makes a lot of our calculations much easier. On the way a couple of problems arise which are mostly due to our own errors, but this is all part of the process. We are also lucky enough to find ourselves in a star system where all planets orbit and rotate in the same direction.

If the mission is successful we want to perform a series of experiments concerning relativity using the broad amount of incredible technology we have been given. We think this will show the extraterrestrials that we have a really good idea of how this universe works. We also find ourselves in a system around a star which is more than twice as massive as our sun, so we want to do some research on the star as well. All this will benefit our own knowledge as well as hopefully impress the aliens.

2. Theory and Method

First we will introduce the relevant scientific theories and the methods we will use in order to accomplish our interplanetary journey. Here we will lay the down the ground work for our journey, explain our assumptions and derive relevant formulae.

2.1. Rocket Mechanics and Satellite Simulation

Our first subject of study was to figure out how to get of the surface of Akkli. To do this we would need to build an engine capable of producing enough thrust to lift our satellite of the surface. Next we will explain rocket mechanics and our subsequent rocket mechanics simulation. After creating a simulation of the launch we will move onto orbital mechanics and explain our work on creating a simulation of our alien given star system. Next we studied the origin of life and the requirements for it. We calculated a rudimentary version of the habitable zone and chose a destination based on our calculations. Next we simulated the atmospheric environment on our chosen planet to make sure our lander was prepared for any eventuality.

2.1.1. Engine Simulation

A rocket gets its momentum from the principle of conservation of momentum. When a gas particle from the engine escapes from the engine nozzle the rocket gains an equal amount of momentum in the opposite direction (forwards). This is the general idea of a rocket engine, a box of hot gas with a hole in it. In this project we will have to make some assumptions to make it feasible to complete.

We will make the following assumptions:

- The density and temperature inside the engine is constant throughout a boost
- The particles in the engine will be H_2
- The gas does not ignite, we will simply give the particles a higher kinetic energy
- The gas is an ideal gas (no collision between)

To simulate the engine chamber we approximated it with a tiny cube with a hole on one of its sides. As one may expect the particles are given random uniform positions throughout the cube. Their velocities are distributed by the Maxwell-Boltzmann distribution function, gotten from forelesningsnotat part 1a.

$$P(v)dv = \left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} e^{-\frac{1}{2}\frac{mv^2}{kT}} \tag{1}$$

This distribution function is a combination of three Gaussian distribution functions with standard deviation: $\sigma = \sqrt{\frac{kT}{m}}$ and a mean value: $\mu = 0$. We now have an engine partition filled with particles each with their own velocity vector according to 1, thus we have everything we need to run the engine simulation to see what kind of force it outputs.

Checking numerical solution vs. analytical

To check if our simulation was satisfyingly calculated, we can compare the analytical and numerical kinetic energy and pressure. According to Newton's second law, we have

$$f = \frac{dp}{dt} \approx \frac{\Delta p}{\Delta t} = \frac{N2p_x}{\Delta t} \tag{2}$$

where Δp is the total change in momentum and N is the number of particles leaving the hole. Each particle has momentum p_x before it hits the top wall of the box and $-p_x$ when it leaves the hole, so the total change in momentum from one particle leaving the hole is $2p_x$, and then we multiply that by the amount of particles. Using this, we can find the numerical pressure using

$$P = \frac{F}{A} \tag{3}$$

By substituting $\frac{N2p_x}{\Delta t}$ for F and L^2 for A(Where L is the length of each side in the box.), we get

$$P = \frac{\frac{N2p_x}{\Delta t}}{L^2} = \frac{N2p_x}{\Delta t L^2} \tag{4}$$

and we have an expression for the numerical pressure. For the analytical pressure we already have the equation of state for an ideal gas,

$$P = nkT \tag{5}$$

To check the kinetic energy, we can calculate numerical kinetic energy for one particle by summing $\frac{1}{2}mv^2$ for every particle, and dividing by the amount of particles to get the average value.(Assuming that $\frac{1}{2}m$ is the same for every particle we can take that out of the sum)

$$E_k = \frac{1}{2}m\frac{1}{n}\sum_{i=1}^{n}(v_x^2 + v_y^2 + v_z^2) \tag{6}$$

For the analytical expression for kinetic energy, we define a time $\Delta t' = \frac{L}{v_x}$ where v_x is the x-velocity of a particle, and $\Delta t'$ is the time it takes for that particle to move a distance L. From probability we know that half of the particles are moving in the positive v_x-direction, and during one $\Delta t'$ they will all have reached the other side. The other half moves the opposite direction, so if we define $\Delta t = 2\Delta t'$, we know that every particle will have hit the same wall within this time. If we substitute in for $\Delta t'$ we get

$$\Delta t = 2\Delta t' = \frac{2L}{v_x} \tag{7}$$

Another way to think about it is that every particle hits a wall once every Δt. The force because of this particle is

$$F = \frac{\Delta p}{\Delta t} = \frac{mv_x^2}{L} \tag{8}$$

and thus, the total force on the wall is

$$F = \frac{nm\overline{v_x^2}}{L} \tag{9}$$

where n is the amount of particles, and $\overline{v_x^2}$ is the average speed. From pythagoras we know that $\overline{v^2} = \overline{v_x^2} + \overline{v_y^2} + \overline{v_z^2}$, but since the average velocity is equally distributed along each dimension, we have $\overline{v_x^2} = \overline{v_y^2} = \overline{v_z^2}$ This gives us

$$\overline{v^2} = 3\overline{v_x^2} \tag{10}$$

$$\overline{v_x^2} = \frac{\overline{v^2}}{3} \tag{11}$$

So now we can write the force as

$$F = \frac{nm\overline{v^2}}{3L} \tag{12}$$

This force works upon an area L^2. So now we can find another expression for the pressure,

$$P = \frac{F}{L^2} = \frac{nm\overline{v^2}}{3L^3} \tag{13}$$

We already know that $PV = nkT$, where V is the volume of the gas, L^3. So if we substitute this for P, we get

$$\frac{nkT}{L^3} = \frac{nm\overline{v^2}}{3L^3} \tag{14}$$

Divide by n and multiply by $3L^3$ and we get

$$3kT = mv^2 \tag{15}$$

Finally, dividing by 2 gives

$$\frac{1}{2}mv^2 = E_k = \frac{3}{2}kT \tag{16}$$

and we have an expression for the analytical kinetic energy.

2.1.2. Launch Simulation

Engine variables

With the engine simulation complete we are ready to move onto a launch simulation. The escape velocity of a given planet is

$$v = \sqrt{\frac{2GM_p}{r}} \tag{17}$$

One engine partition will yield far from sufficient thrust to give us this Δv. To calculate the needed amount of engine partitions, we assumed a constant partition force $\rightarrow$ constant fuel rate $\rightarrow$ constant loss of mass which implies that the acceleration $\propto \frac{1}{m}$. For a first estimate of the number of partitions needed we decided on balancing the force produced by the engine and the force due to gravity at the surface of our home planet. We already know that the gravity force at the surface of our planet is

$$\frac{GMm}{r^2} \tag{18}$$

where G is the gravitational constant, M is the mass of the planet, m is the mass of the rocket and r is the distance from the planet center to the rocket(radius of the planet when at surface). Using this exact force will of course never change our velocity, so we increased the engine force by 1000N to be able to overcome gravity. Considering the fact that the mass of the rocket will decrease over time because of fuel use, and the distance to planet center will increase, we know that the gravitational pull will decrease and our rockets acceleration will increase over time. So as long as our force exceeds the gravitational pull at the surface enough so that it moves at a decent speed, that's all we need. Having a number for the total force we wanted, we could calculate a number for the amount of partitions needed in the rocket engine.

$$\text{Engine partitions needed} = \frac{\text{Force wanted}}{\text{Partition force}} \tag{19}$$

In turn this can be used to calculate how much fuel our full engine will use over time during launch, by multiplying particles per second per partition by the amount of partitions.

Launch simulation

With the specifications of our engine in place our next move was to make a launch simulation. The equation for the acceleration of the rocket which we had to integrate was the following

$$\frac{d^2r}{dt^2} = \frac{F_E}{m_C} - \frac{M_p g}{r^2} \tag{20}$$

where F_E is the engine force, m_C is the current mass, M_p is the planet mass and r is the radial distance from the center of the planet. The reason for the current mass term is that the fuel is constantly being expended and thus the total mass of rocket + fuel is decreasing. In order to solve 20 we integrated numerically using the Euler-Cromer scheme

$$v_{n+1} = v_n + a_n \Delta t \tag{21}$$

$$x_{n+1} = x_n + v_{n+1} \Delta t \tag{22}$$

Making sure that for each iteration we reduced the mass by the amount of fuel consumed in the corresponding amount of time.

Calculating fuel needed for boosts in space

The next step is to figure out how much fuel the engine requires to change the spacecrafts velocity by Δv. From the Tsiolkovsky rocket equation which is derived in [**rocketequation**] we have

$$\Delta v = v_e ln\Big(\frac{m_0}{m_f}\Big) = v_e\Big(ln(m_0) - ln(m_f)\Big) \tag{23}$$

Where v_e is the exhaust velocity, m_0 the initial mass, and m_f is the final mass after the boost.

$$e^{\Delta v} = e^{v_e}(m_0 - m_f) \tag{24}$$

The fuel used from a boost will be equal to $m_0 - m_f$.

$$(m_0 - m_f) = \frac{e^{\Delta v}}{e^{v_e}} \tag{25}$$

2.1.3. Star System Simulation

To create an environment where we could simulate our satellite's journey, we first had to simulate the movement of the planets in our star system. To do this we made some more assumptions to make things easier but still somewhat realistic:

- The solar system is two-dimensional(only moves in the XY-plane)
- The gravitational forces from each planet on the other planets can be neglected(and forces from the planets on the sun.)
- All the planets orbit and rotate in the same direction, counterclockwise.

To calculate the orbits of the planets, we use Newton's law of universal gravitation.

$$\vec{F_G} = -\frac{Gm_p m_s}{r^3}\vec{r} \tag{26}$$

Where m_p and m_s are the masses of the objects(planet and star), and r is the distance between the two objects. Seeing that the masses are considered constant and we knew the distances between the planets and the star at time = 0, we now have everything we need to simulate the planet movements over time. So by solving the following differential equation

$$\frac{d^2\vec{r}}{dt^2} = \frac{\vec{F_G}}{m_p} \tag{27}$$

we now have the acceleration of each of the planets, which we can use to find both velocity and position at each time step, resulting in a complete overview of how the planets are moving.

The next step was to create a numerical integration method that could solve these differential equations for us over time. We decided on using the leapfrog-method which is a lightweight, stable and decently accurate integration-method. Usually the RungeKutta4-method is the more popular choice, which is more accurate, but results in an error that increases when you integrate over long periods of time, which is exactly what we were planning on doing.

So now we had to decide over how many time steps we wanted to use for our simulation. Too many and the program takes hours to run, too few and the calculations won't be accurate enough for our purposes. At first we tried to use 10000 time steps per year, but this resulted in too much of a deviation from where the planets were supposed to be. For a "typical" star system this would probably have been enough, but our star system contains a huge gas giant which orbits at a very low altitude around the star, inside the orbit of our home planet Akkli. This results in small changes in how the planets move, and 10000 time steps per year was not enough to accommodate for this. We tried with 50000 time steps per year, and even this wasn't enough. After this we decided on 100000, but the simulation took so long to run, so we eventually landed on 70000 time steps per year, which seemed to be accurate enough for our calculations.

Checking our simulation's accuracy using Kepler's laws

To check whether the orbits were satisfyingly calculated, we used Kepler's second law of planetary motion, which states that the area swept out by the vector between the planet and the star over a time dt should be constant. The area swept out by this line over a infinitesimal movement over dr and $d\theta$ is

$$dA = \frac{1}{2}r^2 d\theta \tag{28}$$

We divide this by dt and get an expression for $\frac{dA}{dt}$

$$\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt} \tag{29}$$

$\frac{d\theta}{dt}$ is equal to the tangential velocity v_θ

$$\frac{dA}{dt} = \frac{1}{2}r^2 v_\theta \tag{30}$$

so by checking if this area is the same at a planet orbit's periapsis and apoapsis, we could know if our simulations calculated the orbits accurately enough to satisfy Kepler's law.

We can also check our simulation by plotting the analytical positions on top of the numerical ones. For the analytical positions, we used the general solution for the two-body problem,

$$r = \frac{p}{1 + ecos(f)} \tag{31}$$

where e is the eccentricity of the planet's orbit and $f = \theta - \omega$. θ is the angular position of the planet and ω is the initial angle of the planet relative to our home planet.

2.1.4. Deciding Where to Land

Before we embark on our simulated journey we need to decide what planet to go to. We need a formula to roughly calculate the surface temperature of the planets.

Surface Temperature of Driddu

We start by approximating our home star as a point light source with energy released per unit time L, also known as luminosity. Since the energy of a point source spreads radially in all directions another we define the radiative flux of a star to be the energy received per unit time square meter. Because our approximation has both rotational symmetry and reflective symmetry the total luminosity produced by our home star is the surface area of the radiative flux on the sphere with radius r. Meaning the flux is equal to the total luminosity divided by the surface area of a sphere with radius. The symmetry of the problem ensures that the flux is constant over the integral and thus we obtain the relation

$$F = \frac{L}{A} = \frac{L}{4\pi r^2}. \tag{32}$$

If we assume an object is a black body, meaning it does not reflect, we are able to obtain an expression for the energy received from a point source at an arbitrary distance r. To do this we use the Stefan-Boltzmann equation for a black body.

$$L = \sigma A T^4 \tag{33}$$

where A is the area of the black body's surface. The energy received at the planet's surface must be equal to the energy radiated away from the planet. This leans on the assumption that the energy generation from the planet's core is negligible compared to the energy absorbed from the home star. Using 33 for the planet and 32 with the spherical shell with radial distance $r = a_{\text{planet}}$ gives us an estimate for the lowest temperature during a planet-year.

$$4\pi R_p^2 \sigma T_p^4 = \frac{4\pi R_*^2 \sigma T_*^4}{4\pi a^2} \cdot \pi R_p^2 \tag{34}$$

where the left hand side represents the energy flowing out of the planet and the right hand representing the the energy absorbed by the planet from the star. We assumed the absorption area of the planet to be that of a circle with radius equal to that of the planet. Additionally we rewrote the flux of the star using the same argument as for the energy flowing out of the planet. which reduces to

$$T_p^4 = \frac{R_*^2 T_*^4}{4a^2} \tag{35}$$

where a is the semi-major axis of the orbit. Taking the fourth root of both sides gives us our expression for the surface temperature of the planet.

$$T_p = \sqrt[4]{\frac{R_*^2 T_*^4}{4r^2}} \tag{36}$$

Where T_p is the surface temperature of the planet, R_* is the radius of the star, T_* is the surface temperature of the star and r is the distance from the star to the planet. This formula assumes circular orbits, so we have to keep in mind that when using the semi-major axis, we get the coldest possible temperature for each planet. Calculating the temperatures of the planets in the solar system yields only one possible planet to go to, Driddu our next door neighbor.

The Habitable Zone

We use the expression above, 35, to obtain a minimum and maximum distance for the habitable zone. We solve for r which yields

$$r = \frac{R_* T_*^2}{2T_p^2} \tag{37}$$

With this formula we can plug in a minimum and maximum surface temperature T_p and get a minimum and maximum distance from the sun for a planet to sustain life.

Planet gravity dominance

Now that we have a target planet, we need to calculate how close to the planet we need to get for that planets gravitational force to take over and be dominant over the star's gravity. This means that we need to find out how much bigger the gravitationall pull from the planet is compared to the star. So we set the the planet force equal to k times the star force

$$\frac{GM_p m}{r^2} = \frac{GM_* m}{R^2} k \tag{38}$$

where M_* is the star mass, M_p is the planet mass, R is the star-satellite distance, and r is the planet-satellite distance. Solve for r and we get

$$r = R\sqrt{\frac{M_p}{M_*}\frac{1}{k}} \tag{39}$$

We can now use this equation to find out how close we need to be by plugging in the masses and the distance from the star, and choose a suitable k which will make sure the planet gravity is dominant over the star gravity. Lowering k will lead to a higher threshold for the radius, so it's important to pick a large enough k. Using a k larger than or equal to 10 should yield a distance which makes a stable orbit possible.

2.1.5. Orbital Transfer Simulation

Now that we have the movement of the planets in our star system simulated, we have got an environment in which to simulate our satellite's journey from Akkli to Driddu.

To check if the simulation was working as we wanted it to, we first tried to have our satellite take the simplest(but in no way efficient) route to Driddu. We placed our rocket so that it launched radially outwards, straight away from the star, and programmed it to perform an orbital insertion boost around the sun when it reached the orbital height of the target planet Driddu. We made sure its orbit was in the opposite direction of Driddu, so that they would eventually meet. When it was close enough we could then perform another boost to put the satellite in orbit around Driddu. This route is of course ridiculous, seeing that both boosts completely change the direction of the satellite, and it would require a crazy amount of Δv and therefore also fuel. But all that aside, we had proven that our simulations were working and were able to have the satellite launch from Akkli and get to orbit around Driddu.

Now we had to figure out how to actually get to Driddu. We decided on using the most popular and efficient method, called a Hohmann-transfer. A Hohmann-transfer includes boosting prograde(forward) until the apoapsis(tallest point) of your orbit is at the same height as the orbit you are targeting. Then you wait until the satellite is at apoapsis, and boost prograde again to increase your periapsis(lowest point) until the orbit is circular, and then you are in the same orbit as your target. This is the most fuel-efficient way to change your orbital height. More on the Hohmann-transfer can be found in C

Of course, doing this at a random time will never get you an actual encounter with the planet. If the planet is at a different place in the orbit when you perform the second boost, you will never get close to that planet, seeing that you are going in the same orbit at the same speed, but at a different place in the orbit. So before we actually performed the orbital transfer, we had to make sure that our home planet and the target planet were at a specific angle to one another relative to the star. Since our target planet is at a higher orbit around the sun, it has a lower angular velocity than our home planet. So we had to make sure that the target planet was somewhere "in front" of our planet in its orbit, so that when our satellite reaches apoapsis during the Hohmann-transfer, the target planet will also be there at that time. In order to do this, we had to calculate at which angle this would happen. When I say angle here, I am talking about the angle you get between the lines if you draw a line from the two planets to the star.

To calculate this angle, we had to find a formula for how long the transfer would take using Kepler's laws: (This is the same as calcuating half the Period of the elliptical orbit you have after the first boost, when apoapsis is at the target height, and periapsis is still at your original orbit.)

$$t = \pi\sqrt{\frac{(r_1 + r_2)^3}{8\mu}} \tag{40}$$

Where t is time, r_1 and r_2 are the heights of the orbits(height of periapsis and apoapsis when you're in the elliptical orbit), and μ is the standard gravitiational parameter of the primary body.

We also know that the angular velocity we need is

$$\omega_2 = \sqrt{\frac{\mu}{r_2^3}} \tag{41}$$

Knowing the time of transfer and target angular velocity from equations 40 and 41, the angular alignment we need for the Hohmann-transfer should be

$$\alpha = \pi - \omega_2 t \tag{42}$$

Which gives

$$\alpha = \pi\left(1 - \frac{1}{2\sqrt{2}}\sqrt{\left(\frac{r_1}{r_2} + 1\right)^3}\right) \tag{43}$$

So now that we know what angle we need to "be at" when we launch, all that is left to figure out our optimal launch-time is to write a simple python function that will find the time at which the two planets are alligned correctly. Seeing that we already have a code which simulates our star system, all we need to do is run the simulation over a long time, check the angle between the planets at every time step, and if the angle is correct - mark the time.

Using this function we were able to find the correct time at which our planets were aligned at the angle calculated above.

Lastly, there is the calculation of the boosts our satellite has to perform to do the Hohmann-transfer. This can be done using the following formulas,

$$\Delta v_1 = \sqrt{\frac{\mu}{r_1}}\left(\sqrt{\frac{2r_2}{r_1 + r_2}} - 1\right) \tag{44}$$

and

$$\Delta v_2 = \sqrt{\frac{\mu}{r_1}}\left(1 - \sqrt{\frac{2r_2}{r_1 + r_2}}\right) \tag{45}$$

Where $\mu = GM$(Assuming M_{planet} is a lot bigger than $M_{\text{satellite}}$, which it is in this case), r_1 and r_2 are the radii of the origin orbit and the target orbit.

However, we didn't actually use these Δv-equations for anything but a ballpark estimate of how big the boost had to be. We also didn't use the exact launch time given from equation the python script above, we had to use a mix of trial and error, and intuition to find our way to a working Hohmann-transfer in our star system. This is because of quite a lot of assumptions lying behind these equations, especially the assumption of circular orbits, which we do not have. We will go more into this in the discussion-section(4.1.4).

To get a perfect hohmann transfer we wanted to only perform two boosts. The launch boost and the orbital insertion boost at Driddu, the target planet. Using rough numbers from the calculations above, and after a lot of trial and error, we managed to get an encounter close enough to Driddu using only the launch boost to escape velocity and then immediately another boost in the same direction to increase our periapse to Driddu's height. Now the only thing we needed to do was calculate the required Δv for getting into orbit around Driddu.

The orbital velocity around a given planet with mass m is easy to calculate: By setting the sentripetal acceleration equal to the acceleration from gravity, we get

$$\frac{Gm}{r^2} = \frac{v^2}{r} \tag{46}$$

Where G is the gravitational constant, m is the mass of the planet, r is the distance from the planet center, and v is velocity. By multiplying by r and taking the square root of both sides, we get an expression for the orbital velocity needed at a certain height r.

$$v = \sqrt{\frac{Gm}{r}} \tag{47}$$

where r is the distance from the planet's center to (in this case) the satellite. We use python to create a vector symbolizing the velocity we need to achieve, using the value from the formula above and a unit vector pointing parallel to the planet's surface. From the simulation in the code we already know both the satellite and the planet's velocities relative to the sun. If we subtract these two vectors we get the satellites velocity relative to the planet.

