Vector Spaces (I)

1. *Introduction to Vector spaces*
2. *Column-space and Nullspace*
3. *Solving Ax = 0. Pivot variables, special functions*

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1. Intro to Vector Spaces

A new level of understanding for Matrix calculation

❖ For the newcomer - involves lots of **NUMBERS**

❖ For the beginner - involves the use of **VECTORS**, i.e. **Ax** and **AB** are linear combinations of n vectors, the columns of A

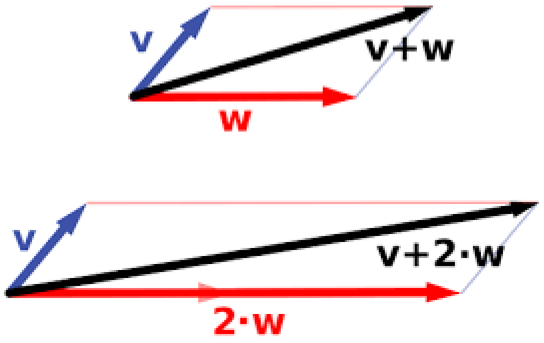
❖ For the initiated - the third level of understanding – **SPACES** of vectors

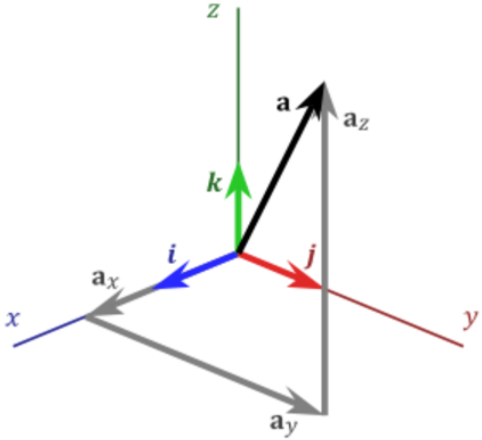
→ completes the understanding of **Ax** = **b**

1. Intro to Vector Spaces

VECTOR SPACES

- One can add vectors & multiply them by scalars → linear combinations
- Define Vector Spaces
- Example – vector space $\Re^2$ – set of all vectors with 2 real nb. components
- Vector $\begin{bmatrix} v \\ w \end{bmatrix}$ represented by an arrow from origin to (a, b) – call $\Re^2$ the x-y plane





1. Intro to Vector Spaces

The 8 rules

❖ Given 2 vectors **x** and **y**, both vector addition and multiplication should obey the following rules:

(1) $x + y = y + x$

(2) $x + (y + z) = (x + y) + z$

(3) There is a unique "zero vector" such that $x + \mathbf{0} = x$ for all x

(4) For each x there is a unique vector $-x$ such that $x + (-x) = \mathbf{0}$

(5) 1 times x equals x

(6) $(c_1 c_2)x = c_1(c_2 x)$

(7) $c(x + y) = cx + cy$

(8) $(c_1 + c_2)x = c_1 x + c_2 x$.

1. Intro to Vector Spaces

SUBSPACES

❖ A vector space that is contained inside of another vector space is called a subspace of that space.

❖ For example, take any non-zero vector v in $\Re^2$. Then the set of all vectors c**v**, where c is a real number, forms a subspace of $\Re^2$.

❖ A line in $\Re^2$ that does not pass through the origin $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is *not a subspace of R^2.*

