Solutions Notebook for
Ordinary Differential Equations
With Applications
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Contents

Chapter 1

Basic Concepts

1.1 How Differential Equations Originate

Exercise 1.1.1. The radium in a piece of lead decomposes at a rate which is proportional to the amount present. If 10 percent of the radium decomposes in 200 years, what percent of the original amount of radium will be present in a piece of lead after 1000 years?

Solution. The differential equation relation the rate of decay to the amount of radium is first-order and linear.

$$\frac{dy}{dt} = -Ky$$

To solve for y, we separate the differentials and integrate both sides.

$$\begin{aligned}
\int \frac{1}{y}\ dy &= \int -K\ dt \\
\ln|y| &= -Kt + C \\
e^{\ln y} &= e^{-Kt+C} \\
y &= e^{-Kt+C} \\
y &= Ce^{-Kt}
\end{aligned}$$

Since 0% of the radium has decayed at time $t = 0$, we can assume $C = 1$.

$$y = e^{-Kt} \tag{1.1}$$

To find the value of the proportionality constant, we use the 10% decay over 200 years.

$$\begin{aligned}
0.9 &= e^{-200K} \\
\ln 0.9 &= \ln e^{-200K} \\
\ln 0.9 &= -200K \\
-\frac{\ln 0.9}{200} &= K
\end{aligned}$$

We now plug in this value for the proportionality constant into Equation 1.1 with $t = 1000$ to get our final answer.

$$\begin{aligned}
y(t) &= e^{-\left(\frac{-\ln 0.9}{200}\right)(1000)} \\
&= e^{5 \ln 0.9} \\
&= e^{\ln 0.9^5} \\
&= 0.9^5 \\
&= 0.59049 \\
&\approx 0.5905
\end{aligned}$$

Therefore, approximately 59.05% of the radium is still present in the piece of lead after 1000 years.

Exercise 1.1.2. Assume that the half-life of the radium in a piece of lead is 1600 years. How much radium will be lost in 100 years?

Solution. Again, the equation relating the decay of radium to the amount of radium present in the piece of lead is

$$\frac{dy}{dt} = -Ky.$$

We again separate the differentials and integrate to find $y(t)$.

$$\begin{aligned}
\int \frac{1}{y}\, dy &= \int -K\, dt \\
\ln|y| &= -Kt + C \\
e^{\ln y} &= e^{-Kt+C} \\
y &= Ce^{-Kt}
\end{aligned}$$

Since no radium has decayed at time $t = 0$, we can assume $C = 1$.

$$y = e^{-Kt} \tag{1.2}$$

Given that the half-life of radium is 1600 years, we can use this to find the proportionality constant K.

$$\begin{aligned}
\frac{1}{2} &= e^{-1600K} \\
\ln\frac{1}{2} &= \ln e^{-1600K} \\
\ln\frac{1}{2} &= -1600K \\
-\frac{\ln\frac{1}{2}}{1600} &= K
\end{aligned}$$

We now use this value of K in Equation 1.2 with $t = 100$ for our final answer.

$$\begin{aligned}
y &= e^{-\left(-\frac{\ln\frac{1}{2}}{1600}\right)(100)} \\
&= e^{\frac{\ln\frac{1}{2}}{16}} \\
&= 0.9576
\end{aligned}$$

Since this is the amount of radium present after 100 years, and we are looking for the amount *lost*, we must subtract from 1 to get our final answer.

$$\begin{aligned}
\text{amount of radium lost} &= 1 - 0.9576 \\
&= 0.0424 \\
&\approx 4.24\%
\end{aligned}$$

Therefore, after 100 years, approximately 4.24% of the radium in the piece of lead has been lost.

Exercise 1.1.3. The following item appeared in a newspaper. "The expedition used the carbon-14 test to measure the amount of radioactivity still present in the organic material found in the ruins, thereby determining that a town existed there as long ago as 7000 B.C." Using the half-life figure of $\mathbf{C}^{14}$ as given in the text, determine the approximate percentage of $\mathbf{C}^{14}$ still present in the organic material at the time of the discovery.

Remark. The text uses a $\mathbf{C}^{14}$ half-life of 5600 years[1].