Now we have a vector for the velocity we need relative to the planet, and the velocity we currently have relative to the planet. Lastly, we subtract these to vectors and get a vector which corresponds to our needed Δv to get into orbit. The code for this function can be found in the results section(3.1.5). The good thing about this method is that we now don't need to calculate the second boost of the Hohmann-transfer to get into a cirucular orbit(Equation 45), because the code does that for us in addition to giving it the nessesary Δv to orbit Driddu at the same time.

2.1.6. Landing Simulation

This next simulation was made for the purpose of deciding a sequence of boosts to bring the lander part of the satellite down to Driddu's surface. Firstly we need to maneuver into a much closer orbit around Driddu. Once in the orbit we will survey Driddu's surface for a suitable landing location. Looking out for noteworthy features. The landing simulation will have a couple of differences from the previous simulations. We disregard every other celestial body in the star system, we will model the atmosphere as explained in Section 2.3, we will simulate in three dimensions, motion is relative to Driddu and our units are no longer astronomical, but SI.

Low-Driddu Orbit

Our target orbit has to be outside Driddu's atmosphere or else the satellite would smash into the gases and disintegrate. The ratio between gravity and the atmospheric drag was decided to be $F_G/F_D > 1000$ for a safe recognizance orbit. Since our greatest source of error is the atmosphere analysis we decided to lower our orbit distance to $r = r_{\rho=1e-10}$. Using equation 91 with this boundary condition and solving for r gave us a height for our orbit. Driddu's atmosphere proved to be very dense, with $\rho_0 = 7.98$. The term 88 is very useful for these kinds of calculations. Its meaning is the radial distance over which the density and pressure fall by a factor of $1/e$

Elevation	Density
0	ρ_0
h_0	$(1/e)\rho_0$
$4h_0$	$(1/e^4)\rho_0$
$8h_0$	$(1/e^8)\rho_0$
$16h_0$	$(1/e^{16})\rho_0$

Now that we had a safe height for our low Driddu orbit, we had to actually change orbit. As explained in section 2.1.5 we can do this from an arbitrary orbit to another.

Lander Power Consumption

Before landing on Driddu we needed to make sure the lander had a power supply. Our power supply of choice was the home star. The lander consumes 40W during daytime to run its instruments. With solar panels converting at 12% efficiency we needed to figure out how big of an area they had to cover.

$$P = FA \tag{48}$$

where P is power, F is flux and A is the area of the solar panels.

$$F = \frac{L}{4\pi R^2} \tag{49}$$

where L is luminosity This last relation can be used for calculating the flux received at a given distance R. If we set $R = R_{\text{star}}$ then the expression will have the physical interpretation of the flux output of the star. However, if we set $R = R_{\text{DridduOrbit}}$ then the physical interpretation is the flux received at Driddu. When $R = R_{\text{star}}$ we use 49 in addition to Stefan-Boltzmann law of flux; $F = \sigma T^4$ and solve to obtain an expression for the luminosity.

$$L = 4\pi R_{\text{star}}^2 \sigma T_{\text{star}}^4 \tag{50}$$

Solving 48 for the area needed a distance $R = R_{\text{Dridduorbit}}$ to generate 40W, and combining with 50 to obtain an expression for the area of the solar panel.

$$A = P\frac{R_{\text{DridduOrbit}}^2}{R_{\text{star}}^2 \sigma T_{\text{star}}^4} \tag{51}$$

where σ is the Stefan-Boltzmann constant.

Scouting the Surface

Once we had gotten the satellite into our recognizance orbit we started scouting the surface of Driddu for landing locations of interest. We used the on board camera to take these pictures. In order to translate

2.2. Navigation and Orientation

The satellite is able to gather some information about its whereabouts using its on-board navigation system. The navigation system consists of a camera, a radar and a spectrograph. The camera takes pictures within a field of view and is used to determine orientation in the star system. The radar finds the distance to the planets and star within the star system. The spectograph is used to deduce the satellites velocity, it does this using the Doppler shifts of two distant stars.

2.2.1. Orientation

Using the camera the satellite takes pictures of the sky in front of it every couple of seconds. In order to compare the pictures these pictures to the spherical background sky, the sky has to be projected onto a flat surface. To do this we use stereographic projection, this method maps each point on the sphere onto a point on the tangent plane of the sphere. Positions on the sphere are denoted with the angles θ: azimuth angle, and ϕ: polar angle. Since our problem is restricted to the x-y plane we will only do projections where $\theta_0 = \frac{\pi}{2}$. The projection is given by the transformation equations,

$$x_{\text{picture}} = k \sin\theta \sin(\phi - \phi_0) \tag{52}$$

$$y_{\text{picture}} = k(\sin\theta_0 \cos\theta - \cos\theta_0 \sin\theta \cos(\phi - \phi_0)) \tag{53}$$

where

$$k = \frac{2}{1 + \cos\theta_0 \cos\theta + \sin\theta_0 \sin\theta \cos(\phi - \phi_0)} \tag{54}$$

We need the inverse of these transformation functions to use with our software, which takes in a θ and ϕ and outputs a corresponding pixel index. The inverse functions are as follow,

$$\theta = \frac{\pi}{2} - \sin^{-1}\left(\cos c \cos\theta_0 + \frac{y_{\text{picture}} \sin c \sin\theta_0}{\rho}\right) \tag{55}$$

$$\phi = \phi_0 + \arctan\left(\frac{x_{\text{picture}} \sin c}{\rho \sin\theta_0 \cos c - y_{\text{picture}} \cos\theta_0 \sin c}\right) \tag{56}$$

where

$$\rho = \sqrt{x_{\text{picture}}^2 + y_{\text{picture}}^2} \tag{57}$$

$$c = 2\arctan\left(\frac{\rho}{2}\right) \tag{58}$$

The camera on board has a field of view α_ϕ in the polar direction and α_θ in the azimutal direction. To figure out the size of our pictures we set both $x = 0$ and $y = 0$ for each equation (3) and (4). To figure out the size in the x-direction, y is set to zero with $\theta_0 = 0$ which yields

$$y = 0 \implies \frac{2\cos\theta}{1 + \sin\theta\cos(\phi - \phi_0)} = 0 \tag{59}$$

Solving this equation for θ, we found $\theta = \frac{\pi}{2}$. Plugging this into (3) to find its min/max values

$$x_{\text{min/max}} = \pm\frac{2\sin(\phi - \phi_0)_{\text{max}}}{1 + cos(\phi - \phi_0)_{\text{max}}} \tag{60}$$

We now set the angle $\frac{\alpha_\phi}{2} = (\phi - \phi_0)_{max}$, and thus the expression for x min/max becomes

$$x_{\text{min/max}} = \pm\frac{2\sin(\frac{\alpha_\phi}{2})}{1 + \cos(\frac{\alpha_\phi}{2})} \tag{61}$$

The same procedure is done for the y min/max which yields a corresponding expression

$$y_{\text{min/max}} = \pm\frac{2\sin(\frac{\alpha_\theta}{2})}{1 + \cos(\frac{\alpha_\theta}{2})} \tag{62}$$

Next we use these min/max values to generate a range of pixels to iterate over in order to generate a projection for each degree $\in [0, 360]$. Next we used this collection of projections and iterated to compute the best fitting angle. The algorithm we used to find the best fitting orientation angle for a satellite picture was as follows

1. For every integer degree $d \in [0, 360]$ generate the projection corresponding to that d

2. Calculate the error between the picture taken from the satellite and the projection
3. If the error is smaller than the previous least error, replace it

2.2.2. Velocity

The figure out the velocity of the satellite the on board spectograph is used. We take advantage of the Doppler effect to deduce the velocity to great precision $\sim \pm 1m/s$. By measuring the position of the H_{alpha} (656.3nm) spectral line in two non-collinear stars relative to the satellite, as well as relative to a rest frame, one can deduce the radial velocity component of the satellite with respect to each reference star,

$$v_{\text{sat}} = v_{\text{refstar}} - v_{\text{rel}} \tag{63}$$

where v_{sat} is the velocity of the satellite in the direction of the reference star. v_{refstar} is the radial velocity of the reference star with respect to a rest frame, and v_{rel} is the radial velocity of the satellite relative to the reference star. If v_{sat} is known with respect to both stars, the velocity of the satellite in the x-y plane can be obtained. To begin we define the unit vectors in the direction of both the stars

$$\vec{u_1} = \begin{pmatrix} \cos\phi_1 \\ \sin\phi_1 \end{pmatrix} \text{ and } \vec{u_2} = \begin{pmatrix} \cos\phi_2 \\ \sin\phi_2 \end{pmatrix} \tag{64}$$

The radial velocity of the satellite with respect to each of the stars is the following,

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \cos\phi_1 & \sin\phi_1 \\ \cos\phi_2 & \sin\phi_2 \end{pmatrix} \begin{pmatrix} v_x \\ v_y \end{pmatrix} \tag{65}$$

The next step is to solve this equation for the x and y components of the satellite velocity. To do this we need to cancel out the matrix on the right hand side. Canceling matrices requires knowing the inverse of the matrix, since performing a linear transformation and then performing the inverse of the transformation returns the original vector. In our case: the x and y velocities of the satellite. Using the inverse of a 2×2-matrix formula

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \tag{66}$$

on our matrix gives

$$\begin{pmatrix} \cos\phi_1 & \sin\phi_1 \\ \cos\phi_2 & \sin\phi_2 \end{pmatrix}^{-1} = \frac{1}{\cos\phi_1 \sin\phi_2 - \cos\phi_2 \sin\phi_1} \begin{pmatrix} \sin\phi_2 & -\sin\phi_1 \\ -\cos\phi_2 & \cos\phi_1 \end{pmatrix} \tag{67}$$

which simplifies to

$$\frac{1}{\sin(\phi_2-\phi_1)}\begin{pmatrix}\sin\phi_2 & -\sin\phi_1\\ -\cos\phi_2 & \cos\phi_1\end{pmatrix} \tag{68}$$

Using the inverse transformation to solve the above equation gives us the x and y components of the satellite's velocity.

$$\begin{pmatrix}v_x\\ v_y\end{pmatrix}=\frac{1}{\sin(\phi_2-\phi_1)}\begin{pmatrix}\sin\phi_2 & -\sin\phi_1\\ -\cos\phi_2 & \cos\phi_1\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix} \tag{69}$$

Using this formula for the satellite velocity vector given the angles and corresponding velocities of two distant stars we could at any point calculate the velocity of the satellite. We tested our implementation against the two trivial cases where the satellite had a zero velocity with respect to the home star and where it had a zero velocity with respect to a reference star.

2.2.3. Position

Using the on board radar of the satellite we are able to deduce our position relative to our home star. This radar returns the distance from the satellite to each of the other major bodies in the star system. With this information we are able to trilaterate the satellite position. Knowing one distance d informs us that the satellite position is on the circle centered at that planet's position with radius $r = d$. Calculating the position of the satellite thus boils down to the following algorithm

1. create the circle centered at $(0, 0)$ with $r = d_{\text{star}}$
2. for each planet pair i, j create the circle centered at $(x_{\text{planet}_{i,j}}, y_{\text{planet}_{i,j}})$ with $r = d_{\text{star}}$
3. the satellite position will be at the intersection of these circles

With a method to acquire complete knowledge of our position, velocity and orientation we are ready to embark on our interplanetary journey to our next door neighbor Driddu.

2.3. Simulating the Atmosphere

During our journey to Driddu we made flux measurements for the wavelengths 600nm through 3000nm, in order to identify which molecular compounds are present in its atmosphere. These fluxes were normalized such that the background flux is 1. The measurements were subjected to various sources of noise. Therefore we had to correct for this noise before we could say which compounds were present. After we knew which compounds were present we could model Driddu's atmosphere. This model is essential to our mission because without it we would not know how big a parachute we need to land safely on the surface.

2.3.1. Chemical Composition of Driddus Atmosphere

To work out the chemical composition of Driddus atmosphere we had to analyze the flux / wavelength data to look for potential absorption lines of various chemical compounds. These compounds are indicator of either geological processes or natural processes. Notable compounds we looked for are:

- Oxygen O_2 has absorption lines $\lambda \in (630, 690, 760)$nm
- Water H_2O has absorption lines $\lambda \in (720, 820, 940)$nm
- Carbon Dioxide CO_2 has absorption lines $\lambda \in (1400, 1600)$nm
- Methane CH_4 has absorption lines $\lambda \in (1660, 2200)$nm
- Carbon Monoxide CO has absorption line $\lambda = 2340$nm
- Nitrous Oxide N_2O has absorption line $\lambda = 2870$nm

Our spectral data had $1.0 \cdot 10^7$ flux measurements each with their own individual measurement noise σ_i. To manually look for dips in the flux to look for absorption would be unfeasible. We have assumed the noise to be independent between observations and a $\langle\sigma_i\rangle = 0$. Then we only need the standard deviation $\sigma(t)$ to be able to model the uncertainty caused by the noise. In addition to the measurement noise, the data also contained two different Doppler shifts: one induced by the velocity of the satellite and another induced by temperature. The first one is the position of the absorption lines mentioned above, and the other is a widening of the spectral line.

Since we cannot locate the absorption line by-eye we will make an estimate of the spectral line position using a $\chi^2(\sigma, \lambda_0, \mathrm{F}_{\min})$ minimization.

To do this we need ranges of probable values for each of these variables. We knew that the satellite would have a maximum velocity, w.r.t the planet, of $10\frac{\mathrm{km}}{\mathrm{s}}$. With this knowledge we calculated the maximum shift in λ_0

$$\Delta\lambda_{0,\mathrm{max/min}} = \pm 10\frac{\lambda_0}{c} \tag{70}$$

We used 300 guesses for the location of the absorption line in our χ^2-minimization. The temperature of the planet was between 150K and 450K which, as mentioned above, smudges the absorption line. We would therefore have to figure out what the range of probable values for the standard wavelength deviation from λ_0 so we could make our guesses. When measuring the Doppler effect only the motion parallel with the radial vector between two observers induces a shift in wavelength. In order to make a good educated guess we would need to figure out the most probable radial velocities, w.r.t the satellite, in the atmosphere. We use the Maxwell-Boltzmann probability distribution to make our guess,

$$P(v)dv = \left(\frac{m}{2\pi kT}\right)^{3/2} e^{-\frac{1}{2}\frac{mv^2}{kT}} 4\pi v^2 dv \tag{71}$$

The number density is

$$n(v)dv = nP(v)dv \tag{72}$$

where n is the total number of particles per volume of the gass. This new distribution $n(v)dv$ is also Gaussian but its meaning has been transformed from probability of finding an atom within a velocity range to the total number of particles per volume within a velocity range. Ergo the velocity with the highest number of particles is

$$\frac{dn(v)}{dv} = 0 \implies v_{mp} = \sqrt{\frac{2kT}{m}} \tag{73}$$

As mentioned above the only the motion parallel to the line of sight induces a Doppler shift. Therefore the most probable radial velocities w.r.t the satellite will be in the range $v_{r,mp} \in [-v_{mp}, v_{mp}]$. This is because the atoms in the atmosphere have random directions of motion. Using this range along with the Doppler formula we obtain a range of probable smudging of the absorption line

$$|\Delta\lambda_{\text{around}\lambda_0}| = \frac{2\lambda_0 v_{mp}}{c} = \frac{2\lambda_0}{c}\sqrt{\frac{2kT}{m}} \tag{74}$$

This model of the absorption line is not good enough for our purposes. We wanted a better estimation of the absorption line, as discussed in forelesningsnotater 1D C, it is possible to model the absorption line as a Gaussian function

$$F_{\text{model}}(\lambda_i) = F_{\text{cont}}(\lambda) + (F_{\text{min}} - F_{\text{cont}}(\lambda))e^{-(\lambda_i-\lambda_0)^2/(2\sigma^2)} \tag{75}$$

This model will return $F = F_{\text{min}}$ when $\lambda = \lambda_0$, and F_{cont} when λ is far from the center of the absorption line. The problem at hand is to figure out which $lambda_0$ is the is the center of the shifted absorption line. Which will give our model the best fit to the data. Because we had a model of an absorption line we could figure out what λ_0 that was most probably the real center of the line, using the Gaussian probability of observing an absorption line in the data.

$$P(F_{\text{obs},0}, F_{\text{obs},1}, ..., F_{\text{obs,n}}, \lambda_0, F_{\text{min}}, \sigma) = \frac{1}{\Pi_{i=1}^{n}(2\pi)^{n/2}\sigma_i} e^{-\frac{1}{2}\Sigma_{i=1}^{n}\frac{(F_{\text{obs}}-F_{\text{model}}(\sigma,\lambda_0,F_{\text{min}}))^2}{\sigma_i^2}} \tag{76}$$

Maximizing the probability function gives us the most probable values for σ, λ_0 and F_{min} given the data. This approach is however computationally expensive and thus time consuming. Notice how the returned value of the function depends upon the exponential term. If the sum in the exponent grows large then the probability of a line drops fast. This means that maximizing the probability function is the same as minimizing the function

$$\chi^2(\sigma, \lambda_0, \mathrm{F_{min}}) = \sum_{i=1}^{N} \frac{(F_{\text{obs}} - F_{\text{model}}(\sigma, \lambda_0, \mathrm{F_{min}}))^2}{\sigma_i^2} \tag{77}$$

because if this sum is small the probability will be big. σ_i is the measurement noise for each measurement.

The line profile's standard deviation is given by

$$\sigma = \frac{\lambda}{c}\sqrt{\frac{kT}{m}}. \tag{78}$$

Notice that the standard deviation of the line profile only depends upon the temperature which we already have a range of values for. So we have our second range to make our guess. Due to computational constraints we opted for a guess with 30 values between $\sigma_{\text{max/min}}$. All that remained was a range for F_{min}. We knew that $0.7 < \text{F}_{\text{min}} < \text{F}_{\text{background}} = 1$ so we had the range of probable for our unknown variables.

Complete with all our variable ranges we were ready to do our χ^2-minimization of the line profile for each of the absorption lines mentioned at the beginning of this section.

2.3.2. Modelling the Atmosphere

We now had χ^2-minimizing variables for the line profile for each molecule in tabel: 2.3.1. Visual study in addition to analytic analysis for each line profile was used to decide whether there was an absorption line in the spectra or if our line profile had a false positive. More on this in section 3. Next we assumed that there was a one to one relationship between the gases in the atmosphere. This allowed us to calculate the mean molecular weight μ

$$\mu = \frac{1}{N}\sum_{i=1}^{N} m_i \tag{79}$$

N being the number of different gases and m_i being the molecular mass of each gas found. Our next step towards modelling the atmosphere is finding the surface temperature of Driddu. We modelled Driddu as a black body, and assumed that the surface temperature was a function of only the power Driddu receives from the home star. At any given time only half of Driddu is facing the star, and the distribution of the power over Driddu's surface is not uniform. However, we do not have to calculate the power density per area. Instead we look at the size of the effective absorption area of Driddu. Thus the surface temperature is the temperature for which the power in and power out is in equilibrium.

$$P_{\text{in}} = L\frac{r_p^2}{4R_{\text{Driddu}}^2} \tag{80}$$

where we have used the effective absorption area $A = \pi r_p^2$. Since the power going into Driddu has to equal the power out we set up the relation

$$P_{\text{out}} = 4\pi\sigma T_p^4 r_p^2 \tag{81}$$

$$P_{\text{in}} = P_{\text{out}} \tag{82}$$

Using this relation along with equation 50 we obtain an expression for the surface temperature of the planet.

$$T_p = \sqrt{\frac{R_{\text{Star}}}{2R_{\text{Driddu}}}} T_{\text{Star}} \tag{83}$$

This expression for the surface temperature of a planet, in this case Driddu, does not depend upon the size of the planet in question. Rather it depends on the planet's proximity to its home star R_{Driddu}, the size of the star R_{Star} as well as the temperature of the star T_{Star}. Thus we had an approximate term for the surface temperature of Driddu.