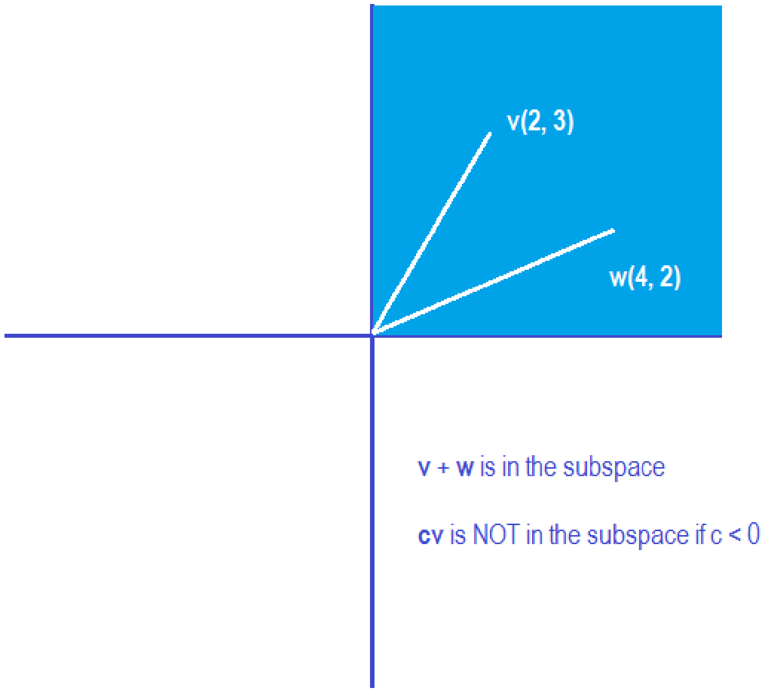
❖ *Multiplying any vector on that line by 0 gives the zero vector, which does not lie on the line*.

❖ Every subspace *must contain the zero vector* because vector spaces are closed under multiplication.

❖ A set of vectors is “**closed**” under addition v + w & multiplication cv (and cw), if these operations do NOT leave the subspace!

❖ **CLOSURE -** if collection of vectors is “closed” under linear combinations

Example: the collection of vectors with exactly 2 positive real valued components is NOT a vector space.



1. Intro to Vector Spaces

Examples of Subspaces

- The subspaces of $\Re^2$:
  - all of $\Re^2$
  - any line through $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$
  - the zero vector alone **Z**
- The subspaces of $\Re^3$:
  - all of $\Re^3$
  - any plane through the origin $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$
  - any line through the origin
  - the zero vector alone **Z**

Key points:

- Every Subspace contains the ZERO vector
- Lines through the origin are also subspaces
- $\Re^N$ is also a valid subspace

1. Intro to Vector Spaces

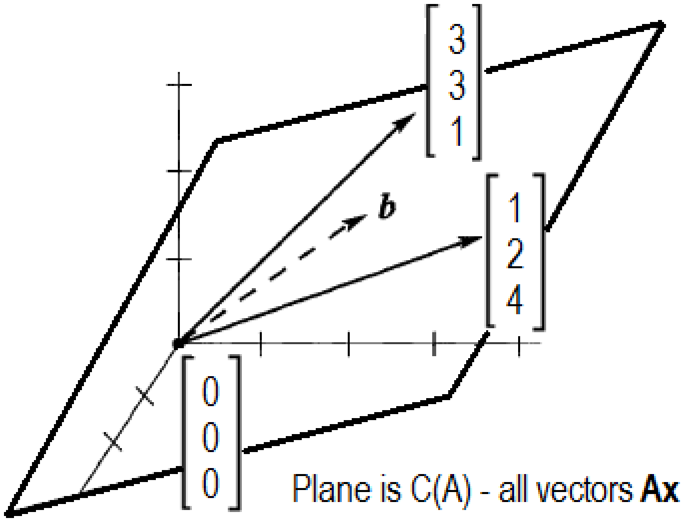
COLUMN SPACE

❖ Given a matrix **A** with columns in $\mathfrak{R}^3$ – these columns and all their linear combinations form a subspace in $\mathfrak{R}^3$ – Column space **C(A)**

❖ The column space of **A** is the plane through the origin of $\mathfrak{R}^3$ that contains both $\begin{bmatrix}1\\2\\4\end{bmatrix}$ and $\begin{bmatrix}3\\3\\1\end{bmatrix}$

❖ The goal for this lecture → understand **Ax = b** in terms of subspaces & column space.

$$\mathbf{Ax} = \mathbf{b} \qquad \mathbf{A} = \begin{bmatrix}1 & 3\\2 & 3\\4 & 1\end{bmatrix} \qquad \mathbf{b} = x_1\begin{bmatrix}1\\2\\4\end{bmatrix} + x_2\begin{bmatrix}3\\3\\1\end{bmatrix}$$



Plane is C(A) - all vectors **Ax**

COLUMN SPACE

- The most important subspaces are tied directly to a matrix **A.**
- The goal is still to solve **Ax = b**
- If **A** is not invertible, the system is solvable for some **b**' but not for the others
- **The** "good" **b** – vectors that can be written as linear combinations of **A** columns

→these **b**'s form the column space of matrix **A**

- To solve **Ax** = **b** is equivalent to *expressing **b** as a combination of **A**'s columns*
- When **b** is in the column space – it is a combination of the columns of **A**
- The coefficients in that combination is the **SOLUTION** for **Ax** = **b**

COLUMN SPACE example

- Does **Ax = b** always has a solution for any RHS b?

