Solutions Notebook for Principles of Mathematical Analysis by Walter Rudin

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Contents

Chapter 1

The Real and Complex Number Systems

1.1 Exercises

Exercise 1.1. If r is rational and x is irrational, prove that $r+x$ and rx are irrational.

We will proceed by proving the irrationality of the product rx first, from which the irrationality of the sum $r+x$ will naturally follow.

Proposition. For any rational number r and irrational number x, the product rx is irrational.

Proof. Suppose to the contrary that the product rx is rational, implying the existence of some integers m and $n \neq 0$ such that $r = \frac{m}{n}$ and $rx = \frac{m}{n} \cdot x = \frac{mx}{n}$. If rx was rational, we could multiply by $\frac{1}{r} = \frac{n}{m}$, giving us $\frac{mx}{n} \cdot \frac{n}{m} = x$. Since we have simply multiplied two (allegedly) rational numbers and the rational numbers are a field and thus closed under multiplication, the result must be a rational number. Since the result is x, and we began by supposing x was irrational, we've arrived at our contradiction.
The product of a rational number r and an irrational number x yields an irrational number rx. $\square$

Proposition. For any rational number r and irrational number x, the sum $r+x$ is irrational.

Proof. Suppose again to the contrary that $r + x$ is rational. Then there exist some integers m and n such that $n \neq 0$ and $r = \frac{m}{n}$. By the standard rules of adding rational numbers, this implies the following.

$$\begin{aligned} r + x &= \frac{m}{n} + x \\ &= \frac{m}{n} + x \cdot \frac{n}{n} \\ &= \frac{m}{n} + \frac{xn}{n} \\ &= \frac{m + xn}{n} \end{aligned}$$

Since we've supposed $r+x$ to be a rational number and the field of rational numbers $\mathbb{Q}$ is by definition closed under arithmetic operations, we should be able to add, subtract, multiply, or divide $r+x$ by any non-zero rational number and still get a rational number as a result. To verify this, we begin with the expression $\frac{(r+x)n-m}{n}$.

$$\begin{aligned} \frac{(r + x)\, n - m}{n} &= \frac{\left(\frac{m+nx}{n}\right) n - m}{n} \\ &= \frac{m + nx - m}{n} \\ &= \frac{nx}{n} \\ &= x \end{aligned}$$

Since the above expression simplifies to x, an irrational number, the supposition that $r + x$ is rational would imply that the field of rational numbers $\mathbb{Q}$ is not closed under its addition and multiplication operations. This is false, by the definition of a field, and this contradiction leads us to conclude that for any rational number r and irrational number x, the sum $r + x$ is irrational. □

Exercise 1.2. Prove that there is no rational number whose square is 12.

Suppose there did exist some rational number r such that its square was 12. From the assumption that $r^2 = 12$, we deduce that $r = \sqrt{12}$, and therefore $r = \sqrt{4 \cdot 3} \equiv 2\sqrt{3}$. We will show that $\sqrt{3}$ is irrational, and by Proposition 1.1 the product $2\sqrt{3}$ is as well.

Proposition. The real number $\sqrt{3}$ is irrational.

Proof. Suppose to the contrary there exists some rational number r such that its square is 3. This implies the existence of some non-zero integers

m and n such that $r = \frac{m}{n}$ and $\frac{m^2}{n^2} = 3$. Since m and n are in reduced form, they cannot both be even.

We thus have $m^2 = 3n^2$. ... TODO: Finish proof ☐

Proposition. There is no rational number whose square is 12.

Exercise 1.18. If $k \geq 2$ and $\mathbf{x} \in \mathbb{R}^k$, prove that there exists $\mathbf{y} \in \mathbb{R}^k$ such that $\mathbf{y} \neq \mathbf{0}$ but $\mathbf{x} \cdot \mathbf{y} = 0$. Is this also true if $k = 1$?

Proposition. Let $k \geq 2$ and $\mathbf{x} \in \mathbb{R}^k$. There exists some $\mathbf{y} \in \mathbb{R}^k$ where $\mathbf{y} \neq \mathbf{0}$ such that $\mathbf{x} \cdot \mathbf{y} = 0$.

