Oblig 1 - FYS2140

1.

a)

i) $z = i$

$z^* = -i$

$|z| = 1$

$|z|^2 = 1$

$zz^* = -i^2$
$= -(-1)$
$= 1 \quad = |z|^2$

ii) $z = 3+4i$

$z^* = 3-4i$

$|z| = \sqrt{3^2+4^2} = \sqrt{25} = 5$

$|z|^2 = 25$

$zz^* = (3+4i)(3-4i)$
$= 9 - 12i + 12i - 16i^2$
$= 9 - 16(-1)$
$= 9 + 16$
$= 25$

iii) $z = -3$

$z^* = -3$

$|z| = 3$

$|z|^2 = 9$

$zz^* = 9$

iv) $z = 1+i$

$z^* = 1-i$

$|z| = \sqrt{2}$

$|z|^2 = 2$

$zz^* = (1+i)(1-i)$
$= 1 - i + i - i^2$
$= 1 - (-1)$
$= 1 + 1$
$= 2$

$$\frac{z_1}{z_2} = \frac{z_1 z_2^*}{|z_2|^2}$$

b)

i) $$\frac{3+4i}{1-2i} = \frac{(3+4i)(1+2i)}{3}$$

$$= \frac{3+6i+4i+8i^2}{3}$$

$$= \frac{3+10i-8}{3}$$

$$= \frac{-5+10i}{3} = \underline{\underline{-\frac{5}{3} + \frac{10}{3}i}}$$

ii) $$\frac{\sqrt{3}+i}{(1-i)(\sqrt{3}-i)} = \frac{\sqrt{3}+i}{\sqrt{3}-i-\sqrt{3}i-1}$$

$$= \frac{\sqrt{3}+i}{(\sqrt{3}-1)-(\sqrt{3}i+i)} \Longrightarrow$$

$$\boxed{\begin{aligned} z_1 &= \sqrt{3}+i \\ z_2 &= (\sqrt{3}-1)-(\sqrt{3}+1)i \\ z_2^* &= (\sqrt{3}-1)+(\sqrt{3}+1)i \\ |z_2| &= \sqrt{3-\cancel{2\sqrt{3}}+1+3+\cancel{2\sqrt{3}}+1} \\ &= \sqrt{8} \\ |z_2|^2 &= 8 \end{aligned}}$$

$$= \frac{(\sqrt{3}+i)((\sqrt{3}-1)+(\sqrt{3}+1)i)}{8}$$

$$= \frac{(\sqrt{3}+i)(\sqrt{3}-1)+(\sqrt{3}+i)(\sqrt{3}i+i)}{8}$$

$$= \frac{3-\sqrt{3}+\sqrt{3}i-i+3i+\sqrt{3}i-\sqrt{3}-1}{8}$$

$$= \frac{2-2\sqrt{3}+(2+2\sqrt{3})i}{8}$$

$$= \underline{\underline{\frac{1-\sqrt{3}}{4} + \frac{1+\sqrt{3}}{4}i}}$$

c)

i) $z = 2i$

$r = 2$

$\theta = \frac{\pi}{2}$

$$\underline{\underline{z = 2e^{\left(i\frac{\pi}{2}\right)}}}$$

ii)

$z = -6 + 6\sqrt{3}\,i$

$r = \sqrt{6^2 + 36 \cdot 3}$

$= \sqrt{144}$

$= 12$

$\cos\theta = \frac{-6}{12} = -\frac{1}{2}$

$\theta = \frac{2}{3}\pi$

$$\underline{\underline{z = 12e^{\left(i\frac{2}{3}\pi\right)}}}$$

iii) $z = -1$ $\qquad r = 1$

$\theta = \pi$

$\Downarrow$

$$\underline{\underline{z = e^{i\pi}}}$$

d)

i) $z_1 = 2e^{-i\pi} \qquad z_2 = 3e^{i\frac{\pi}{3}}$

$$z_1 z_2 = 6e^{\frac{\pi}{3}i - \pi i} = 6e^{\frac{\pi}{3}i - \frac{3\pi}{3}i} = \underline{\underline{6e^{-\frac{2\pi}{3}i}}}$$

ii) $z_1 = e^{-i\frac{\pi}{5}} \qquad z_2 = 3e^{i\frac{\pi}{5}}$

$$z_1 z_2 = 3e^{i\frac{\pi}{5} - i\frac{\pi}{5}} = 3e^{0} = 3 \cdot 1 = \underline{\underline{3}}$$

Dersom man multipliserer et komplekst tall med $e^{\frac{\pi}{2}i}$ vil det geometrisk sett rotere 90° mot klokka.

