STAT 531: Assignment # 3

Due on November 5, 2020

Dr. Lu TTh 11:00 am

By: Jana Osea (30016679)

Contents

Problem 1

(a) We want to calculate the 2-step transition probability matrix.

$$P^2 = \begin{pmatrix} 0.4 & 0.6 \\ 0.6 & 0.4 \end{pmatrix} \begin{pmatrix} 0.4 & 0.6 \\ 0.6 & 0.4 \end{pmatrix} = \begin{pmatrix} 0.52 & 0.48 \\ 0.48 & 0.52 \end{pmatrix}$$

(b) We want to find $P(X_3 = I|X_0 = I)$.

$$P^3 = \begin{pmatrix} 0.4 & 0.6 \\ 0.6 & 0.4 \end{pmatrix} \begin{pmatrix} 0.4 & 0.6 \\ 0.6 & 0.4 \end{pmatrix} \begin{pmatrix} 0.4 & 0.6 \\ 0.6 & 0.4 \end{pmatrix} = \begin{pmatrix} 0.496 & 0.504 \\ 0.504 & 0.496 \end{pmatrix}$$

And so $P(X_3 = I|X_0 = I) = 0.496$

Problem 2

(a) We want to find $P(X_2 = 2|X_0 = 1)$.

$$P^2 = \begin{pmatrix} 0.8 & 0.2 \\ 0.4 & 0.6 \end{pmatrix} \begin{pmatrix} 0.8 & 0.2 \\ 0.4 & 0.6 \end{pmatrix} = \begin{pmatrix} 0.72 & 0.28 \\ 0.56 & 0.44 \end{pmatrix}$$

And so $P(X_2 = 2|X_0 = 1) = 0.28$.

(b) We want to find π_2 in $\boldsymbol{\pi} = \lim_{n\to\infty} \boldsymbol{\pi}^{(n)} = (\pi_1 \quad \pi_2)$. We start with

$$\begin{aligned}
\begin{pmatrix} \pi_1 & \pi_2 \end{pmatrix} \begin{pmatrix} 0.8 & 0.2 \\ 0.4 & 0.6 \end{pmatrix} &= \begin{pmatrix} \pi_1 & \pi_2 \end{pmatrix} \\
\to 0.8\pi_1 + 0.4\pi_2 &= \pi_1 \\
0.2\pi_1 + 0.6\pi_2 &= \pi_2 \\
\pi_1 + \pi_2 &= 1 \\
\to \underbrace{\begin{pmatrix} -0.2 & 0.4 \\ 1 & 1 \end{pmatrix}}_{A} \begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
det(A) = -0.2 - 0.4 &= -0.6
\end{aligned}$$

We calculate each part as

$$\begin{aligned}
\pi_1 &= \frac{1}{-0.6} det \begin{pmatrix} 0 & 0.4 \\ 1 & 1 \end{pmatrix} \\
&= \frac{-0.4}{-0.6} \\
&= \frac{2}{3}
\end{aligned}$$

$$\begin{aligned}
\pi_2 &= \frac{1}{-0.6} det \begin{pmatrix} -0.2 & 0 \\ 1 & 1 \end{pmatrix} \\
&= \frac{-0.2}{-0.6} \\
&= \frac{1}{3}
\end{aligned}$$

Therefore, the probability that Nov. 1 next year Rainbow will rain is $\pi_2 = \frac{1}{3}$.

Problem 3

(a) We want to find the distribution π_2 of X_2. We find

$$
\begin{aligned}
P^{(2)} &= (0 \;\; 1)P^2 \\
\rightarrow P^{(2)} &= (0 \;\; 1)\begin{pmatrix} 0.8 & 0.2 \\ 0.1 & 0.9 \end{pmatrix}\begin{pmatrix} 0.8 & 0.2 \\ 0.1 & 0.9 \end{pmatrix} \\
&= (0 \;\; 1)\begin{pmatrix} 0.66 & 0.34 \\ 0.17 & 0.83 \end{pmatrix} \\
&= (0.17 \;\; 0.83)
\end{aligned}
$$

Hence, the distribution of X_2 is given by $X_2 \sim (0.17 \;\; 0.83)$.

