MAT 1110 # Oblig 2 Joakim Flatby

1.

a) x gir 0.5 til y:

	x
x_1	0.5
y_1	0.5
z_1	0

y gir 0.5 til z:

	y
x_1	0
y_1	0.5
z_1	0.5

z gir 0.5 til x:

	z
x_1	0.5
y_1	0
z_1	0.5

$\Rightarrow$

	x	y	z
x_1	0.5	0	0.5
y_1	0.5	0.5	0
z_1	0	0.5	0.5

dette gir matrisen: $\begin{pmatrix} 0.5 & 0 & 0.5 \\ 0.5 & 0.5 & 0 \\ 0 & 0.5 & 0.5 \end{pmatrix}$

som beskriver forflytningen av kulene, men for å oppfylle $\vec{x}_{n+1} = \frac{1}{2} M x_n$, deler vi på $\frac{1}{2}$, og får:

$$M = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

b) <u>Egenverdier:</u>

$$\lambda I_3 = \lambda \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$$

$$\det(\lambda I_3 - M) = 0$$

$$\det\left(\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} - \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \right) = 0$$

$$\det \begin{pmatrix} \lambda-1 & 0 & -1 \\ -1 & \lambda-1 & 0 \\ 0 & -1 & \lambda-1 \end{pmatrix} = 0$$

$$\lambda - 1(\lambda^2 - 2\lambda + 1) - (1-0) = 0$$

$$\lambda^3 - 2\lambda + \lambda - \lambda^2 + 2\lambda - 1 - 1 = 0$$

$$\lambda^3 - 3\lambda^2 + 3\lambda - 2 = 0$$

$$(\lambda - 2)(\lambda^2 - \lambda + 1) = 0$$

$$(\lambda - 2) = 0 \implies \lambda_1 = 2$$

$$(\lambda^2 - \lambda + 1) = 0$$

$$\Downarrow$$

$$\lambda = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1 \pm \sqrt{-3}}{2} = \frac{1 \pm i\sqrt{3}}{2}$$

$$\lambda_1 = 2 \qquad \lambda_2 = \frac{1 - i\sqrt{3}}{2} \qquad \lambda_3 = \frac{1 + i\sqrt{3}}{2}$$

<u>Egenvektorer</u>:

Bruker $M\vec{v} = \lambda\vec{v}$, der λ er en av egenverdiene, og $\vec{v}$ er den tilhørende egenvektoren.

Setter inn $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ for $\vec{v}$,

dette gir et ligningsett:

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \lambda_n \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

som tilsvarer:

$$x + z = \lambda_n x$$

$$x + y = \lambda_n y$$

$$y + z = \lambda_n z$$

dette løses for x, y og z, og $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ vil være egenvektoren som hører til egenverdien λ_n.

Dette gjentas for alle egenverdiene, og får:

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad v_2 = \begin{pmatrix} \frac{-1 + i\sqrt{3}}{2} \\ \frac{-1 - i\sqrt{3}}{2} \\ 1 \end{pmatrix} \qquad v_3 = \begin{pmatrix} \frac{-1 - i\sqrt{3}}{2} \\ \frac{-1 + i\sqrt{3}}{2} \\ 1 \end{pmatrix}$$

Dette stemmer med verdiene jeg får ved å bruke Python også:

```
import numpy as np

M = np.array([[1, 0, 1],
              [1, 1, 0],
              [0, 1, 1]])

eigenvalues, eigenvectors = np.linalg.eig(M)

print "Egenvektorer: "
print eigenvectors
```

```
Joakims-MacBook-Pro:oblig2 joakimflatby$ python oppg1.py
Egenverdier:
[ 0.5+0.8660254j  0.5-0.8660254j  2.0+0.j        ]
Joakims-MacBook-Pro:oblig2 joakimflatby$ python oppg1.py
Egenvektorer:
[[-0.28867513+0.5j -0.28867513-0.5j -0.57735027+0.j ]
 [ 0.57735027+0.j   0.57735027-0.j  -0.57735027+0.j ]
 [-0.28867513-0.5j -0.28867513+0.5j -0.57735027+0.j ]]
Joakims-MacBook-Pro:oblig2 joakimflatby$
```

c)

$x_0 = 100, \quad y_0 = 0, \quad z_0 = 0$

$$\vec{x}_0 = \begin{pmatrix} 100 \\ 0 \\ 0 \end{pmatrix}$$

Uttrykk $\vec{x}_0$ som en linearkombinasjon av egenvektorene:

Må finne tall α, β, γ, slik at

$$\alpha \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} \frac{-1+i\sqrt{3}}{2} \\ \frac{-1-i\sqrt{3}}{2} \\ 1 \end{pmatrix} + \gamma \begin{pmatrix} \frac{-1-i\sqrt{3}}{2} \\ \frac{-1+i\sqrt{3}}{2} \\ 1 \end{pmatrix} = \begin{pmatrix} 100 \\ 0 \\ 0 \end{pmatrix}$$

dette tilsvarer et ligningsystem vi kan skrive som en matrise:

$$\begin{array}{c} \mathrm{I} \\ \mathrm{II} \\ \mathrm{III} \end{array} \left(\begin{array}{ccc|c} 1 & \frac{-1+i\sqrt{3}}{2} & \frac{-1-i\sqrt{3}}{2} & 100 \\ 1 & \frac{-1-i\sqrt{3}}{2} & \frac{-1+i\sqrt{3}}{2} & 0 \\ 1 & 1 & 1 & 0 \end{array} \right)$$

Radreduserer denne matrisen, og får:

$$\begin{pmatrix} 1 & 0 & 0 & \frac{100}{3} \\ 0 & 1 & 0 & \frac{-50}{3} - \frac{50i}{\sqrt{3}} \\ 0 & 0 & 1 & \frac{-50}{3} + \frac{50i}{\sqrt{3}} \end{pmatrix}$$

$$\|$$

$$\Downarrow$$

$$\alpha = \frac{100}{3}$$

$$\beta = \frac{-50}{3} - \frac{50i}{\sqrt{3}}$$

$$\gamma = \frac{-50}{3} + \frac{50i}{\sqrt{3}}$$

Setter inn α β og γ i ligningen:

$$\frac{100}{3}\begin{pmatrix}1\\1\\1\end{pmatrix} + \left(\frac{-50}{3} - \frac{50i}{\sqrt{3}}\right)\begin{pmatrix}\frac{-1+i\sqrt{3}}{2}\\ \frac{-1-i\sqrt{3}}{2}\\ 1\end{pmatrix} + \left(\frac{-50}{3} + \frac{50i}{\sqrt{3}}\right)\begin{pmatrix}\frac{-1-i\sqrt{3}}{2}\\ \frac{-1+i\sqrt{3}}{2}\\ 1\end{pmatrix} = \begin{pmatrix}100\\0\\0\end{pmatrix}$$

og får:

$$\begin{pmatrix}100\\0\\0\end{pmatrix} = \begin{pmatrix}100\\0\\0\end{pmatrix}$$

Finn $\vec{x}_5$:

$$\vec{x}_n = \left(\tfrac{1}{2}\right)^n M^n \vec{x}_0 = \left(\tfrac{1}{2}\right)^n \left(\alpha \lambda_1^n \vec{v}_1 + \beta \lambda_2^n \vec{v}_2 + \gamma \lambda_3^n \vec{v}_3\right)$$

$$\vec{x}_5 = \frac{100}{3}\vec{v}_1 + \frac{16}{2^{10}}\left(1+\sqrt{3}i\right)\begin{pmatrix} \frac{100}{3} \\ \frac{50}{3}\left(-1+\sqrt{3}i\right) \\ \frac{50}{3}\left(-1-\sqrt{3}i\right) \end{pmatrix}$$

$$+ \frac{-16}{2^{10}}\left(-1+\sqrt{3}i\right)\begin{pmatrix} \frac{100}{3} \\ -\frac{50}{3}\left(1+\sqrt{3}i\right) \\ \frac{50}{3}\left(-1+\sqrt{3}i\right) \end{pmatrix}$$

$$= \frac{100}{3}\vec{v}_1 + \frac{16}{2^{10}}\begin{pmatrix} \frac{100}{3} + \frac{100}{\sqrt{3}}i \\ -\frac{200}{3} \\ \frac{100}{3} - \frac{100}{\sqrt{3}}i \end{pmatrix} - \frac{16}{2^{10}}\begin{pmatrix} -\frac{100}{3} + \frac{100}{\sqrt{3}}i \\ \frac{200}{3} \\ -\frac{100}{3} + \frac{100i}{\sqrt{3}} \end{pmatrix}$$

$$= \frac{100}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} \frac{25}{24} \\ -\frac{25}{12} \\ \frac{25}{24} \end{pmatrix} = \begin{pmatrix} \frac{100}{3} \\ \frac{100}{3} \\ \frac{100}{3} \end{pmatrix} + \begin{pmatrix} \frac{25}{24} \\ -\frac{25}{12} \\ \frac{25}{24} \end{pmatrix} = \begin{pmatrix} \frac{275}{8} \\ \frac{125}{4} \\ \frac{275}{8} \end{pmatrix}$$

$$\underline{\underline{\vec{x}_5 = \begin{pmatrix} 34.375 \\ 31.25 \\ 34.375 \end{pmatrix}}}$$

d)

$$\lim_{n\to\infty} \vec{x}_n = \lim_{n\to\infty} \left(\frac{1}{2}\right)^n M^n \vec{x}_0$$

