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AST3310 PROJECT 2 - NUMERICAL MODEL OF A STAR

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1. INTRODUCTION

To properly model a star, we need to expand our model to include convection. The convection of energy from the stellar core out to the outer layers control the temperature of the core, and hence affects the energy production. In principle there is also heat conduction at some level, but we will disregard that in this project.

2. THEORY

We consider a rising parcel of gas in the star. We know that the total energy flux is given by

$$F_R + F_C = \frac{16\sigma T^4}{3\kappa\rho H_P}\nabla_{\text{stable}} \tag{1}$$

where F_R and F_C are radiative and convective flux, σ is the Stefan Boltzmann constant, T is temperature, κ is the opacity, ρ is the density, H_P is the pressure scale height and ∇_{stable} is the temperature gradient needed if all energy were to be carried by radiation.

The radiative flux is

$$F_R = \frac{16\sigma T^4}{3\kappa\rho H_P}\nabla^* \tag{2}$$

where ∇^* is the actual temperature gradient. The convective flux is

$$F_C = \rho c_P T\sqrt{g\delta}H_P^{-\frac{3}{2}}\left(\frac{l_m}{2}\right)^2(\nabla^* - \nabla_p)^{\frac{3}{2}} \tag{3}$$

where c_P is the specific heat at constant pressure, g is the gravitational acceleration, $\delta = -\left(\frac{\partial \ln\rho}{\partial \ln T}\right)_P$ which simplifies to 1 using the ideal gas law, l_m is the mixing length, and ∇_p is the gradient of the rising parcel.

We use these to get an expression containing $(\nabla^* - \nabla_p)$ as a function of ∇^* and ∇_{stable}

$$\frac{16\sigma T^4}{3\kappa\rho H_P}\nabla^* + \rho c_P T\sqrt{g\delta}H_P^{-\frac{3}{2}}\left(\frac{l_m}{2}\right)^2(\nabla^* - \nabla_p)^{\frac{3}{2}} = \frac{16\sigma T^4}{3\kappa\rho H_P}\nabla_{\text{stable}} \tag{4}$$

$$\rho c_P T\sqrt{g\delta}H_P^{-\frac{3}{2}}\left(\frac{l_m}{2}\right)^2(\nabla^* - \nabla_p)^{\frac{3}{2}} = \frac{16\sigma T^4}{3\kappa\rho H_P}(\nabla_{\text{stable}} - \nabla^*) \tag{5}$$

$$(\nabla^* - \nabla_p)^{\frac{3}{2}} = \frac{64\sigma T^3}{3l_m^2\kappa\rho^2 c_P}\sqrt{\frac{H_P}{g\delta}}(\nabla_{\text{stable}} - \nabla^*) \tag{6}$$

If we now take this equation

$$(\nabla_p - \nabla_{ad}) = (\nabla^* - \nabla_{ad}) - (\nabla^* - \nabla_p) \tag{7}$$

and insert

$$(\nabla_p - \nabla_{ad}) = \frac{32\sigma T^3}{3\kappa\rho^2 c_P v}\frac{S}{Qd}(\nabla^* - \nabla_p) \tag{8}$$

we get

$$\frac{32\sigma T^3}{3\kappa\rho^2 c_P v}\frac{S}{Qd}(\nabla^* - \nabla_p) = (\nabla^* - \nabla_{ad}) - (\nabla^* - \nabla_p) \tag{9}$$

We want to find a second order equation for $(\nabla^* - \nabla_p)^{\frac{1}{2}}$. We have

$$v = \sqrt{\frac{g\delta l_m^2}{4H_P}}(\nabla^* - \nabla_p)^{\frac{1}{2}} \tag{10}$$

and insert into Eq. 9 to get

$$\frac{32\sigma T^3}{3\kappa\rho^2 c_P}\sqrt{\frac{4H_P}{g\delta l_m^2}}\frac{S}{Qd}(\nabla^* - \nabla_p)^{\frac{1}{2}} = (\nabla^* - \nabla_{ad}) - (\nabla^* - \nabla_p)$$
$$\frac{64\sigma T^3}{3\kappa\rho^2 c_P l_m}\sqrt{\frac{H_P}{g\delta}}\frac{S}{Qd}(\nabla^* - \nabla_p)^{\frac{1}{2}} = (\nabla^* - \nabla_{ad}) - (\nabla^* - \nabla_p) \tag{11}$$