We model the atmosphere as adiabatic up to the height where $T = T_0/2$ and isothermal for heights above this. We then solved the density profile of the Driddu's atmosphere using the equation of hydrostatic equilibrium coupled with assuming the gas to be ideal,

$$P = \frac{\rho k T}{\mu m_H} \tag{84}$$

$$\frac{dP}{dr} = -\rho(r)g(r) = -\rho(r)\frac{GM(r)}{r^2} \tag{85}$$

where $\rho(r)$ is the density at radius r and $M(r)$ is the total mass inside a radius r.

$$M(r) = \int_0^r dr' 4\pi (r')^2 \rho(r') \tag{86}$$

Isothermal Region

First we modelled the isothermal region of the atmosphere, $r > T_/2$, this is a simpler case than the adiabatic case because the temperature is constant. Solving equation 84 for the density and inserting in equation 85 yields, combined with approximating gravity constant throughout the region yields

$$\frac{dP}{dr} = -Ph_0 \tag{87}$$

where r is the height above Driddu's surface and h_0 is the scale height of Driddu's atmosphere

$$h_0 = \frac{kT}{g\mu m_H} \tag{88}$$

which is a first order separable differential equation. With a solution of the form

$$P(r) = Ce^{-\frac{r}{h_0}} \implies P = P(r_{T_0/2})e^{-\frac{r}{h_0}} \tag{89}$$

Combining this expression with the equation 84 we obtain the density profile of the atmosphere to be

$$\rho(r) = \frac{P(r_{T_0/2})}{h_0}e^{-\frac{r}{h_0}} = \rho_0 e^{-\frac{r}{h_0}} \tag{90}$$

$$\rho(r) = \rho_0 e^{-\frac{r}{h_0}} \tag{91}$$

Adiabatic Region

In the adiabatic region of the atmosphere $r < r_{T_0/2}$ the gas is able to change temperature without loosing or gaining heat from the environment. We utilize the adiabatic law from thermodynamics to solve for the pressure and density profile

$$P^{1-\gamma}T^{\gamma} = C \tag{92}$$

where γ is the adiabatic index calculated as $\gamma = \frac{C_P}{C_V} = \frac{f+2}{f}$, C_P being the specific heat for constant pressure, C_V being the specific heat for constant volume and f is the number of degrees of freedom: three translational and two rotational. Differentiating equation 92,

$$\frac{dP}{P} = \frac{\gamma}{\gamma - 1}\frac{dT}{T} \tag{93}$$

Combining with equation 84 solved for density ρ

$$\rho = \frac{dP}{dT}\frac{\mu m_H}{k}\frac{\gamma - 1}{\gamma} \tag{94}$$

Inserting into equation 85

$$\frac{dT}{dr} = -\frac{\gamma - 1}{\gamma}\frac{g\mu m_H}{k} \tag{95}$$

Again solving a separable differential equation with the boundary condition $r = 0 \rightarrow T = T_0$

$$T(r) = T_0\left(1 - \frac{\gamma - 1}{\gamma}\frac{r}{h_0}\right) \tag{96}$$

The pressure profile is found using equation 92 $\implies P \propto T^{\gamma/(\gamma-1)}$. It follows that

$$P(r) = P_0\left(1 - \frac{\gamma - 1}{\gamma}\frac{r}{h_0}\right) \tag{97}$$

Now all that remains is figuring out the density profile of the adiabatic region; substituting P into equation 84

$$\rho(r) = \frac{P(r)}{T(h)}\frac{\mu m_H}{k} = \frac{P_0}{T_0}\left(1 - \frac{\gamma - 1}{\gamma}\frac{r}{h_0}\right)^{(\frac{\gamma}{\gamma-1}-1)} \tag{98}$$

$$\rho(r) = \rho_0\left(1 - \frac{\gamma - 1}{\gamma}\frac{r}{h_0}\right)^{\frac{1}{(\gamma-1)}} \tag{99}$$

The only term not defined is ρ_0 for the isothermal region i.e. the pressure at $r = r_{T_0/2}$. Solving for r when the adiabatic temperature profile $T(r) = T_0/2$ yields

$$T_0\left(1 - \frac{\gamma - 1}{\gamma}\frac{r}{h_0}\right) \implies r_{T_0/2} = \frac{\gamma}{2(\gamma - 1)}h_0 \tag{100}$$

Inserting this value for the radius into the adiabatic density profile we finally have all the pieces needed for our model

$$\rho_0^{\text{iso}} = \rho(r_{T_0/2}) = (\frac{\rho_0}{2})^{\frac{1}{(\gamma+1)}} \tag{101}$$

To summarize our model of Driddu's atmosphere:

- adiabatic density profile: $\rho(r) = \rho_0 \left(1 - \frac{\gamma-1}{\gamma}\frac{r}{h_0}\right)^{\frac{1}{(\gamma-1)}}$
- adiabatic temperature profile: $T(r) = T_0 \left(1 - \frac{\gamma-1}{\gamma}\frac{r}{h_0}\right)$
- isothermal density profile: $\rho(r) = \rho_0^{\text{iso}} e^{-\frac{r}{h_0}}$
- isothermal temperature profile: $T(r) = T_0/2$
- The boundary between the adiabatic and isothermal region is at $r = 25939$m

2.3.3. Air Resistance and Terminal Velocity

Once the satellite enters Driddu's atmosphere it will be subject to frictional forces. We will model this friction using the drag equation

$$F_D = \frac{1}{2}\rho C_D A v^2 \tag{102}$$

where A is cross sectional area, v is the velocity relative to the atmosphere and C_D is the drag coefficient. We will do calculation with $C_D = 1$ for simplicity. When the lander travels through the atmosphere it will gradually slow down until its velocity is the terminal velocity. Once the satellite reaches the terminal velocity its acceleration stays equal to zero. This implies that the forces on the satellite are balanced

$$G\frac{M_p m_l}{r^2} = \frac{1}{2}\rho C_D A v^2 \tag{103}$$

If the altitude of the satellite is negligible compared to the radius of the planet, that is $r = r_p + r_{\text{sat}} \implies r = r_p$, we can solve for the terminal velocity close to Driddu.

$$v_t = \sqrt{\frac{2GM_p m_l}{\rho_0 A r_p^2}} \tag{104}$$

Our landing craft could sustain up to a velocity of $3\frac{\text{m}}{\text{s}}$ at landing without breaking. In other words we wish the satellite to have a terminal velocity of $3\frac{\text{m}}{\text{s}}$ close to Driddu. To compute the required size of the landing parachute we can now solve, with $v_t = 3\frac{\text{m}}{\text{s}}$, for the cross sectional area A.

$$A = \frac{2GM_p m_l}{\rho v^2 r_p^2} \tag{105}$$

3. Results

Next we will go through our collection of results from our calculations and simulations.

3.1. Rocket / Satellite Simulation

3.1.1. Engine Simulation

Simulation

To make sure the particles behaved as we wanted them to with regards to collisions with the walls we first plotted a small engine partition containing only 100 particles.(Even though the actual simulation contains 100000 particles per partition, but it would be impossible to see what's going on in the plot and take too much time.)

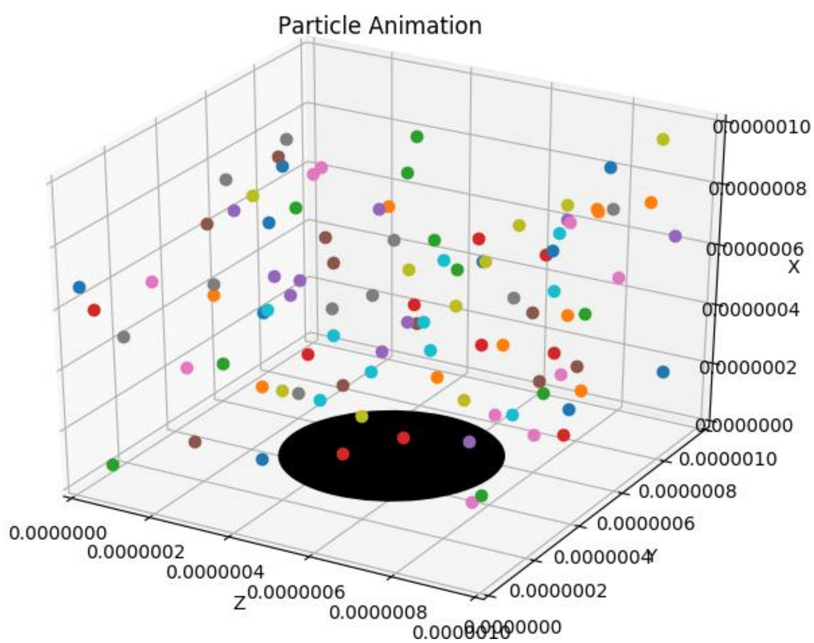


Figure 1: One small partition of our engine simulated using just 100 particles. X, Y and Z-axis are all in meters.

3.1.2. Launch Simulation

Engine variables

Running our engine partition simulation with 100000 particles at 10000 Kelvin, using 1000 time steps, $dt = 10^{-12}$ and with a engine partition size of $10^{-6}m \times 10^{-6}m \times 10^{-6}m$, gave us the following result:

Force per partition	3.4e-09 N
Particles per second per partition	$6.32 \cdot 10^{13}$

Force wanted	$1.07 \cdot 10^{6}$ N
Engine partitions needed	$3.15 \cdot 10^{14}$

Launch simulation

Escape velocity at surface	12735 m/s
Escape velocity at height reached	10789 m/s

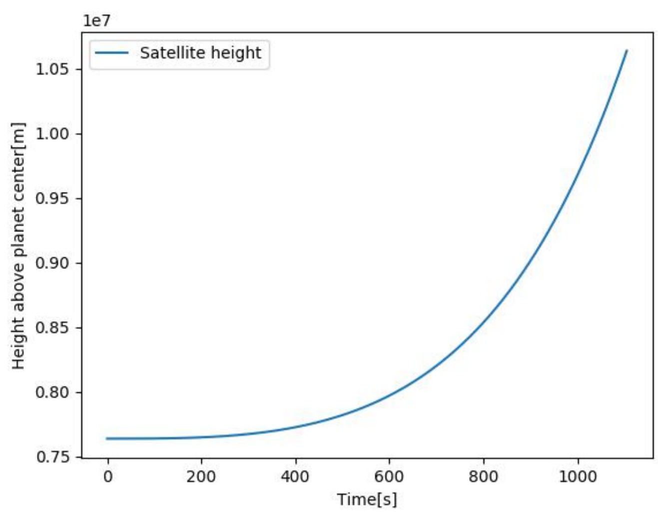


Figure 2: The radial position of the rocket plotted over time.

3.1.3. Star System Simulation

As you can see, the result of our simulations show that we have a pretty big star system. Even though this simulation was run over 100 years, the outermost planet didn't even make it around the star once. "Luckily" though, we start out in the habitable zone and won't have to go very far(in relation to the rest of the star system) in order to get to our target. Here is a closer look at the lower-orbit planets who are important to our journey:

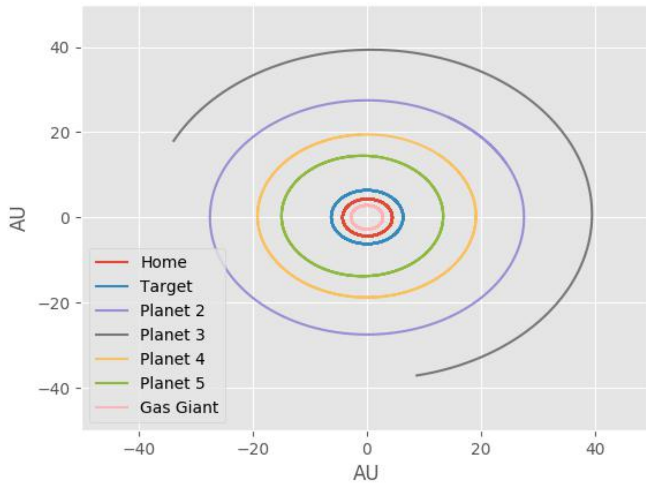


Figure 3: The simulated paths of the planets in our star system. Units are AU on both axes and the simulation was run for 100 years.

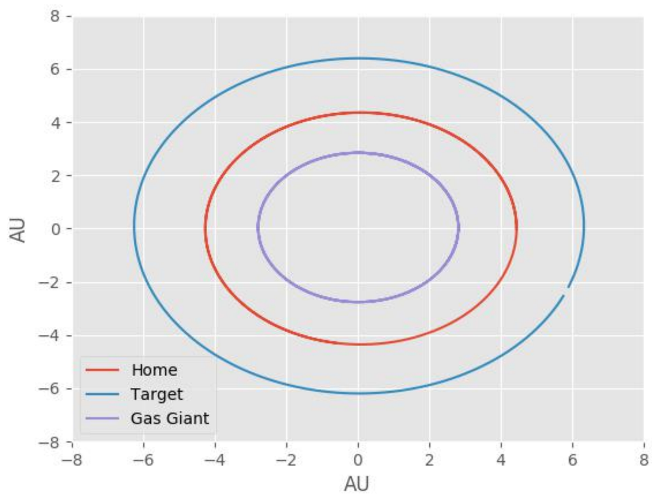


Figure 4: Simulated paths of planets 0, 1 and 6.

In comparison, this simulation was only run for 10 years instead of 100, and all 3 of the planets made it around anyway. As you can see, the gas giant in our star system orbits at the "low" altitude of 2-3 AU above the sun.

As mentioned in the theory section(2.1.3), we checked if our simulated orbits were satisfyingly approximated using Kepler's laws.

Checking our simulations accuracy using Kepler's laws

Using our simulation we printed out each planet's apoapsis and periapsis, as seen here

Apoapsis:

Planet	Time(Years)	Apoapsis height(AU)
0	0.0	4.44345620
1	2,5748	6.39799144
2	10.397	27.4951134
3	52.829	39.6877237
4	0,3646	19.4768310
5	7,5376	15.0277315
6	0,8063	2.84413133

Periapsis:

Planet	Time(Years)	Periapsis height(AU)
0	2.9059	4.26463310
1	7.6232	6.18645634
2	27.083	27.4548341
3	129.65	37.6782026
4	27.083	18.7422095
5	24.559	13.2690266
6	2.3057	2.75790904

Using this, we were able to check the Kepler-area at apoapsis and periapsis for each planet:

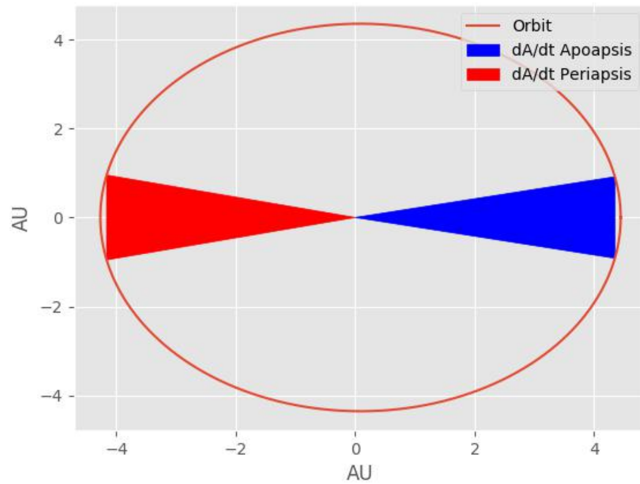


Figure 5: dA/dt plotted at both periapsis and apoapsis of planet 0

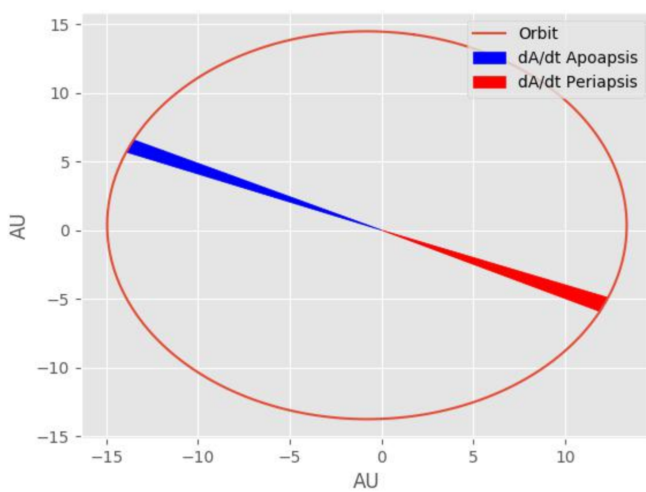


Figure 6: dA/dt plotted at both periapsis and apoapsis of planet 5

Here are the sizes of $\frac{dA}{dt}$ compared in numbers, and as we can see it's not perfect but very close.

$\frac{dA}{dt}$ periapsis	1494346.77518
$\frac{dA}{dt}$ apoapsis	1494301.99978

Lastly here is the analytical and numerical positions of planets 0 and 5 plotted on top of each other, and as you can see they are accurate enough to the point where we can't differentiate between numerical and analytical orbits unless we zoom in.

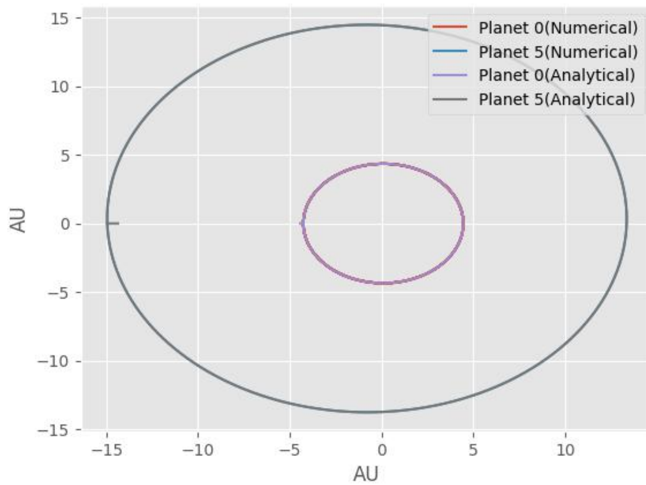


Figure 7: Simulated paths of planets 0 and 5 compared with analytical paths.

3.1.4. Deciding Where to Land

Surface temperature of a planet

Planet	Surface Temperature [K]
0	325
1	271
2	129
3	109
4	155
5	180
6	406

The Habitable Zone

Planet	Distance from star[m]	Distance from star[AU]
r_{min}	$4.547 \cdot 10^{11}$	3.039
r_{max}	$1.023 \cdot 10^{12}$	6.839

Only planet 0(our home planet) and planet 1(our target planet) fall within the habitable zone.

3.1.5. Orbital Transfer Simulation

These first two plots are the first attempts at a rendezvous we had using our simulation, and just for testing purposes.

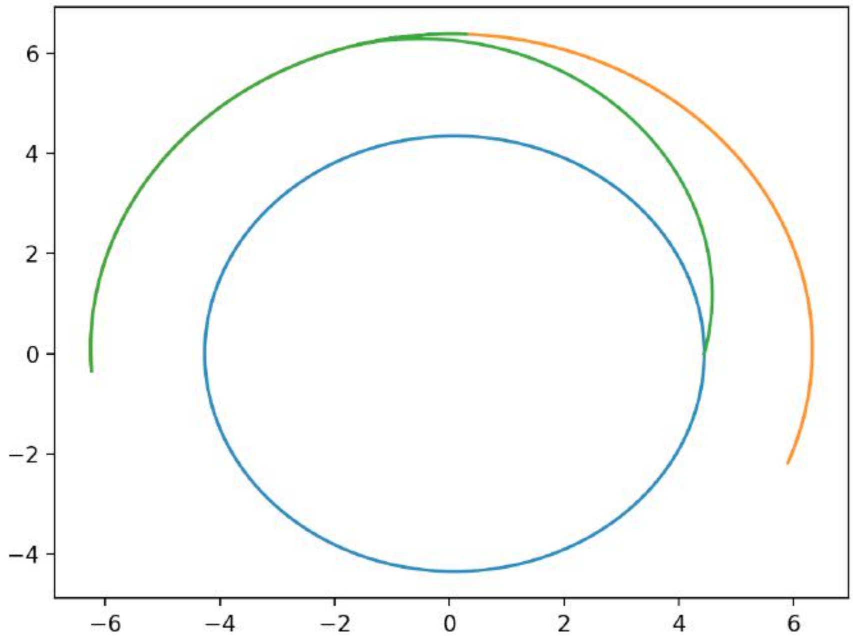


Figure 8: First successful transfer to orbit, only to test the simulation

As explained in the theory section we used python to pinpoint the time at which the planets were alligned correctly.

Launch time needed for Hohmann-transfer is after	11.3336428571 years

Actual simulation of Hohmann-transfer

Here is the final simulated journey, containing one launch-boost and one orbital insertion-boost at Driddu, which is zoomed in in the next figure()

Fuel used during transfer

	kg
Initial mass	101100
Mass left after launch	27342.0912214
Mass after first boost	26891.6435053
Mass after orbital insertion(second boost)	23412.1651052

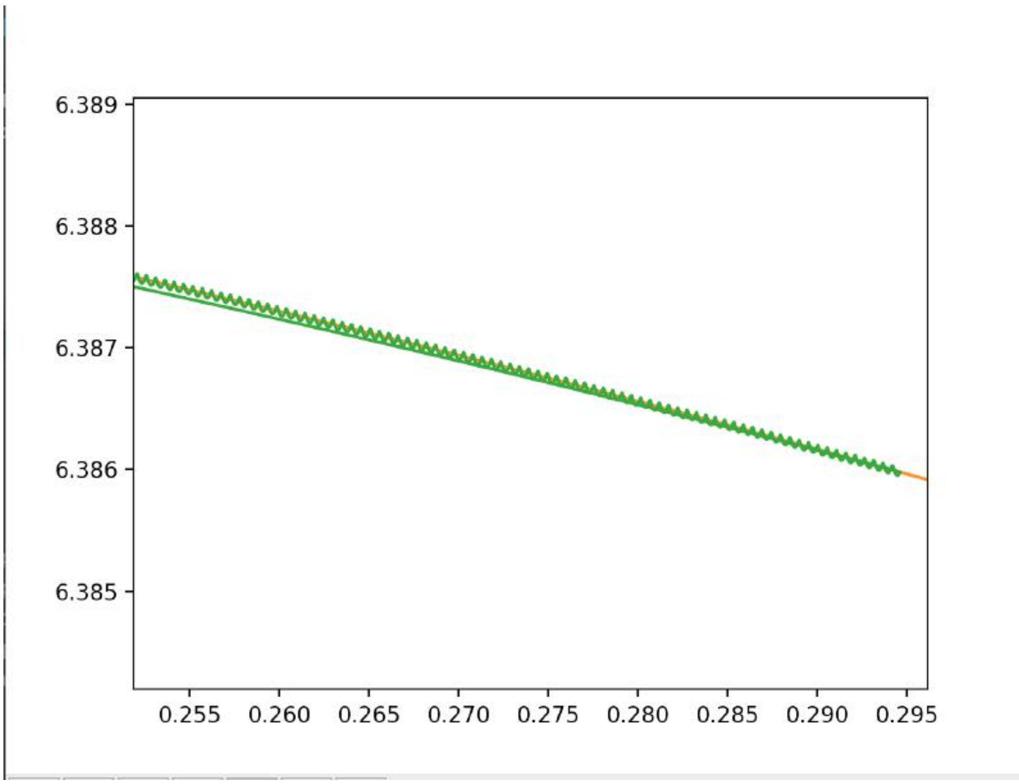


Figure 9: Figure 8 zoomed in at orbital insertion

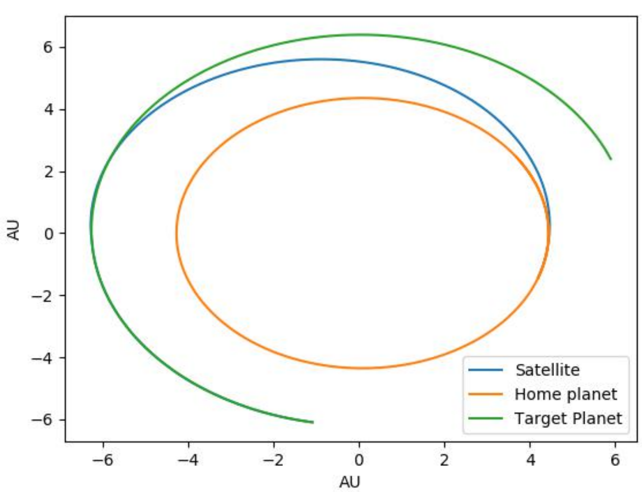


Figure 10: Final simulated journey from Akkli to Driddu.

