Robust Tube MPC Using Gain-Scheduled Policies for a Class of LPV Systems

James Fleming Qusay Hawari Mark Cannon

63rd IEEE Conference on Decision and Control, Milan, 2024
(Also accepted to L-CSS in joint submission)

Loughborough University
UNIVERSITY OF OXFORD

UKRI Engineering and Physical Sciences Research Council

Intro and Outline

An online QP-based robust MPC method for “LPV-A” systems like

$$x_{t+1} = A(\theta_t)x_t + Bu_t$$

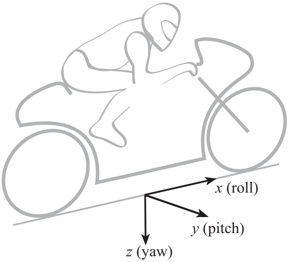
where θ_t is time-varying but measurable at time t, and may have rate bounds.

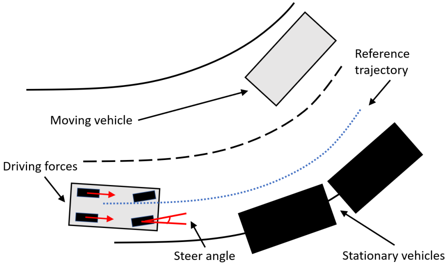
Unlike previous online MPC algorithms for this, we optimise over policies $u_t = Kx_t + c_t(\theta_t)$ that are affine functions of θ_t, which reduces conservatism.

Outline:

Motivation - road vehicles

- The lateral dynamics of road vehicles depend strongly on forward speed, which leads naturally to a LPV or quasi-LPV system.
- Applications: motorcycle stability assistance, steering control of autonomous vehicles.
- Ongoing EPSRC research grant in the UK (EP/X015459/1: "Learning of safety critical model predictive controllers for autonomous systems")





Problem formulation

Consider an LPV-A system with parameter $\theta_t \in \mathbb{R}^q$ available at time t

$$x_{t+1} = A(\theta_t)x_t + Bu_t, \quad A(\theta_t) = A^{(0)} + \sum_{i=1}^{q} \Delta_A^{(i)} \theta_{t,i}, \tag{1}$$

where $\theta_t \in \Theta$ (a polytope). Minimise a worst-case quadratic function:

$$J_0 = \max_{\theta_0, \theta_1, \ldots} \sum_{t=0}^{\infty} \left(x_t^T Q x_t + u_t^T R u_t \right)$$

subject to the constraints $Fx_t + Gu_t \leq \underline{1}$.

Assumptions:

- The system is quadratically stabilisable by some $u_t = K(\theta_t)x_t$
- For some $\theta \in \Theta$ the pair $(Q^{1/2}, A(\theta))$ is observable.

Parameter-dependent control policy

The MPC optimises over a control policy $u_{k|t}(\theta_k) = K(\theta_k)x_{k|t} + c_{k|t}(\theta_k)$:

$$u_{k|t}(\theta_k) = K(\theta_k)x_{k|t} + \underbrace{c_{k|t}^{(0)} + \sum_{i=1}^{q} c_{k|t}^{(i)}\theta_{k,i}}_{c_{k|t}(\theta_k)} \tag{2}$$

where $c_{k|t}^{(0)}$ and $c_{k|t}^{(i)}$ are nonzero for $k < t + N$ and determined online via a QP.

- This is an ‘affine-in-the-parameter’ control policy.
- Conceptually similar to ’affine-in-the-disturbance’: planned control action depends on parameters that have not been observed yet.

By substituting in (1), (2) can write dynamics, constraint in $x_{k|t}$ and $c_{k|t}$:

$$x_{k+1|t} = \bar{A}(\theta_{k|t})x_{k|t} + Bc_{k|t}(\theta_{k|t})$$

$$\bar{F}(\theta_k)x_t + Gc_{k|t}(\theta_k) \leq \underline{1}$$

where $\bar{A}(\theta) = A(\theta) + BK(\theta)$ and $\bar{F}(\theta) = F + BK(\theta)$.

Autonomous LPV formulation of predictions

Concatenate $c_{k|t}^{(i)}$ into a vector $d_{k|t}$ that specifies policy at time k, then:

Concatenate $d_{k|t}$ into a vector $\underline{d}_t$ that specifies policy over $k = t \dots t + N$:

$$d_{k|t} = \begin{bmatrix} c_{k|t}^{(0)} \\ c_{k|t}^{(1)} \\ \dots \\ c_{k|t}^{(q)} \end{bmatrix}, \quad \underline{d}_t = \begin{bmatrix} d_{t|t} \\ d_{t+1|t} \\ \vdots \\ d_{t+N-1|t} \end{bmatrix}$$

It is possible to write the prediction dynamics in the equivalent form

$$\begin{bmatrix} x_{k+1|t} \\ \underline{d}_{k+1|t} \end{bmatrix} = \begin{bmatrix} \bar{A}(\theta_{k|t}) & BS(\theta_{k|t}) \\ 0 & T \end{bmatrix} \begin{bmatrix} x_{k|t} \\ \underline{d}_{k|t} \end{bmatrix} \tag{3}$$

where $S(\theta)$ 'selects' the required combination of $c_{k|t}^{(i)}$ values and T 'shifts' up the components of $\underline{d}_{k|t}$ (shown in paper).