(b) We want to find the steady-state distribution of X_n.

$$
\begin{aligned}
\begin{pmatrix} \pi_1 & \pi_2 \end{pmatrix}\begin{pmatrix} 0.8 & 0.2 \\ 0.1 & 0.9 \end{pmatrix} &= \begin{pmatrix} \pi_1 & \pi_2 \end{pmatrix} \\
\rightarrow 0.8\pi_1 + 0.1\pi_2 &= \pi_1 \\
0.2\pi_1 + 0.9\pi_2 &= \pi_2 \\
\pi_1 + \pi_2 &= 1 \\
\rightarrow \underbrace{\begin{pmatrix} -0.2 & 0.1 \\ 1 & 1 \end{pmatrix}}_{A}\begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
det(A) = -0.2 - 0.1 &= -0.3
\end{aligned}
$$

We calculate each part as

$$
\begin{aligned}
\pi_1 &= \frac{1}{-0.3} det \begin{pmatrix} 0 & 0.1 \\ 1 & 1 \end{pmatrix} \\
&= \frac{-0.1}{-0.3} \\
&= \frac{1}{3}
\end{aligned}
$$

$$
\begin{aligned}
\pi_2 &= \frac{1}{-0.3} det \begin{pmatrix} -0.2 & 0 \\ 1 & 1 \end{pmatrix} \\
&= \frac{-0.2}{-0.3} \\
&= \frac{2}{3}
\end{aligned}
$$

Therefore, the distribution of X_n is given by $X_n \sim \left(\frac{1}{3} \;\; \frac{2}{3}\right)$

Problem 5

(a) In order to show that P is a regular, there must exist some power n such that all entries in the P^n are greater than but not equal to 0.

$$P = \begin{pmatrix} 0.8 & 0.14 & 0.04 & 0.02 \\ 0 & 0.6 & 0.3 & 0.1 \\ 0 & 0 & 0.65 & 0.35 \\ 0.9 & 0 & 0 & 0.1 \end{pmatrix}$$
$$\to P^3 = \begin{pmatrix} 0.5678 & 0.20972 & 0.15012 & 0.07236 \\ 0.2295 & 0.2286 & 0.35535 & 0.18655 \\ 0.48825 & 0.0441 & 0.287225 & 0.180425 \\ 0.6732 & 0.189 & 0.0936 & 0.0442 \end{pmatrix}$$

Since all entries of P^3 greater than but not equal to 0, P is a regular Markov chain.

(b) We want to find the steady state distribution $\boldsymbol{\pi}$.

$$\begin{pmatrix} \pi_1 & \pi_2 & \pi_3 & \pi_4 \end{pmatrix} \begin{pmatrix} 0.8 & 0.14 & 0.04 & 0.02 \\ 0 & 0.6 & 0.3 & 0.1 \\ 0 & 0 & 0.65 & 0.35 \\ 0.9 & 0 & 0 & 0.1 \end{pmatrix} = \begin{pmatrix} \pi_1 & \pi_2 & \pi_3 & \pi_4 \end{pmatrix}$$
$$\begin{aligned} \to 0.8\pi_1 + 0.9\pi_4 &= \pi_1 \\ 0.14\pi_1 + 0.6\pi_2 &= \pi_2 \\ 0.04\pi_1 + 0.3\pi_2 + 0.65\pi_3 &= \pi_3 \\ 0.02\pi_1 + 0.1\pi_2 + 0.35\pi_3 + 0.1\pi_4 &= \pi_4 \\ \pi_1 + \pi_2 + \pi_3 + \pi_4 &= 1 \end{aligned}$$
$$\to \underbrace{\begin{pmatrix} -0.2 & 0 & 0 & 0.9 \\ 0.14 & -0.4 & 0 & 0 \\ 0.04 & 0.3 & -0.35 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}}_{A} \begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \\ \pi_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$
$$det(A) = -0.2503$$

We calculate each part as

$$\begin{aligned} \pi_1 &= \frac{1}{-0.2503} det \begin{pmatrix} 0 & 0 & 0 & 0.9 \\ 0 & -0.4 & 0 & 0 \\ 0 & 0.3 & -0.35 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \\ &= 0.503 \end{aligned}$$

$$\begin{aligned} \pi_2 &= \frac{1}{-0.2503} det \begin{pmatrix} -0.2 & 0 & 0 & 0.9 \\ 0.14 & 0 & 0 & 0 \\ 0.04 & 0 & -0.35 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \\ &= 0.176 \end{aligned}$$

$$\pi_3 = \frac{1}{-0.2503} det \begin{pmatrix} -0.2 & 0 & 0 & 0.9 \\ 0.14 & -0.4 & 0 & 0 \\ 0.04 & 0.3 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$
$$= 0.209$$

$$\pi_4 = \frac{1}{-0.2503} det \begin{pmatrix} -0.2 & 0 & 0 & 0 \\ 0.14 & -0.4 & 0 & 0 \\ 0.04 & 0.3 & -0.35 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$
$$= 0.112$$

Therefore, the steady state distribution of X_n is given by $\boldsymbol{\pi} = \begin{pmatrix} 0.503 & 0.176 & 0.209 & 0.112 \end{pmatrix}$

Problem 8

The Markov Chain is given by the figure below.