3.1.6. Trilateration Algorithm

Following is the result of our trilateration algorithm, described in 2.2.3.

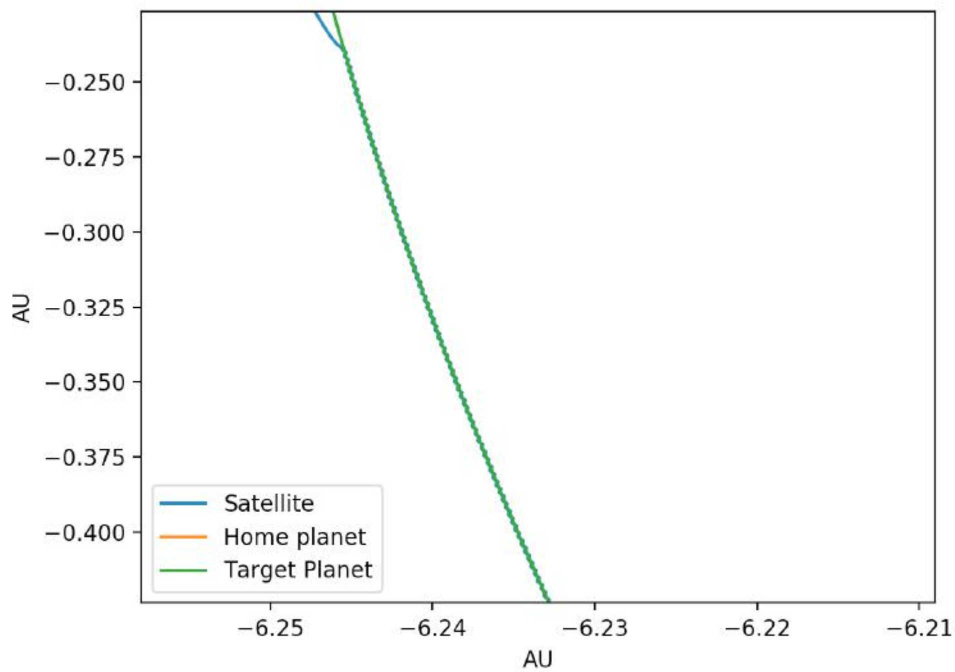
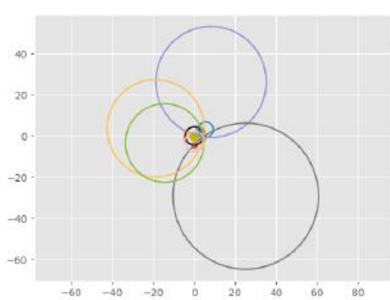
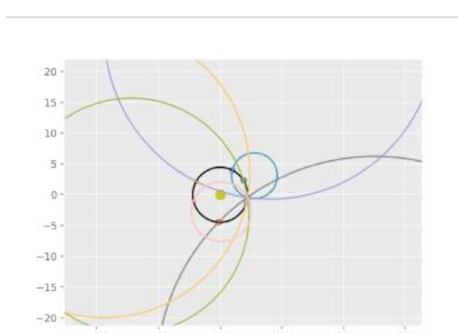


Figure 11: Figure 10 zoomed in at orbital insertion. (It looks like the satellite is coming in at a sharper angle than it actually does because of uneven zooming on each axis).



Circles drawn at corresponding planet's position with radius equal to the distance returned from the satellite radar



Zoomed in version of the trilateration

3.2. Simulating the Atmosphere

3.2.1. Surface Temperature

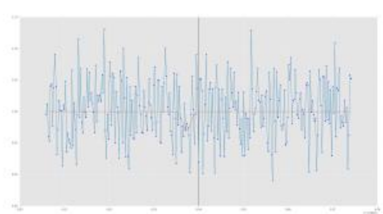
Calculating the temperature on the surface of Driddu using the absorption area approximation along with equation 83

$$T_0 = \sqrt{\frac{1296759948}{941304992815}} \cdot 10328 = 271K \tag{106}$$

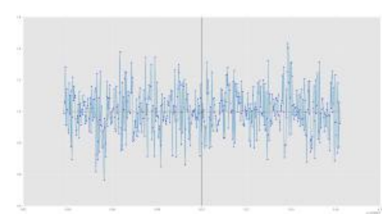
So the temperature is slightly below zero on the Celsius scale.

3.2.2. Chemical Composition of Driddu's Atmosphere

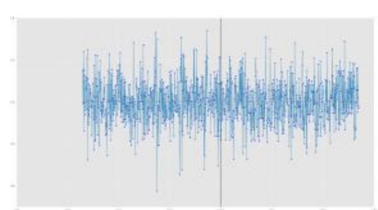
Following is the results of our chi square minimization. We found potential lines for H_2O, CH_4 and N_2O. We have also included an example for an absorption line which was found to be not present; CO_2 Using our potential absorption lines we



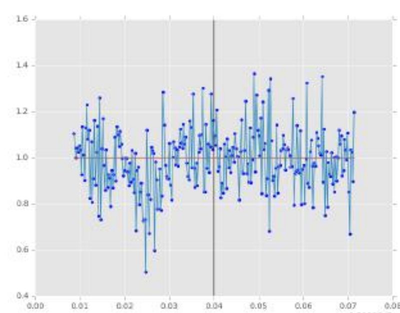
Possible Absorption line for H_2O



Possible Absorption line for CH_4



Possible Absorption line for N_2O



Example of no absorption line for CO_2

had to decide whether the candidate lines really were absorption lines and not false positives. The values which minimized χ^2 for each of the absorption lines were:

- H_2O: $\sigma = 0.00085463$, $\lambda_c = 719.99735117$, $F_{min} = 0.96896552$
- CH_4: $\sigma = 0.00154593$, $\lambda_c = 1659.95523846$, $F_{min} = 0.86551724$
- N_2O: $\sigma = 0.00338083$, $\lambda_c = 2869.91143779$, $F_{min} = 0.95862069$

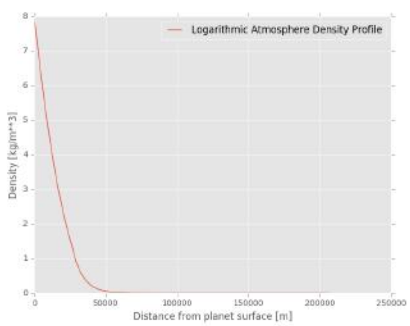
Next we used our expression for the standard deviation, 78, of the absorption line due to Driddu's temperature to evaluate our potential absorption lines. The temperatures corresponding to our best fit of the smudging of the absorption lines are:

- H_2O: $T = 276.4$K
- CH_4: $T = 151.5$K
- N_2O: $T = 665.0$K

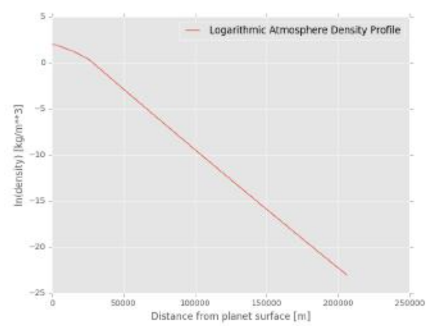
Comparing these with the surface temperature of Driddu, found here: 3.2, the lines for H_2O and CH_4 are more likely than N_2O. The mean molecular weight in Driddu's atmosphere, depending on whether we presume the CH_4 line as real, is either 18.015amu or 17.02amu. We opted to assume the CH_4 as real, because CH_4 is a marker for extraterrestrial life.

3.2.3. Modelling the Atmosphere

Following is our plot of the density profile of the Driddu's atmosphere. We see that the density decreases sharply and then levels out close to zero.



Density vs. distance from surface



Logarithmic Density vs. distance from surface

Figure 15: Adiabatic region density up to $r_{T_0/2} = 25939$m and isothermal region after.

3.2.4. Air Resistance and Terminal Velocity

Calculation of parachute area on lander in order to have a terminal velocity of 3m/s at the surface. Using expression 105

$$A = 22.2m^2 \tag{107}$$

3.2.5. Landing

The landing was successful on the first try, and we touched down with 2.98m/s on top of a small mountain on Driddu's surface. This means our calculations were successful, seeing that the area of 22.2m^2 was calculated for the purpose of touching down with 3 m/s

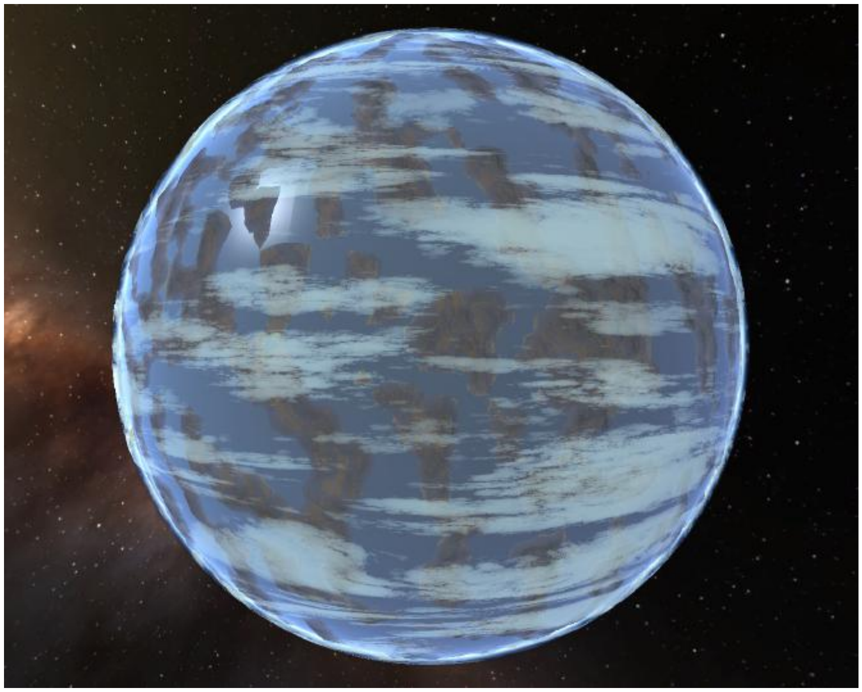


Figure 16: Photo from orbit around Driddu before descending

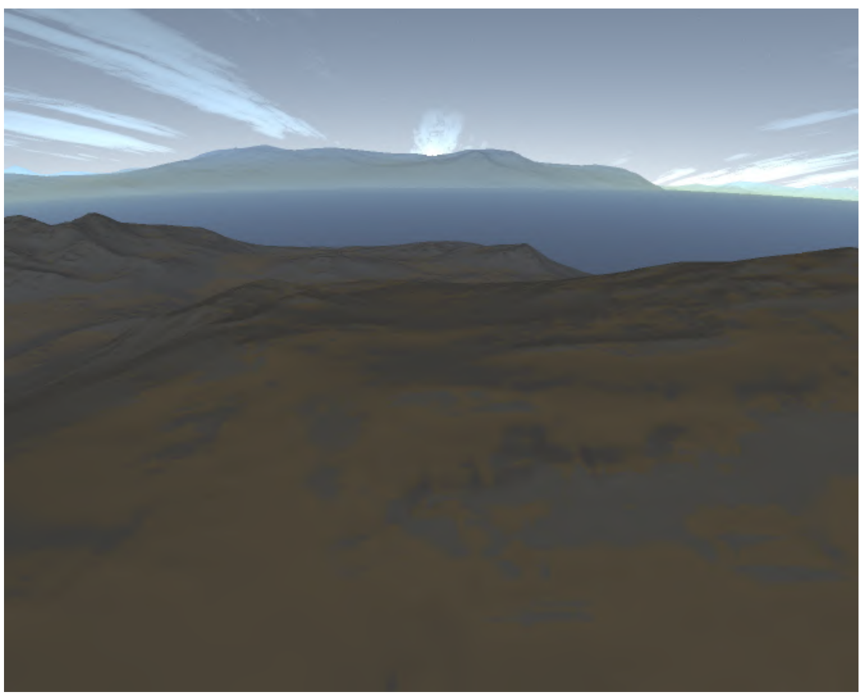


Figure 17: Photo from touchdown

4. Discussion

4.1. Rocket / Satellite Simulation

4.1.1. Engine Simulation

From what we can tell, the simulation is working as expected and our results look correct. Of course the simulation isn't 100% correct according to the real world, but with the limitations we have set and the assumptions we have made in the theory section(2.1.1) this seems like a decent result.

4.1.2. Launch Simulation

Engine variables

Our engine and fuel tank sizes are unrealistic, but it will theoretically work with the assumptions we have made. We have "created" a rocket that will overcome the force of gravity and reach escape velocity within a realistic amount of time.

Launch simulation

As we expected, the launch started off slowly but picked up speed at an increasing pace when the mass of the rocket and the gravitational pull decreased. We also checked that our calculations of the gravitational force were correct, by giving the rocket the exact thrust equal to the force of gravity, and the rocket (almost) didn't move.

At first we had programmed our code to calculate the exact escape velocity at the current height for each time step, and conclude the launch when that escape velocity was reached. This is what gave us 10789.9m/s. However, when we got to the orbital transfer simulation(2.1.5), we realized that letting our rocket have the exact escape velocity wasn't enough to actually escape properly. So for a simple fix we ended up programming it so that our rocket was boosting until it reached **escape velocity at surface**, which was 12735.1m/s

4.1.3. Star System Simulation

As mentioned in the theory-section(2.1.3), the first thing we found out was that our star system has a huge gas giant in a low orbit around our star. It has the lowest orbit of all the planets, even though it's also the biggest planet in the system by far. We think this might be one of the big reasons for our simulation having to use so many time steps to achieve the desired accuracy.

Since this gas giant was there we also discussed at this stage whether we wanted to change our target destination to a planet far out, and use the gas giant to perform a slingshot maneuver. In the end, though, we decided that we would rather stay in the habitable zone and hope for life at Driddu. Seeing that higher accuracy would require even more time steps per year, we decided to stick with 70 000. There is enough other inaccurate parts of this simulation as well, so even if we used a huge amount of time steps, it wouldn't actually be 100% accurate to real life. From this point on we are prepared to have to redo our calculations when we know the actual positions of both planets and satellite during the real journey.

Checking our simulation's accuracy using Kepler's laws

As you can see in Figure 6 dA/dt was very close to being the same for both periapsis and apoapsis. This shows us that our simulation is working pretty well. The little difference probably comes partly from simulation accuracy not being high enough, and a little error from calculating between meters and AU.

4.1.4. Orbital Transfer Simulation

When we implemented the code which calculated the Δv needed to get in to orbit around Driddu in the first test transfer in Figure 8, the satellite automatically got into a stable orbit and stayed there for the entire simulation, even over a big amount of years. We originally expected to have to run a second simulation with a larger amount of time steps per year to get into orbit, but seeing as we ended up having to use 70 000 time steps per year to simulate the planets movements in Section 2.1.3, this simulation turned out to be accurate enough.

The calculation of the angle for the Hohmann-transfer and the launch time returned from the code in 3.1.5 showed to be correct right away. We automatically got close to the planet at apoapsis, but of course nowhere near close enough for an encounter right away. This is mostly because equation 44 assumes circular orbits, and our planets orbits are not completely circular. Some error factor could also be because of the huge gas giant which is in orbit below us. Since it wasn't completely accurate anyways, we ended up using 11.3 years as start time even though the exact value we got was 11.3336428571, and then make tiny tweaks to the launch boost until we got a close enough encounter. When we did, the code we had from making the first simulated journey(8) automatically put our satellite into orbit. So now we had an actual fuel efficient simulated route for our satellite to take, and it was time to move away from simulations.

4.2. Simulating the Atmosphere

4.2.1. Modelling the Atmosphere

As shown in figure 15 the density of the atmosphere sharply drops of with increasing radial distance in the adiabatic region of the atmosphere, and levels out after reaching the isothermal region. This is in line with how we expected our model to behave, and does seem quite realistic. We initially though assuming the gravitational force to be constant throughout the atmosphere would be a big source of uncertainty, however after further discussion we have arrived at the conclusion that the error at most would be a few percent.

4.2.2. Air Resistance and Terminal Velocity

When simulating the drag we initially had some problems with our acceleration calculation returning NaN. We zeroed in on the cause to be us using the expression for the drag even when we were far above the atmosphere. We fixed the problem by implementing an if-test to assure that the drag force was zero when above a critical radial distance from the surface. We also made some mistakes when calculating the unit vector for the drag force, which cost us a lot of time, but we eventually figured out what was wrong. After fixing these errors we finally got a result that seemed likely.

5. Conclusion

Through our work and effort we managed to please the extraterrestrial race of beings with our scientific progress. We successfully created a working rocket engine, tweaking the properties of the engine to match our given planets gravity. We modelled our solar system using a symplectic integrator and used this model to figure out how, when and where to launch from Akkli, in the process showing that we were able to perform a transfer using as little fuel as possible. Further along on our journey we learned how to analyze atmospheres and sort through messy data, a big milestone for any civilization. We managed to achieve the correct descent speed by using the analysis of the atmosphere, and took pictures along the way. Upon arriving at Driddu, we learned a lot about relativity and the life of a star through calculations and simulations done in the laboratory of our lander. We solved several complex problems, most of which will surely gain the attention of our fellow space explorers. From all of this we have gained a significant amount of knowledge about several aspects of our universe, and hopefully in the process impressed a couple aliens.

APPENDIX

A. Special Relativity

A.1. Laser Battle

The first relativistic situation we will be discussing is two battle ships moving with zero velocity relative to each other. During their battle they pass close to Driddu with velocity relative to Driddu of 0.882c.

For the Space Ship Frame

In the frame of the space ships they each fire a laser directed at the other simultaneously. We organize the laser shooting as event A and B, A is the leftmost ship firing. Once the laser hits the other ship, the ship explodes and we will call these event C and D. C is the leftmost explosion.

In the space ship frame of reference the distance between the space ships is measured to be: $L' = 1248.37\text{km}$.

We wanted to figure out how long it took for the lasers to hit the other craft, using the distance between them L' we could find just that. Since the speed of light is constant in all reference frames we convert L' into seconds through the speed of light.

$$L' = \frac{1248.37\text{km}}{c} = 0.00416\text{s} = 4.16\text{ms} \rightarrow 1248.37\text{km} \tag{A1}$$

This time is the time at which event C and D happen in the space ship frame.

Ground Frame

The situation we already discussed looks quite different from another point of view. This time we will discuss the events A, B, C and D as seen from an observer on the surface of Driddu. When we discussed the series of events as seen from the space ships we said that pairs of events happened simultaneously, in the ground frame this is no longer the case. Seen from the ground event A will happen first, then event B. The beams will still meet up at halfway between the ships, but event C will happen before event D.

We now imagine the rightmost space ship survives the explosion. Light from the explosion travels from the left ship to the right ship. We call the observation of the leftmost explosion event E. We wanted to figure out at what time event E happens in the ground frame. We took some measurements of the timing of the four first events, they were as following: $t_{\text{A}} = 0.0\text{ms}$, $t_{\text{B}} = 7.550\text{ms}$, $t_{\text{C}} = 8.8\text{ms}$ and $t_{\text{D}} = 16.2\text{ms}$.

Our thinking is that if event C is made into our new start time, then the light from the explosion should use the same amount of time as the time it took the light from event A to reach the rightmost ship. Therefore $t_{\text{E}} = t_{\text{C}} + t_{\text{D}} = 25.0\text{ms}$ is the time at which event E takes place in according to the ground reference frame. The position at which event E takes place is found by realizing that the time it takes for any light coming from ship A to ship B, while they are both moving, will always be t_{D}. The next part is the movement of the ships before the leftmost explosion, this length will be vt_{C}, and thus the position of event E can be written as $x_{\mathrm{E}} = vt_{\mathrm{C}} + ct_{\mathrm{D}} = 23.96\text{ms}, 7.184 \cdot 10^6\text{m}$

Since we have both the position and time of event E we can use the fact that the spacetime interval is invariant between reference frames to figure out the spacetime coordinates of event E in the space ship frame. For simplicity we chose the spacetime interval between event A and event E.

$$(\Delta s)^2 = (\Delta t)^2 - (\Delta x)^2 \tag{A2}$$

The distance from event A to event E in spacetime becomes

$$\Delta s = \sqrt{t_{\mathrm{E}}^2 - x_{\mathrm{E}}^2} = \sqrt{(25.0)^2 - (23.96)^2} = 7.135\text{ms} \tag{A3}$$

or equivalently in meters $\Delta s = 2.139 \cdot 10^6\text{m}$. Now that we know the spacetime distance between event A and E in the ground frame we can deduce the when event E happens in the space ship frame.

$$(\Delta t')^2 = (\Delta s)^2 + (\Delta x')^2 \tag{A4}$$

We notice that since we are in the space ship frame Driddu looks to be shooting of in the negative x direction while both the space ships are still, $\Delta x' = 0$. This means that the time event E happens in the ship's frame is at $t = 7.135$ms. This result could have been achieved without any calculations at all because the space ship frame is a free float frame. The light from the explosion has to use the same amount of time as the laser light, this means that event E in the space ship frame happens after $2t_{\mathrm{C}}$. Comparing our result from the spacetime interval with our distance L' we realize that our timing measurements must have been slightly inaccurate.

