STAT 531: Assignment # 3

Due on November 5, 2020

Dr. Lu TTh 11:00 am

By: Jana Osea (30016679)

Contents

Problem 1

(a) We want to calculate the 2-step transition probability matrix.

$$P^2 = \begin{pmatrix} 0.4 & 0.6 \\ 0.6 & 0.4 \end{pmatrix} \begin{pmatrix} 0.4 & 0.6 \\ 0.6 & 0.4 \end{pmatrix} = \begin{pmatrix} 0.52 & 0.48 \\ 0.48 & 0.52 \end{pmatrix}$$

(b) We want to find $P(X_3 = I | X_0 = I)$.

$$P^3 = \begin{pmatrix} 0.4 & 0.6 \\ 0.6 & 0.4 \end{pmatrix} \begin{pmatrix} 0.4 & 0.6 \\ 0.6 & 0.4 \end{pmatrix} \begin{pmatrix} 0.4 & 0.6 \\ 0.6 & 0.4 \end{pmatrix} = \begin{pmatrix} 0.496 & 0.504 \\ 0.504 & 0.496 \end{pmatrix}$$

And so $P(X_3 = I | X_0 = I) = 0.496$

Problem 2

(a) We want to find $P(X_2 = 2 | X_0 = 1)$.

$$P^2 = \begin{pmatrix} 0.8 & 0.2 \\ 0.4 & 0.6 \end{pmatrix} \begin{pmatrix} 0.8 & 0.2 \\ 0.4 & 0.6 \end{pmatrix} = \begin{pmatrix} 0.72 & 0.28 \\ 0.56 & 0.44 \end{pmatrix}$$

And so $P(X_2 = 2 | X_0 = 1) = 0.28$.

(b) We want to find π_2 in $\boldsymbol{\pi} = \lim_{n\to\infty} \boldsymbol{\pi}^{(n)} = (\pi_1 \quad \pi_2)$. We start with

$$\begin{aligned} \begin{pmatrix} \pi_1 & \pi_2 \end{pmatrix} \begin{pmatrix} 0.8 & 0.2 \\ 0.4 & 0.6 \end{pmatrix} &= \begin{pmatrix} \pi_1 & \pi_2 \end{pmatrix} \\ \to 0.8\pi_1 + 0.4\pi_2 &= \pi_1 \\ 0.2\pi_1 + 0.6\pi_2 &= \pi_2 \\ \pi_1 + \pi_2 &= 1 \\ \to \underbrace{\begin{pmatrix} -0.2 & 0.4 \\ 1 & 1 \end{pmatrix}}_{A} \begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ det(A) = -0.2 - 0.4 &= -0.6 \end{aligned}$$

We calculate each part as

$$\begin{aligned} \pi_1 &= \frac{1}{-0.6} det \begin{pmatrix} 0 & 0.4 \\ 1 & 1 \end{pmatrix} \\ &= \frac{-0.4}{-0.6} \\ &= \frac{2}{3} \end{aligned}$$

$$\begin{aligned} \pi_2 &= \frac{1}{-0.6} det \begin{pmatrix} -0.2 & 0 \\ 1 & 1 \end{pmatrix} \\ &= \frac{-0.2}{-0.6} \\ &= \frac{1}{3} \end{aligned}$$

Therefore, the probability that Nov. 1 next year Rainbow will rain is $\pi_2 = \frac{1}{3}$.

Problem 3

(a) We want to find the distribution π_2 of X_2. We find

$$\begin{aligned} P^{(2)} &= (0 \;\; 1)P^2 \\ \rightarrow P^{(2)} &= (0 \;\; 1)\begin{pmatrix} 0.8 & 0.2 \\ 0.1 & 0.9 \end{pmatrix}\begin{pmatrix} 0.8 & 0.2 \\ 0.1 & 0.9 \end{pmatrix} \\ &= (0 \;\; 1)\begin{pmatrix} 0.66 & 0.34 \\ 0.17 & 0.83 \end{pmatrix} \\ &= (0.17 \;\; 0.83) \end{aligned}$$

Hence, the distribution of X_2 is given by $X_2 \sim (0.17 \;\; 0.83)$.