Proof. Let $\mathbf{x} = (1, 0)$ and $\mathbf{y} = (0, 1)$. By the definition of the inner product of two vectors,

$$\begin{aligned}
\mathbf{x} \cdot \mathbf{y} &= \sum_{i=1}^{2} x_i y_i \\
&= x_1 y_1 + x_2 y_2 \\
&= (1 \cdot 0) + (0 \cdot 1) \\
&= (0) + (0) \\
&= 0.
\end{aligned}$$

Therefore, there exists some $\mathbf{y} \in \mathbb{R}^k$ where $\mathbf{y} \neq \mathbf{0}$ such that $\mathbf{x} \cdot \mathbf{y} = 0$. ☐

This is not true for vector spaces of dimension one.

Chapter 2

Basic Topology

Theorem 2.12. Let $\{E_n\}$ for $n = 1, 2, 3, \ldots$ be a sequence of countable sets, and put

$$S = \bigcup_{n=1}^{\infty} E_n. \tag{2.1}$$

Then S is countable.

Corollary. Suppose A is at most countable, and, for every $\alpha \in A$, B_α is at most countable. Put

$$T = \bigcup_{\alpha \in A} B_\alpha.$$

Then T is at most countable.

Theorem 2.13. Let A be a countable set, and let B_n be the set of all n-tuples $(a_1, \ldots, a_n)$, where $a_k \in A$ for $(k = 1, \ldots, n)$, and the elements $a_1, \ldots, a_n$ need not be distinct. Then B_n is countable.

Theorem 2.19. Every neighborhood is an open set.

Theorem 2.20. If p is a limit point of a set E, then every neighborhood of p contains infinitely many points of E.

Theorem 2.23. A set E is open if and only if its complement is closed.

Corollary. A set F is closed if and only if its complement is open.

Theorem 2.28. Let E be a nonempty set of real numbers which is bounded above. Let $y = \sup E$. Then $y \in \overline{E}$. Hence $y \in E$ if E is closed.

Definition 2.31. By an **open cover** of a set E in a metric space X we mean a collection $\{G_\alpha\}$ of open subsets of X such that $E \subset \cup_\alpha G_\alpha$.

Definition 2.32. A subset K of a metric space X is said to be **compact** if every open cover of K contains a *finite* subcover.

Theorem 2.35. Closed subsets of compact sets are compact.

Theorem 2.41. If a set in $\mathbb{R}^k$ has one of the following three properties, then it has the other two:

(a) E is closed and bounded.

(b) E is compact.

(c) Every infinite subset of E has a limit point in E.

Theorem 2.47. A subset E of the real line $\mathbb{R}^1$ is connected if and only if it has the following property: if $x \in E$, $y \in E$, and $x < z < y$, then $z \in E$.

2.1 Exercises

Exercise 2.1. Prove that the empty set is a subset of every set.

Proposition. The empty set, denoted by $\emptyset$, is a subset of every set.

Proof. Let A be an arbitrary set. In order to prove that the empty set is a subset of A, we must show that for any element x in $\emptyset$, x is also an element of A. Since –by definition– the empty set contains no elements, this statement is vacuously true and thus not very interesting.

Consider instead the contrapositive: for any element x, if x is not an element of A, then x is not an element of $\emptyset$. Again, since $\emptyset$ contains no elements, this statement is true for all elements $x \notin A$, as required. □

Exercise 2.2. A complex number z is said to be *algebraic* if there are integers $a_0, \ldots, a_n$, not all zero, such that

$$a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = 0.$$

Prove that the set of all algebraic numbers is countable. *Hint:* For every positive integer N there are only finitely many equations with

$$n + |a_0| + |a_1| + \cdots + |a_n| = N.$$

Proposition. The set of all algebraic numbers is countable.

Exercise 2.3. Prove that there exist real numbers which are not algebraic.

Exercise 2.4. Is the set of all irrational real numbers countable?

Exercise 2.12. Let $K \subset \mathbb{R}^1$ consist of 0 and the numbers $\frac{1}{n}$, for $n = 1, 2, 3, \ldots$. Prove that K is compact directly from the definition (without using the Heine-Borel Theorem).

Proposition. Let $K \subset \mathbb{R}^1$ consist of 0 and the numbers $\frac{1}{n}$ for $n = 1, 2, 3, \ldots$. The space K is compact.