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

- Possible b's that give solutions:

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$$

Subspaces – Union & Intersection

- A **vector space** is a collection of vectors which is closed under linear combinations - for any two vectors ***v*** and w in the space and any two real numbers *c* and *d*, the vector *c**v*** + *d**w*** is also in the vector space.
- A plane **P** containing (0,0,0) and a line **L** containing (0,0,0) are both subspaces of $\mathcal{R}^3$.
- The union **P∪L** is generally NOT a subspace of $\mathcal{R}^3$.
- The intersection **P∩L** is **always a subspace** of $\mathcal{R}^3$.

COLUMN SPACE – Other Examples

- Let's try to describe the column spaces (as subspaces of $\Re^2$) for:

$$\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \end{bmatrix}$$

- The column space of **I** is the WHOLE space $\Re^2$ – every vector is a combination of the columns of I → **C(I) is $\Re^2$**
- The column space of **A** is only a LINE – the column space contains (1,2) and (2,4) as well as other vectors (c, 2c) but they are along the same line → **C(A)** – is a line
- The column space of **B** is the WHOLE space $\Re^2$. Every b is attainable! Ex: b = (5,4) is col-2 + col-3, or 2col-1 + col-3.

2. The Nullspace of A

Recap – Column spaces

❖ A **column space** of a matrix **A** is the vector space made up of all linear combinations of the columns of **A**.

❖ **Ax** = **b**

❖ Given a matrix **A**, for what vectors **b** does **Ax** = **b** have a solution **x**?

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix}$$

❖ **Ax** = **b** does not have a solution for every choice of **b** b/c solving the eq. is equivalent to solving four linear equations in three unknowns.

2. The Nullspace of A

Recap – Column spaces

❖ If there is a solution **x** to **Ax** = **b**, then ***b** must be a linear combination of the columns of **A**.*

❖ Only three columns cannot fill the entire four dimensional vector space – some vectors **b** cannot be expressed as linear combinations of columns of **A**.

❖ what **b**'s allow **Ax** = **b** to be solved?

❖ A useful approach is to choose **x** and find the vector **b** = ***Ax*** corresponding to that solution. The components of x are just the coefficients in a linear combination of columns of **A**.

❖ The system of linear equations **Ax = b** is solvable exactly when **b** is a vector in the column space of **A**.

2. The Nullspace of A

Recap – Column spaces

- For our example matrix **A**, what can we say about the column space of **A**?
- *Are the columns of **A** independent*?
- In other words, does each column *contribute something new* to the subspace?
- The third column of **A** is the sum of the first two columns, so does not add anything to the subspace – *throw it away*?
- The column space of our matrix **A** is a two dimensional subspace of $\Re^4$.

2. The Nullspace of A

Definition

- The **nullspace N(A)** of a matrix **A** is the *collection of all solutions* $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ to the equation $\mathbf{Ax} = \mathbf{0}$. These vectors $\mathbf{x}$ are in $\mathcal{R}^N$.
- **A**(m, n) square or rectangular → one immediate solution is $\mathbf{x} = 0$
- If **A** is **invertible** then $\mathbf{x} = 0$ is the **only solution**
- For **non-invertible A**, there are also **non-zero** solutions to $\mathbf{Ax} = \mathbf{0}$

→ each of these solutions belong to the nullspace of **A** → **N(A)**

2. The Nullspace of A

The possible solutions for the Nullspace:

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} c \\ c \\ -c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$c = 0$, or any scalar