$\left(\frac{1}{2}\right)^n \cdot M^n$ gir

$$\frac{1}{\infty} \cdot \begin{pmatrix} \infty & \infty & \infty \\ \infty & \infty & \infty \\ \infty & \infty & \infty \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$\lim_{n\to\infty} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 100 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 100 \\ 100 \\ 100 \end{pmatrix}$$

Som vil si at når antall forflytninger går mot uendelig, vil kulene i skålene gå mot å være jevnt fordelt.

2.

a)

$$A_5 = \begin{pmatrix} -2 & 1 & 0 & 0 & 1 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 1 & 0 & 0 & 1 & -2 \end{pmatrix}$$

Hvis vi ser på A_5 som et ligningssett, får vi:

$$\begin{aligned} -2 + 1 + 0 + 0 + 1 &= 0 \\ 1 - 2 + 1 + 0 + 0 &= 0 \\ 0 + 1 - 2 + 1 + 0 &= 0 \\ 0 + 0 + 1 - 2 + 1 &= 0 \\ 1 + 0 + 0 + 1 - 2 &= 0 \end{aligned}$$

kolonnene er derfor lineart avhengige, noe som betyr at $\det(A_n) = 0$, og derfor må 0 være en egenverdi for A_n.

$$v_n^0 = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

b)

$$A_n \cdot \vec{v}^{\,k} = \begin{pmatrix} w_1^k \\ w_2^k \\ \vdots \\ w_n^k \end{pmatrix}$$

der

$$w_i^k = \begin{cases} -2v_1^k + v_2^k + v_n^k & \text{hvis } i = 1 \\ v_{i-1}^k - 2v_i^k + v_{i+1}^k & \text{hvis } 1 < i < n \\ v_1^k + v_{n-1}^k - 2v_n^k & \text{hvis } i = n \end{cases}$$

For $1 < i < n$ får vi

$$w_i^k = \sqrt{2}\, \sin\left(\frac{2\pi(i-1)k}{n} + \frac{\pi}{4}\right) - 2\sqrt{2}\, \sin\left(\frac{2\pi ik}{n} + \frac{\pi}{4}\right)$$

$$+ \sqrt{2}\, \sin\left(\frac{2\pi(i+1)k}{n} + \frac{\pi}{4}\right)$$

$$= 2\sqrt{2}\left(\cos\left(\frac{2\pi k}{n}\right) - 1\right) \sin\left(\frac{2\pi ik}{n} + \frac{\pi}{4}\right)$$

$$= 2\left(\cos\left(\frac{2\pi k}{n}\right) - 1\right) v_i^k$$

dette stemmer også for:

$$w_1^k = 2\left(\cos\left(\frac{2\pi k}{n}\right) - 1\right) v_1^k$$

og

$$w_n^k = 2\left(\cos\left(\frac{2\pi k}{n}\right) - 1\right) v_n^k$$

og $\vec{v}^k$ er dermed en egenvektor til A_n

med egenverdi $2\left(\cos\left(\frac{2\pi k}{n}\right) - 1\right)$

c)

Spektralteoremet for symmetriske matriser sier at dersom A er en symmetrisk matrise, finnes det en ortonormal basis i $\mathbb{R}^n$ som består av egenvektorene til A

d)

3.