For Both Frames

If we now consider a third observer in a spaceship located in the middle between A and B, we can try to determine that time has to run differently in the two frames, without actually building on any relativistic ideas other than the fact that the speed of light is constant in all reference frames.

If we first consider the spaceship frame, we know that the laser beams were emitted simultaneously and that our observer is located in the middle between the two firing spaceships. Since all three spaceships are moving at the same, constant speed, we can consider this a free float frame. So this situation is not really that complicated. We have two lasers which move at the same speed, and have to travel the same amount of distance. Obviously they will cross this distance in the same amount of time, and cross in the middle.

Now we consider the planet frame of reference, still assuming we don't know about things like length contractions and time dilation. The only things we know are that according to the spaceship observer, the beams both emitted from their spaceships at the same time, and crossed exactly through the middle spaceship's position. The fact that they cross through the middle must be correct for the planet frame as well, seeing that we know the beams travel at a constant speed c in any frame of reference, so none of them can reach the middle before or after the other. With this in mind, we can conclude that the beams were in fact not emitted at the same time, in the planets frame of reference. This is because the three spaceships are moving relative to the planet. When the left laser is emitted, the middle spaceship is moving away from it with velocity v. You would think that this would be compensated for by the laser-firing spaceship also having velocity v(that's what happens in the spaceship frame, the velocity v is just considered 0 in that frame.), but since the speed of light is always constant c, then the laser cannot move with $c+v$, but will still move with a constant speed c. This will result in the left laser having to take longer to reach the middle spaceship, because the middle spaceship is moving away from it with speed v. The opposite happens from the right spaceship. The laser still moves at a constant speed c, but the middle spaceship is moving towards it, so it will take less time to reach it, than it did in the spaceship frame of reference. Since we know the two lasers crossed at the same place at the same time, but also that they both took a different time to get there, then we can only conclude that in this frame of reference the lasers were emitted at different times. More precisely, the left laser had to be fired before the right laser, for them to meet at the same place at the same time.

Using what we know at this point we can write the positions of the space ships and laser beams as functions of time. We know that the left spaceship starts at $X = 0$ and moves with velocity v to the right. So the function for the leftmost spaceship will just be

$$x_{\text{ship1}}(t) = vt \tag{A5}$$

For the right spaceship, we have the same situation except it starts at x = L So for the right spaceship we have

$$x_{\text{ship2}}(t) = vt + L \tag{A6}$$

The third ship is in the middle of ship 1 and 2,

$$x_{\text{ship3}}(t) = vt + \frac{L}{2} \tag{A7}$$

The left ship's laser is fired at the speed of light at $t = 0$, so to follow it as a function of time we get

$$x_{\text{laser1}}(t) = ct \tag{A8}$$

The laser from the rightmost ship is a little more complicated. We know the time t_B at which it is fired. So all we need to do is follow the ship's movement up until the laser i emitted (t_B), and then star following the laser beam, which moves at speed c the opposite direction. We get

$$x_{\text{laser2}}(t) = L + vt_B - c(t - t_B) \tag{A9}$$

And by setting the units such that $c = 1$, we get

$$x_{\text{laser1}}(t) = t \tag{A10}$$

$$x_{\text{laser2}}(t) = L + vt_B - (t - t_B) \tag{A11}$$

With these functions in hand, we can find the time t_x at which the lasers meet at the middle. We set the position of laser 1 equal to the position of the middle space ship.

$$t_x = vt_x + \frac{L}{2} \tag{A12}$$

$$t_x - vt_x = \frac{L}{2} \tag{A13}$$

$$t_x = \frac{\frac{L}{2}}{1 - v} \tag{A14}$$

Since we know that the position of laser 2 also is equal to the position of spaceship 3 at $t = t_x$, we can use this to find an expression for t_B

$$vt_x + \frac{L}{2} = L + vt_B - (t_x - t_B) \tag{A15}$$

$$t_x + vt_x - \frac{L}{2} = t_B + vt_B \tag{A16}$$

$$(1 + v)t_B = (1 + v)t_x - \frac{L}{2} \tag{A17}$$

$$t_B = t_x - \frac{L/2}{1 + v} \tag{A18}$$

To find the time of the left explosion, we can use that the position of the left spaceship is equal to the position of the laser beam from the right ship.

$$vt_C = L + vt_B - (t_C - t_B) \tag{A19}$$

$$vt_C = L + vt_B - t_C + t_B \tag{A20}$$

$$t_C + vt_C = t_B + vt_B + L \tag{A21}$$

$$(1 + v)t_C = (1 + v)t_B + L \tag{A22}$$

$$t_C = t_B + \frac{L}{1 + v} \tag{A23}$$

From A18 we can then substitute for t_B and get

$$t_C = t_x - \frac{L/2}{1+v} + \frac{L}{1+v} \tag{A24}$$

$$t_C = t_x + \frac{L/2}{1+v} \tag{A25}$$

Now that we know the time at which spaceship 1 explodes, we can track the position of the light coming from the explosion as well. To do this, we only need to set the position equal to the position of the spaceship up until t_C and add $c(t - t_C) = t - t_C$ to follow the light's movement after that

$$x_{\text{light1}} = vt_C + (t - t_C) \tag{A26}$$

Next we want to find the time t_{x2} when the two light beams from each explosion meet in the middle. The beams meet when the position of light 1, x_{light1}, is equal to the position of space ship 3, the middle ship.

$$vt_C + t_{x2} - t_C = vt_{x2} + L/2 \tag{A27}$$

$$vt_{x2} - t_{x2} = vt_C - t_C - L/2 \tag{A28}$$

$$t_{x2}(v-1) = t_C(v-1) - L/2 \tag{A29}$$

$$t_{x2} = t_C - \frac{L/2}{v-1} \tag{A30}$$

Now we can use the expression we found for t_C, A25

$$t_{x2} = t_x + \frac{L/2}{1+v} - \frac{L/2}{v-1} \tag{A31}$$

$$t_{x2} = t_x + L/2\left(\frac{1}{1+v} - \frac{1}{v-1}\right) \tag{A32}$$

By finding the common denominator we get

$$t_{x2} = t_x + L/2\left(\frac{-2}{v^2-1}\right) \tag{A33}$$

Multiply by -1 on both sides of the fraction and by multiplying $L/2$ back in the parenthesis we get

$$t_{x2} = t_x + \frac{L}{1-v^2} \tag{A34}$$

Lastly we look at the time interval $\Delta t = t_{x2} - t_x$, how long does it take from the observer in the middle ship sees the lasers crossing, until he sees the explosions?

We substitute for $t_x 2$ with the expression we just made(A34)

$$\Delta t = t_x + \frac{L}{1-v^2} - t_x \tag{A35}$$

and we see that the time it takes is

$$\Delta t = \frac{L}{1-v^2} \tag{A36}$$

A.2. Relativistic Ping Pong

For the next situation the ships are equipped with mirrors such that they can bounce a laser beam between them. As with the last situation they are a fixed distance apart. The leftmost ship fires it's laser when it passes $x = 0$ in the ground frame, at which there is a space station with zero velocity with respect to Driddu. We are first interested in four distinct events A, B, C and D. The emission of the beam is event A, the first reflection is event B, the second reflection is event D, and the space station exploding is event C. Event B and C happen at the same time in the space ship frame. We will use coordinates (x, y) for the planet frame and (x', t') for the space ship frame.

The positions and times of all the events in the space ship frame were measure to be: We notice how the time between bounces are equal. This is because the speed of

$$\begin{array}{c|c} x'_{\mathrm{A}} = 0.0\mathrm{m} & t'_{\mathrm{A}} = 0.0\mathrm{s} \\ x'_{\mathrm{B}} = 4.00 \cdot 10^5\mathrm{m} & t'_{\mathrm{B}} = 1.34\mathrm{ms} \\ x'_{\mathrm{C}} = 2.32 \cdot 10^5\mathrm{m} & t'_{\mathrm{C}} = 1.34\mathrm{ms} \\ x'_{\mathrm{D}} = 0.0\mathrm{m} & t'_{\mathrm{D}} = 2.67\mathrm{ms} \end{array}$$

light is constant in all reference frames. We wish to know what the situation looks like from Driddu's frame of reference. Event A happens at the same place and time in both frames.

The time of event B in the ground frame is found by setting the position of the right ship $x_{\mathrm{rs}}(t) = -vt + L$ equal to the position of the laser beam $x_{\mathrm{l}} = ct$ giving

$$t_{\mathrm{B}} = \frac{L}{(1+v)} \tag{A37}$$

Since we have the length between the ships in the ship frame $L' = x'_{\mathrm{B}}$ we can find the length between the ships in the planet frame $L = x_{\mathrm{B}}$ through the Lorentz transformation

$$\begin{pmatrix} t' \\ x' \end{pmatrix} = \begin{pmatrix} \gamma & -v\gamma \\ -v\gamma & \gamma \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix} \tag{A38}$$

Since we are only interested in the length contraction at this moment we use the scalar equation for the contraction. The ship velocity is in terms of the speed of light so the expression becomes

$$L = L'\sqrt{1-v^2} = 3.034 \cdot 10^5\text{m} \tag{A39}$$

we see that the length has contracted when comparing the point of view of the space ship and the ground. Using this value we can see that the time at which event B happens is $t_\text{B} = 0.61$ms.

To figure out when event D happens we set the position function for the leftmost ship equal to the position function of the reflected light. The position of the reflected light is $x_\text{r,l} = L - ct$ and the position of the leftmost ship, going in negative x direction, is $x_\text{D}(t) = -vt$. To find the intersection time, collision, between these position functions we set them equal to each other and solve for t

$$t_\text{D} = \frac{L}{(1-v)} \tag{A40}$$

. Using our value for L we calculated the time of event D in the ground frame to be $t_\text{D} = 2.89$ms. We see that in the ground frame $\Delta t_\text{AB} < \Delta t_{BD}$ contradictory to what is happening in the space ship reference frame. The two ships relative velocity is zero and therefore the beam of light has to spend an equal amount of time going from the left to the right ship as from the right to the left. Because of the space ships velocity in the ground frame the time from event A to B has to be smaller than the time from B to D. This is because the right ship meets up with the light and the left ship moves away from the light.

Event C is now the only event which we have not found the spacetime coordinates for in the ground frame. We remember how event B and C were measured to be simultaneous in the space ship frame. We use the invariance of the spacetime interval to figure out the coordinates of event C.

$$(\Delta s'_{AC}) = \sqrt{(1.34\text{ms})^2 - (0.774\text{ms})^2} = 1.094\text{ms} \tag{A41}$$

$$(\Delta s'_{AC}) = \sqrt{(\Delta t_{AC}) - (\Delta x_{AC})} \tag{A42}$$

However we keep in mind that the station is at rest with respect to the planet and thus $(\Delta x_{AC}) = 0$ giving us the complete spacetime coordinates of event C: $x_C = 0.0$m and $t_C = 1.196$ms. Here we see one of the principle effects of special relativity, in the planet frame of reference the two events B and C happen at different times. However in the space ships frame of reference the two events B and C happen simultaneously. This shows that simultaneity is a relative concept just as velocity is, and depends upon the chosen reference system.

This result is not unexpected and could be arrived at through the following reasoning. Because B and C are simultaneous in the space ships frame we imagine we place a third observer at rest in the space ships frame between the space station and the rightmost space ship. This observer will see the light from the explosion and the reflected beam at the same time in the space ship frame. If event B and C were to be simultaneous in the planet frame then this third observer would see the light from the explosion first and then the reflected beam from his point of view. But he is per definition supposed to see the explosion and the reflection at the same time because the events were defined to happen simultaneously in the space ship frame. This is a contradiction with the result being that event C necessarily must happen after event B in the planet frame.

A.3. The Twin Paradox

Now we look at a situation where we imagine that we are in a spaceship which is travelling from planet A to planet B, while a second spaceship travels from planet C to planet A in the opposite direction. The planets are all at rest relative to each other and aligned with 200 light years in between, so that there is 400 light years from planet A to planet C. Spaceship 1 is travelling at $v = 0.99$ and spaceship 2 at $v = -0.99$.

First of all, we only look at the first two planets and our spaceship. Since all the planets are at rest relative to each other they are all in the same frame of reference, and we will call this the planet frame. In the planet frame it took $\Delta t = \frac{200}{0.99} \approx 202$ years to arrive at P2, but for us in the spaceship frame the same trip will take less time due to time dilation, by a factor of γ. So to figure out how much time the trip would take if we measured it on a wristwatch on the spaceship, we use

$$\Delta t' = \frac{\Delta t}{\gamma} = \Delta t\sqrt{1 - v^2} = 202\sqrt{1 - 0.99^2} = 28.5\text{years} \tag{A43}$$

We now imagine the situation where we embark on a return journey, the calculations when going this way would be the exact same, except that the velocity $v = -0.99$ would be negative, but since the equation has v^2 we end up with the same answers. So in total, the full journey will take 404 years in the planet frame, and 57 years in the spaceship frame.

If we do these calculations again, but this time switching so that the spaceship is the laboratory frame and the planets are moving by at $v = 0.99$, we can plug the 28.5 years we already know it takes from A to B in the spaceship frame for Δt, and calculate how much time passes in the moving planet frame, $\Delta t'$.

$$\Delta t' = \frac{\Delta t}{\gamma} = 28.5\sqrt{1 - 0.99^2} \approx 4 \tag{A44}$$

Again this will be the same on the way back(Or in this case, when planet 1 is on it's way back to us). So the full journey in the spaceship frame will take 57 years, and in the planet frame, 8 years.

At first glance, this might look like a paradox, because we now have two times for the same frame of reference, and if we can do this, can't we just switch between the two frames of reference over and over again until Δt and $\Delta t'$ are miniscule? The answer is that we can't do this. The reason is that this violates one of the principles of special relativity which states that the laws of physics are only equal in all **inertial** frames of reference. Since we know that the spaceship had to accelerate in order to change its velocity from $v = 0.99$ to $v = -0.99$, that reference frame is not inertial. Planet 1 was never accelerated like that, and therefore we were never allowed to switch to the spaceship as our laboratory frame, when our previous calculations were made using the planets as a laboratory frame.

In order to properly calculate situations as the one above and handle acceleration correctly, one would use general relativity. However, let's see if we can solve it using special relativity anyway. What we did wrong before was to switch between reference frames when we knew that one of them went through acceleration during the trip we were looking at. So to make sure we don't include any acceleration in the spaceship reference frame, we can split the journey up into two reference frames, one where the spaceship has a positive constant velocity going from planet 1 to planet $2(x', t')$, and one where it has a negative constant velocity going back(x'', t''). The laboratory frame is the planet frame as it was before(x, t). We define event A as the departure from planet 1, which happens at $x_A = x'_A = 0$ and $t_A = t'_A = 0$. Event B we define as the moment we arrive at planet 2. So how do we find t_B in the planet frame? We already know the distance L_0 and the velocity v the spaceship is travelling at, so all we need to do is divide the distance by the velocity to get the time it took. We then have $(x_B, t_B) = (L_0, \frac{L_0}{v})$ So the time t_B is equal to $\frac{L_0}{v}$. In the outgoing spaceship frame we have $t'_B = \frac{L_0}{v}\sqrt{1 - v^2} \approx 28.5$ years.

Now we define event B', where another observer in another spaceship, but in the same reference frame as us(The outgoing spaceship frame), checks what the time is on planet 1. In this frame of reference, event B' and event B happens at the same time, $t'_{B'} = t'_B$. We want to figure out the time $t_{B'}$ when event B' happens in the planet frame. To this, we first find $x'_{B'}$, the position of event B' in the outgoing reference frame. We know that the distance between planet 1 and planet 2 in the planet frame is equal to L_0, so all we need to do is to account for the length contraction by a factor of γ

$$x'_{B'} = L_0\sqrt{1 - v^2} \tag{A45}$$

Now that we have $x'_{B'}$, we can find $t_{B'}$ by using the spacetime interval invariance. The spacetime interval between two events will be the same in all reference frames. Hence we have

$$\Delta s^2_{AB'} = \Delta s'^2_{AB'} \tag{A46}$$

If we split these into components we get

$$\Delta t^2_{AB'} - \Delta x^2_{AB'} = \Delta t'^2_{AB'} - \Delta x'^2_{AB'} \tag{A47}$$

Where $\Delta t^2_{AB'} = (t_{B'} - t_A)^2 = t^2_{B'}$ because we start at time $t_A = 0$

$\Delta x^2_{AB'} = (x_{B'} - x_A)^2 = 0$ because both $x_{B'}$ and x_A are zero
$\Delta t'^2_{AB'} = (t'_{B'} - t'_A)^2 = t'^2_{B'}$, and
$\Delta x'^2_{AB'} = (x'_{B'} - x'_A)^2 = x'_{B'}$, which we just found. So if we substitute we get

$$t^2_{B'} = t'^2_{B'} - x'_{B'} \tag{A48}$$

And thus we have

$$t_{B'} = \sqrt{t'^2_{B'} - x'_{B'}} \tag{A49}$$

We already know that $t'_B = \frac{L_0}{v}\sqrt{1-v^2}$, and as stated earlier $t'_{B'} = t'_B$, so by substituting we get

$$t_{B'} = \sqrt{\frac{L_0^2}{v^2}(1-v^2) - L_0^2(1-v^2)} \tag{A50}$$

$$t_{B'} = \sqrt{L_0^2 v^2 - 2L_0^2 + \frac{L_0^2}{v^2}} \tag{A51}$$

$$t_{B'} = \sqrt{\frac{L_0^2}{v^2}(v^4 - 2v^2 + 1)} \tag{A52}$$

$$t_{B'} = \sqrt{\frac{L_0^2}{v^2}(1 - v^2)^2} \tag{A53}$$

$$t_{B'} = \frac{L_0}{v} - L_0 v \tag{A54}$$

and now we have an expression for $t_{B'}$.

If we plug in 200 light years for L_0 and 0.99 for v, we get

$$t_{B'} \approx 4 \tag{A55}$$

So why do we now get $t_{B'} = 4$ years when we already know that event B happens at 202 years in the planet frame? This is because in the planet frame, event B and event B' are not simultaneous even though they are in the outgoing spaceship frame.

Now we consider a person P in the returning spaceship frame(x'', t''). Person P starts in a spaceship at planet 3 with velocity $v = -0.99$. We define person P's departure from planet 3 as event D. In the planet frame, event A and event D happen simultaneously. We want to know what time person P records on his watch at event B, when our spaceship meets with his. Let's define some spacetime intervals.

$$\Delta s^2_{BD} = \Delta t^2_{BD} - \Delta x^2_{BD} \tag{A56}$$

$$\Delta s^2_{BD} = (t_B - t_D)^2 - (x_B - x_D)^2 \tag{A57}$$

We know that $t_D = t_A = 0$ in the planet frame.

$$\Delta s^2_{BD} = (\frac{L_0}{v} - 0)^2 - (L_0 - 2L_0)^2 \tag{A58}$$

$$\Delta s_{BD}^2 = \frac{L_0^2}{v^2} - L_0^2 + 4L_0^2 - 4L_0^2 \tag{A59}$$

$$\Delta s_{BD}^2 = L_0^2(\frac{1}{v^2} - 1) \tag{A60}$$

$$\Delta s_{BD}''^2 = \Delta t_{BD}''^2 - \Delta x_{BD}''^2 = (t_B'' - t_D'')^2 - (x_B'' - x_D'')^2 \tag{A61}$$

We know that for the returning spaceship frame, $(t_D'', x_D'') = (0, 0)$, and we also know that $x_B'' = 0$ because from person P's perspective, event D and event B both happen right outside his window. Event D is his spaceship departing from planet 3, and event B is his spaceship meeting ours. Since we are in person P's frame of reference, both event B and D happen at $x = 0$. So now we have

$$\Delta s_{BD}''^2 = t_B''^2 \tag{A62}$$

and with this in hand, we can use the invariance of the spacetime interval to find an expression for t_B''.

$$\Delta s_{BD}''^2 = \Delta s_{BD}^2 \tag{A63}$$

$$L_0^2(\frac{1}{v^2} - 1) = t_B''^2 \tag{A64}$$

$$t_B''^2 = \frac{L_0^2}{v^2}(1 - v^2) \tag{A65}$$

$$t_B'' = \frac{L_0}{v}\sqrt{1 - v^2} \tag{A66}$$

$$t_B'' = \frac{L_0}{v\gamma} \tag{A67}$$

Now let's look at an observer in the returning spaceship frame who is at planet 1's position and records the time at planet 1 when event B happens in his frame. We define the following space and time intervals

$$\Delta x_{DB''} = 2L_0 - 0 = 2L_0 \tag{A68}$$

$$\Delta t_{DB''} = 0 - t_{B''} = t_{B''} \tag{A69}$$

$$\Delta x_{DB''}'' = 0 - \frac{L_0}{\gamma} \tag{A70}$$

Here $x_{B''}'' = \frac{L_0}{\gamma}$ because the observer is positioned at planet 1, which is L_0 away from planet 2 in the planet frame. Person P is at planet 2 undergoing event B, and we are in his frame of reference, so to account for the velocity v we have relative to the planet frame, we have to shrink L_0 by a factor of *gamma*.

