Team-Based Inquiry Learning - Linear Algebra

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This is a port of the first standard covered in Clontz and Lewis's Team-Based Inquiry Learning module on Linear Transformations.

Contents

A linear transformation (or linear map) is a rule that describes how to transform vectors in one vector space into vectors in another vector space. Here I am adding a "quote" just for practice.

Chapter 1

Module A: Algebra of Linear Transformations

1.1 Determine if a map between vector spaces of polynomials is linear or not.

This is where I can write a short introductory slide (or not) to the section.

Definition 1.1.1. A *linear transformation* (also known as a *linear map*) is a map between vector spaces that preserves the vector space operations.

More precisely, if V and W are vector spaces, a map $T : V \to W$ is called a linear transformation if

1. $T(\vec{v} + \vec{w}) = T(\vec{v}) + T(\vec{w})$ for any $\vec{v}, \vec{w} \in V$.

2. $T(c\vec{v}) = cT(\vec{v})$ for any $c \in \mathbb{R}, \vec{v} \in V$.

In other words, a map is linear when vector space operations can be applied before or after the transformation without affecting the result.

Example 1.1.2. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x - z \\ 3y \end{bmatrix}$$

To show that T is linear, we must verify...

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} u \\ v \\ w \end{bmatrix}\right) = T\left(\begin{bmatrix} x + u \\ y + v \\ z + w \end{bmatrix}\right) = \begin{bmatrix} (x + u) - (z + w) \\ 3(y + v) \end{bmatrix}$$

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) + T\left(\begin{bmatrix} u \\ v \\ w \end{bmatrix}\right) = \begin{bmatrix} x - z \\ 3y \end{bmatrix} + \begin{bmatrix} u - w \\ 3v \end{bmatrix} = \begin{bmatrix} (x + u) - (z + w) \\ 3(y + v) \end{bmatrix}$$

And also...

$$T\left(c\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = T\left(\begin{bmatrix} cx \\ cy \\ cz \end{bmatrix}\right) = \begin{bmatrix} cx - cz \\ 3cy \end{bmatrix} \text{ and } cT\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = c\begin{bmatrix} x - z \\ 3y \end{bmatrix} = \begin{bmatrix} cx - cz \\ 3cy \end{bmatrix}$$

Therefore T is a linear transformation.

Example 1.1.3. Let $T : \mathbb{R}^2 \to \mathbb{R}^4$ be given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x + y \\ x^2 \\ y + 3 \\ y - 2^x \end{bmatrix}$$

To show that T is not linear, we only need to find one counterexample.

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) = T\left(\begin{bmatrix} 2 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} 6 \\ 4 \\ 7 \\ 0 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) + T\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 4 \\ -1 \end{bmatrix} + \begin{bmatrix} 5 \\ 4 \\ 6 \\ -5 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 10 \\ -6 \end{bmatrix}$$

Since the resulting vectors are different, T is not a linear transformation.

Fact 1.1.4. *A map between Euclidean spaces $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear exactly when every component of the output is a linear combination of the components of $\mathbb{R}^n$.*

For example, the following map is definitely linear because $x - z$ and $3y$ are linear combinations of x, y, z:

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x - z \\ 3y \end{bmatrix} = \begin{bmatrix} 1x + 0y - 1z \\ 0x + 3y + 0z \end{bmatrix}$$

But this map is not linear because x^2, $y + 3$, and $y - 2^x$ are not linear combinations (even though $x + y$ is):

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x + y \\ x^2 \\ y + 3 \\ y - 2^x \end{bmatrix}$$

Activity 1.1.1. (*5 min*)
Recall the following rules from calculus, where $D : \mathcal{P} \to \mathcal{P}$ is the derivative map defined by $D(f(x)) = f'(x)$ for each polynomial f.

$$D(f + g) = f'(x) + g'(x)$$

$$D(cf(x)) = cf'(x)$$

What can we conclude from these rules?

A $\mathcal{P}$ is not a vector space

B D is a linear map

C D is not a linear map

Activity 1.1.2. (*10 min*)
Let the polynomial maps $S : \mathcal{P}^4 \to \mathcal{P}^3$ and $T : \mathcal{P}^4 \to \mathcal{P}^3$ be defined by

$$S(f(x)) = 2f'(x) - f''(x) \qquad T(f(x)) = f'(x) + x^3$$

Compute $S(x^4 + x)$, $S(x^4) + S(x)$, $T(x^4 + x)$, and $T(x^4) + T(x)$. Which of these maps is definitely not linear?

Fact 1.1.5. *If $L : V \to W$ is linear, then*

$$L(\vec{z}) = L(0\vec{v}) = 0L(\vec{v}) = \vec{z}$$

where $\vec{z}$ is the additive identity of the vector spaces V, W.

Put another way, an easy way to prove that a map like $T(f(x)) = f'(x) + x^3$ can't be linear is because

$$T(0) = \frac{d}{dx}[0] + x^3 = 0 + x^3 = x^3 \neq 0.$$

Note 1.1.6. Showing $L : V \to W$ is not a linear transformation can be done by finding an example for any one of the following.

- Show $L(\vec{z}) \neq \vec{z}$ (where $\vec{z}$ is the additive identity of L and W).

- Find $\vec{v}, \vec{w} \in V$ such that $L(\vec{v} + \vec{w}) \neq L(\vec{v}) + L(\vec{w})$.

- Find $\vec{v} \in V$ and $c \in \mathbb{R}$ such that $L(c\vec{v}) \neq cL(\vec{v})$.

Otherwise, L can be shown to be linear by proving the following in general.

- For all $\vec{v}, \vec{w} \in V$, $L(\vec{v} + \vec{w}) \neq L(\vec{v}) + L(\vec{w})$.

- For all $\vec{v} \in V$ and $c \in \mathbb{R}$, $L(c\vec{v}) \neq cL(\vec{v})$.

Note the similarities between this process and showing that a subset of a vector space is/isn't a subspace.

Activity 1.1.3. *(15 min)*

Continue to consider $S : \mathcal{P}^4 \to \mathcal{P}^3$ defined by

$$S(f(x)) = 2f'(x) - f''(x)$$

(a) Verify that

$$S(f(x) + g(x)) = 2f'(x) + 2g'(x) - f''(x) - g''(x)$$

is equal to $S(f(x)) + S(g(x))$ for all polynomials f, g.

(b) Verify that $S(cf(x))$ is equal to $cS(f(x))$ for all real numbers c and polynomials f.

(c) Is S linear?

Activity 1.1.4. *(20 min)*

Let the polynomial maps $S : \mathcal{P} \to \mathcal{P}$ and $T : \mathcal{P} \to \mathcal{P}$ be defined by

$$S(f(x)) = (f(x))^2 \qquad T(f(x)) = 3xf(x^2)$$

(a) Note that $S(0) = 0$ and $T(0) = 0$. So instead, show that $S(x + 1) \neq S(x) + S(1)$ to verify that S is not linear.

(b) Prove that T is linear by verifying that $T(f(x) + g(x)) = T(f(x)) + T(g(x))$ and $T(cf(x)) = cT(f(x))$.