We define

$$U = \frac{64\sigma T^3}{3\kappa\rho^2 c_P}\sqrt{\frac{H_P}{g\delta}} \tag{12}$$

so it becomes

$$U\left(\frac{S}{Qdl_m}\right)(\nabla^* - \nabla_p)^{\frac{1}{2}} = (\nabla^* - \nabla_{ad}) - (\nabla^* - \nabla_p) \tag{13}$$

We now define $(\nabla^* - \nabla_p)^{\frac{1}{2}} = \xi$ and write this as a second order equation

$$U\left(\frac{S}{Qdl_m}\right)\xi = (\nabla^* - \nabla_{ad}) - \xi^2 \tag{14}$$

$$\nabla^* = \xi^2 + U\left(\frac{S}{Qdl_m}\right)\xi + \nabla_{ad} \tag{15}$$

lastly we can use that

$$\frac{S}{Qdl_m} = \frac{4}{l_m^2} \tag{16}$$

and call it K.

$$\nabla^* = \xi^2 + UK\xi + \nabla_{ad} \tag{17}$$

We will now plug this in for ∇^* in Eq. 6 and replace $(\nabla^* - \nabla_p)^{\frac{3}{2}} = \xi^3$

$$\xi^3 = \frac{1}{l_m^2}U(\nabla_{\text{stable}} - (\xi^2 + UK\xi + \nabla_{ad})) \tag{18}$$

$$\xi^3 - \frac{U}{l_m^2}\Big(\xi^2 + UK\xi - (\nabla_{\text{stable}} - \nabla_{ad})\Big) = 0 \tag{19}$$

We can now use this to find ξ, and plug it into Eq. (17) to get the temperature gradient. With that in hand we can find $\frac{dT}{dm}$ by

$$\begin{aligned}\nabla &= \frac{\partial lnT}{\partial lnP} \\ \nabla &= \frac{P}{T}\frac{\partial T}{\partial P} \\ \nabla &= \frac{P}{T}\frac{\partial m}{\partial P}\frac{\partial T}{\partial m} \\ \rightarrow \frac{\partial T}{\partial m} &= -\frac{GmT}{4\pi r^4 P}\nabla\end{aligned} \tag{20}$$

3. IMPLEMENTING IN PYTHON

I use a modified version of the code from the last project which has been heavily improved since then. Now we want to add convection. To do that, we need to check for convective instability at any given step in our integration. If our stellar core is not able to transport enough energy away by radiation we will have convection, which will affect the temperature of the system. So we want include convection if

$$\nabla_{stable} > \nabla_{ad} \tag{21}$$

that is, if the temperature gradient needed for all energy to be transported by radiation is larger than the temperature gradient we would have if we had adiabatic expansion. If this is true we have an unstable system and need to include convective transport of energy.

Inside the main loop, we calculate ∇_{stable} using

$$\nabla_{stable} = \frac{3\kappa LP}{64\pi\sigma GmT^4} \tag{22}$$

∇_{ad} using

$$\nabla_{ad} = \frac{P\delta}{T\rho C_P} \tag{23}$$

and total flux using

$$F = \frac{L}{4\pi r^2} \tag{24}$$

We check for convective instability using Eq. (21).

If the instability criterion is true, we calculate ∇^* by solving for ξ in equation (19), and inserting that value into Eq. (17). We then use ∇^* to calculate the new $\frac{dT}{dm}$ using Eq. (20). We also use ∇^* to calculate radiative flux F_R with Eq. (2), and find convective flux by

$$F_C = F - F_R \tag{25}$$

If the instability criterion is false, we set $\nabla^* = \nabla_{stable}$, $F_R = F$ and calculate dT/dm with the usual equation we used in last project

$$\frac{\partial T}{\partial m} = -\frac{3\kappa L}{256\pi^2\sigma r^4 T^3} \tag{26}$$

I use an adaptive mass step in this project as well, as described in [1], mostly with $p = 0.01$ throughout this project.