Upper bound on cost

This autonomous formulation as an LPV system allows us to bound the cost using a set of Lyapunov-like LMIs. If there is a $W \succeq 0$ such that:

$$W - \begin{bmatrix} \bar{A}(\theta) & BS(\theta) \\ 0 & T \end{bmatrix}^T W \overbrace{\begin{bmatrix} \bar{A}(\theta) & BS(\theta) \\ 0 & T \end{bmatrix}}^{autonomous\ \ LPV\ \ prediction\ \ dynamics}$$
$$\succeq \begin{bmatrix} Q + (K(\theta))^T RK(\theta) & 0 \\ 0 & (S(\theta))^T RS(\theta) \end{bmatrix} \qquad (4)$$

for all $\theta \in \Theta$. Then the cost is upper bounded as

$$\max_{\theta_0, \theta_1, \ldots} \sum_{k=t}^{\infty} \left(x_{k|t}^T Q x_{k|t} + u_{k|t}^T R u_{k|t} \right) \leq \begin{bmatrix} x_t \\ \underline{d}_t \end{bmatrix}^T W \begin{bmatrix} x_t \\ \underline{d}_t \end{bmatrix}$$

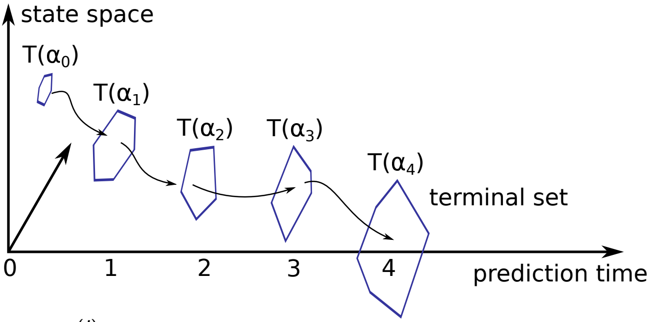
Outline of proof: Pre- and post- multiply (4) by $[x_t \ \underline{d}_t]^T$ and its transpose, then sum over $k = t, ..., \infty$.

Tube MPC

To apply constraints, bound the $x_{k|t}$ within polyhedra ("tube cross-sections"):

$$\mathcal{T}(\alpha_{k|t}) = \{x \in \mathbb{R}^n \mid Vx \leq g(\alpha_{k|t})\}$$

where $\alpha_{k|t}$ are parameters of the "tube", and $g(\alpha)$ is an affine function.



Optimize over $c_{k|t}^{(i)}$ AND $\alpha_{k|t}$.

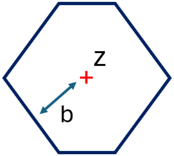
Possible tube parameterisations

From $\mathcal{T}(\alpha_{k|t}) = \{x \in \mathbb{R}^n \mid Vx \leq g(\alpha_{k|t})\}$ the tube cross-sections have facet normals in fixed directions, but several parameterisations are possible.

Homothetic:

$$\mathcal{T}(z_{k|t}, b_{k|t}) = \{x \in \mathbb{R}^n \mid Vx \leq Vz_{k|t} + b_{k|t}\underline{1}\}$$

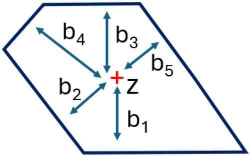
(translate and scale a polyhedra around the state space, low complexity, few variables required)



Different distances to each facet:

$$\mathcal{T}(z_{k|t}, \underline{b}_{k|t}) = \{x \in \mathbb{R}^n \mid Vx \leq Vz_{k|t} + \underline{b}_{k|t}\}$$

(many more variables, but also less conservatism in resulting MPC)

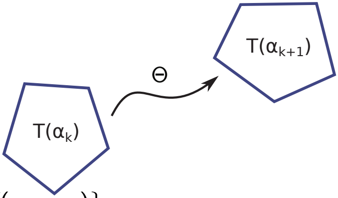


Applying constraints to tube

Need to ensure that the tube contains the state, and satisfies constraints:

To contain all predicted states:

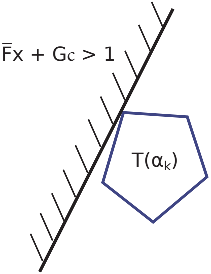
$$x_{0|t} \in \mathcal{T}(\alpha_{0|t})$$
$$x_{k|t} \in \mathcal{T}(\alpha_{k|t}) \Rightarrow x_{k+1|t} \in \mathcal{T}(\alpha_{k+1|t})$$



i.e. must be subset of the preimage:

$$\mathcal{T}(\alpha_{k|t}) \subseteq \{x \in \mathbb{R}^n \mid \bar{A}(\theta_k)x_t + Bc_{k|t}(\theta_k) \in \mathcal{T}(\alpha_{k+1|t})\}$$

To satisfy constraints:



$$x_{k|t} \in \mathcal{T}(\alpha_{k|t}) \Rightarrow x_{k+1|t} \in \mathcal{T}(\alpha_{k+1|t})$$

i.e. must be subset of the feasible set:

$$\mathcal{T}(\alpha_{k|t}) \subseteq \{x \in \mathbb{R}^n \mid \bar{F}(\theta_k)x_t + Gc_{k|t}(\theta_k) \leq \underline{1}\}$$

From subset conditions to linear constraints

We give a lemma to reformulate these subset conditions as linear constraints:

Lemma 1 (subsets of affinely-parameterised sets)

Let $\mathcal{P}_1(\beta_1) = \{x : V_1 x \leq g_1(\beta_1)\}$ and $\mathcal{P}_2(\beta_2) = \{x : V_2 x \leq g_2(\beta_2)\}$

and, for any particular value of β_1^*, let H^* be the matrix with rows h_i^T:

$$h_i^T = \underset{h \geq 0,\ h^T V_1 = v_{2,i}^T}{\arg\min} h^T g_1(\beta_1^*)$$

Then $\mathcal{P}_1(\beta_1) \subseteq \mathcal{P}_2(\beta_2)$ if the inequalities $H^* g_1(\beta_1) \leq g_2(\beta_2)$ hold.

Outline of proof: Consider the LP $\max_{x \in \mathcal{P}_1(\beta_1)} \ v_{2,i}^T x$ which has the dual:

$$\min_{h \geq 0,\ h^T V_1 = v_{2,i}^T} h^T g_1(\beta_1)$$

The rows of H^* correspond to feasible (suboptimal) points for this dual, so if $H^* g_1(\beta_1) \leq g_2(\beta_2)$ then $\max_{x \in \mathcal{P}_1(\beta_1)} \ v_{2,i}^T x \leq g_{2,i}(\beta_2)$ for all rows $v_{2,i}^T$ of V_2.

Tube constraints

We wished for the tube cross-sections to satisfy the subset conditions:

$$\{x \in \mathbb{R}^n \mid Vx \le g(\alpha_{k|t})\} \subseteq \{x \in \mathbb{R}^n \mid V(\bar{A}_{k|t}(\theta_k)x + Bc_{k|t}(\theta_k)) \le g(\alpha_{k+1|t})\}$$

$$\{x \in \mathbb{R}^n \mid Vx \le g(\alpha_{k|t})\} \subseteq \{x \in \mathbb{R}^n \mid \bar{F}(\theta_k)x + Gc_{k|t}(\theta_k) \le \underline{1}\}.$$

By applying Lemma 1 we can obtain linear constraints in the form:

$$H_p(\theta_k)g(\alpha_{k|t}) + VBc_{k|t}(\theta_k) \le g(\alpha_{k+1|t})$$

$$H_f(\theta_k)g(\alpha_{k|t}) + Gc_{k|t}(\theta_k) \le \underline{1},$$

We can also obtain conditions for the terminal tube cross-section by putting $\alpha_{k|t} = \alpha_{t+N|t}$ and $c_{k|t}(\theta_k) = 0$ for all $k \ge t + N$.

(this forces the final tube cross-section to be a positively invariant set satisfying the constraints)

MPC Algorithm

Putting everything together, the online MPC optimisation is a QP:

$$
\begin{aligned}
\min_{x,\alpha_{k|t},c^{(i)}_{k|t}} \quad & \begin{bmatrix} x_t \\ \underline{d}_t \end{bmatrix}^T W \begin{bmatrix} x_t \\ \underline{d}_t \end{bmatrix} \\
\text{subject to} \quad & Vx_t \leq g(\alpha_t), \\
& H_p(\theta_k) g(\alpha_{k|t}) + VBc_{k|t}(\theta_k) \leq g(\alpha_{k+1|t}), \\
& H_f(\theta_k) g(\alpha_{k|t}) + Gc_{k|t}(\theta_k) \leq \underline{1}, \\
& H_p(\theta_N) g(\alpha_{N|t}) \leq g(\alpha_{N|t}), \\
& H_f(\theta_N) g(\alpha_{N|t}) \leq \underline{1}, \\
& \textit{for all } \theta_k \in \Theta \\
& \textit{for all } k = t, \ldots, t\text{+}N\text{-}1
\end{aligned}
$$

If desired, can apply rate bounds via pre-processing of Θ_k (details in paper).