We know that event B and event B'' happen at the same time in the returning spaceship frame, so we can use $t''_{B''} = t''_B$

$$\Delta t''_{DB''} = 0 - \frac{L_0}{v\gamma} \tag{A71}$$

Now we have the components of $\Delta s''^2_{DB''}$ and $\Delta s^2_{DB''}$ we can use invariance of the spacetime interval which means we can set

$$\Delta s^2_{DB''} = \Delta s''^2_{DB''} \tag{A72}$$

which gives

$$t^2_{B''} - (2L_0)^2 = (\frac{L_0}{v\gamma})^2 - (\frac{L_0}{\gamma})^2 \tag{A73}$$

$$t^2_{B''} = \frac{L_0^2}{v^2}(1 - v^2) - L_0^2(1 - v^2) + 4L_0^2 \tag{A74}$$

$$t^2_{B''} = \frac{L_0^2}{v^2} - L_0^2 - L_0^2 + L_0^2 v^2 + 4L_0^2 \tag{A75}$$

$$t^2_{B''} = \left(\frac{L_0}{v}\right)^2 + 2L_0^2 + L_0^2 v^2 \tag{A76}$$

$$t^2_{B''} = \left(\frac{L_0}{v} + L_0 v\right)^2 \tag{A77}$$

$$t_{B''} = \frac{L_0}{v} + L_0 v \tag{A78}$$

And now we have an expression for $t_{B''}$. If we insert numbers into this we get $t_{B''} \approx 400$ years. This definitely seems like a paradox. An observer in the outgoing spaceship frame recorded 4 years having passed on planet 1 at the time of event B' in our frame and passed the information over to us. But right after we switch over to the other spaceship, we record how much time has passed on planet 1 again, and suddenly it says 400 years? This is again, just like earlier because of the violation of the rule of inertial frames discussed earlier. We change our speed (hence we accelerate) and our reference frame when we jump from the outgoing to the returning spaceship frame, and thus our comparisons of the different times are not valid.

Let's dive even deeper into this subject. We imagine that the spaceship we return in decelerates at a constant rate g measured in the planet frame. We want to figure out what time ΔT the spaceship uses to come to a full stop. We know that the acceleration is constant, so we can use $a = \frac{\Delta v}{\Delta T}$. We know the change in velocity($\Delta v = (-v_0) - 0$) and we know the acceleration $a = g$, so all we need to do is solve this for ΔT.

$$\Delta T = \frac{\Delta v}{a} = \frac{-v_0}{g} \tag{A79}$$

When the spaceship has decelerated to zero it keeps accelerating in the same direction, so it turns around and comes back to planet 2. We want to find out what velocity it has when it comes back to planet 2. Since the acceleration is constant, we know that the distance it travels before and after the velocity reaches zero is the same. Therefore the change in velocity Δv will also be the same, so the velocity of the ship will be $v = 0.99$ when it returns to planet 2(If the original velocity was $v = -0.99$) in the planet frame.

To handle the constant acceleration, we divide the entire accelerated part of the trip into a series of events Y and Y' which correspond to repeating events B and B'. $T_Y = 0$ when the spaceship starts to accelerate, and $T_Y = \Delta t$ when it has reached velocity $v = 0$ and turns around. Similarly to earlier in this section, Y' will now be an event that happens at the same time as Y in our spaceship frame, but at the location of planet 1.

Let's define the times and positions for events Y and Y'. We know that in the planet frame, the spaceship starts at $x = L_0$, moves in the negative x-direction with velocity -0.99 and accelerates at a constant rate. We have this common formula for constant acceleration,

$$x_Y = L_0 + vT_Y + \frac{1}{2}gT_Y^2 \tag{A80}$$

Since $t = s/v$, and we know T_Y is the time lapsed since event B, we have

$$t_Y = \frac{L_0}{v} + T_Y \tag{A81}$$

The time $t_{Y'}$ in the planet frame will be the same as t_Y with added compensation for length contraction effects. This is done by dividing the distance by γ and multiplying the velocity by γ.

$$t_{Y'} = \frac{L_0}{v\gamma^2} + T_Y' \tag{A82}$$

In the planet frame, $x = 0$ is at the position of planet 1, so since event Y' happens at the position of planet 1,

$$x_{Y'} = 0 \tag{A83}$$

Now let's look at the returning spaceship frame. Event Y is at the position of our spaceship, so

$$x_Y' = 0 \tag{A84}$$

$$t_Y' = t_Y' \tag{A85}$$

We know that the distance from our spaceship to planet 1 is the same as the distance from the planet to our spaceship in the planet frame with added length contraction.

$$x'_{Y'} = \frac{L_0 + vT_Y + \frac{1}{2}gT_Y^2}{\gamma} \tag{A86}$$

Where γ is using the velocity at T_Y. As we defined earlier, events Y and Y' happen simultaneously in the spaceship frame of reference.

$$t'_{Y'} = t'_Y \tag{A87}$$

Using the invariance of the spacetime interval again, we can set $\Delta s^2_{YY'} = \Delta s'^2_{YY'}$, and keen $t_{Y'} = t_{Y'}$ even though we already have an expression for it. This gives

$$t_{Y'} = -(L_0 + vT_Y + \frac{1}{2}gT_Y^2)(1 - \frac{1}{\gamma}) + \frac{L_0}{v} + T_Y \tag{A88}$$

By substituting $T_Y = \Delta T = \frac{v}{g}$ we end up with

$$t_{Y'} = -L_0 v + \frac{L_0}{v} + T_Y \tag{A89}$$

Now we'll assume the constant acceleration is $g = -0.1\mathrm{m/s}^2$. If units for distance and time are both measured in seconds, the acceleration will then be

$$a = \frac{0.1}{c} = 3.34 \cdot 10^{-10}\mathrm{s}^{-1} \tag{A90}$$

Let's put in the numbers for t_Y and $t_{Y'}$ at time $T_Y = \Delta T$.

$$t_Y = \frac{L_0}{v} + \frac{v}{g} = \frac{200}{0.99} + \frac{0.99}{3.34 \cdot 10^{-10}} \approx 94\mathrm{years} \tag{A91}$$

$$t_{Y'} = -L_0 v + \frac{L_0}{v} + \frac{v}{g} = -200 * 0.99 + \frac{200}{0.99} - \frac{0.99}{3.34 \cdot 10^{-10}} \approx 94\mathrm{years} \tag{A92}$$

A.4. Space Time Diagram

For the next situation we will consider 4 space ships travelling in the positive x-direction past Driddu. Three of the space ships travel at a constant velocity whereas the last one is accelerating. We started of with plotting out a spacetime diagram of the situation as seen from Driddu In order to draw the diagram we had to figure out

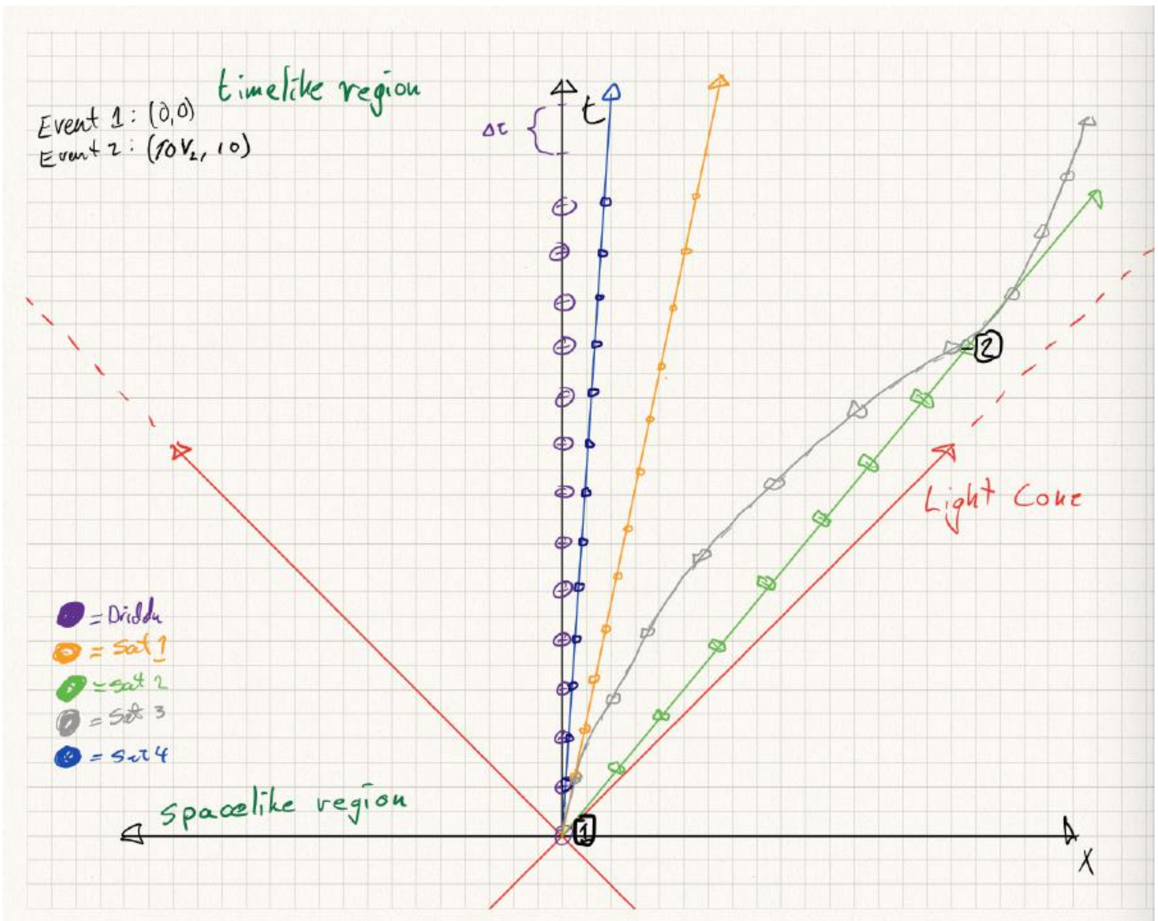


Figure 18: Ships 1, 2 and 4 move with constant velocity while ship 3 is accelerating. The ticks seen on each ship is plotted with $\Delta t_{\text{Driddu}} < \Delta t_{s4} < \Delta t_{s1} < \Delta t_2$. While ship 3 is accelerating the distance between ticks becomes greater.

whether the astronaut in sat 3 would experience more or less time than the astronaut in sat 2. To do this we utilized the principle of maximal aging which states that: a particle in free float will follow the worldline which corresponds to the longest possible longest possible proper time interval between the two events. Since sat 2 is assumed to be in a free float frame: constant velocity. We know that sat 2 should have registered more ticks on its wristwatch than sat 3's wristwatch.

A.5. Consecutive Lasers

We now consider a space ship moving with velocity $v = 0.979c$ with respect to Driddu. The spaceship moves in a straight line at a constant velocity in the positive x direction, during its flyby it fires two consecutive burst of light in the positive x direction. In a spacetime diagram light moves at 45 degrees because $\Delta t = \Delta x$. Going

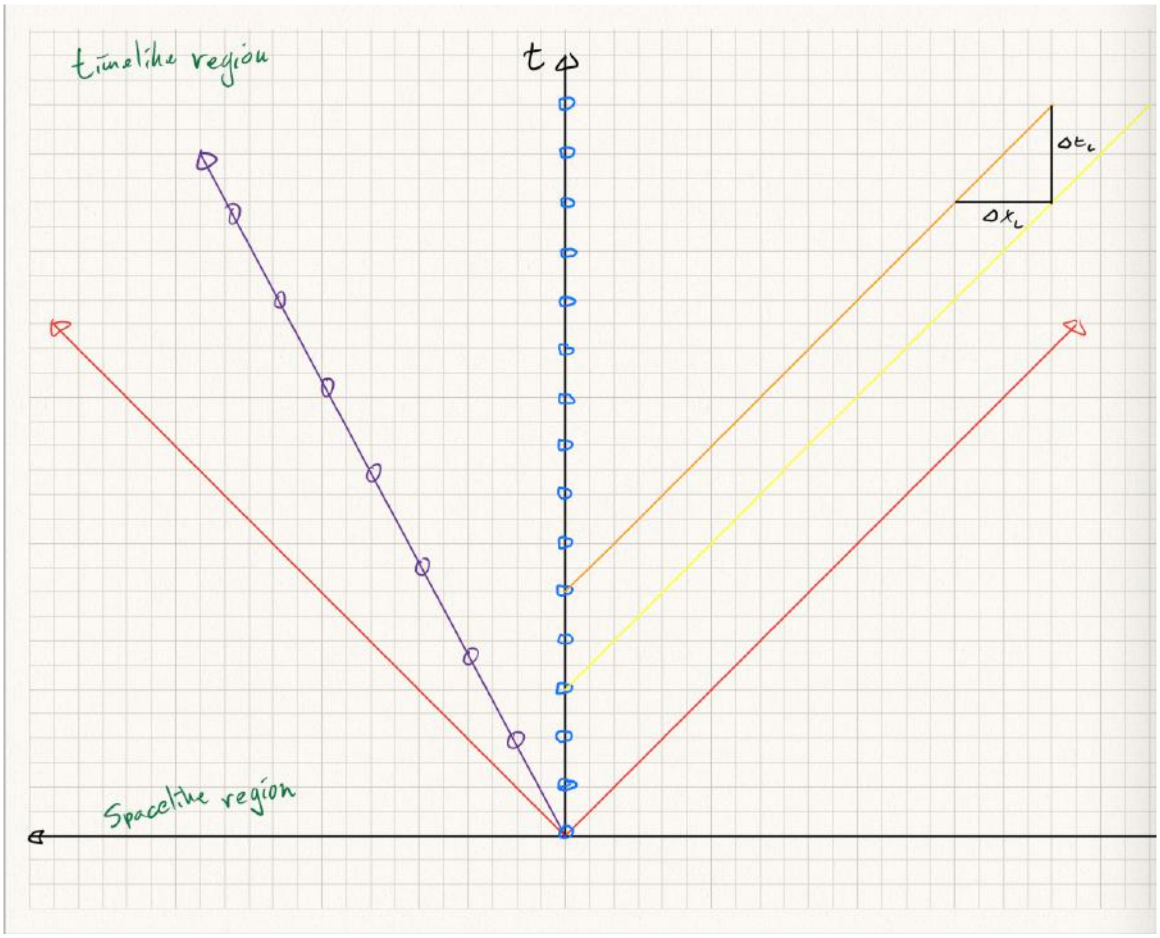


Figure 19: The space craft is stationary at x=0 travelling only through time while Driddu is moving in the negative x direction. It fires two lasers represented by the yellow and orange lines.

from one frame of reference to another is achieved through the Lorentz Transformation A38, which is a transformation devised to preserve the 45 degree nature of light. This preservation is the mathematical formulation of the idea that light has an invariant velocity regardless of reference frame. This means that regardless of reference frame the time between emitting the light will equal the spacial distance between the light, this is seen in the spacetime diagram below as the sides of the triangle formed between the world lines of the photons being equal. We can thus deduce how the spacetime diagram in the planet frame, through this 45 degree line preservation of the Lorentz transformation.

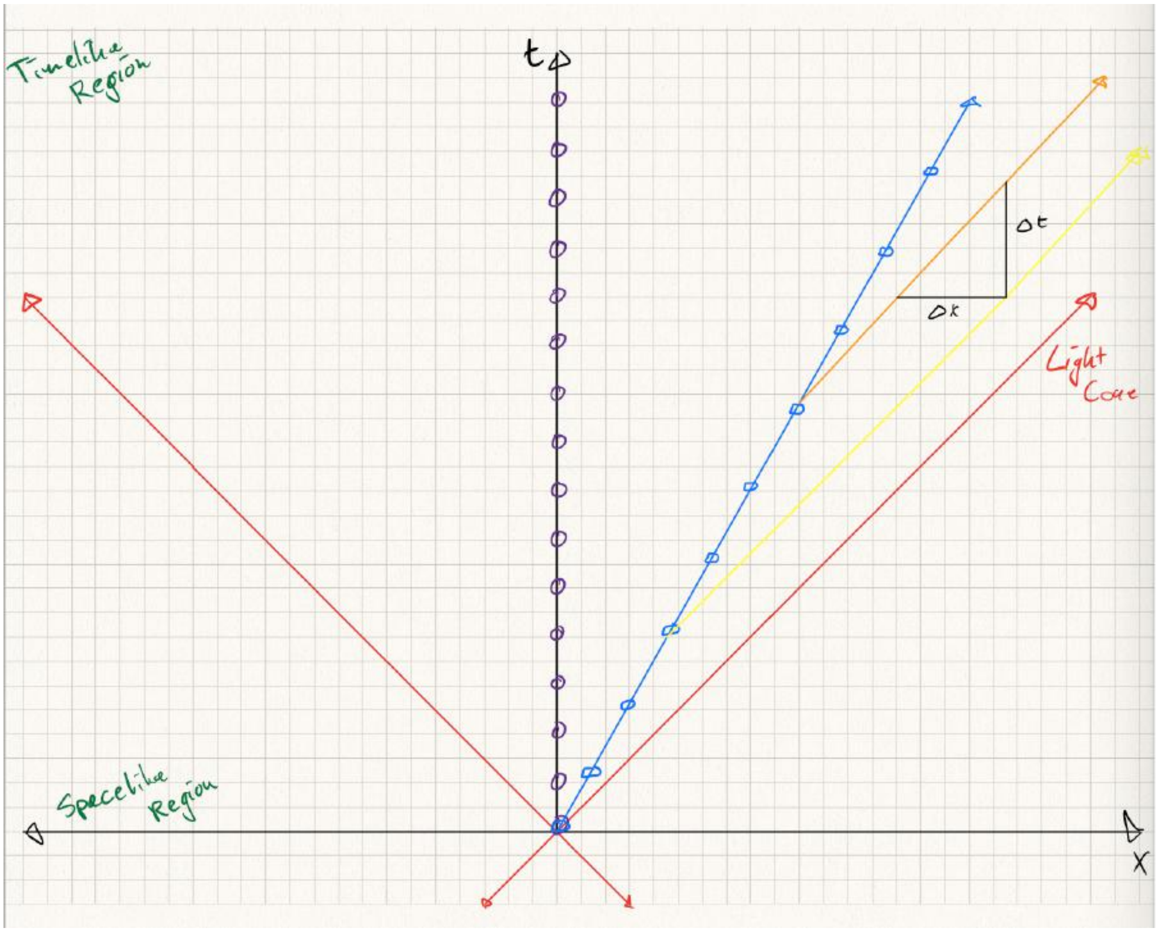


Figure 20: The craft is moving in the positive x-direction away from Driddu and fires two lasers: yellow and orange.

In the planet frame we see that the ship first fires of a laser which travels in front of the ship at the speed of light and that the ship travels in space between emitting the first laser and the second. We use the inverse Lorentz transformation of velocity

$$v_x = \frac{v'_x + v_{\text{rel}}}{1 + v_{\text{rel}} v'_x} \tag{A93}$$

$$v_x = \frac{1 + v_{\text{rel}}}{1 + v_{\text{rel}}} = 1 \tag{A94}$$

to show that the speed of the laser is the speed of light in both frames of reference. Which is expected since we know that light travels at the speed of light in all reference frames from experiments like the Michelson-Morley experiment.

We measured the distance between the lasers in the space ship frame to be $L' = 208$km. In the planet frame we measured the distance between the light beams to be $L = 10$km. The ratio between these is $\frac{L'}{L} = 20.8$, comparing this with what the formula for length contraction gives

$$L = \frac{L'}{\gamma} \rightarrow \frac{L'}{L} = \gamma \tag{A95}$$

In this situation gamma would be $\gamma = \frac{1}{\sqrt{1-v^2}} = 4.9$. This however disagrees with our observations which says the ratio of the length between the lasers in the two frames is $\sim 4\gamma$. The reason for this can be seen in the diagram, we have assumed the length measured between the beams to be a physical object which adheres to the Lorentz transformation of length. However the spacial distance between the lasers in the planet frame is equal to the temporal distance in the planet frame. This is because light always travels at the speed of light, 45 degrees in spacetime diagram, so the distance between them does not get contracted according to length contraction.

A.6. The Space Ship Hotel

A small ship is bringing passengers from the planet to the big space ship hotel. The passenger ship is travelling with the velocity v_A, the hotel ship is travelling with velocity v_B and the hotel ship combined with the passenger ship is travelling with a velocity v_C, all velocities with respect to the planet. The mass of the ships was $m_A = 10^5$kg, $m_B = 10^6$kg and $m_C = 1.1282 \cdot 10^6$kg. Our goal is to describe the velocity of both the space ships in the planet frame.

First we wrote the momenergy four vector $P'_{\mu,A}$ in the combined ship frame in terms of the velocity and mass of ship A. Which is just the transpose of the row vector $(E_{\text{rel}}, p_{\text{rel,x}}, p_{\text{rel,y}}, p_{\text{rel,z}})$.

$$P'_{\mu,A} = \begin{pmatrix} \gamma m_A \\ \gamma m_A v'_{A,x} \\ \gamma m_A v'_{A,y} \\ \gamma m_A v'_{A,z} \end{pmatrix} \tag{A96}$$

We continued with writing the momenergy four vectors for space ship B and the combined ship C after the docking.

$$P'_{\mu,B} = \begin{pmatrix} \gamma m_B \\ \gamma m_B v'_{B,x} \\ \gamma m_B v'_{B,y} \\ \gamma m_B v'_{B,z} \end{pmatrix} \tag{A97}$$

For the combined ship we have

$$P'_{\mu,C} = \begin{pmatrix} \gamma m_C \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{A98}$$

We can use these four vectors to find the velocities of ship A and B before the collision in the combined ship frame. The conservation of momenergy in the combined frame gives

$$P'_{\mu,C} = P'_{\mu,A} + P'_{\mu,B} \tag{A99}$$

$$\begin{pmatrix} \gamma m_C \\ \\ \gamma m_C v'_{C,x} \end{pmatrix} = \begin{pmatrix} \gamma m_A \\ \\ \gamma m_A v'_{A,x} \end{pmatrix} + \begin{pmatrix} \gamma m_B \\ \\ \gamma m_B v'_{B,x} \end{pmatrix} \tag{A100}$$

We see from this vector equation that $\gamma m_C = \gamma m_A + \gamma m_B$ and $\gamma m_A v'_A = -\gamma m_B v'_B$. We know that $v'_C = 0.0$c, $v_C = -0.383$c, $m_A = 10^5$kg, $m_B = 10^6$kg and the combined mass is $m_C = 1.1282 \cdot 10^6$kg.