Lastly we set the values for the next step as usual, where $\frac{\partial T}{\partial m}$ is now the appropriate value depending on whether we are convectively unstable or not.

This is pretty much all that has to be done. Calculating F_C, F_R and F wasn't strictly necessary to complete the simulation, however we want to visualize our results by displaying a cross section plot of our star. To do this we use the code given in [2], which uses the values of F_C to conclude whether we have convection or not at any given radius in the star. The sun is then plotted by drawing a circle of the "current" radius every (eg.)10 steps, and coloring the circle differently depending on F_C.

4. INITIAL PARAMETERS

We start with the initial parameters of the sun, with a few approximations. They are listed in the task for this project [3]. From here on, when I say for example $\rho_\odot$, I am referring to the value $\rho_0 = 1.42 \cdot 10^{-7}\rho_\odot$ given in this task. Here is the result using those values

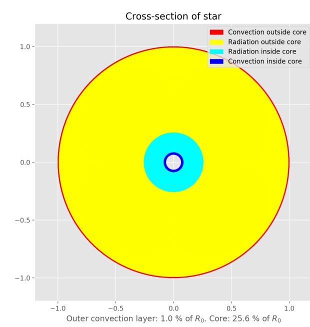


FIG. 1.—: Cross section plot of a star with the parameters of the solar surface as initial values

We see that there is a thin convection layer on the surface, and a decently sized core with another convection layer inside. However, the radius clearly doesn't reach zero, it stops at 6.6% as we can see in the final values:

	Mass	Radius	Luminosity	Density	Temperature
Value	0	4.6e+07	1.4e+25	1.1e+05	1.4e+07
$\frac{x}{x_0}$	0	0.066	0.036	5.707e+08	2400

TABLE 1: Final values for star model with the initial values of the surface of the sun.

We can also look at the temperature gradients and see that where ∇_{stable} crosses ∇_{ad} corresponds nicely with the cross section plot.

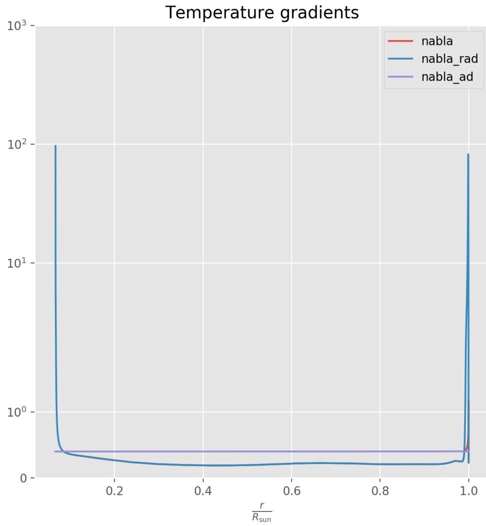


Fig. 2.—: Temperature gradients with the parameters of the solar surface as initial values

Now we want to find our own parameters. If we start by increasing $\rho_0 = 5\rho_\odot$ we see that we get a little more convection, but not close to 15%. We also see that the luminosity ends at a higher value with a higher initial density. But since the task has given us free reigns in regards to the upper limit of ρ_0, lets set $\rho_0 = 50\rho_\odot$ right away.

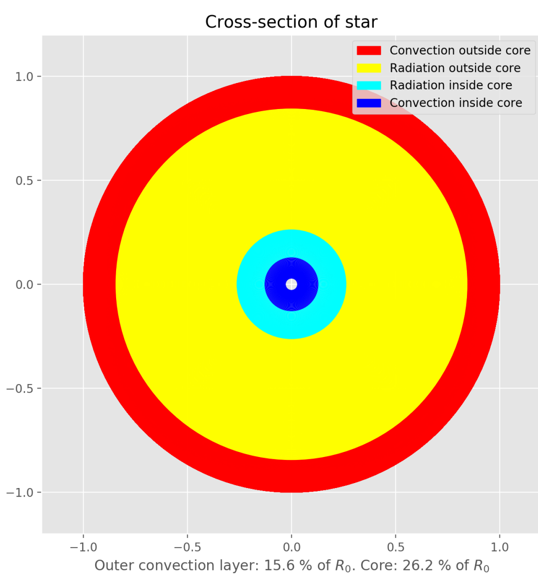
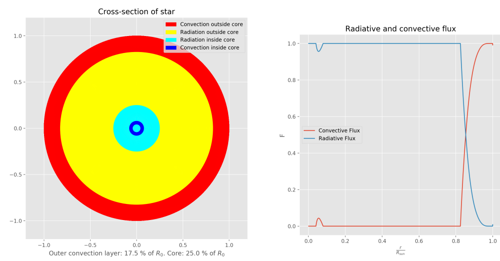