This Nullspace **N(A)** is a line in $\mathcal{R}^3$

2. The Nullspace of A

Check that solution vectors form a subspace

- Suppose **x** and **y** are in the nullspace → **Ax** = 0 and **Ay** = 0
- The rules of matrix multiplication **A**(x_1 + x_2)= **0** + **0** and **A**(c **x**)= c**Ax** = c**0**
- Since RHS are zero → **x** + **y** and c**x** are in the nullspace **N(A)**
- The solution vectors **x** have ***n*** components → they are vectors in $\mathfrak{R}^N$ → the **nullspace N(A) is a subspace of $\mathfrak{R}^N$**
- The **column space C(A)** is a **subspace of $\mathfrak{R}^M$**.
- If the right side **b** is not zero, the solutions of **Ax** = **b** do not form a subspace.
- The vector **x** = 0 is only a solution if **b** = 0.
- *When the set of solutions does not include **x** = 0, it cannot be a subspace!!!*

2. The Nullspace of A

Example (1)

❖ Given an SLE $\begin{cases} x_1 + 2x_2 = 0 \\ 3x_1 + 6x_2 = 0 \end{cases}$ $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ matrix **A** is singular

❖ What is the **Nullspace** of A?

❖ In the row picture line $x_1 + 2x_2 = 0$ is the same as $3x_1 + 6x_2 = 0$ (x3)

❖ This line is the **N(A)** and it contains all solutions (x_1, x_2)

❖ Best way to describe a nullspace → choose one point – "***special solution***", i.e. for $x_2 = 1$, $x_1 = -2$ from first equation.

❖ Conclusion: the **Nullspace N(A)** contains all multiples of $s = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

2. The Nullspace of A

Example (2)

❖ Given $x + 2y + 3z = 0$ the corresponding matrix $\mathbf{A} = [1 \ \ 2 \ \ 3]$

❖ The equation $\mathbf{Ax} = 0$ produces a ***plane*** *through the origin* (0, 0, 0) → this plane is a subspace of $\mathcal{R}^3$ and *it is the nullspace of **A***

❖ The plane $x + 2y + 3z = 0$ has 2 special solutions:

$$s_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \qquad s_2 = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

$$[1 \ \ 2 \ \ 3]\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

❖ These vectors $\mathbf{s_1}$ and $\mathbf{s_2}$ lie on the plane $x + 2y + 3z = 0$ which is N(A).

❖ All vectors in the plane are combinations of $\mathbf{s_1}$ & $\mathbf{s_2}$ (zeros in col 2 & 3 - ***free***)

❖ Col-1 contains the pivot – so first component **x** is not "***free***"

2. The Nullspace of A

Example (3)

❖ Given 3 matrices $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix}$ $\mathbf{B} = \begin{bmatrix} A \\ 2A \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 8 \\ 2 & 4 \\ 6 & 16 \end{bmatrix}$ $\mathbf{C} = [A \;\; 2A] = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 3 & 8 & 6 & 16 \end{bmatrix}$

→let's describe their nullspaces

❖ **Ax** = 0 has only the zero solution → **N(A) = Z**

❖ Elimination: $\begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 = 0 \\ x_2 = 0 \end{bmatrix}$

A is invertible – all columns of **A** have pivots

❖ The rectangular matrix **B** has the same nullspace Z - by adding extra eq., the **N(B)** cannot become larger →*the extra rows impose more conditions on the vectors **x** in the nullspace.*

2. The Nullspace of A

Example (3)

❖The rectangular matrix **C** is very different – has extra columns vs. rows

❖ The solution vector x has 4 components – elimination will produce pivots in the first 2 columns of C – the other 2 columns are free:

$$\mathbf{C} = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 3 & 8 & 6 & 16 \end{bmatrix} \Rightarrow \mathbf{U} = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & 2 & 0 & 4 \end{bmatrix}$$

$$\begin{array}{cc} \text{pivot columns} & \text{free columns} \end{array}$$

❖ For the free variables $\mathbf{x_3}$ & $\mathbf{x_4}$ we make special choices of ones and zeros

❖ The pivot variables $\mathbf{x_1}$ & $\mathbf{x_2}$ are determined by **Ux** = 0

$$s_1 = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad s_2 = \begin{bmatrix} 0 \\ -2 \\ 0 \\ 1 \end{bmatrix} \begin{array}{l} \text{pivot variables} \\ \\ \text{free variables} \end{array}$$

2. The Nullspace of A

Example (3)

❖The Elimination procedure will NOT stop at the upper triangular matrix U!