B. General Relativity

B.1. Gravitational Doppler Effect

For this though experiment we imagine a shell observer at the spherical shell with radius r. The shell observer is pointing a laser pen radially outwards from the central mass at the center of the shell. The laser beam has wavelength λ and we will try to figure out the wavelength λ' a far away observer measures. The frequency of the laser light emitted by the laser pen is $\nu = 1/\Delta t$ while the frequency of the light received by the far away observer is $\nu' = 1/\Delta t'$. Here Δt and $\Delta t'$ is the time between the peaks of the laser light.

We want to find an expression relating the time between the peaks of light for the two observers. Both of the observers are at rest, one on the shell and the other far away. We define two consecutive ticks of the shell clock as events A and B. Next we use the invariance of the Schwarzschild line element to relate the far away time $\Delta t'$ to the shell time Δt.

$$(\Delta s)^2 = \left(1 - \frac{2M}{r}\right)(\Delta t)^2 - \frac{1}{\left(1 - \frac{2M}{r}\right)}(\Delta r)^2 - r^2(\Delta\phi)^2 \tag{B101}$$

Because the observers are at rest $(\Delta r)^2 = 0$ and $(\Delta\phi)^2 = 0$ the line element becomes

$$\Delta s_{AB} = \sqrt{\left(1 - \frac{2M}{r}\right)}\Delta t' \tag{B102}$$

Since the ticks of the shell clock happen at the same location in space the line element between them is just the temporal distance between the events. Because we defined our events to be consecutive ticks of the clock the proper time is just the interval between A and B, i.e: Δt. Since the line element is invariant we obtain

$$\Delta t = \sqrt{\left(1 - \frac{2M}{r}\right)}\Delta t' \tag{B103}$$

Solving for the time between the peaks for the far away observer gives

$$\Delta t' = \frac{\Delta t}{\sqrt{\left(1 - \frac{2M}{r}\right)}} \tag{B104}$$

Using this relation between for the interval between the lasers peaks we can deduce the gravitational Doppler formula which gives the wavelength observed by the far away observer.

$$\frac{\lambda' - \lambda}{\lambda} \tag{B105}$$

We know that the wavelength is inversely proportional to the frequency through the speed of light $\lambda = c/\nu$. We plug this into the Doppler expression

$$\frac{\Delta\lambda}{\lambda} = \frac{c\Delta t' - c\Delta t}{c\Delta t} = \frac{1}{\sqrt{\left(1 - \frac{2M}{r}\right)}} - 1 \tag{B106}$$

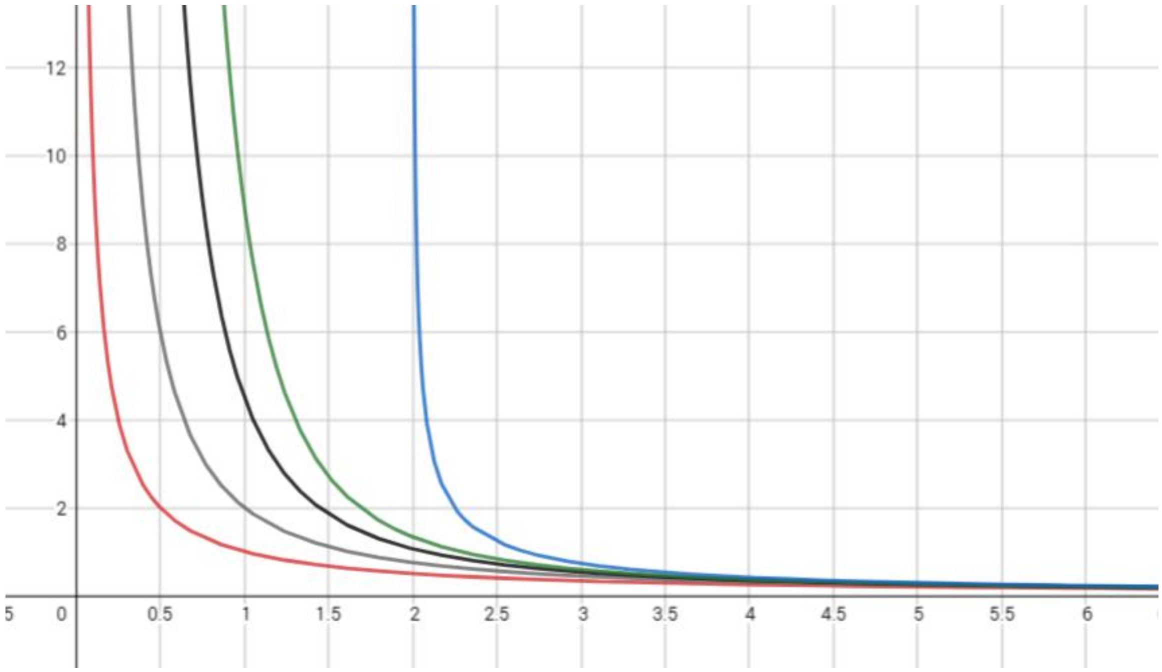


Figure 21: A plot of the first through fourth order Taylor polynomial of B106 . The polynomials get progressively closer to our expression (blue). We used $M = 1$

Where we used the relation between Δt and $\Delta t'$ in the last step. If we Taylor expand the Doppler formula we get

$$\frac{1}{\sqrt{\left(1-\frac{2M}{r}\right)}} - 1 = \frac{M}{r} + \frac{3M^2}{2r^2} + \frac{5M^3}{2r^3} + \frac{35M^4}{8r^4} \tag{B107}$$

Looking at our Taylor approximations we see that they all converge to the real function once r grows sufficiently large. Therefore when the value of $|r| >> 2|M|$ we can use the first order approximation and the gravitational Doppler formula simplifies to

$$\frac{\Delta\lambda}{\lambda} = \frac{M}{r} \tag{B108}$$

Next we will discuss what wavelength of light an observer far away from the Sun will observe for the light with wavelength $\lambda_{\text{max}} = 500\text{nm}$ emitted from the surface of the Sun. The mass of the Sun in meters is $M_m = M_{kg}G/c^2 = 1477\text{m}$. The ratio between the mass of the Sun and the radius is $M/r = 2.123 \cdot 10^{-6}$. From this comparison we see that it is fine to use our simplified Doppler expression and the red shift measured by the far away observer is $N_{\text{redshift}} = 2.123 \cdot 10^{-6}$. This redshift will alter the Sun's true color, but will be too tiny for any change in the apparent color. Now we discuss the other side of the gravitational Doppler effect, gravitational blueshift. For light coming from far away and entering the gravitational field of the Earth the distance between the peaks of the light becomes smaller. The mass of the Earth in meters is $M_m = M_{kg}G/c^2 = 4.435 \cdot 10^{-3}\text{m}$. Comparing to the Earth's radius we find that $4.435 \cdot 10^{-3}/r_{Earth} = 6.9535 \cdot 10^{-10}$. Again we can use our expression for the gravitational Doppler effect. $N_{\text{blueshift}} = 6.9535 \cdot 10^{-10}$ and the Doppler effect will again alter the Star's true color, however to a much lesser degree than the redshift of the Sun.

A quasar is one of the most powerful energy sources in the universe. From one quasar we found evidence for an emission line at $\lambda = 2150$nm which we recognized as the line $\lambda = 600$nm in the laboratory. This observation supports the hypothesis that there is a black hole at the center of most quasars because you would be hard pressed to find somewhere else that could produce a redshift this high. From the observation we can figure out how far from the center of the quasar the radiation is emerging from. We use our gravitational Doppler formula B106 and solve for the radius

$$r = \frac{\lambda M}{\Delta\lambda} \tag{B109}$$

B.2. Two Shell Observers

In this part we will consider two shell observers, one close to and another one farther away from a black hole. The black hole has a mass $M_{kg} = 1171.24$ Solar masses, converted to meters: $M_m = 1.729 \cdot 10^6$m. The clocks of the observers are synced at $t = 0$. The two astronauts routines were recorded in the frame of reference close to the black hole.

For the astronaut close to the black hole the schedule went like: wake up 6am, breakfast 7am, lunch 12pm, dinner 6pm, brush teeth 11:15pm and good night 23:30. His partners schedule went like: wake up 2:52am, breakfast 3:11am, lunch 4:19am, dinner 6am, brush teeth 7:12am and good night at 7:15am.

Seen from the frame of reference close to the black hole the far away astronauts day passes in the interval between 2am and 7am. In the last section we converted between times of a shell observer and a far away observer, in this situation we wish to convert between the time of different shell observers. The way we will do this is to first convert to a far away observer and then convert to our partner astronaut. We use B103 to figure out when a far away observer observes the events of our partners day to happen.

The time intervals of the outermost astronauts seen from the innermost given in minutes is: $\Delta t_{B,0} = 19$m, $\Delta t_{B,1} = 68$m, $\Delta t_{B,2} = 101$m, $\Delta t_{B,3} = 72$, $\Delta t_{B,4} = 3$m. First we convert these intervals to the intervals observed by a far away observer.

Now all that remains is to convert from the far away into the farthest shell frame. We still use the B103 relation to convert.

Where in both convertions we have used the distances $r_{\text{close}} = 3792.5$km and $r_{\text{far}} = 30911.4$km for the shell observers. From our second table we see that astronaut 2's schedule goes as: wake up 10 hours after going to bed, spend one hour waking up before breakfast, 3hours 35min between breakfast and lunch, 5 hours 20 min between lunch and dinner, 3hours 48min before brushing his teeth, nine minutes from brushing to going to bed and sleep 10 hours again. This matches with what we see from

Close frame	Far Away frame
$\Delta t_{B,0} = 19\text{m}$	$\Delta t^*_{B,0} = 64\text{m}$
$\Delta t_{B,1} = 68\text{m}$	$\Delta t^*_{B,1} = 229\text{m}$
$\Delta t_{B,2} = 101\text{m}$	$\Delta t^*_{B,2} = 340\text{m}$
$\Delta t_{B,3} = 72\text{m}$	$\Delta t^*_{B,3} = 242\text{m}$
$\Delta t_{B,4} = 3\text{m}$	$\Delta t^*_{B,4} = 10\text{m}$
$\Delta t_{B,5} = 189\text{m}$	$\Delta t^*_{B,5} = 636\text{m}$

Table 1: We converted the intervals between the events in astronaut 2's day as seen in the close frame to the far away frame. We used B103

to transform the shell clock to the far away clock. $2M/r = 0.9118$

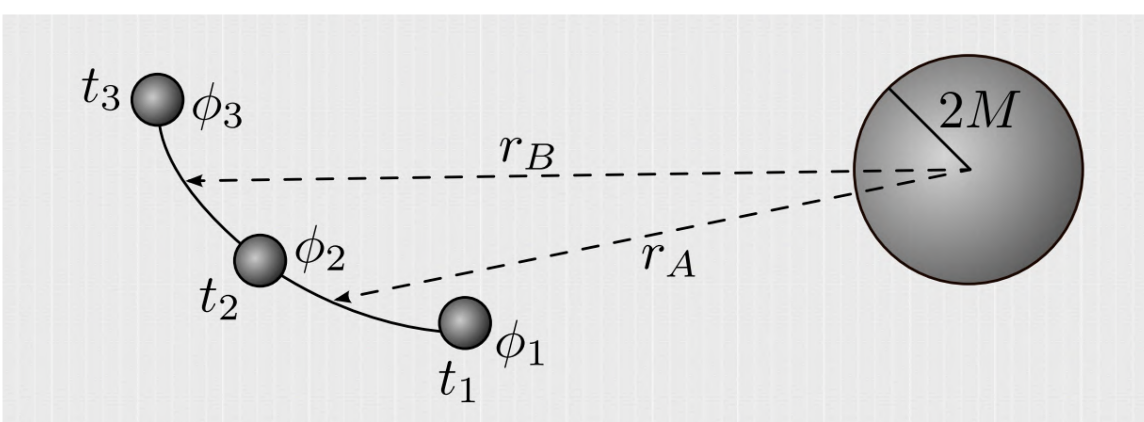
Far Away frame	Middle frame
$\Delta t^*_{B,0} = 64\text{m}$	$\Delta t'_{B,0} = 60\text{m}$
$\Delta t^*_{B,1} = 229\text{m}$	$\Delta t'_{B,1} = 215\text{m}$
$\Delta t^*_{B,2} = 340\text{m}$	$\Delta t'_{B,2} = 320\text{m}$
$\Delta t^*_{B,3} = 242\text{m}$	$\Delta t'_{B,3} = 228\text{m}$
$\Delta t^*_{B,4} = 10\text{m}$	$\Delta t'_{B,4} = 9\text{m}$
$\Delta t^*_{B,4} = 636\text{m}$	$\Delta t'_{B,4} = 599\text{m}$

Table 2: We converted the intervals of astronaut 2's day as seen in the far away frame to the middle frame. $2M/r = 0.1119$

astronaut 2's reference frame. Astronaut 2's day: wake up 9am, breakfast 10am, lunch 13:30, dinner 19:00, brush teeth 22:30, good night 23:00.

B.3. Conservation of Angular Momentum

In this section we will deduce the law of conservation of angular momentum in general relativity. We will study the motion of an object which passes through three points (r_1, ϕ_1), (r_2, ϕ_2) and (r_3, ϕ_3) at times t_1, t_2, t_3. We fix $t_1, t_2, t_3, r_1, r_2, r_3, \phi_1, \phi_3$



leaving ϕ_2 as the only free variable. We assume that between (r_1, ϕ_1) and (r_2, ϕ_2) the radius is $r = r_A$ and between (r_2, ϕ_2) and (r_3, ϕ_3) the radius is $r = r_B$. We use the Schwarzschild line element B101 to figure out the proper time interval between ϕ_1 and ϕ_3

$$\tau_{1,3} = (\tau_{1,2}) + (\tau_{2,3}) \tag{B110}$$

$$\tau 1, 2 = \sqrt{(1 - \frac{2M}{r_A})t_{1,2}^2 - \frac{r_{1,2}^2}{1 - \frac{2M}{r_A}} - r_A^2\phi_{1,2}^2} \tag{B111}$$

$$\tau_{2,3} = \sqrt{(1 - \frac{2M}{r_B})t_{2,3}^2 - \frac{r_{2,3}^2}{1 - \frac{2M}{r_B}} - r_B^2\phi_{2,3}^2} \tag{B112}$$

Therefore the total proper time from 1 to 3 is the sum of the proper time between 1,2 and 2,3.

With the line element in hand we can use the principle of maximal aging to figure out what the which ϕ_2 that maximizes the total proper time. We find the maximum of a function with respect to a variable the regular way, setting the partial derivative to zero.

$$\frac{\tau_{1,3}}{d\phi_2} = \frac{d\tau_{1,2}}{d\phi_2} + \frac{\tau_{2,3}}{\phi_2} = 0 \tag{B113}$$

$$-\frac{r_A^2\phi_{1,2}^2}{\tau_{1,2}}\frac{d\phi_{1,2}}{d\phi_2} - \frac{r_B^2\phi_{2,3}^2}{\tau_{2,3}}\frac{d\phi_{2,3}}{d\phi_2} = 0 \tag{B114}$$

because $\phi_{1,2}$ and $\phi_{2,3}$ both contain a $\pm\phi_2$ their derivatives with respect to ϕ_2 becomes 1 and -1. We can rewrite as

$$\frac{r_A^2\phi_{1,2}^2}{\tau_{1,2}} = \frac{r_B^2\phi_{2,3}^2}{\tau_{2,3}} \tag{B115}$$

This tells us that an object moving in curved spacetime will have the longest proper time when ϕ_2 follows this expression above. If we take the limit where $\tau_{1,2}$, $\tau_{2,3}$, $\phi_{1,2}$ and $\phi_{2,3}$ are so small they can be expressed as infinitesimally small periods of time and angles. We rewrite our expression for these infinitesimals and notice that r_A and r_B will be 'the same' when dealing with infinitesimal sizes $d\tau$ and $d\phi$

$$\frac{r^2 d\phi}{d\tau} = constant \tag{B116}$$

C. Home Star

Categorizing our Home Star

In section 2.1.6 we deduced the expression for the luminosity of a black body: 50. The luminosity of a black body depends upon the radius and the temperature of the body, because these factors are proportional to the energy being radiated from

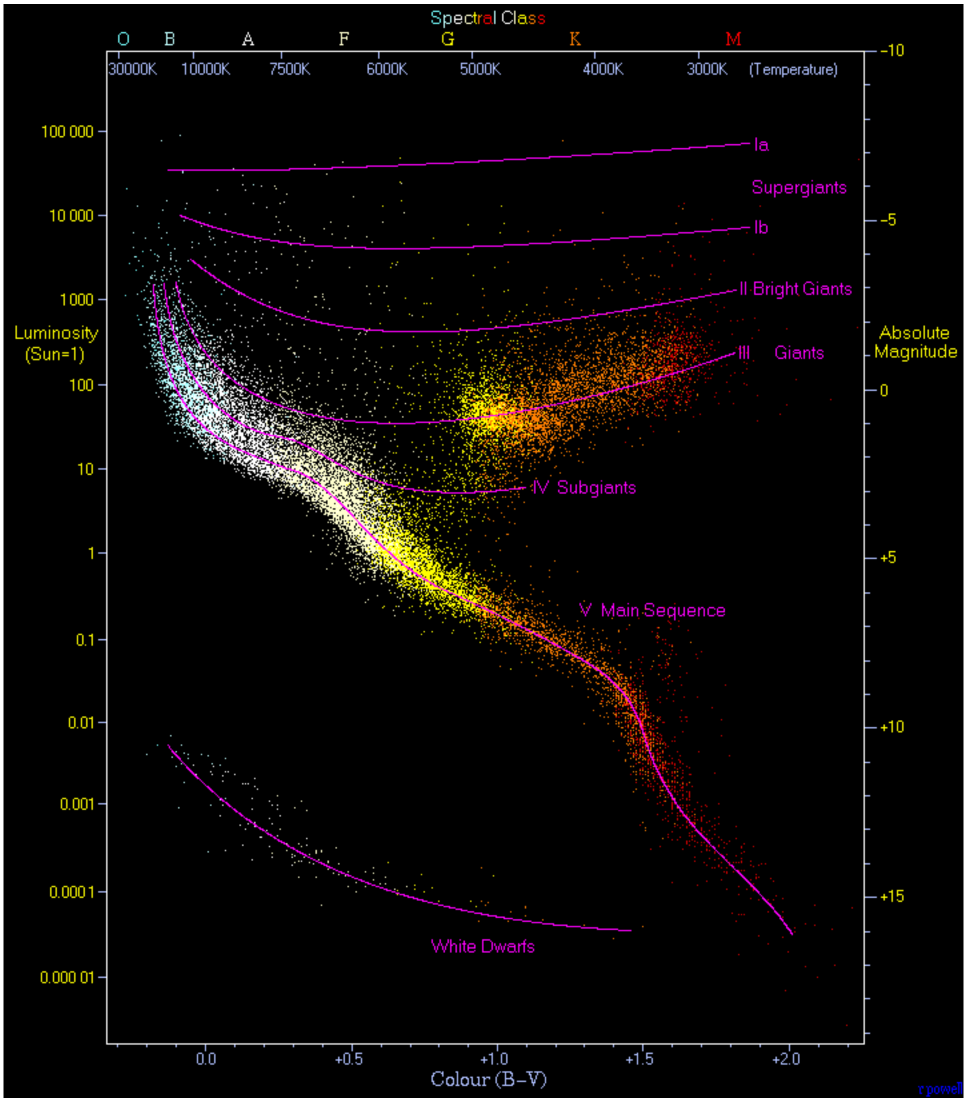


Figure 22: HR-diagram made from 22000 stars from the Hipparcos Catalogue and 1000 from the Gliese Catalogue of nearby stars.

the body's core. A smaller radius will increase the pressure in the core which will increase the magnitude of fusions happening. Similarly an increase in temperature will increase the fusion rate and result in a brighter star.