Proof. Any open cover must include the interval $a < 0 < b$. By the Archimedean property of $\mathbb{R}$, there are an infinite number of points $k \in K$ between 0 and b, irrespective of the value of b. The covering set for the rest of the points in K from b to 1 is therefore finite.

Since any cover of K contains some finite subcover, K is compact. □

Exercise 2.16. Regard $\mathbb{Q}$, the set of all rational numbers, as a metric space, with $d(p, q) = |p - q|$. Let E be the set of all $p \in \mathbb{Q}$ such that $2 < p^2 < 3$. Show that E is closed and bounded in $\mathbb{Q}$, but that E is not compact. Is E open in $\mathbb{Q}$?

Chapter 3

Numerical Sequences and Series

3.1 Exercises

Exercise 3.25. Let X be the metric space whose points are the rational numbers, with the metric $d(x, y) = |x - y|$. What is the completion of this space? (Compare Exercise 24.)

Solution. The completion of this space is the real numbers, within which the rational numbers are dense. In fact, one of the axiomatic constructions of the real numbers is precisely the completion of the rational numbers by the use of Cauchy sequences.

Chapter 4

Continuity

Theorem 4.15. If $\mathbf{f}$ is a continuous mapping of a compact metric space X into $\mathbb{R}^k$, then $\mathbf{f}(X)$ is closed and bounded. Thus, $\mathbf{f}$ is bounded.

Theorem 4.16. Suppose f is a continuous real function on a compact metric space X, and

$$M = \sup_{p \in X} f(p), \qquad m = \inf_{p \in X} f(p). \tag{4.1}$$

Then there exist points $p, q \in X$ such that $f(p) = M$ and $f(q) = m$.

Theorem 4.22. If f is a continuous mapping of a metric space X into a metric space Y, and if E is a connected subset of X, then $f(E)$ is connected.

Theorem 4.23. Let f be a continuous real function on the interval $[a, b]$. If $f(a) < f(b)$ and if c is a number such that $f(a) < c < f(b)$, then there exists a point $x \in (a, b)$ such that $f(x) = c$.

4.1 Exercises

Exercise 4.14. Let $I = [0, 1]$ be the closed unit interval. Suppose f is a continuous mapping of I into I. Prove that $f(x) = x$ for at least one $x \in I$.

Proposition. Let $I = [0, 1]$ be the closed unit interval and let $f : I \to I$ be a continuous function. There exists some $x \in I$ such that $f(x) = x$.

Proof. content... TODO: Finish proof $\square$

Chapter 5

Differentiation

5.1 Exercises

Exercise 5.1. Let f be defined for all real x, and suppose that

$$|f(x) - f(y)| \leq (x - y)^2$$

for all real x and y. Prove that f is constant.

Exercise 5.6. Suppose

(a) f is continuous for $x \geq 0$,

(b) $f'(x)$ exists for $x > 0$,

(c) $f(0) = 0$,

(d) f' is monotonically increasing.

Put

$$g(x) = \frac{f(x)}{x} \quad (x > 0)$$

and prove that g is monotonically increasing.

Exercise 5.9. Let f be a continuous real function on $\mathbb{R}^1$, of which it is known that $f'(x)$ exists for all $x \neq 0$ and that $f'(x) \to 3$ as $x \to 0$. Does it follow that $f'(0)$ exists?

Solution. If the left- and right-hand derivatives are both equal to 3 as x approaches zero, then $f'(0)$ exists and is equal to 3. However, the function

$g(x) = 3|x|$ is an example of a function whose derivative is defined for all $x \neq 0$ and whose right-hand derivative approaches 3 as x^+ approaches zero. However, the derivative of $g(x)$ at $x = 0$ does not exist.

Chapter 6

The Riemann-Stieltjes Integral

6.1 Exercises

Exercise 6.1. Suppose α increases on $[a,b]$, $a \leq x_0 \leq b$, α is continuous at x_0, $f(x_0) = 1$, and $f(x) = 0$ if $x \neq x_0$. Prove that $f \in \mathscr{R}(\alpha)$ and that $\int f d\alpha = 0$.

Chapter 7

Sequences and Series of Functions

7.1 Exercises

Exercise 7.1. Prove that every uniformly convergent sequence of bounded functions is uniformly bounded.

