Solutions Notebook for
The Real Numbers and Real Analysis
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22 March, 2020 – April 6, 2020

Contents

Chapter 1

Construction of the Real Numbers

Chapter 2

Properties of the Real Numbers

Chapter 3

Limits and Continuity

3.3 Continuity

Theorem 3.3.5. Let $A \subseteq \mathbb{R}$ be a non-empty set, let $c \in A$, let $f, g : A \to \mathbb{R}$ be functions and let $k \in \mathbb{R}$. Suppose that f and g are continuous at c.

1. $f + g$ is continuous at c.

2. $f - g$ is continuous at c.

3. kf is continuous at c.

4. fg is continuous at c.

5. If $g(c) \neq 0$, then $\frac{f}{g}$ is continuous at c.

3.4 Uniform Continuity

Definition 3.4.1. Let $A \subseteq \mathbb{R}$ be a set, and let $f : A \to \mathbb{R}$ be a function. The function f is **uniformly continuous** if for each $\epsilon > 0$, there is some $\delta > 0$ such that $x, y \in A$ and $|x - y| < \delta$ imply $|f(x) - f(y)| < \epsilon$.

Lemma 3.4.2. Let $A \subseteq \mathbb{R}$ be a set, and let $f : A \to \mathbb{R}$ be a function. If f is uniformly continuous, then f is continuous.

Theorem 3.4.4. Let $C \subseteq \mathbb{R}$ be a closed bounded interval, and let $f : C \to \mathbb{R}$ be a function. If f is continuous, then f is uniformly continuous.

3.5 Two Important Theorems

Theorem 3.5.1 (Extreme Value Theorem). Let $C \subseteq \mathbb{R}$ be a closed bounded interval, and let $f : C \to \mathbb{R}$ be a function. Suppose that f is continuous. Then there are $x_{\text{min}}, x_{\text{max}} \in C$ such that

$$f(x_{\text{min}}) \leq f(x) \leq f(x_{\text{max}})$$

for all $x \in C$.

Theorem 3.5.2 (Intermediate Value Theorem). Let $[a, b] \subseteq \mathbb{R}$ be a closed bounded interval, and let $f : [a, b] \to \mathbb{R}$ be a function. Suppose that f is continuous. Let $r \in \mathbb{R}$. If r is strictly between $f(a)$ and $f(b)$, then there is some $c \in (a, b)$ such that $f(c) = r$.

3.5.1 Exercises

Exercise 3.5.4. Let $[a, b] \subseteq \mathbb{R}$ be a closed bounded interval, and let $f : [a, b] \to [a, b]$ be a function. Suppose that f is continuous. Prove that there is some $c \in [a, b]$ such that $f(c) = c$. The number c is called a **fixed point** of f.

Proposition (Existence of a Fixed Point in a Compact Interval of $\mathbb{R}$). Let $[a, b] \subseteq \mathbb{R}$ be a closed bounded interval, and let $f : [a, b] \to [a, b]$ be a continuous function. There exists a **fixed point** $c \in [a, b]$ such that $f(c) = c$.

Proof. The function f is an endomorphism, as its codomain is exactly its domain, and thus the image of f is a subset of $[a, b]$. We can infer from this that $f(a) \geq a$ and $f(b) \leq b$.

We now define a new function, $g : [a, b] \to \mathbb{R}$, where $g(x) = f(x) - x$ for all $x \in [a, b]$.[1] We began by supposing f is continuous, and we know x is continuous from a to b, since $[a, b]$ is a closed bounded interval. Therefore, by Theorem 3.3.5, g is continuous.

Using our inference that $f(a) \geq a$ and $f(b) \leq b$ we deduce that $g(a) \geq 0$ and $g(b) \leq 0$.

$$\begin{aligned} f(a) &\geq a & f(b) &\leq b \\ f(a) - a &\geq a - a & f(b) - b &\leq b - b \\ f(a) - a &\geq 0 & f(b) - b &\leq 0 \\ g(a) &\geq 0 & g(b) &\leq 0 \end{aligned}$$

By the Intermediate Value Theorem (Theorem 3.5.2), there exists some $\xi \in [a, b]$ such that $g(\xi) = 0$. By the definition of g, this means that $f(\xi) = \xi$, completing the proof. $\square$

Chapter 4

Differentiation

4.2 The Derivative

Definition 4.2.1. Let $I \subseteq \mathbb{R}$ be an open interval, let $c \in I$, and let $f : I \to \mathbb{R}$ be a function.