(b) We want to find the steady-state distribution of X_n.

$$\begin{aligned} \begin{pmatrix} \pi_1 & \pi_2 \end{pmatrix}\begin{pmatrix} 0.8 & 0.2 \\ 0.1 & 0.9 \end{pmatrix} &= \begin{pmatrix} \pi_1 & \pi_2 \end{pmatrix} \\ \rightarrow 0.8\pi_1 + 0.1\pi_2 &= \pi_1 \\ 0.2\pi_1 + 0.9\pi_2 &= \pi_2 \\ \pi_1 + \pi_2 &= 1 \\ \rightarrow \underbrace{\begin{pmatrix} -0.2 & 0.1 \\ 1 & 1 \end{pmatrix}}_{A}\begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ det(A) = -0.2 - 0.1 &= -0.3 \end{aligned}$$

We calculate each part as

$$\begin{aligned} \pi_1 &= \frac{1}{-0.3} det\begin{pmatrix} 0 & 0.1 \\ 1 & 1 \end{pmatrix} \\ &= \frac{-0.1}{-0.3} \\ &= \frac{1}{3} \end{aligned}$$

$$\begin{aligned} \pi_2 &= \frac{1}{-0.3} det\begin{pmatrix} -0.2 & 0 \\ 1 & 1 \end{pmatrix} \\ &= \frac{-0.2}{-0.3} \\ &= \frac{2}{3} \end{aligned}$$

Therefore, the distribution of X_n is given by $X_n \sim \left(\frac{1}{3} \;\; \frac{2}{3}\right)$

Problem 5

(a) In order to show that P is a regular, there must exist some power n such that all entries in the P^n are greater than but not equal to 0.

$$P = \begin{pmatrix} 0.8 & 0.14 & 0.04 & 0.02 \\ 0 & 0.6 & 0.3 & 0.1 \\ 0 & 0 & 0.65 & 0.35 \\ 0.9 & 0 & 0 & 0.1 \end{pmatrix}$$

$$\rightarrow P^3 = \begin{pmatrix} 0.5678 & 0.20972 & 0.15012 & 0.07236 \\ 0.2295 & 0.2286 & 0.35535 & 0.18655 \\ 0.48825 & 0.0441 & 0.287225 & 0.180425 \\ 0.6732 & 0.189 & 0.0936 & 0.0442 \end{pmatrix}$$

Since all entries of P^3 greater than but not equal to 0, P is a regular Markov chain.

(b) We want to find the steady state distribution $\boldsymbol{\pi}$.

$$\begin{pmatrix} \pi_1 & \pi_2 & \pi_3 & \pi_4 \end{pmatrix} \begin{pmatrix} 0.8 & 0.14 & 0.04 & 0.02 \\ 0 & 0.6 & 0.3 & 0.1 \\ 0 & 0 & 0.65 & 0.35 \\ 0.9 & 0 & 0 & 0.1 \end{pmatrix} = \begin{pmatrix} \pi_1 & \pi_2 & \pi_3 & \pi_4 \end{pmatrix}$$

$$\begin{aligned} \rightarrow 0.8\pi_1 + 0.9\pi_4 &= \pi_1 \\ 0.14\pi_1 + 0.6\pi_2 &= \pi_2 \\ 0.04\pi_1 + 0.3\pi_2 + 0.65\pi_3 &= \pi_3 \\ 0.02\pi_1 + 0.1\pi_2 + 0.35\pi_3 + 0.1\pi_4 &= \pi_4 \\ \pi_1 + \pi_2 + \pi_3 + \pi_4 &= 1 \end{aligned}$$

$$\rightarrow \underbrace{\begin{pmatrix} -0.2 & 0 & 0 & 0.9 \\ 0.14 & -0.4 & 0 & 0 \\ 0.04 & 0.3 & -0.35 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}}_{A} \begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \\ \pi_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$det(A) = -0.2503$$