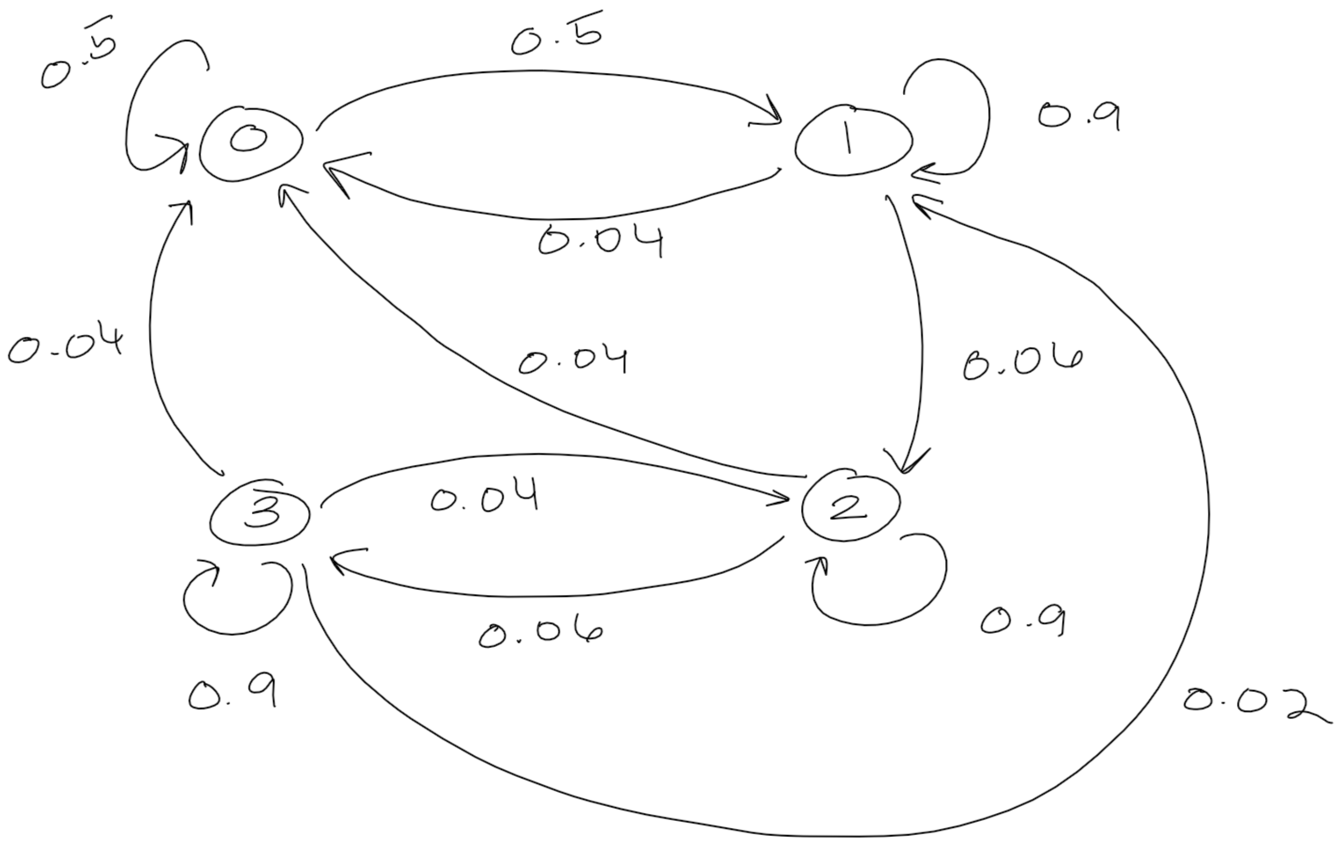

Figure 1: Markov Chain for Lan car reports

The transition matric P is given below:

$$P = \begin{pmatrix} 0.5 & 0.5 & 0 & 0 \\ 0.04 & 0.9 & 0.06 & 0 \\ 0.04 & 0 & 0.9 & 0.06 \\ 0.04 & 0.02 & 0.04 & 0.9 \end{pmatrix}$$

Problem 9

The Markov Chain is given by the figure below.