1. The function f is **differentiable** at c if
$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} \tag{4.1}$$
exists; if this limit exists, it is called the **derivative** of f at c, and it is denoted $f'(c)$.

2. The function f is **differentiable** if it is differentiable at every number in I. If f is differentiable, the **derivative** of f is the function $f' : I \to \mathbb{R}$ whose value at x is $f'(x)$ for all $x \in I$.

Lemma 4.2.2. Let $I \subseteq \mathbb{R}$ be an open interval, let $c \in I$, and let $f : I \to \mathbb{R}$ be a function. Then f is differentiable at c if and only if
$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h} \tag{4.2}$$
exists, and if this limit exists it equals $f'(c)$.

4.2.1 Exercises

Exercise 4.2.1. Using only the definition of derivatives and Lemma 4.2.2, find the derivative of each of the following functions.

1. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 3x - 8$ for all $x \in \mathbb{R}$.
2. Let $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = x^3$ for all $x \in \mathbb{R}$.

Chapter 5

Integration

Chapter 6

Limits to Infinity

Chapter 7

Transcendental Functions

7.2 Logarithmic and Exponential Functions

Definition 7.2.1. The **natural logarithm** function is the function $\ln : (0, \infty) \to \mathbb{R}$ defined by

$$\ln x = \int_1^x \frac{1}{t}\, dt \tag{7.1}$$

for all $x \in (0, \infty)$.

Theorem 7.2.2.

1. The function ln is differentiable, and $\frac{d}{dx}(\ln) = \frac{1}{x}$ for all $x \in (0, \infty)$.

2. The function ln is strictly increasing.

7.2.1 Exercises

Exercise 7.2.1. Prove that $\ln 2 > 0$.

Proof. By the Fundamental Theorem of Calculus, the derivative of $\ln x$ is $\frac{1}{x}$, which is positive for all values in the interval $(0, \infty)$. Since its first derivative is positive for all values in $(0, \infty)$, the function $\ln x$ is strictly increasing. By the Archimedean property of the real numbers, there exists at least one real number c such that $1 < c < 2$, and we can therefore conclude that $\ln 1 < \ln c < \ln 2$. Since $0 = \ln 1$, we have that $0 < \ln 2$, as required. $\square$

Exercise 7.2.7. Prove that $\exp x \geq 1 + x$ for all $x \in (0, \infty)$.

Proposition. If $\ln^{-1}(x) = \exp x$, then $\exp x \geq 1 + x$ for all $x \in (0, \infty)$.

Proof. While the value of $\exp x$ when x equals zero is outside the domain we are given, 0 is the greatest lower bound of our interval. By Theorem 7.2.2, $\exp x$ is strictly increasing, and therefore $\exp x \geq \exp 0$ for all $x \in (0, \infty)$.

By Definition 7.2.1, $\ln 1 = 0$, and therefore $\exp 0 = 1$. We now differentiate both sides of the inequality to determine which expression increases more with respect to x. Since both expressions are equal when x equals zero, the derivative is the sole factor in determining whether the inequality is true.

$$\frac{d}{dx}(\exp x) \geq \frac{d}{dx}(1 + x)$$
$$\exp x \geq 1$$

Once again, since we know that $\exp 0 = 1$ and $\exp x$ is strictly increasing, we can conclude that $\exp x \geq 1 + x$ for all $x \in (0, \infty)$. $\square$

Chapter 8

Sequences

Chapter 9

Series

Chapter 10

Sequences and Series of Functions

Appendix A

Functions

Definition A.0.1. [2] Let A and B be sets. A **function** (also called a **map**) f from A to B, denoted $f : A \to B$, is a subset $F \subseteq A \times B$ such that for each $a \in A$, there is one and only one pair in F of the form (a, b). The set A is called the **domain** of f and the set B is called the **codomain** of f.

Bibliography

[1] K. G. Binmore. *Mathematical Analysis*. 2nd ed. Cambridge University Press, 1983.

[2] Ethan D. Bloch. *Proofs and Fundamentals*. 2nd ed. Undergraduate Texts in Mathematics. Springer, 2011.

Index