$$f(x,y,z) = \log(x) + \log(y) + 3\log(z),$$

$$x>0,\ y>0,\ z>0,\ x^2+y^2+z^2=5r^2$$

$$\begin{pmatrix} \frac{1}{x} \\ \frac{1}{y} \\ \frac{3}{z} \end{pmatrix} \qquad \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix}$$

$$\frac{1}{x} = \lambda 2x, \quad \frac{1}{y} = \lambda 2y, \quad \frac{3}{z} = \lambda 2z$$

$$\frac{\frac{1}{x}}{\frac{1}{y}} = \frac{2\lambda x}{2\lambda y} \Rightarrow \frac{y}{x} = \frac{x}{y} \Rightarrow x^2 = y^2 \Rightarrow x = y$$

$$\frac{\frac{3}{z}}{\frac{1}{y}} = \frac{2\lambda z}{2\lambda y} \Rightarrow \frac{3y}{z} = \frac{z}{y} \Rightarrow 3y^2 = 3z \Rightarrow z = \sqrt{3}y$$

$$x^2+y^2+z^2 = 5r^2$$

$$\Downarrow$$

$$2y^2 + (\sqrt{3}y)^2 = 5r^2 \Rightarrow 5y^2 = 5r^2 \Rightarrow y = r$$

Maksimum er punktet:

$$\underline{(r,\ r,\ \sqrt{3}r)}$$

Vis ulikheten:

Har:

$$\log(x) + \log(y) + 3\log(z) \leq 2\log(r) + 3\log(\sqrt{3}r)$$

$$e^{\log(x)+\log(y)+3\log(z)} \leq e^{2\log(r)+3\log(\sqrt{3}r)}$$

$$xyz^3 \leq r^2(\sqrt{3}r)^3$$

$$\leq \sqrt{27}\,r^5$$

$$xyz^3 \leq \sqrt{27}\left(\frac{x^2+y^2+z^2}{5}\right)^{\frac{5}{2}}$$

Siden vi har betingelsene $x, y, z > 0$, kan vi opphøye alt i 2

$$x^2y^2z^6 \leq 27\left(\frac{x^2+y^2+z^2}{5}\right)^5$$

Setter $x = \sqrt{a}$, $y = \sqrt{b}$, $z = \sqrt{c}$, og får:

$$\underline{\underline{abc^3 \leq 27\left(\frac{a+b+c}{5}\right)^5}}$$

4.

a) $f(\vec{x}_n) = \begin{pmatrix} x^2 - y^2 + \alpha \\ 2xy + \beta \end{pmatrix} \Rightarrow \begin{array}{l} x = x^2 - y^2 + \alpha \\ y = 2xy + \beta \end{array}$

$$0 = x^2 - x - y^2 + \alpha \Rightarrow 0 = x(x-1) - y^2 + \alpha$$

$$0 = 2xy - y + \beta \Rightarrow 0 = (2x-1)y + \beta$$

$$\text{I} \quad 0 = x(x-1) - \left(\frac{\beta}{1-2x}\right)^2 + \alpha$$

$$\text{II} \quad y = \frac{-\beta}{(2x-1)} = \frac{\beta}{1-2x}$$

$$\text{I} \quad x(x-1)(1^2 - 4x^2) = \beta^2 - \alpha(1-4x^2)$$

Osv.

⋮

Røtter: $x = -\frac{1}{2}$, $x = 0$, $x = \frac{1}{2}$, $x = 1$

Løsning blir da:

$$\vec{x}^{\pm} = \begin{pmatrix} x^{\pm} \\ y^{\pm} \end{pmatrix}, \quad x_- < \frac{1}{2} < x_+, \quad y^{\pm} = \frac{\beta}{1-2x^{\pm}}$$

b)

Vis at $|\vec{x}_{n+1}| \geq |\vec{x}_n|^2 - |c|$, der $c = (\alpha, \beta)$

$$x_{n+1} = f(\vec{x}_n) = \begin{pmatrix} \vec{x}_n - \vec{y}_n \\ 2x_n y_n \end{pmatrix} + \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$\left| \begin{pmatrix} x_n^2 - y_n^2 + \alpha \\ 2x_n y_n + \beta \end{pmatrix} \right| \geq \left| \begin{pmatrix} x_n \\ y_n \end{pmatrix} \right|^2 - \left| \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \right|$$

Trekantulikheten:

$$\left| \begin{pmatrix} x_n^2 - y_n^2 \\ 2x_n y_n \end{pmatrix} \right| + \left| \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \right| \geq \left| \begin{pmatrix} x_n \\ y_n \end{pmatrix} \right|^2 - \left| \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \right|$$

$$\sqrt{(x_n^2 - y_n^2)^2 + (2x_n y_n)^2} + \sqrt{\alpha^2 + \beta^2} \geq x_n^2 + y_n^2 - \sqrt{\alpha^2 + \beta}$$

$$\Downarrow$$

$$\underline{\underline{x_n^2 + y_n^2 + \sqrt{\alpha^2 + \beta^2} \geq x_n^2 + y_n^2 - \sqrt{\alpha^2 + \beta^2}}}$$