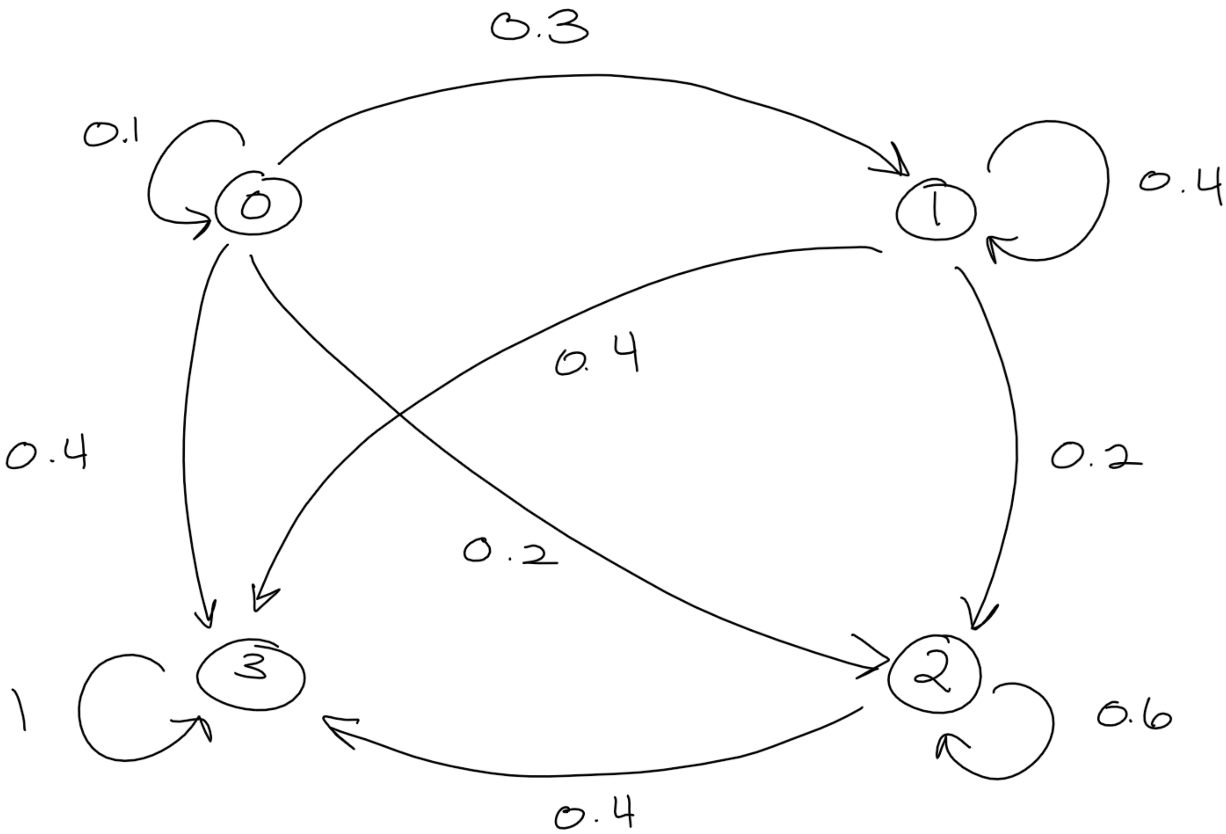

Figure 2: Markov Chain for Problem 9

The transition matric P is given below:

$$P = \begin{pmatrix} 0.1 & 0.3 & 0.2 & 0.4 \\ 0 & 0.4 & 0.2 & 0.4 \\ 0 & 0 & 0.6 & 0.4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Problem 13

(a) We want to show that the complete conditional for μ is normal and the complete conditiomal for σ^2 is inverse gamma. We also want to find the corresponding parameters.

First note that,

$$\begin{aligned} p(\mu, \phi|x) &= \frac{p(x, \mu, \phi)}{p(x)} \\ &\propto p(x, \mu, \phi) \qquad\qquad (*) \end{aligned}$$

since denominator does not involve μ or ϕ.