We calculate each part as

$$\begin{aligned} \pi_1 &= \frac{1}{-0.2503} det \begin{pmatrix} 0 & 0 & 0 & 0.9 \\ 0 & -0.4 & 0 & 0 \\ 0 & 0.3 & -0.35 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \\ &= 0.503 \end{aligned}$$

$$\begin{aligned} \pi_2 &= \frac{1}{-0.2503} det \begin{pmatrix} -0.2 & 0 & 0 & 0.9 \\ 0.14 & 0 & 0 & 0 \\ 0.04 & 0 & -0.35 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \\ &= 0.176 \end{aligned}$$

$$\pi_3 = \frac{1}{-0.2503} det \left(\begin{array}{cccc} -0.2 & 0 & 0 & 0.9 \\ 0.14 & -0.4 & 0 & 0 \\ 0.04 & 0.3 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{array} \right)$$
$$= 0.209$$

$$\pi_4 = \frac{1}{-0.2503} det \left(\begin{array}{cccc} -0.2 & 0 & 0 & 0 \\ 0.14 & -0.4 & 0 & 0 \\ 0.04 & 0.3 & -0.35 & 0 \\ 1 & 1 & 1 & 1 \end{array} \right)$$
$$= 0.112$$

`Therefore, the steady state distribution of $X_n$ is given by $\boldsymbol{\pi} = \left(\begin{array}{cccc} 0`

Problem 8

The Markov Chain is given by the figure below.

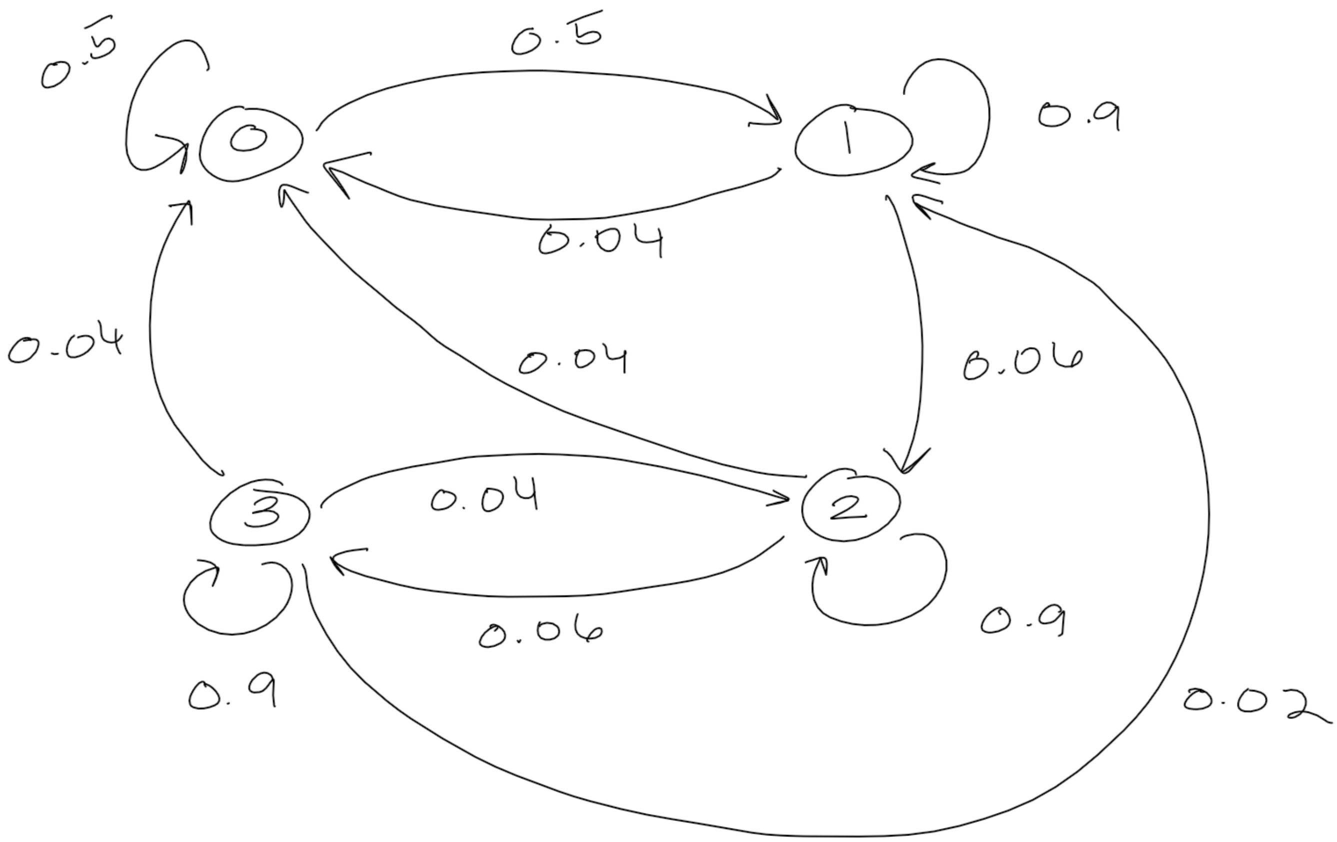


Figure 1: Markov Chain for Lan car reports

The transition matric P is given below:

$$P = \begin{pmatrix} 0.5 & 0.5 & 0 & 0 \\ 0.04 & 0.9 & 0.06 & 0 \\ 0.04 & 0 & 0.9 & 0.06 \\ 0.04 & 0.02 & 0.04 & 0.9 \end{pmatrix}$$

Problem 9

The Markov Chain is given by the figure below.

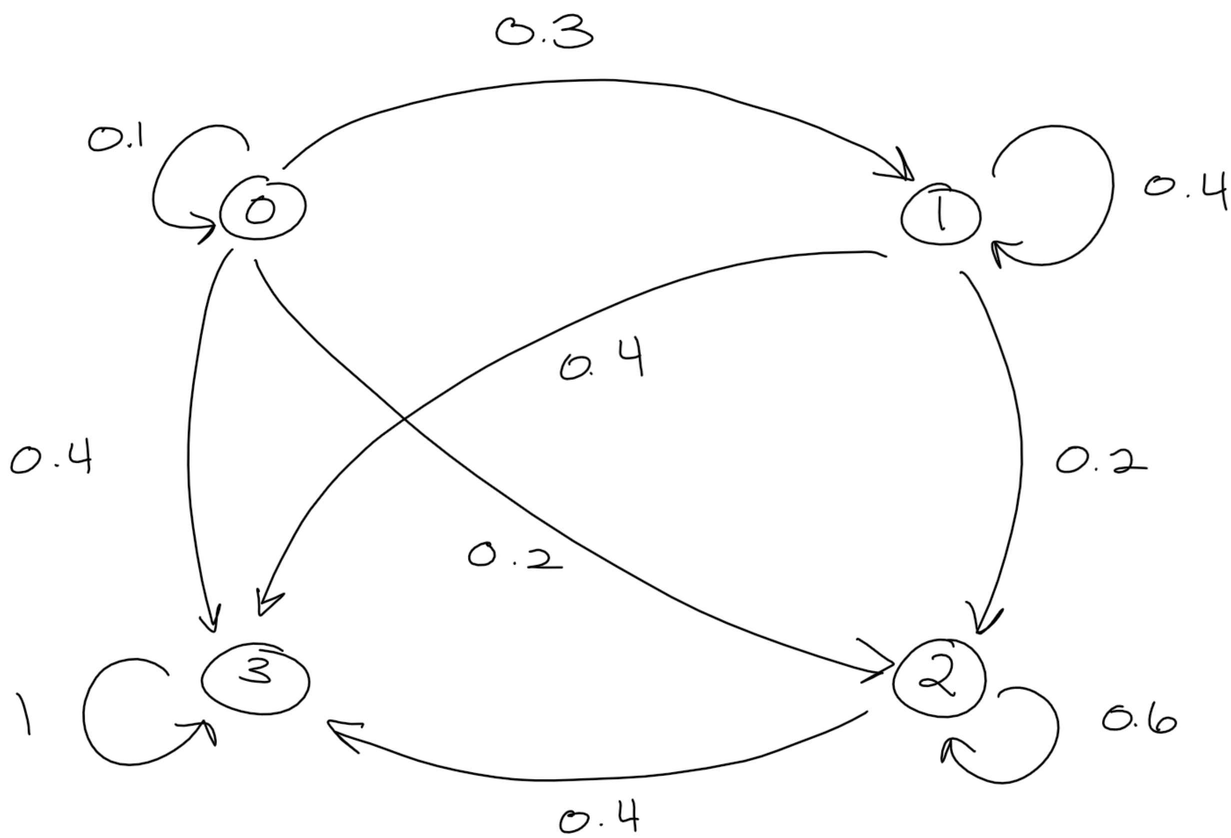


Figure 2: Markov Chain for Problem 9

The transition matric P is given below:

$$P = \begin{pmatrix} 0.1 & 0.3 & 0.2 & 0.4 \\ 0 & 0.4 & 0.2 & 0.4 \\ 0 & 0 & 0.6 & 0.4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Problem 13

(a) We want to show that the complete conditional for μ is normal and the complete conditiomal for σ^2 is inverse gamma. We also want to find the corresponding parameters.