2.

a)

$$\frac{df(x)}{dx} = bf(x) \qquad \text{init: } f(0)=1, \quad f'(0)=3$$

$$\frac{\frac{df(x)}{dx}}{f(x)} = b$$

$$\int \frac{f'(x)}{f(x)}\,dx = \int b\,dx$$

$$\ln|f(x)| = bx + C$$

$$f(x) = e^{bx+c}$$

$$= e^{bx} \cdot e^{c}$$

Generell løsn: $\underline{f(x) = e^{bx} \cdot C}$

$$f(0) = 1$$

$$\Downarrow$$

$$1 \cdot C = 1 \Rightarrow \underline{C = 1}$$

$$f(x) = e^{bx}$$

$$f'(x) = b e^{bx}$$

$$f'(0) = 3$$

$$\Downarrow$$

$$3 = b \cdot 1 \Rightarrow \underline{b = 3}$$

$$\Rightarrow \underline{\underline{f(x) = e^{3x}}}$$

b)

$$\frac{d^2 f(x)}{dx^2} = a f(x)$$

$$\frac{d^2 (e^{bx})}{dx^2} = a e^{bx}$$

$$b^2 e^{bx} = a e^{bx}$$

$$b^2 e^{bx} - a e^{bx} = 0$$

$$e^{bx}(b^2 - a) = 0 \quad \Big| \quad e^{bx}$$ kan aldri bli 0, så:

$$b^2 - a = 0$$

$$b^2 = a$$

$$b = \sqrt{a} \quad \vee \quad b = -\sqrt{a}$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$f(x) = e^{\sqrt{a}x} \cdot C_1 \quad \Big| \quad f(x) = e^{-\sqrt{a}x} \cdot C_2$$

Kaller vi $C_1 = A$ og $C_2 = B$ og setter sammen løsningene, får vi:

$$f(x) = A e^{\sqrt{a}x} + B e^{-\sqrt{a}x}$$

$$f(x) = Ae^{\sqrt{a}x} + \frac{B}{e^{\sqrt{a}x}}$$

Dersom $\lim_{x \to \infty} f(x) = 0$ må $A = 0$

Hvis $\lim_{x \to -\infty} f(x) = 0$ må $B = 0$

Lar $A = \frac{C+D}{2}$ og $B = \frac{C-D}{2}$, da får vi

$$f'(x) = \frac{C+D}{2} e^{\sqrt{a}x} + \frac{C-D}{2} e^{-\sqrt{a}x}$$

$$= \frac{C}{2} e^{\sqrt{a}x} + \frac{D}{2} e^{\sqrt{a}x} + \frac{C}{2} e^{-\sqrt{a}x} - \frac{D}{2} e^{-\sqrt{a}x}$$

$$= \frac{C}{2}\left(e^{\sqrt{a}x} + e^{-\sqrt{a}x}\right) + \frac{D}{2}\left(e^{\sqrt{a}x} - e^{-\sqrt{a}x}\right)$$

$$= C \frac{e^{\sqrt{a}x} + e^{-\sqrt{a}x}}{2} + D \frac{e^{\sqrt{a}x} - e^{-\sqrt{a}x}}{2}$$

Bruker at $\cosh(u) = \frac{e^{u} + e^{-u}}{2}$ og $\sinh(u) = \frac{e^{u} - e^{-u}}{2}$

og får at

$$\underline{f(x) = C\cosh(\sqrt{a}x) + D\sinh(\sqrt{a}x)}$$

c) $\underline{a < 0}$

$$f(x) = Ae^{i\sqrt{a}x} + Be^{-i\sqrt{a}x}$$

Euler's identity:

$$f(x) = A\left(\cos(\sqrt{a}x) + i\sin(\sqrt{a}x)\right) + B\left(\cos(-\sqrt{a}x) + i\sin(-\sqrt{a}x)\right)$$