❖ Continue the procedure to make the matrix simpler:

➢ Produce '0' above pivots by **eliminating upward**

➢ Produce '1' in the pivots by **dividing whole row by its pivot**

❖ RHS Zero vector does not change → N(C) stays the same – easier to be see when one reaches the **Reduced Row Echelon Form** – **R**:

$$\mathbf{U} = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & 2 & 0 & 4 \end{bmatrix} \Rightarrow \mathbf{R} = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$

(1) R1 – R2

(2) R2 times ½

❖ Special solution are MUCH easier to find with **Rx** = 0

2. The Nullspace of A

<u>Recap remarks</u>

❖ For many matrices, the only solution to $\mathbf{A}\mathbf{x} = 0$ is $\mathbf{x} = 0$.

❖ Their nullspaces $\mathbf{N(A)} = \mathbf{Z}$ contain only that zero vector.

❖ The only combination of the columns that produces $\mathbf{b} = 0$ is then the "zero combination" or "trivial combination".

❖ The solution is trivial (just $x = 0$) but the idea is not trivial.

❖ This case of a zero nullspace Z is of the greatest importance →meaning is that the columns of **A** are independent. No combination of columns gives the zero vector (except the zero combination).

❖ All columns have pivots, and no columns are free.

3. Solving Ax = 0. Pivots & Special solutions

- A way to do elimination on Rectangular matrices!
- Allowing all matrices - not just “nice” square matrices with inverses
- Pivots are still nonzero
- The columns below the pivots are still zero
- But it might happen that a column has no pivot
- That free column doesn't stop the calculation

→ *Go on to the next column!*

3. Solving Ax = 0

Example

- Given a 3 by 4 rectangular matrix:

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 2 & 2 & 8 & 10 \\ 3 & 3 & 10 & 13 \end{bmatrix} \Rightarrow \begin{bmatrix} \boxed{1} & 1 & 2 & 3 \\ 0 & 0 & \boxed{4} & 4 \\ 0 & 0 & 4 & 4 \end{bmatrix} \begin{matrix} \\ \text{substract 2 x row 1} \\ \text{substract 3 x row 1} \end{matrix}$$

- Trouble for pivot 2 → got to next column – second pivot is "4"

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 2 & 2 & 8 & 10 \\ 3 & 3 & 10 & 13 \end{bmatrix} \Rightarrow \begin{bmatrix} \boxed{1} & 1 & 2 & 3 \\ 0 & 0 & \boxed{4} & 4 \\ 0 & 0 & 4 & 4 \end{bmatrix} \Rightarrow \underbrace{\begin{bmatrix} \boxed{1} & 1 & 2 & 3 \\ 0 & 0 & \boxed{4} & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{U} \begin{matrix} \text{substract row 2} \\ \text{from row 3} \end{matrix}$$

- Only 2 pivots and last equations: 0 = 0

3. Solving Ax = 0

<u>Back substitution for Ux = 0</u>

- ❖ We have 4 unknowns and just 2 pivots → many possible solutions!
- ❖ Separate **pivot** variables from **free** variables
- ❖ When **A** is invertible, **all** variables are **pivot** variables
- ❖ Free variables x_2 & x_4 could be given ANY values → then back substitute into the pivot variables x_1 & x_3
- ❖ Simplest choice for free variables – '0' and '1'
- ❖ Special solutions: $x_1 + x_2 + 2x_3 + 3x_4 = 0$ and $4x_3 + 4x_4 = 0$
  - ➢ set $x_2 = 1$ & $x_4 = 0$; back substitution → $x_3 = 0$ & $x_1 = -1$
  - ➢ set $x_2 = 0$ & $x_4 = 1$; back substitution → $x_3 = -1$ & $x_1 = -1$

$$\begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

3. Solving Ax = 0

Complete solution

- $s_1 \rightarrow x_2 = 1$ & $x_4 = 0$
- $s_2 \rightarrow x_2 = 0$ & $x_4 = 1$
- All solutions are

linear combinations of s_1 & s_2

$$x = x_2 \underbrace{\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{\text{special}} + x_4 \underbrace{\begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix}}_{\text{special}} = \underbrace{\begin{bmatrix} -x_2 - x_4 \\ x_2 \\ -x_4 \\ x_4 \end{bmatrix}}_{\text{complete}}$$