First, note that Bayes' theorem gives,

$$p(\mu|\sigma^2, x) = \frac{p(\mu, \sigma^2, x)}{p(\sigma^2, x)} = \frac{L(x_1, \cdots, x_n|\mu, \sigma^2)p(\mu)p(\sigma^2)}{p(\sigma^2, x)} \propto L(x_1, \cdots, x_n|\mu, \sigma^2)p(\mu) \tag{13.1}$$

Since the denominator does not involve μ.

$$p(\sigma^2|\mu, x) = \frac{p(\mu, \sigma^2, x)}{p(\sigma^2, x)} = \frac{L(x_1, \cdots, x_n|\mu, \sigma^2)p(\mu)p(\sigma^2)}{p(\mu, x)} \propto L(x_1, \cdots, x_n|\mu, \sigma^2)p(\sigma^2) \tag{13.2}$$

Since the denominator does not involve σ^2.

Second, the priors are given by:

$$p(\mu) = \frac{1}{\sqrt{2\pi s^2}} exp\left(-\frac{1}{2s^2}(\mu - m)^2\right) \tag{13.3}$$

$$p(\sigma^{-2}) = \frac{1}{\Gamma(a)b^a}(\sigma^{-2})^{a-1} exp\left(\frac{1}{\sigma^2 b}\right) \tag{13.4}$$

Third, the likelihood function is given by

$$L(x_1, \cdots, x_n|\mu, \sigma^2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} exp\left(-\frac{1}{2\sigma^2}(x_i - \mu)^2\right) = \frac{1}{(\sqrt{2\pi\sigma^2})^n} exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \mu)^2\right) \tag{13.5}$$

(1) **Complete Conditional for μ**

$$\begin{aligned}
p(\mu|\sigma^2, x) &\propto L(x_1, \cdots, x_n|\mu, \sigma^2) \times p(\mu) && \text{by (13.1)} \\
&\propto \frac{1}{(\sqrt{2\pi\sigma^2})^n} exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \mu)^2\right) \times \frac{1}{\sqrt{2\pi s^2}} exp\left(\frac{1}{2s^2}(\mu - m)^2\right) && \text{by (13.3, 13.5)} \\
&\propto exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \mu)^2 - \frac{1}{2s^2}(\mu - m)^2\right) \\
&\propto exp\left(-\frac{1}{2}\frac{\sigma^2 + ns^2}{\sigma^2 s}\left(\mu - \frac{s\sum_{i=1}^{n} x_i + \sigma^2 m}{\sigma^2 + ns^2}\right)^2\right)
\end{aligned}$$

Hence, $(\mu|\sigma^2, x) \sim N\left(\frac{s^2\sum_{i=1}^{n} x_i + \sigma^2 m}{\sigma^2 + ns^2}, \frac{\sigma^2 s}{\sigma^2 + ns^2}\right)$

(2) **Complete Conditional for σ^2**

$$\begin{aligned}
p(\sigma^2|\mu, x) &\propto L(x_1, \cdots, x_n|\mu, \sigma^2) \times p(\sigma^2) && \text{by (13.2)} \\
&\propto \frac{1}{(\sqrt{2\pi\sigma^2})^n} exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \mu)^2\right) \times \frac{1}{\Gamma(a)b^a}(\sigma^{-2})^{a-1} exp\left(\frac{1}{\sigma^2 b}\right) && \text{by (13.4, 13.5)} \\
&\frac{1}{\sigma^{2(n/2+a-1)}} exp\left(-\frac{1}{\sigma^2}\left(b + \frac{1}{2}\sum_{i=1}^{n}(x_i - \mu)^2\right)\right)
\end{aligned}$$