Second, note that the likelihood function is given by

$$L(\mu, \phi) = (2\pi)^{-n/2}\phi^{n/2} exp\left(-\frac{\phi}{2}\sum_{i=1}^{n}(x_i - \mu)^2\right) \qquad (**)$$

Third, the joint density of (x, μ, ϕ) is given by

$$p(x, \mu, \phi) = (constant) \times \phi^{n/2} exp\left(-\frac{\phi}{2}\sum_{i=1}^{n}(x_i - \mu)^2\right) \times exp\left(-\frac{1}{2s}(\mu - m)^2\right) \times \phi^{a-1} exp\left(-b\phi\right) \qquad (***)$$

(1) Complete conditional for μ:

$$\begin{aligned} p(\mu|\phi, x) &= \frac{p(\mu, \phi|x)}{p(\phi|x)} \\ &\propto \frac{p(x, \mu, \phi)}{p(\phi|x)} \text{ according to (*)} \\ &\propto p(x, \mu, \phi) \text{ since denominator does not involve } \mu \end{aligned}$$

Hence, we perform some algebra to find the parameters

$$\begin{aligned} p(\mu|\phi, x) &\propto p(x, \mu, \phi) \\ &\propto \phi^{n/2} exp\left(-\frac{\phi}{2}\sum_{i=1}^{n}(x_i - \mu)^2\right) \times exp\left(-\frac{1}{2s}(\mu - m)^2\right) \times \phi^{a-1} exp\left(-b\phi\right) \text{ from (***)} \\ &\propto exp\left(-\frac{\phi}{2}\sum_{i=1}^{n}(x_i - \mu)^2\right) \times exp\left(-\frac{1}{2s}(\mu - m)^2\right) \\ &\propto exp\left(-\frac{1}{2}\left(\phi\sum x^2 - \phi 2\sum x\mu + \phi n\mu^2 + \frac{1}{s}\mu^2 - \frac{1}{s}2\mu m + \frac{1}{s}m^2\right)\right) \\ &\propto exp\left(-\frac{1}{2}\left(\mu^2\left(n\phi + \frac{1}{s}\right) + \phi\sum x^2 - \phi 2\sum x\mu - \frac{1}{s}2\mu m + \frac{1}{s}m^2\right)\right) \\ &\propto exp\left(-\frac{1}{2}\left(n\phi + \frac{1}{s}\right)\left(\mu - \frac{1}{\phi + s^{-1}}\left(\phi\sum x + \frac{1}{s}m\right)\right)^2\right) \end{aligned}$$

Hence, $(\mu|\phi, x) \sim N\left(\frac{1}{n\phi + s^{-1}}\left(\phi\sum_{i=1}^{n} x_i + \frac{1}{s}m\right), n\phi + \frac{1}{s}\right)$

(2) Complete conditional for σ^2:

$$\begin{aligned}
p(\phi|\mu, x) &= \frac{p(\mu, \phi|x)}{p(\mu|x)} \\
&\propto \frac{p(x, \mu, \phi)}{p(\mu|x)} \text{ according to (*)} \\
&\propto p(x, \mu, \phi) \text{ since denominator does not involve } \phi
\end{aligned}$$

Hence, we perform some algebra to find the parameters

$$\begin{aligned}
p(\phi|\mu, x) &\propto p(x, \mu, \phi) \\
&\propto \phi^{n/2} exp\left(-\frac{\phi}{2}\sum_{i=1}^{n}(x_i - \mu)^2\right) \times exp\left(-\frac{1}{2s}(\mu - m)^2\right) \times \phi^{a-1} exp\left(-b\phi\right) \text{ from (***)} \\
&\propto \phi^{n/2} exp\left(-\frac{\phi}{2}\sum_{i=1}^{n}(x_i - \mu)^2\right) \times \phi^{a-1} exp\left(-b\phi\right) \\
&\propto \phi^{n/2+a-1} exp\left(-\phi\left(b + \frac{1}{2}\sum(x - \mu)^2\right)\right)
\end{aligned}$$

And so, $(\phi|\mu, x) \sim Gamma\left(\frac{n}{2} + a - 1, b + \frac{1}{2}\sum_{i=1}^{n}(x_i - \mu)^2\right)$.

Therefore, $(\sigma^2|\mu, x) \sim InvGamma\left(\frac{n}{2} + a - 1, \frac{1}{b + \frac{1}{2}\sum_{i=1}^{n}(x_i - \mu)^2}\right)$

(b)

(c)

(d)