Exercise 7.2. If $\{f_n\}$ and $\{g_n\}$ converge uniformly on a set E, prove that $\{f_n + g_n\}$ converges uniformly on E. If, in addition, $\{f_n\}$ and $\{g_n\}$ are sequences of bounded functions, prove that $\{f_n g_n\}$ converges uniformly on E.

Chapter 8

Some Special Functions

8.1 Exercises

Exercise 8.7. If $0 < x < \frac{\pi}{2}$, prove that

$$\frac{2}{\pi} < \frac{\sin x}{x} < 1.$$

Exercise 8.30. Use Stirling's formula to prove that

$$\lim_{x \to \infty} \frac{\Gamma\left(x+c\right)}{x^c \Gamma\left(x\right)} = 1$$

for every real constant c.

Chapter 9

Functions of Several Variables

9.1 Exercises

Exercise 9.1. If S is a nonempty subset of a vector space X, prove (as asserted in Sec. 9.1) that the span of S is a vector space.

Exercise 9.2. Prove (as asserted in Sec. 9.6) that BA is linear if A and B are linear transformations. Prove also that A^{-1} is linear and invertible.

Exercise 9.3. Assume $A \in L\left(x, y\right)$ and $A\mathbf{x} = \mathbf{0}$ only when $\mathbf{x} = \mathbf{0}$. Prove that A is then $1 - 1$.

Exercise 9.4. Prove (as asserted in Sec. 9.30) that null spaces and ranges of linear transformations are vector spaces.

Chapter 10

Integration on Differential Forms

10.1 Exercises

Exercise 10.1. Let H be a compact convex set in $\mathbb{R}^k$, with nonempty interior. Let $f \in \mathscr{C}(H)$, put $f(\mathbf{x}) = 0$ in the complement of H, and define $\int_H f$ as in Definition 10.3.

Prove that $\int_H f$ is independent of the order in which the k integrations are carried out. *Hint:* Approximate f by functions that are continuous on $\mathbb{R}^k$ and whose supports are in H, as was done in Example 10.4.

Chapter 11

The Lebesgue Theory

11.1 Exercises

Exercise 11.1. If $f \geq 0$ and $\int_E f d\mu = 0$, prove that $f(x) = 0$ almost everywhere on E. *Hint:* Let E_n be the subset of E on which $f(x) > \frac{1}{n}$. Write $A = \bigcup E_n$. Then $\mu(A) = 0$ if and only if $\mu(E_n) = 0$ for every n.

Exercise 11.11. If $f, g \in \mathscr{L}(\mu)$ on X, define the distance between f and g by

$$\int_X |f - g| \, d\mu.$$

Prove that $\mathscr{L}(\mu)$ is a complete metric space.

Appendix A

Continuity

Definition A.1. [1] Let $A \subseteq \mathbb{R}$ be a set, and let $f : A \to \mathbb{R}$ be a function.

1. Let $c \in A$. The function f is **continuous at** c if for each $\epsilon > 0$, there is some $\delta > 0$ such that $x \in A$ and $|x-c| < \delta$ imply $|f(x) - f(c)| < \epsilon$. The function f is **discontinuous** at c if f is not continuous at c; in that case we also say that f has a **discontinuity at** c.

2. The function f is **continuous** if it is continuous at every number in A. The function f is **discontinuous** if it is not continuous.

Theorem A.2 (Intermediate Value Theorem)**.** [1] Let $[a, b] \subseteq \mathbb{R}$ be a closed bounded interval, and let $f : [a, b] \to \mathbb{R}$ be a function. Suppose that f is continuous. Let $r \in \mathbb{R}$. If r is strictly between $f(a)$ and $f(b)$, then there is some $c \in (a, b)$ such that $f(c) = r$.

Bibliography

[1] Ethan D. Bloch. *The Real Numbers and Real Analysis*. 1st ed. Springer, 2011. ISBN: 978-0-387-72176-7.

Glossary

compact A subset K of a metric space X is said to be *compact* if every open cover of K contains a *finite* subcover. 5

open cover An *open cover* of a set E in a metric space X is a collection $\{G_\alpha\}$ of open subsets of X such that $E \subset \cup_\alpha G_\alpha$. 5

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