Fig. 3.—: Cross section of star with regular initial values except for $\rho_0 = 50\rho_\odot$

We now have an outer convection layer of about 15% of R_0, which is our goal. However, the luminosity now only goes down to 50% at $m = 0$, and we have an inner convective layer with just as high convective flux as the outer layer. If we now decrease the temperature, the luminosity drops faster. When we choose $T_0 = 0.9T_\odot$ and $\rho_0 = 50\rho_\odot$ we actually get a system where all three values L, R and M go to 0.00% of their initial values on the last step. However, there is still an inner convective layer with a high convective flux, so this is not the final model. Through trial and error I noticed that decreasing the initial mass by a little reduces the amount of convection in the core. Decreasing it too much has the effect that mass and radius don't reach 0, but if we lower it a little bit, to $0.95M_\odot$ we get the following plots



(a) Cross section (b) Flux

where we can see that the convective flux is now much smaller in the core than on the surface. This is a prefectly fine model, well within the 5% limit for the final values, however I continued to tweak the parameters until I got a model in which all values ended up smaller than 1% of the original value and with no inner convection layer at all. Long story short, lowering the radius to $0.9_\odot$ removed the inner convection layer and let us integrate down to 1% mass, but increased our final radius to 5% of $_\odot$, so lastly I decreased T even more to $T = 0.7T_\odot$, and these are the final parameters I chose, shown in Table 2.

Obviously the road to finding these values wasn't as short and tidy as explained here, but it's a summary of some of the observations I made along the way.

5. RESULTS

	M	R	L	ρ	T
Initial	$0.95M_\odot$	$0.9R_\odot$	$L_\odot$	$54\rho_\odot$	$0.7T_\odot$
Final	1.5e+28	5.3e+05	0.0	8.5e+06	4.1e+07
$\frac{\text{Final}}{\text{Initial}}$	0.0081M_0	0.0008R_0	0.0L_0	$7.841\text{e+}08\rho_0$	$1.0\text{e+}04\text{T}_0$

TABLE 2: Initial and final values for the best model. Bottom row is the fraction of the original value, and as we can see M, R and L all go to under 1% of their original values.

In Table 2 we see the initial and final paramaters of my best model, along with final values relative to the initial value. Luminosity goes to 0, radius to 0.08% and mass to 0.81% of the original values.

If we take a look at the cross section plot in Figure 5, we can see that the convection layer takes up 17.5% of R_0 and the core has a radius of 27.5% of R_0. There's continuous radiation throughout the core, no inner convection layer. As we can see from Figure 11 we have a high convective flux close to the surface which gets taken over by radiative flux as we move inwards away from the convective layer.

This corresponds well to Figure 12, where we can also see that ∇_{stable} is really close to ∇_{ad} in the inner layer but we never actually reach convective instability.

If we take a look at Figure 13, we see that the PPI chain dominates throughout this entire star. However, as we get to higher temperatures in the innermost parts of the core we can see that the PPII and PPIII energy levels are on their way to rise rapidly, if the simulation had continued. This is easier to see in the zoomed in version in Figure 14. One of the goals of this project was to have the PPII and PPIII chains dominate in the higher temperatures, it seems like they would if it were to continue but in the selected model they unfortunately don't.

Comparing our model to the actual sun, we are still quite off on the width of the convection layer, as the real sun has convection in more like the outer 30% of its radius. The temperature seems pretty accurate, though it shoots up to 40MK in the last couple steps. The temperature of the actual suns core is about 15MK, and this is right around the value of the temperature in our core before the extremes of the last steps.

