Short questions

There may not be true/false questions on the exam, but it's a good idea to review these:

1. Given a square matrix A whose nullspace is just $\{\mathbf{0}\}$, what is the nullspace of A^T?

   $N(A^T)$ is also $\{\mathbf{0}\}$ because A is square.

2. Do the invertible matrices form a subspace of the vector space of 5 by 5 matrices?

   No. The sum of two invertible matrices may not be invertible. Also, 0 is not invertible, so is not in the collection of invertible matrices.

3. True or false: If $B^2 = 0$, then it must be true that $B = 0$.

   False. We could have $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

4. True or false: A system $A\mathbf{x} = \mathbf{b}$ of n equations with n unknowns is solvable for every right hand side $\mathbf{b}$ if the columns of A are independent.

   True. A is invertible, and $\mathbf{x} = A^{-1}\mathbf{b}$ is a (unique) solution.

5. True or false: If $m = n$ then the row space equals the column space.

   False. The dimensions are equal, but the spaces are not. A good example to look at is $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

6. True or false: The matrices A and $-A$ share the same four spaces.

   True, because whenever a vector $\mathbf{v}$ is in a space, so is $-\mathbf{v}$.

7. True or false: If A and B have the same four subspaces, then A is a multiple of B.

   A good way to approach this question is to first try to convince yourself that it isn't true – look for a counterexample. If A is 3 by 3 and invertible, then its row and column space are both $\mathbb{R}^3$ and its nullspaces are $\{\mathbf{0}\}$. If B is any other invertible 3 by 3 matrix it will have the same four subspaces, and it may not be a multiple of A. So we answer "false".

   It's good to ask how we could truthfully complete the statement "If A and B have the same four subspaces, then ..."

8. If we exchange two rows of A, which subspaces stay the same?

   The row space and the nullspace stay the same.

question 1

Suppose $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are non-zero vectors in $\mathbb{R}^7$. They span a subspace of $\mathbb{R}^7$. What are the possible dimensions of that vector space?

The answer is 1, 2 or 3. The dimension can't be higher because a basis for this subspace has at most three vectors. It can't be 0 because the vectors are non-zero.

question 2

Suppose a 5 by 3 matrix R in reduced row echelon form has $r = 3$ pivots.

1. What's the nullspace of R?

   Since the rank is 3 and there are 3 columns, there is no combination of the columns that equals $\mathbf{0}$ except the trivial one. $N(R) = \{\mathbf{0}\}$.

2. Let B be the 10 by 3 matrix $\begin{bmatrix} R \\ 2R \end{bmatrix}$. What's the reduced row echelon form of B?

   Answer: $\begin{bmatrix} R \\ 0 \end{bmatrix}$.

3. What is the rank of B?

   Answer: 3.

4. What is the reduced row echelon form of $C = \begin{bmatrix} R & R \\ R & 0 \end{bmatrix}$?

   When we perform row reduction we get:

$$\begin{bmatrix} R & R \\ R & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} R & R \\ 0 & -R \end{bmatrix} \longrightarrow \begin{bmatrix} R & 0 \\ 0 & -R \end{bmatrix} \longrightarrow \begin{bmatrix} R & 0 \\ 0 & R \end{bmatrix}.$$

   Then we might need to move some zero rows to the bottom of the matrix.

5. What is the rank of C?

   Answer: 6.

6. What is the dimension of the nullspace of C^T?

   $m = 10$ and $r = 6$ so $\dim N(C^T) = 10 - 6 = 4$.

question 3

Suppose we know that $A\mathbf{x} = \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}$ and that:

$$\mathbf{x} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + c \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + d \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

is a complete solution.

Note that in this problem we don't know what A is.

1. What is the shape of the matrix A?

   Answer: 3 by 3, because $\mathbf{x}$ and $\mathbf{b}$ both have three components.

2. What's the dimension of the row space of A?

   From the complete solution we can see that the dimension of the nullspace of A is 2, so the rank of A must be $3 - 2 = 1$.

3. What is A?

   Because the second and third components of the particular solution $\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$ are zero, we see that the first column vector of A must be $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

   Knowing that $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is in the nullspace tells us that the third column of A must be $\mathbf{0}$. The fact that $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ is in the nullspace tells us that the second column must be the negative of the first. So,

   $$A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & -2 & 0 \\ 1 & -1 & 0 \end{bmatrix}.$$

   If we had time, we could check that this A times $\mathbf{x}$ equals $\mathbf{b}$.

4. For what vectors b does $A\mathbf{x} = \mathbf{b}$ have a solution $\mathbf{x}$?

   This equation has a solution exactly when $\mathbf{b}$ is in the column space of A, so when $\mathbf{b}$ is a multiple of $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$. This makes sense; we know that the rank of A is 1 and the nullspace is large.

   In contrast, we might have had $r = m$ or $r = n$.

question 4

Suppose:

$$B = CD = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Try to answer the questions below without performing this matrix multiplication CD.

1. Give a basis for the nullspace of B.

   The matrix B is 3 by 4, so $N(B) \subseteq \mathbb{R}^4$. Because $C = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ is invertible, the nullspace of B is the same as the nullspace of $D = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Matrix D is in reduced form, so its special solutions form a basis for $N(D) = N(B)$:

   $$\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

   These vectors are independent, and if time permits we can multiply to check that they are in $N(B)$.

2. Find the complete solution to $B\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

   We can now describe any vector in the nullspace, so all we need to do is find a particular solution. There are many possible particular solutions; the simplest one is given below.

   One way to solve this is to notice that $C \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and then find a vector $\mathbf{x}$ for which $D\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Another approach is to notice that the first column of $B = CD$ is $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. In either case, we get the complete solution:

   $$\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + d \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

   Again, we can check our work by multiplying.