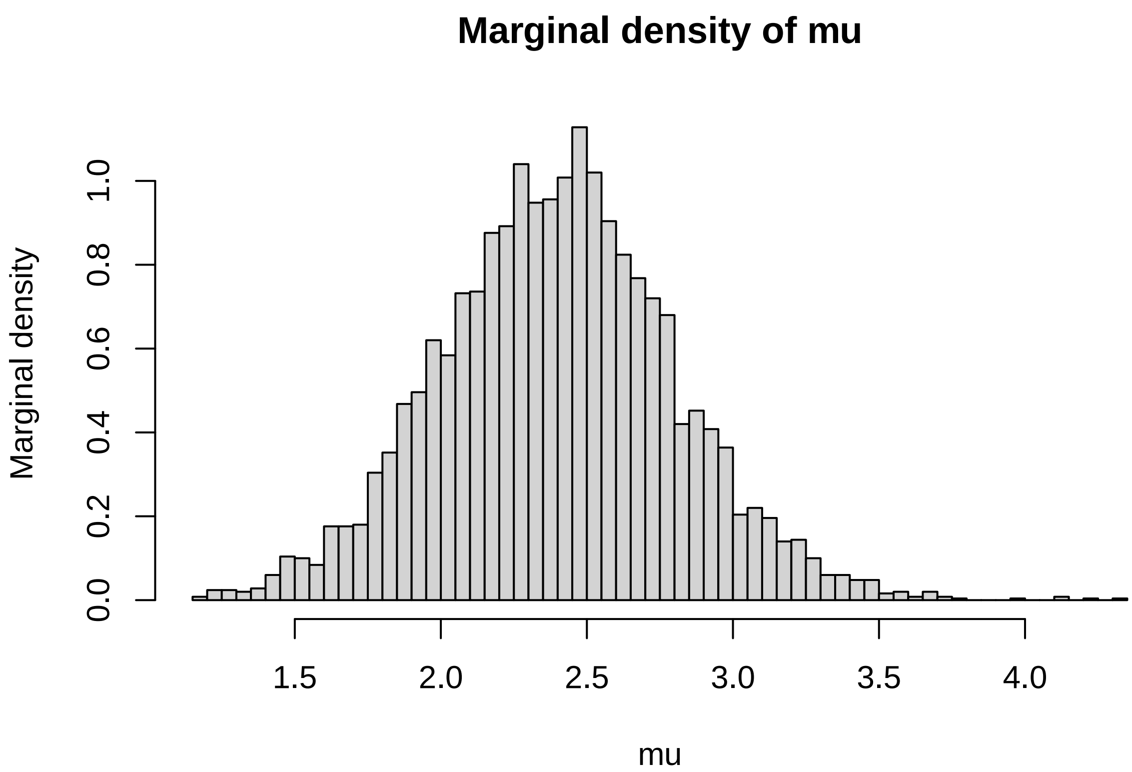
Hence, $(\sigma^2|\mu, x) \sim invGamma\left(\frac{n}{2} + a, b + \frac{1}{2}\sum_{i=1}^{n}(x_i - \mu)^2\right)$

(b) Given the conditional distributions above (a), the Gibbs sampling algorithm is specified below:

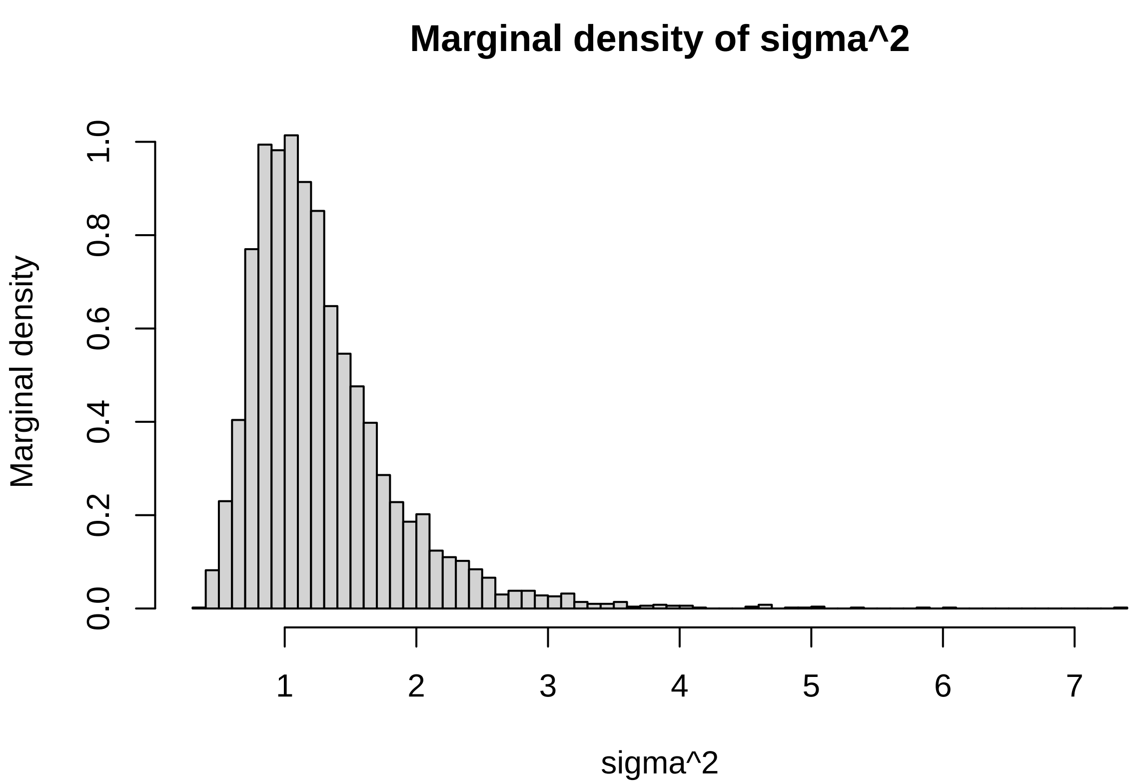
1. Initialize σ_0^2 and generate μ_1 from $p(\mu|\sigma^2, x = \sigma_0^2)$.
2. Using μ_1, generate σ_1^2 from $p(\sigma^2|\mu, x = \mu_1)$.
3. Repeat step 1 and 2 arbitrary n times (eg. $n = 10e5$). Throw away burn in from sampler $p\%$ (eg. 40%).

(c) Below is an R program

```
library(invgamma)

x = c(1.45, 2.08, 1.62, 1.51, 1.94, 1.43, 1.49, 1.10, 2.14, 2.29)
n = 10; m = 1; s = 0.2; s2 = s^2; a = 2; b = 1
Nsim = 10000
mu1 = 0.6; sigma1 = 0.4

mu = rep(0, Nsim)
sigma2 = rep(0, Nsim)

sigma2[1]=sigma1^2
mu[1] = mu1

set.seed(1)
for (i in 2:Nsim) {
  mu[i] = rnorm(1,
    mean = (1/sigma2[i-1]+n*s2) * (s2*sum(x)+sigma2[i-1]*m),
    sd = sqrt((sigma2[i-1]*s)/(sigma2[i-1] + n*s2))
    )
  sigma2[i] = 1/rgamma(1,
                    shape = (n/2)+a,
                    rate = (1/2)*(sum((x - mu1)^2)) + b
                    )
}

mu.final = tail(mu, 5000)
sigma2.final = tail(sigma2, 5000)
hist(mu.final,nclass=75,freq=FALSE,xlab="mu",ylab="Marginal density",main="Marginal density of mu")
```




```
hist(sigma2.final,nclass=75,freq=FALSE,xlab="sigma^2",ylab="Marginal density",main="Marginal density of
```



Commentary: The plots above show that μ seems to follow a normal distribution. As well, σ^2 seems to follows an inverse Gamma distribution.

(d)
```r
mean(mu.final)
```

```
## [1] 2.399519
```

```r
mean(sigma2.final)
```

```
## [1] 1.283088
```

```r
mean(x)
```

```
## [1] 1.705
```

```r
var(x)
```

```
## [1] 0.1472722
```

The mean of μ is 2.399519 and the mean of σ^2 is 1.283088. The estimates differ from the sample.

Problem 14

(a) We want to derive the posterior distribution $f(\theta|X = (x_1, x_2, x_3, x_4))$ up to a constant factor.

