# Solutions Notebook for Advanced Calculus by James J. Callahan

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# Chapter 1

# Starting Points

## 1.1 Substitution

## 1.2 Work and path integrals

**Definition 1.1.** The **work** done by the constant force $\mathbf{F} = (P, Q)$ in displacing an object along the segment $\Delta x = (\Delta x, \Delta y)$ is

$$W = \mathbf{F} \cdot \Delta x = P\Delta x + Q\Delta y = W_x + W_y.$$

**Definition 1.2.** A **smooth, simple, oriented curve** $\overrightarrow{C}$ in $\mathbb{R}^n$ is the image of a smooth 1–1 map,

$$\mathbf{x} \;:\; [a, b] \to \mathbb{R}^n \;:\; t \mapsto \mathbf{x}(t),$$

where $\mathbf{x}'(t) \neq 0$ for all $a < t < b$. The point $\mathbf{x}(a)$ is the **start** of $\overrightarrow{C}$ and $\mathbf{x}(b)$ is its **end**.

**Definition 1.3** (Smooth path integral)**.**

$$\int_{\overrightarrow{C}} \mathbf{F} \cdot d\mathbf{x} = \lim_{\substack{k \to \infty \\ \text{mesh} \to 0}} \sum_{i=1}^{k} \mathbf{F}(\mathbf{x}_i) \cdot \Delta \mathbf{x}_i,$$

if the limit exists when taken over all ordered partitions $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{k+1}$ of $\overrightarrow{C}$ with **mesh** $= \max_i ||\Delta \mathbf{x}_i||$ and $\Delta \mathbf{x}_i = \mathbf{x}_{i+1} - \mathbf{x}_i$, $i = 1, \ldots, k$.

**Definition 1.4.** The **work** done by the force $\mathbf{F}$ along the smooth oriented path $\overrightarrow{C}$ is

$$W = \int_{\overrightarrow{C}} \mathbf{F} \cdot \Delta \mathbf{x}. \tag{1.1}$$

**Theorem 1.1.** Suppose $\overrightarrow{C}$ is a smooth, simple, oriented curve in $\mathbb{R}^n$ that is parametrized by $\mathbf{x}(t)$, $a \le t \le b$. If $\mathbf{F}(\mathbf{x})$ is a continuous vector function defined on $\overrightarrow{C}$, then the integral of $\mathbf{F}$ over the path $\overrightarrow{C}$ exists, and

$$\int_{\overrightarrow{C}} \mathbf{F} \cdot \Delta\mathbf{x} = \int_a^b \mathbf{F}(\mathbf{x}(t)) \cdot \mathbf{x}'(t)\, dt. \tag{1.2}$$

## 1.3 Polar coordinates

## 1.4 Exercises

**Exercise 1.1.** Evaluate $\displaystyle\int_0^\infty \frac{dx}{1+x^2}$ and $\displaystyle\int_{-\infty}^1 \frac{dx}{1+x^2}$.

**Solution** (1)**.** Let $x = \tan u$.

$$\begin{aligned} x &= \tan u \\ &= \frac{\sin u}{\cos u} \end{aligned}$$

We differentiate $x$ to find the trigonometric substitution factor.

$$\begin{aligned} \frac{d}{du}(x) &= \frac{f'(u)g(u) - g'(u)f(u)}{[g(u)]^2} \\ \frac{dx}{du} &= \frac{\left[\left(\sin(u)'\right) \cdot (\cos u)\right] - \left[\left(\cos(u)'\right) \cdot (\sin u)\right]}{[\cos(u)]^2} \\ &= [\cos u \cdot \cos u] - [-\sin u \cdot \sin u] \end{aligned}$$