Solutions Notebook for Principles of Mathematical Analysis by Walter Rudin

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Chapter 1

The Real and Complex Number Systems

Exercise 1.1. If r is rational and x is irrational, prove that $r + x$ and rx are irrational.

We will proceed by proving the irrationality of the product rx first, from which the irrationality of the sum $r + x$ will naturally follow.

Proposition. For any rational number r and irrational number x, the product rx is irrational.

Proof. Suppose to the contrary that the product rx is rational, implying the existence of some integers m and $n \neq 0$ such that $r = \frac{m}{n}$ and $rx = \frac{m}{n} \cdot x = \frac{mx}{n}$. If rx was rational, we could multiply by $\frac{1}{r} = \frac{n}{m}$, giving us $\frac{mx}{n} \cdot \frac{n}{m} = x$. Since we have simply multiplied two (allegedly) rational numbers and the rational numbers are a field and thus closed under multiplication, the result must be a rational number. Since the result is x, and we began by supposing x was irrational, we've arrived at our contradiction.
The product of a rational number r and an irrational number x yields an irrational number rx. $\square$

Proposition. For any rational number r and irrational number x, the sum $r + x$ is irrational.

Proof. Suppose again to the contrary that $r + x$ is rational. Then there exist some integers m and n such that $n \neq 0$ and $r = \frac{m}{n}$. By the standard rules of adding rational numbers, this implies the following.

$$\begin{aligned} r + x &= \frac{m}{n} + x \\ &= \frac{m}{n} + x \cdot \frac{n}{n} \\ &= \frac{m}{n} + \frac{xn}{n} \\ &= \frac{m + xn}{n} \end{aligned}$$

Since we've supposed $r+x$ to be a rational number and the field of rational numbers $\mathbb{Q}$ is by definition closed under arithmetic operations, we should be able to add, subtract, multiply, or divide $r+x$ by any non-zero rational number and still get a rational number as a result. To verify this, we begin with the expression $\frac{(r+x)n-m}{n}$.

$$\begin{aligned} \frac{(r+x)\,n - m}{n} &= \frac{\left(\frac{m+nx}{n}\right) n - m}{n} \\ &= \frac{m + nx - m}{n} \\ &= \frac{nx}{n} \\ &= x \end{aligned}$$

Since the above expression simplifies to x, an irrational number, the supposition that $r + x$ is rational would imply that the field of rational numbers $\mathbb{Q}$ is not closed under its addition and multiplication operations. This is false, by the definition of a field, and this contradiction leads us to conclude that for any rational number r and irrational number x, the sum $r + x$ is irrational. $\square$

Chapter 2

Basic Topology

Theorem 2.13. Let A be a countable set, and let B_n be the set of all n-tuples $(a_1, \ldots, a_n)$, where $a_k \in A$ for $(k = 1, \ldots, n)$, and the elements $a_1, \ldots, a_n$ need not be distinct. Then B_n is countable.

Exercise 2.1. Prove that the empty set is a subset of every set.

Proposition. The empty set, denoted by $\emptyset$, is a subset of every set.

Proof. Let A be an arbitrary set. In order to prove that the empty set is a subset of A, we must show that for any element x in $\emptyset$, x is also an element of A. Since –by definition– the empty set contains no elements, this statement is vacuously true and thus not very interesting.

Consider instead the contrapositive: for any element x, if x is not an element of A, then x is not an element of $\emptyset$. Again, since $\emptyset$ contains no elements, this statement is true for all elements $x \notin A$, as required. □

Exercise 2.2. A complex number z is said to be *algebraic* if there are integers $a_0, \ldots, a_n$, not all zero, such that

$$a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = 0.$$

Prove that the set of all algebraic numbers is countable. *Hint:* For every positive integer N there are only finitely many equations with

$$n + |a_0| + |a_1| + \cdots + |a_n| = N.$$

Proposition. The set of all algebraic numbers is countable.

Chapter 3

Numerical Sequences and Series

Exercise 3.25. Let X be the metric space whose points are the rational numbers, with the metric $d(x, y) = |x - y|$. What is the completion of this space? (Compare Exercise 24.)

Solution. The completion of this space is the real numbers, within which the rational numbers are dense. In fact, one of the axiomatic constructions of the real numbers is precisely the completion of the rational numbers by the use of Cauchy sequences.

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