Solutions Notebook for Principles of Mathematical Analysis by Walter Rudin

Jose Fernando Lopez Fernandez

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Chapter 1

The Real and Complex Number Systems

Exercise 1.1. If r is rational and x is irrational, prove that $r+x$ and rx are irrational.

We will proceed by proving the irrationality of the product rx first, from which the irrationality of the sum $r+x$ will naturally follow.

Proposition. For any rational number r and irrational number x, the product rx is irrational.

Proof. Suppose to the contrary that the product rx is rational, implying the existence of some integers m and $n \neq 0$ such that $r = \frac{m}{n}$ and $rx = \frac{m}{n} \cdot x = \frac{mx}{n}$. If rx was rational, we could multiply by $\frac{1}{r} = \frac{n}{m}$, giving us $\frac{mx}{n} \cdot \frac{n}{m} = x$. Since we have simply multiplied two (allegedly) rational numbers and the rational numbers are a field and thus closed under multiplication, the result must be a rational number. Since the result is x, and we began by supposing x was irrational, we've arrived at our contradiction.
The product of a rational number r and an irrational number x yields an irrational number rx. $\square$

Proposition. For any rational number r and irrational number x, the sum $r+x$ is irrational.

Proof. Suppose again to the contrary that $r+x$ is rational. Then there exist some integers m and n such that $n \neq 0$ and $r = \frac{m}{n}$. By the standard rules of adding rational numbers, this implies the following.

$$\begin{aligned} r + x &= \frac{m}{n} + x \\ &= \frac{m}{n} + x \cdot \frac{n}{n} \\ &= \frac{m}{n} + \frac{xn}{n} \\ &= \frac{m + xn}{n} \end{aligned}$$

Since we've supposed $r+x$ to be a rational number and the field of rational numbers $\mathbb{Q}$ is by definition closed under arithmetic operations, we should be able to add, subtract, multiply, or divide $r+x$ by any non-zero rational number and still get a rational number as a result. To verify this, we begin with the expression $\frac{(r+x)n-m}{n}$.

$$\begin{aligned} \frac{(r + x)\, n - m}{n} &= \frac{\left(\frac{m+nx}{n}\right) n - m}{n} \\ &= \frac{m + nx - m}{n} \\ &= \frac{nx}{n} \\ &= x \end{aligned}$$

Since the above expression simplifies to x, an irrational number, the supposition that $r + x$ is rational would imply that the field of rational numbers $\mathbb{Q}$ is not closed under its addition and multiplication operations. This is false, by the definition of a field, and this contradiction leads us to conclude that for any rational number r and irrational number x, the sum $r + x$ is irrational. $\square$

Chapter 2

Basic Topology

Exercise 2.1. Prove that the empty set is a subset of every set.

Proposition. The empty set, denoted by $\emptyset$, is a subset of every set.

Proof. Let A be an arbitrary set. In order to prove that the empty set is a subset of A, we must show that for any element x in $\emptyset$, x is also an element of A. Since –by definition– the empty set contains no elements, this statement is vacuously true and thus not very interesting.

Consider instead the contrapositive: for any element x, if x is not an element of A, then x is not an element of $\emptyset$. Again, since $\emptyset$ contains no elements, this statement is true for all elements $x \notin A$, as required. □

Exercise 2.2. A complex number z is said to be *algebraic* if there are integers $a_0, \ldots, a_n$, not all zero, such that

$$a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = 0.$$

Prove that the set of all algebraic numbers is countable. *Hint:* For every positive integer N there are only finitely many equations with

$$n + |a_0| + |a_1| + \cdots + |a_n| = N.$$

Proposition. The set of all algebraic numbers is countable.

Proof. content... □

Chapter 3

Numerical Sequences and Series