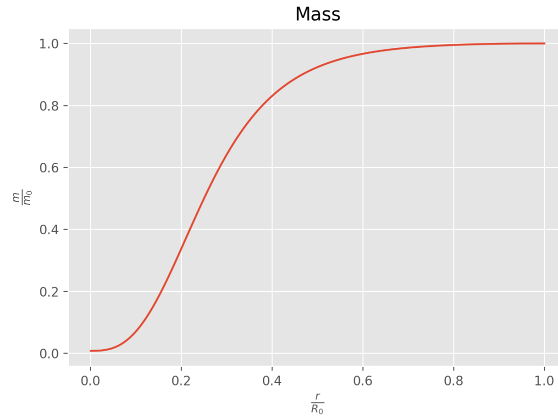


FIG. 7.—: M vs. R for best model

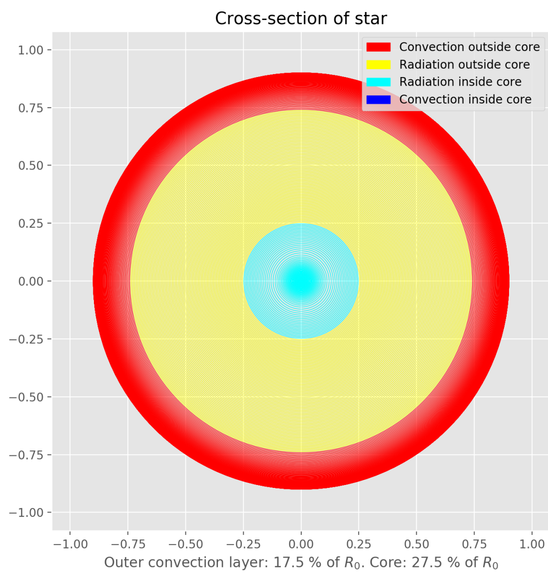


FIG. 5.—: Cross section plot of the best model

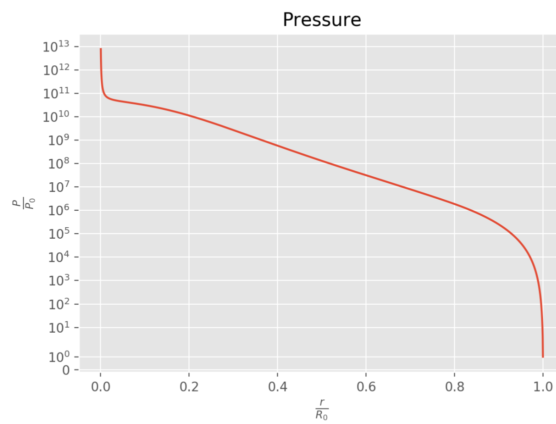


FIG. 8.—: P vs. R

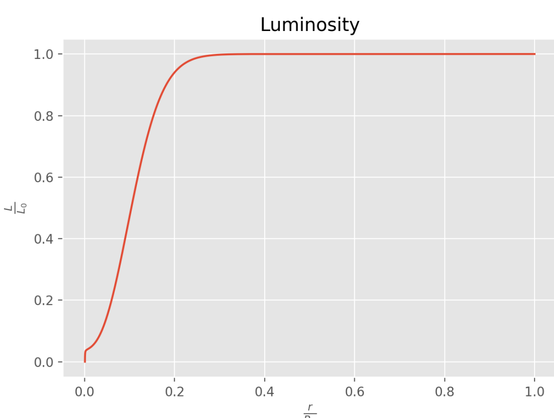


FIG. 6.—: L vs. R for best model

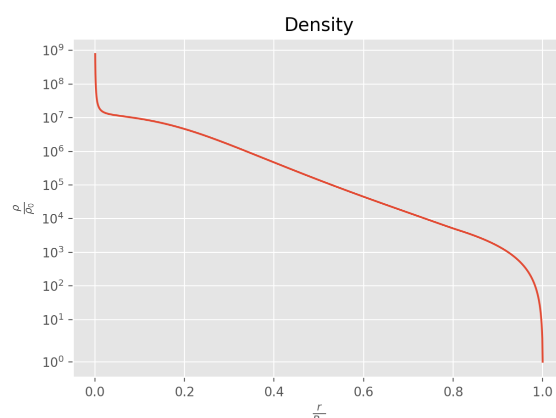


FIG. 9.—: ρ vs. R for best model

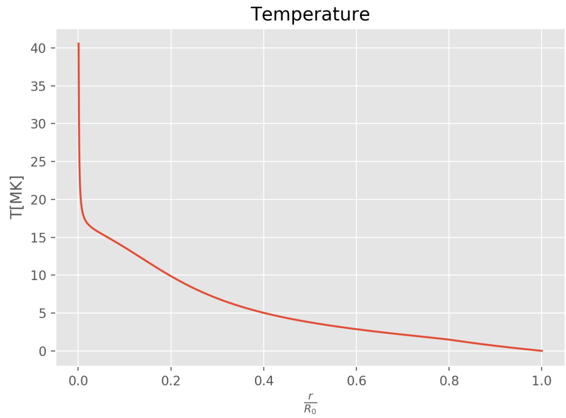


FIG. 10.—: T vs. R for best model