Solution. We begin with the differential equation for exponential decay.

$$\frac{dy}{dt} = -Ky$$

Since Equation 1.1 is a separable linear first-order equation, we divide both sides by y and integrate with respect to dy and dt.

$$\begin{aligned}\int \frac{1}{y}\, dy &= \int -K\, dt \\ \ln|y| &= -Kt + C \\ e^{\ln y} &= e^{-Kt+C} \\ y &= Ce^{-Kt}\end{aligned}$$

Since no carbon-14 has decayed at time $t = 0$, we can assume $C = 1$, and we thus have the general equation for the quantity of carbon-14 with respect to time.

$$y = e^{-Kt}$$

Assuming a half-life of 5600 years, we now solve for the proportionality constant of the rate of decay.

$$\begin{aligned}y(t) &= e^{-K(5600)}, \quad \text{for } y(t) = \frac{1}{2} \\ \frac{1}{2} &= e^{-5600K} \\ \ln\frac{1}{2} &= \ln e^{-5600K} \\ \ln\frac{1}{2} &= -5600K \\ -\frac{\ln\frac{1}{2}}{5600} &= K\end{aligned}$$

We now use this value for the proportionality constant and a value of $t = 9000$[1] for our final answer.

$$\begin{aligned}y(9000) &= e^{-\left(-\frac{\ln\frac{1}{2}}{5600}\right)(9000)} \\ &= e^{\frac{9000\ln\frac{1}{2}}{5600}} \\ &\approx 0.3282\end{aligned}$$

Therefore, about 32.82% of the carbon-14 is still present in the organic material 9000 years later.

[1]Since the town existed during 7000 B.C., the total elapsed time in years is $7000 + 2000$, assuming a current year of somewhere in the early 2000's.

Chapter 24

Operators and Laplace Transforms

24.1 Differential and Polynomial Operators

Exercise 24.1.1. Prove by induction that

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n\,(n+1)\,(2n+1)}{6}.$$

Proof. We begin by proving the base case for $n = 1$.

$$\begin{aligned}
\sum_{n=1}^{1} n^2 &= \frac{n\,(n+1)\,(2n+1)}{6} \\
1^2 &= \frac{(1)\,((1)+1)\,(2\,(1)+1)}{6} \\
1 &= \frac{(2)\,(3)}{6} \\
1 &= \frac{6}{6} \\
1 &= 1
\end{aligned}$$

We now proceed to the inductive step. Having show that the base case holds, we now suppose that the predicate is true for n and proceed to prove for $n+1$.

$$\begin{aligned}
1^2+\cdots+n^2+(n+1)^2 &= \frac{n(n+1)(2n+1)}{6}+(n+1)^2 \\
&= \frac{n(n+1)(2n+1)+6(n+1)^2}{6} \\
&= \frac{\left(n^2+n\right)(2n+1)+6(n+1)(n+1)}{6} \\
&= \frac{\left(2n^3+2n^2+n^2+n\right)+6\left(n^2+2n+1\right)}{6} \\
&= \frac{2n^3+9n^2+13n+6}{6} \\
&= \frac{(n+1)\left(2n^2+7n+6\right)}{6} \\
&= \frac{(n+1)(n+2)(2n+3)}{6} \\
&= \frac{((n)+1)((n+1)+1)(2(n+1)+1)}{6}
\end{aligned}$$

□

Exercise 24.1.2. Find D^0y, Dy, D^2y, D^3y for each of the following:

(a) $y(x)=3x^2$

(b) $y(x)=3\sin 2x$

(c) $y(x)=\sqrt{x}$

Solution.

(a)

$$\begin{aligned}
y(x) &= 3x^2 \\
D^0y &= 3x^2 \\
Dy &= 6x \\
D^2y &= 6 \\
D^3y &= 0
\end{aligned}$$

(b)

$$\begin{aligned} y(x) &= 3\sin(2x) \\ D^0 y &= 3\sin(2x) \\ Dy &= 6\cos(2x) \\ D^2 y &= -12\sin(2x) \\ D^3 y &= -24\cos(2x) \end{aligned}$$

(c)

$$\begin{aligned} y(x) &= \sqrt{x} \\ D^0 y &= \sqrt{x} \\ Dy &= \frac{1}{2\sqrt{x}} \\ D^2 y &= -\frac{1}{4\sqrt{x^3}} \\ D^3 y &= \frac{3}{8\sqrt{x^5}} \end{aligned}$$

Bibliography

[1] Morris Tenenbaum and Harry Pollard. *Ordinary Differential Equations*. 1st ed. Dover Publications, 1985. ISBN: 978-0-496-64940-5.