3.

a)

i)

$$\int_{-\infty}^{\infty} e^{-x^2-4x-1}\,dx = \int_{-\infty}^{\infty} e^{-(x^2+2\cdot 2x+1)}\,dx$$

Rottmann $$= \sqrt{\pi}\, e^{\frac{4-1}{1}}$$

$$= \underline{\underline{\sqrt{\pi}\, e^{3}}}$$

ii)

$$\int_0^\infty x e^{-2x^2}\, dx = \frac{1}{2} \cdot 2^{-\frac{1+1}{2}} \Gamma\left(\frac{1+1}{2}\right)$$

$$= \frac{1}{4} \Gamma(1) \qquad \Big|\ \Gamma(1) = 1$$

$$= \underline{\underline{\frac{1}{4}}}$$

b)

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-2\sqrt{x^2+y^2+z^2}}\, dx\, dy\, dz$$

Gjør om til sfæriske koordinater, der jacobi determinanten er $r^2 \sin\phi$, og får

$$\int_0^{\pi}\int_0^{2\pi}\int_0^{\infty} e^{-2r} r^2 \sin\phi \, dr\, d\theta\, d\phi$$

$$= \int_0^{\pi} \sin\phi \int_0^{2\pi}\int_0^{\infty} r^2 e^{-2r}\, dr\, d\theta\, d\phi$$

Bruker den meget nyttige formelen $\int_0^\infty x^n e^{-\alpha x}\, dx = \frac{1}{\alpha^{n+1}} n!$

$$= \int_0^{\pi} \sin\phi \int_0^{2\pi} \frac{1}{2^3}\, 2!\, d\theta\, d\phi$$

$$= \int_0^{\pi} \sin\phi \int_0^{2\pi} \frac{1}{4}\, d\theta\, d\phi$$

$$= \int_0^{\pi} \frac{\pi}{2} \sin\phi \, d\phi = \underline{\underline{\frac{\pi}{2}}}$$

c)

$$\frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \frac{\sqrt{ma}}{\hbar} e^{-ma|x|/\hbar^2} e^{-ipx/\hbar}\, dx$$

$$f(x) = \frac{\sqrt{ma}}{\hbar\sqrt{2\pi\hbar}} e^{-ma|x|/\hbar^2} e^{-ipx/\hbar}$$

$$\boxed{e^{-\beta x} e^{i\lambda x} = \frac{i}{\lambda + p + i\beta}}$$

$$\frac{\sqrt{ma}}{\hbar\sqrt{2\pi\hbar}} e^{-\left(\frac{ma}{\hbar}\right)x} e^{i\left(-\frac{p}{\hbar}\right)x}$$

$$\beta = \frac{ma}{\hbar}$$

$$\lambda = -\frac{p}{\hbar}$$

$$= \frac{\sqrt{ma}}{\hbar\sqrt{2\pi\hbar}} \frac{i}{-\frac{\rho}{\hbar} + \rho + \frac{ima}{\hbar}}$$

$$= \frac{i\sqrt{ma}}{\hbar\sqrt{2\pi\hbar}\left(-\frac{\rho}{\hbar} + \rho + \frac{ima}{\hbar}\right)}$$

$$= \frac{i\sqrt{ma}}{\hbar\sqrt{2\pi\hbar}\left(-\frac{\rho}{\hbar} + \rho + i\rho_0\right)}$$

$$= \frac{i\sqrt{ma}}{-\rho\sqrt{2\pi\hbar} + \rho\hbar\sqrt{2\pi\hbar} + i\rho_0\hbar\sqrt{2\pi\hbar}}$$

$$= \sqrt{\frac{ma}{\hbar}}\left(\frac{i}{-\rho\sqrt{2\pi} + \rho\hbar\sqrt{2\pi} + i\rho_0\hbar\sqrt{2\pi}}\right)$$

$$= \frac{\sqrt{\rho_0}\, i}{-\rho\sqrt{2\pi} + \rho\hbar\sqrt{2\pi} + i\rho_0\hbar\sqrt{2\pi}}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{\sqrt{\rho_0}\, i}{-\rho + \rho\hbar + i\rho_0\hbar}$$

Fortsetter å prøve på denne, leverer foreløpig....