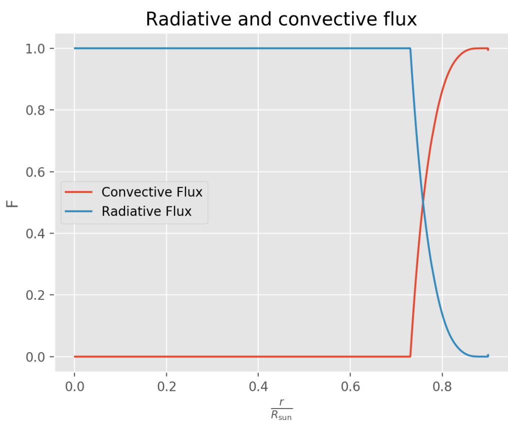


FIG. 11.—: Radiative and convective flux for best model(Relative to total flux)

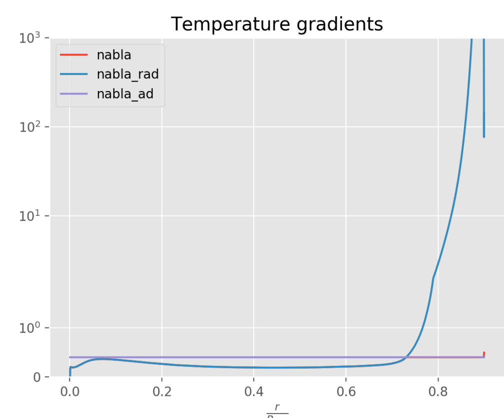


FIG. 12.—: Temperature gradients for the best model

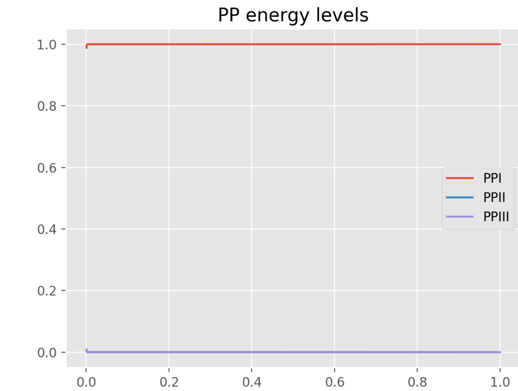


FIG. 13.—

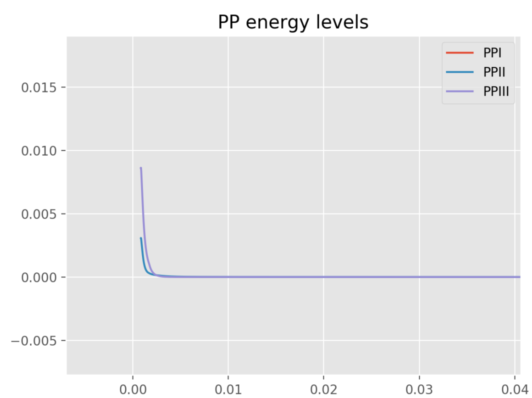


FIG. 14.—

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[3]Modelling a Star - Boris Gudiksen
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