Closed-loop properties of algorithm

Theorem 1 - Closed-loop recursive feasibility If a solution to the optimisation problem exists at time t and the system evolves according to the dynamics (1), then a solution also exists at time $t+1$.

Outline of proof: By construction: if a solution existed at time $t-1$, can use same solution at t appended with $c_{t+N|t}(\theta) = 0$ and $\alpha_{t+N|t} = \alpha_{t+N-1|t-1}$

Theorem 2 - Exponential stability The state x_t and the control parameters $\underline{d}_t$ converge exponentially to zero in closed-loop:

$$\begin{bmatrix} x_t \\ \underline{d}_t \end{bmatrix}^T W \begin{bmatrix} x_t \\ \underline{d}_t \end{bmatrix} \leq \varepsilon^{\frac{t}{n}-1} \begin{bmatrix} x_0 \\ \underline{d}_0 \end{bmatrix}^T W \begin{bmatrix} x_0 \\ \underline{d}_0 \end{bmatrix}, \quad 0 \leq \varepsilon < 1$$

Outline of proof: We can show that the upper bound on the cost is a Lyapunov function, which is guaranteed to decrease over time due to the observability assumption on $(Q^{1/2}, A(\theta))$.

Simulation examples

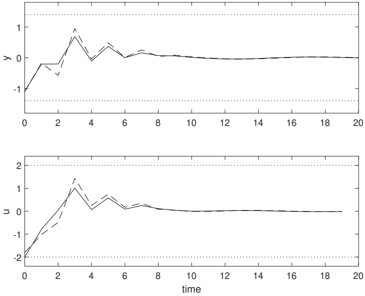
- Tested using two example systems from the existing literature.
- Implemented using Yalmip and the OSQP solver.
- Compared with LMI-based MPC methods and QP-based tube MPC without the parameter-dependant control policy.

Example systems were taken from:

[1] Y. Lu and Y. Arkun, "Quasi-min-max MPC algorithms for LPV systems," *Automatica*, vol. 36, no. 4, pp. 527–540, 2000.

[2] J. Fleming, B. Kouvaritakis, and M. Cannon, "Robust tube MPC for linear systems with multiplicative uncertainty," *IEEE Trans. Autom. Control*, vol. 60, no. 4, pp. 1087–1092, Apr. 2015.

Example 1 - Simulation Results



Dashed: Quasi-min-max MPC [1], Solid: LPV Tube MPC (N=5) with scheduled policy. Constraint boundaries are also shown as dotted lines.

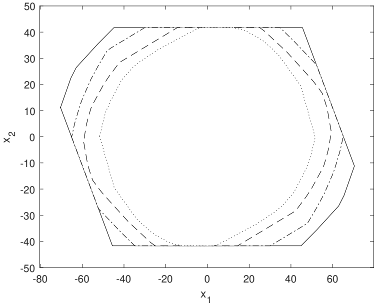
MPC Algorithm	N=5 (Alg. 1)	N=10 (Alg. 1)	Quasi-min-max [1]
Solver	OSQP	OSQP	MOSEK
Closed-loop cost	11.2	11.1	14.8
Computational time /ms	5.41	10.43	7.81

Comparison of cost and computation time for Example 1

Example 2 - Simulation Results

MPC Algorithm	Robust (Alg.1)	LPV (Alg.1)	+ rate bound	+ scheduling	Robust [2]
ROA volume	5863	7731	8979	9824	6198
Closed-loop cost	4530	4522	3045	3002	4546
Computation time /ms	3.08	3.16	3.19	4.37	5.70

Comparison of cost and computation time for Example 2



Dotted: Robust Tube MPC. Dashed: LPV Tube MPC with measurement of θ_t.
Dotdash: Tube MPC with rate bounds. Solid: Tube MPC with scheduled policy.

Conclusion and future work

Summary:

- We presented a Tube MPC for LPV systems using a parameter-dependent control policy,
- Restricted to LPV systems where parameter affects only the A matrix,
- Gives improvements in closed-loop cost and region of attraction compared to some existing methods,

Current/future work:

- Implementation into a MATLAB library (to be shared on github),
- Application on a real-world motorcycle stabilisation problem,
- Further improvements to cost bound (currently only the tube constraints are relaxed based on parameter rate bounds).