- The special solutions are in the nullspace N(A) and their linear combinations are filling out the whole nullspace
- There is **special** solution for **every free** variable
- If no free variables, there are n pivots, only solution is the trivial one **x**= 0.
- The nullspace contains only Z – zero vector

3. Solving Ax = 0

Echelon matrices

- ❖ Forward elimination A → U – acts by row operations (row exchanges)
- ❖ When no pivot available (=0) moves to the next column
- ❖ An echelon matrix is an ***m x n*** "staircase" U matrix - less pivots than columns

$$\mathbf{U} = \begin{bmatrix} p & x & x & x & x & x & x \\ 0 & p & x & x & x & x & x \\ 0 & 0 & 0 & 0 & 0 & p & x \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

3 pivot variables x_1, x_2, x_6

4 free variables x_3, x_4, x_5, x_7

4 special solution in N(U)

❖The columns have 4 components → **C(U)** lies in $\mathcal{R}^4$ - Every vector in **C(U)** has 4th component zero (u_1, u_2, u_3, 0). The **b**' in **Ux** = **b** are combinations of the 7 columns

❖ The nullspace **N(U)** is a subspace of $\mathcal{R}^7$ → The solutions of **Ux** = **0** are all the combinations of the 4 special solutions – one for each free variable

3. Solving Ax = 0

Echelon matrices

- Columns 3-4-5-7 have no pivots → free variables x_3, x_4, x_5, x_7
- Set 1 free variable to '1' and the other free variables to '0'
- Solve $Ux = 0$ for the pivot variables x_1, x_2, x_6
- This gives one of the 4 special solutions in the N(U)

Theorem:

- *If **Ax** = 0 has more unknowns than equations (more columns than rows, n>m), there is at least one free variable, and one special solution – non-zero*
- A short-wide matrix always has non-zero vectors in its nullspace
- The nullspace has the dimension of the number of free variables

3. Solving $Ax = 0$

Reduced Echelon matrix

❖ Go an extra step from an echelon U matrix:

$$\begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) divide second row by 4

(b) subtract 2 times new row from 1st row

$$\mathbf{R} = \mathbf{rref(A)} = \begin{bmatrix} 1 & \boxed{1} & 0 & \boxed{1} \\ 0 & \boxed{0} & 1 & \boxed{1} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

❖ The reduced row echelon matrix **R** has '1' as pivots & '0' above pivots

❖ If **A** in invertible, its **R = I**

❖ **R** makes it very easy to find **special solutions** – directly from R

❖ Special solutions: $x_1 + x_2 + 2x_3 + 3x_4 = 0$ & $4x_3 + 4x_4 = 0$

- set $x_2 = 1$ & $x_4 = 0$; back substitution → $x_3 = 0$ & $x_1 = -1$
- set $x_2 = 0$ & $x_4 = 1$; back substitution → $x_3 = -1$ & $x_1 = -1$

$$x = x_2 \underbrace{\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{\text{special}} + x_4 \underbrace{\begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix}}_{\text{special}} = \underbrace{\begin{bmatrix} -x_2 - x_4 \\ x_2 \\ -x_4 \\ x_4 \end{bmatrix}}_{\text{complete}}$$

Review of Key ideas:

1. The nullspace **N(A)** is a subspace of $\mathbf{R}^n$. It contains all solutions to $\mathbf{A}x = 0$
2. Elimination produces an echelon matrix **U**, and then a row reduced **R**, with pivot columns and free columns
3. Every free column of **U** or **R** leads to a special solution. One free variable could be set to '1' and the others to '0'. Back substitution solves $\mathbf{A}x = 0$
4. The complete solution to $\mathbf{A}x = 0$ is a combination of the special solutions
5. If $\mathbf{n} > \mathbf{m}$ then **A** has at least one column without pivots, giving a special solution. So there are nonzero vectors x in the nullspace of this rectangular **A**