The star in our star system has a radius $r = 1296759$km and a surface temperature $T = 10328$K, which implies, from equation 50, a luminosity $L = 1.36335 \cdot 10^{28}$W. Approximately 35 times the luminosity of our sun:

$$\frac{1.36335 \cdot 10^{28}}{3.828 \cdot 10^{26}} \approx 35 \tag{C117}$$

Our star is therefore located on the left of the main sequence in figure 22, in spectral class B. In astrophysics, the mass-luminosity relation is an equation giving the relationship between a star's mass and its luminosity:

$$\frac{L}{L_{sun}} = \left(\frac{M}{M_{sun}}\right)^{a} \tag{C118}$$

where $1 < a < 6$. The value $a = 3.5$ is commonly used for main-sequence stars such as our system's star. Using our calculated ratio of luminosity as well as the mass ratio of our star to the sun

$$35 \neq \left(2.44\right)^{3.5} = 22.69 \tag{C119}$$

Since the star is larger than two solar masses the relation is a better approximation we set $a = 4$, then we have

$$35 \approx 2.44^4 = 35.44 \tag{C120}$$

Birth of a Star

Assuming that our star started out as a giant molecular cloud, consisting of 75% hydrogen atoms and 25% helium atoms at the temperature 10K we calculated the largest possible radius the gas cloud could have. Coupled with the assumptions that the giant gas cloud was spherically distributed and that no external forces influenced its gravitational collapse. To calculate the largest radius we use the virial theorem.

$$\langle K \rangle = -\frac{1}{2}\langle U \rangle \tag{C121}$$

The virial theorem tells us what the relationship between the kinetic and potential energy has to be in order for the system to stay stable. If the system has a large kinetic energy compared to the potential energy, the cloud expands and will not collapse. However if the potential energy is dominating, the cloud will be gravitationally bound and undergo collapse. Thus we want,

$$2K < |U| \tag{C122}$$

In order to find $\langle K \rangle$ and $\langle U \rangle$ we would initially have to take the average of the kinetic and potential energy in the gas cloud over a long time. However due to the ergodic hypothesis, which states that: averaging a system over a long time period may be equal to averaging the system altogether. Therefore, we can apply the virial theorem to our infant star by taking the mean of the kinetic and potential energy for every particle in the gas cloud.

The potential energy of the giant gas cloud is found by considering the potential du of a mass-element dm at a distance r from the center of mass.

$$du = -G\frac{M(r)dm}{r} \tag{C123}$$

This is integrated over all mass-elements dm in the spherical shell of thickness dr at distance r. We then obtain the potential energy dU between the shell and the spherical mass $M(r)$ inside the shell.

$$dU = -G\frac{M(r)4\pi r^2\rho(r)dr}{r} \tag{C124}$$

Next to find the total potential energy we integrate this over r, so that we find the potential energy of 'each' shell in the whole of the gas cloud.

$$U = -4\pi G \int_0^R M(r)\rho(r) r dr \tag{C125}$$

Next we assumed the density to be constant equal to the mass of the cloud over the volume of the cloud. The total potential energy thus becomes

$$U = -4\pi G \left(\frac{M}{(4/3)\pi R^3}\right)^2 \frac{4\pi}{3} \int_0^R r^4 dr = -\frac{3GM^2}{5R} \tag{C126}$$

We already deduced in section 2.1.1 the expression for the kinetic energy of a gas particle 16. With this relation between the temperature of the gas and the kinetic energy of its particles we find the kinetic energy of the collective gas cloud as

$$K = \frac{3}{2} NkT \tag{C127}$$

Plugging our terms for the total potential and kinetic energy of the giant gas cloud into the condition for collapse: C122.

$$3NkT < \frac{3GM^2}{5R} \tag{C128}$$

rewriting $N = \frac{M}{\mu m_H}$ and solving for R

$$R < \frac{G\mu m_H}{5kT} M \tag{C129}$$

$$R < 1.390 \cdot 10^{12} m \tag{C130}$$

or $1.469 \cdot 10^{-4}$ light years. Since R has to be less than this we assumed the 'true' radius of the giant gas cloud to be $1.460 \cdot 10^{-4}$ly. We were curious as to where this giant gas cloud would fall onto the HR-diagram 22, so we calculated the luminosity using equation 50 and found:

$$L = 4\pi \cdot (1.381 \cdot 10^{12})^2 \cdot 5.6704 \cdot 10^{-8} \cdot 10^4 = 1.35897 \cdot 10^{22} \tag{C131}$$

which is $3.53 \cdot 10^{-5} \cdot L_{sun}$. This puts our giant gas cloud at the lowest tick mark on the left hand side of figure 22. Combining with the temperature of the cloud, sets the position on the HR-diagram around the lowest tick mark all the way to the right where $T = 10$K

Modelling the Star's core

Next we will use the equation for hydrostatic equilibrium 85 to make a rough estimate of the temperature of our star's core. We will assume that the density of the star is uniform throughout its interior, as well as assuming the only source of pressure in the star is the ideal gas pressure, and not the radiation pressure. Our assumption of uniform density $\rho = \rho_0 = m_{star}/V_{star}$ lets us express the mass inside the sphere of radius r to be

$$M(r) = \frac{4\pi\rho r^3}{3} \tag{C132}$$

$$\frac{\rho_0 k}{\mu m_H}\frac{dT}{dr} = -\rho_0\frac{GM(r)}{r^2} \tag{C133}$$

$$\frac{dT}{dr} = -\frac{\mu m_H}{k}\frac{G}{r^2}\frac{4\pi\rho_0 r^3}{3} = -\frac{4\pi}{3}G\rho_0 r\frac{\mu m_H}{k} \tag{C134}$$

Integrating the temperature from the center of the star to the surface, $0 \le r \le R$

$$\int_0^R \frac{dT}{dr} = -\frac{4\pi}{3}G\rho_0\frac{\mu m_H}{k}\int_0^R r dr \tag{C135}$$

$$T(R) = T(0) - \frac{2\pi}{3}GR^2\rho_0\frac{\mu m_H}{k} \tag{C136}$$

Rearranging to solve for $T(0)$ we obtain an expression for the core temperature of the star.

$$T_{core} = T_{surf} + \frac{2\pi}{3}GR^2\rho_0\frac{\mu m_H}{k} \tag{C137}$$

Plugging in all our variables we obtain a core temperature of

$$T_{core} = 2.67 \cdot 10^7 K \tag{C138}$$

Assuming this core temperature is close to the real one implies that the CNO-cycle is the dominant energy source in the star's core.

The CNO-cycle

The Carbon-Nitrogen-Oxygen cycle is one of the two known processes by which stars convert hydrogen to helium. The CNO-cycle fuses four hydrogen atoms into a helium nucleus, using Carbon, Nitrogen and Oxygen isotopes as catalysts. Under the conditions found in our star, the catalytic hydrogen fusion is limited by proton captures. There are four catalytic cycles proposed, cleverly named: CNO-I, CNO-II, CNO-III and CNO-IV. The main reactions of the cycles are written out in the table 3 below. CNO-I is by far the most frequent of these cycles. In the sun, the CNO-II branch occurs 0.04% of the time. The CNO-III branch is only significant in more massive stars. The PP-IV chain has this far only been theorized, but has never been observed as it is extremely improbable. However in the PP-IV chain ^3_2He captures a proton directly to give ^4_2He. The difference in occurrence stems from the cross sectional requirement for the cycle to happen.

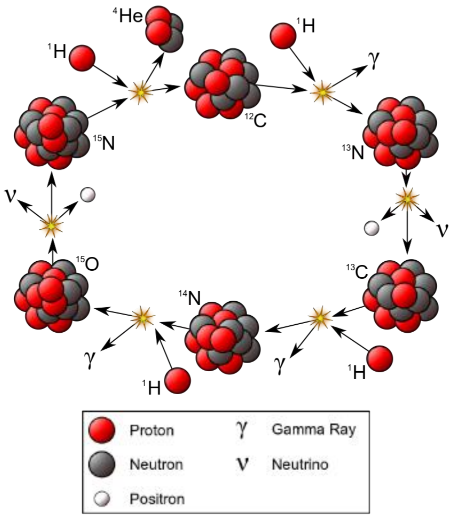
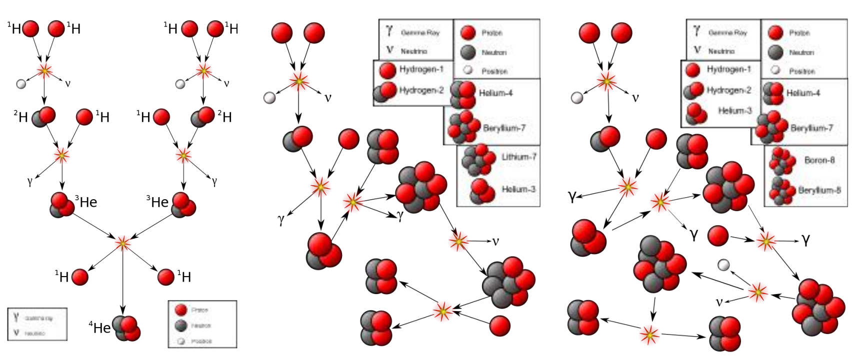


Figure 23: CNO-I fusion cycle in stars core, dominant energy source for core temperatures above 20 million Kelvin

Cycle:	Main reactions
CNO-I	${}^{12}_{6}C \rightarrow^{13}_{7} N \rightarrow^{13}_{6} C \rightarrow^{14}_{7} N \rightarrow^{15}_{8} O \rightarrow^{15}_{7} N \rightarrow^{12}_{6} C$
CNO-II	${}^{15}_{7}N \rightarrow^{16}_{8} O \rightarrow^{17}_{9} F \rightarrow^{17}_{8} O \rightarrow^{14}_{7} N \rightarrow^{15}_{8} O \rightarrow^{15}_{7} N$
CNO-III	${}^{17}_{8}O \rightarrow^{18}_{9} F \rightarrow^{18}_{8} O \rightarrow^{15}_{7} N \rightarrow^{16}_{8} O \rightarrow^{17}_{9} F \rightarrow^{17}_{8} O$
CNO-IV	${}^{19}_{9}F \rightarrow^{16}_{8} O \rightarrow^{17}_{9} F \rightarrow^{17}_{8} O \rightarrow^{18}_{9} F \rightarrow^{18}_{8} O \rightarrow^{19}_{9} F$

Table 3: Reaction Cycles and yields. The energy yield is $\sim 25MeV$ and is calculated from the energy output of each step in the cycle and subtracting the average energy of the neutrinos emitted in the decay process.

The PP-Chain



PP-I PP-II PP-III

The other known process for fusing hydrogen into is called the Proton-Proton-Chain. As was the case with the CNO-cycle the PP-Chain also has subbranches, which have been named, called the PP-I, PP-II, PP-III and PP-IV branches.

Branch	Main Reactions
Start step	${}^1_1H +^1_1 H \rightarrow^2_1 H +^1_1 H \rightarrow^3_2 He$
PP-I	${}^3_2He +^3_2 He \rightarrow^4_2 He + 2^1_1H$
PP-II	${}^3_2He +^4_2 He \rightarrow^7_4 Be \rightarrow^7_3 Li +^1_1 H \rightarrow 2^4_2He$
PP-III	${}^3_2He +^4_2 He \rightarrow^7_4 Be +^1_1 H \rightarrow^8_5 B \rightarrow^8_4 Be \rightarrow 2^4_2He$

Table 4: The various branches of the PP-chain. The total energy yield of each of the chains is 26.73MeV, however due to neutrinos the PP-I, PP-II and PP-III chains lose respectively 2%, 4% and 28.3% of the total energy.

Nuclear Reactions in the Core

Using our calculated value for the core temperature of the star C138 we now wanted to calculate the corresponding luminosity produced and compare it to our previously calculated value C131. We assumed that the core stretched out to $0.2r_{star}$. In addition we also assumed the core to consist of 74.5% Hydrogen, 25.3% Helium and 0.2% Carbon, Oxygen and Nitrogen. We used the expressions for the energy production rate of the PP-Chain and CNO-Cycle.

$$\epsilon_{pp} \approx \epsilon_{0,pp} X_H^2 \rho T_6^4 \tag{C139}$$

$$\epsilon_{CNO} \approx \epsilon_{0,CNO} X_H X_{CNO} \rho T_6^{20} \tag{C140}$$

where $\epsilon_{0,pp} = 1.08 \cdot 10^{-12} Wm^3/kg^2$, $\epsilon_{0,CNO} = 8.24 \cdot 10^{-31} Wm^3/kg^2$, X_H is the mass fraction of Hydrogen, X_{CNO} is the mass fraction of C, N, O and T_6 is the temperature in millions of Kelvin. The units of ϵ_{pp} and ϵ_{CNO} are W/kg With its relation to the luminosity being:

$$\frac{dL}{dm} = \epsilon \tag{C141}$$

To determine the luminosity, L of the star we considered all the energy generated by each layer of the star with thickness dr. The total mass of this infinitesimal layer is:

$$dm = \rho dV = 4\pi r^2 \rho dr \tag{C142}$$

Substituting for dm in C141 we obtain

$$\frac{dL}{dr} = 4\pi r^2 \rho \epsilon \tag{C143}$$

Which we integrate, from the core to the surface, to obtain our new expression for the luminosity

$$L = \frac{4}{3}\pi R^3 \rho \epsilon \tag{C144}$$

Plugging in our star's numbers

$$Lpp = 7.17252 \cdot 10^{24} W \tag{C145}$$

$$L_{CNO} = 4.47165 \cdot 10^{24} W \tag{C146}$$

$$L_{tot} = L_{pp} + L_{CNO} = 1.164417 \cdot 10^{25} W \tag{C147}$$

Compared to our previously calculated luminosity

$$\frac{1.164417 \cdot 10^{25}}{1.36335 \cdot 10^{28}} = 8.54 \cdot 10^{-4} \tag{C148}$$

The newly calculated luminosity is 85400 times smaller than the luminosity we deduced and calculated from the expression for the flux from a point source and the Stefan-Boltzmann flux law. This newer luminosity would correspond to a star with surface temperature

$$T = \sqrt[4]{\frac{L}{4\pi R^2 \sigma}} \tag{C149}$$

$$T = 1765K \tag{C150}$$

The following assumptions were made to get to this value for the surface temperature. When calculating the core temperature we assumed the density to be constant, and the only source of pressure to be the ideal gas pressure. Since this new surface temperature depends upon our calculated core temperature calculation, taking into account the radiation pressure would increase this surface temperature. However the assumption we think caused the luminosity to be off by a factor of 85400 is that we assumed uniform density. The value for our star's density is thus $531 \frac{kg}{m^3}$ throughout the whole star. In a real star $\sim 99\%$ of the fusion power is generated within 24% of the radius. This is because this region is way more dense than the rest off the star, making the cross section for the fusion chains mentioned above much larger. Comparing our core density to the density in the core of the sun we see where the orders of magnitude error stems from. $\rho_{\text{suns core}} = 282\rho_{\text{star}}$, considering the expression for the luminosity above C144 we observe that $L \sim \rho^2$.

A Star's Life on the HR-diagram

One can estimate the lifetime of a star on the main sequence. The time it takes for the Hydrogen available in the core to be burnt up. If the star is in equilibrium, meaning the pressures inside the star and the gravitational collapse are in balance, the energy generated in the core must equal the energy radiated away from the surface. Considering the relation between the luminosity and mass discussed earlier C118 and remembering that luminosity is energy per time, W, lets us make an approximation of the lifetime of the star. We use the mass energy relation to find the total energy generated from the fusion processes: $E = pMc^2$ where p is the fraction of mass converted to energy. Next we assume the luminosity to stay constant throughout the lifetime of the star on the main sequence. Using the mass luminosity relation we find that

$$L = \frac{pMc^2}{t_{\text{mainseq}}} \tag{C151}$$

Thus giving us the first order approximation of the lifetime on the main sequence.

$$t_{\text{mainseq}} \propto \frac{1}{M^3} \tag{C152}$$

After this time has gone by the core has run dry of Hydrogen for the fusion and the gravitational collapse continues. This increases the pressure and temperature in and around the core, such that a shell around the previous Helium core now starts fusing Hydrogen. The energy produced by the fusion in this shell is greater than the previous energy produced in the core. This generates a radiation pressure greatly increasing the radius of the star. Once hydrostatic equilibrium is regained the star has become a Sub-Giant.

Thus the star is located on the Sub-Giants branch in figure 22. While in this state the increasing radius is making the effective temperature lower and lower down to a critical temperature of 2500K. This temperature is critical because once reached convection becomes the dominant mechanism for energy transport out of the star. Convection is the transfer of energy through movement of a medium, and is a lot more efficient than the transfer of energy due to thermal radiation. When convection kicks in at around 2500K the luminosity of the star increases a lot. The star has now become a red giant, with increased luminosity which moves it up to the Giants branch in the HR-diagram.

Since our star has a mass $m = 2.4M_{\text{sun}}$ the pressure in the core will keep increasing in the same manner. More Hydrogen in the shell is being fused into Helium, and eventually the pressure in the core will be high enough for the triple alpha process. This process fuses three alpha particles into Carbon and which collects in the core. The previously contracting core finally expands. This moves the Hydrogen shell outwards and a substantial fraction of it will not be hot enough to continue fusing. This means that the star's luminosity will decrease since the Helium burning is a lot less efficient then the Hydrogen burning. The stars temperature will continue to decrease as this process continues to happen, moving the star to the right along the Giants branch on the HR-diagram. Once the Helium in the core is burnt up it will contract, hydrostatic equilibrium is lost. The Layers come crashing inwards and the core temperature increases. A shell of Helium around the core ignites and the radius of the star increases as an effect of the newly generated energy. Once the effective temperature hits that critical 2500K, we will again see an increase in the luminosity due to convection dominating. The star now moves up on the HR-diagram to somewhere between the Bright Giants branch and Super-Giants branch.

While in this state the density of the core keeps increasing until all quantum states up to the Fermi energy $E_F \approx 0.3\text{MeV}$ are filled. The core is now an electron degenerate gas along with 'floating' positive ions. Degenerate gases are highly resistant to further compression because no two electrons can occupy the same quantum state. This means that the thermal energy cannot be radiated away because the electrons cannot move to lower energy levels, however that means that the pressure fighting against gravity is independent of the temperature. We call this pressure the degeneracy pressure.

$$P_{\text{deg}} = \frac{3}{\pi}^{2/3} \frac{h^2}{20m_e} n_e^{5/3} \tag{C153}$$

where n_e is the number density of electrons per volume, h is Planck's constant and c is the speed of light. This degeneracy pressure is now the pressure fighting the gravitational collapse of the star. The star will continue to burn Hydrogen in the outermost fusion layer which will produce Helium making the Helium layer grow denser until it is partially degenerate. Since a degenerate gas's density is not dependent on the temperature of the gas and a degenerate gas is a good heat conductor, this outer layer of Helium is close to isothermal. The temperature in this layer will gradually increase until it reaches $T \sim 10^8$K at which pretty much all the helium will start to fuse. Since this happens throughout the whole layer, almost at once, there is a huge burst of outward energy as the Hydrogen layer is lifted outwards. The Hydrogen burning stops and the star contracts and the cycle continues. As more and more layers are blown off in subsequent Helium flashes the effective temperature of the star increases. Moving the star to the left on the HR-diagram.

Once all the non-fusion layers have been blown off, and the layers where fusion is taking place start being ejected the luminosity of the star quickly drops off. The remainder of the star is now located at the bottom left of the HR-diagram, and has become a white dwarf. A white dwarf is the electron degenerate Carbon and Oxygen core left over after the Helium flash process. As discussed earlier for a fully degenerate gas there should not be any thermal radiation, but white dwarfs actually have a small convective envelope near the surface which is only partially degenerative. Thus the dead remains of our star will cool off until it no longer glows, meaning all the thermal energy has been radiated away. From the document linked to in 25 from the University of Patras they derive a first order approximation for the time it takes for the white dwarf to cool of

$$t_{\text{cool}} \propto L^{-5/7} \tag{C154}$$

It turns out there are other complicated factors also contributing to the cooling, but we will not go into this here.

We assumed the remaining mass of our star was

$$M_{\text{whitedwarf}} = \frac{M_*}{8M_{\text{sun}}} M_{\text{Chandrasekhar}} \tag{C155}$$

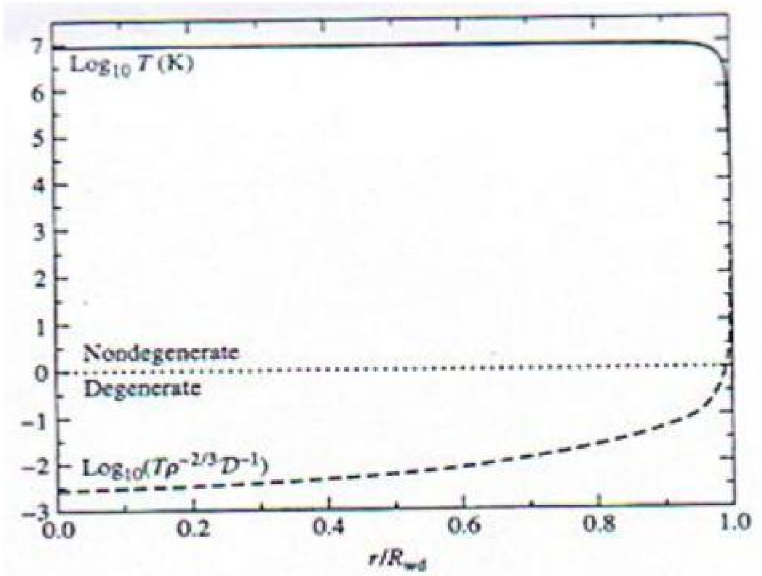


Figure 25: Temperature and Degeneracy vs. Radius. The temperature stays constant through the white dwarf and the drop in temperature corresponds very well with the drop in degeneracy. Source: http://www.physics.upatras.gr/UploadedFiles/course_149_4311.pdf

because we wanted to calculate an approximation of the density of our dead star. We use hydrostatic equilibrium to make an estimation of the radius of the white dwarf.

$$P = \frac{3GM^2}{4\pi R^4} \tag{C156}$$

where the pressure is the degeneracy pressure C153.

$$n_e = \frac{Z}{A}\frac{\rho}{m_H} \tag{C157}$$

We now have to solve this equation for R

$$R_{WD} = \left(\frac{3}{2\pi}\right)^{4/3}\frac{h^2}{20m_eG}\left(\frac{Z}{Am_H}\right)^{5/3}M^{-1/3} \tag{C158}$$

and we obtain 1883879.84612m which is approximately a third of the Earth's radius.

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