The likelihood of θ given x_1, x_2, x_3, x_4 is given by

$$L(\theta|X = (x_1, x_2, x_3, x_4)) = \frac{n!}{x_1!x_2!x_3!x_4!}\left(\frac{1}{2}+\frac{\theta}{4}\right)^{x_1}\left(\frac{1-\theta}{4}\right)^{x_2}\left(\frac{1-\theta}{4}\right)^{x_3}\left(\frac{\theta}{4}\right)^{x_4}$$

We assume that $\theta \sim Unif(0, 1)$, $p(\theta) = 1$, and $(x_1, x_2, x_3, x_4) = (125, 18, 20, 34)$.we find the posterior distribution to a constant factor by

$$\begin{aligned} p(\theta|X = (125, 18, 20, 34)) &\propto L(\theta|X = (125, 18, 20, 34)) \times p(\theta) \\ &\propto \left(\frac{1}{2}+\frac{\theta}{4}\right)^{125}\left(\frac{1-\theta}{4}\right)^{18}\left(\frac{1-\theta}{4}\right)^{20}\left(\frac{\theta}{4}\right)^{34} \times 1 \\ &\propto \left(\frac{1}{2}+\frac{\theta}{4}\right)^{125}\left(\frac{1-\theta}{4}\right)^{38}\left(\frac{\theta}{4}\right)^{34} \qquad (14.1) \end{aligned}$$

(b) We perform a random walk Metropolis Hastings algorithm with theory and R code as follows:

Theory: We consider

$$Y_t = X_t + \epsilon_t$$

where ϵ_t is a derived from distribution g independed of X_t.

The candidate density $q(y|x)$ is now of the $g(y - x)$. And so the Markov chain associated with q is a random walk when the density g is symmetric around zero, i.e. $g(-t) = g(t)$.

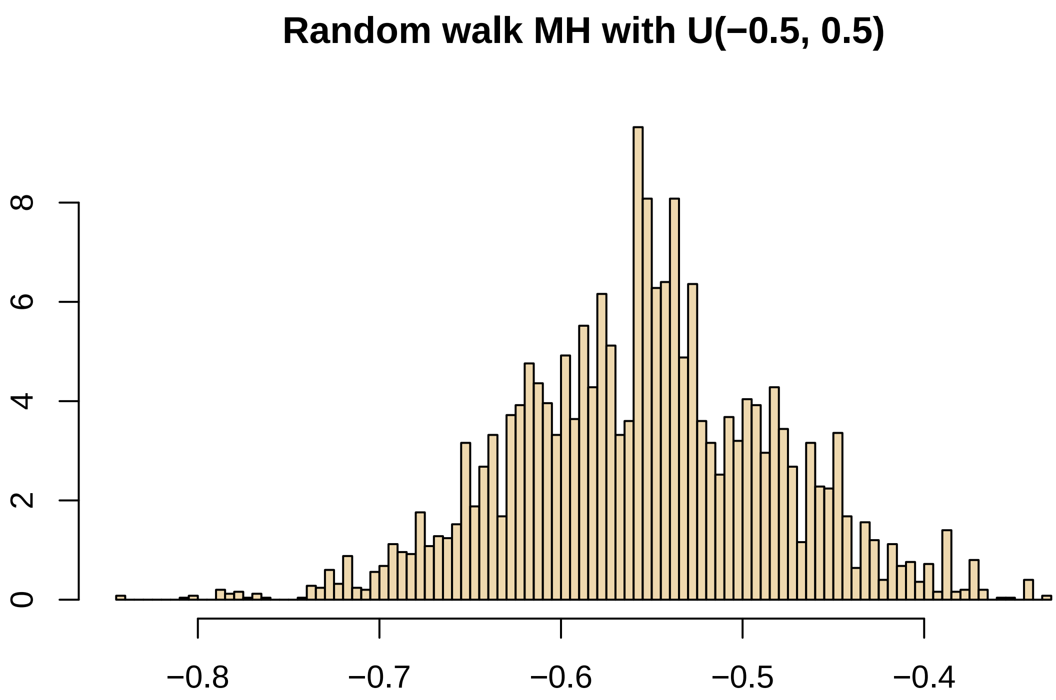
Algorithm: We want to simulate the target θ density proportional to (14.1) with the proposal density $g \sim Unif(-0.5, 0.5)$. Given x_t,

1. Generate $Y_{t+1} \sim g(y - x_t)$
2. Take

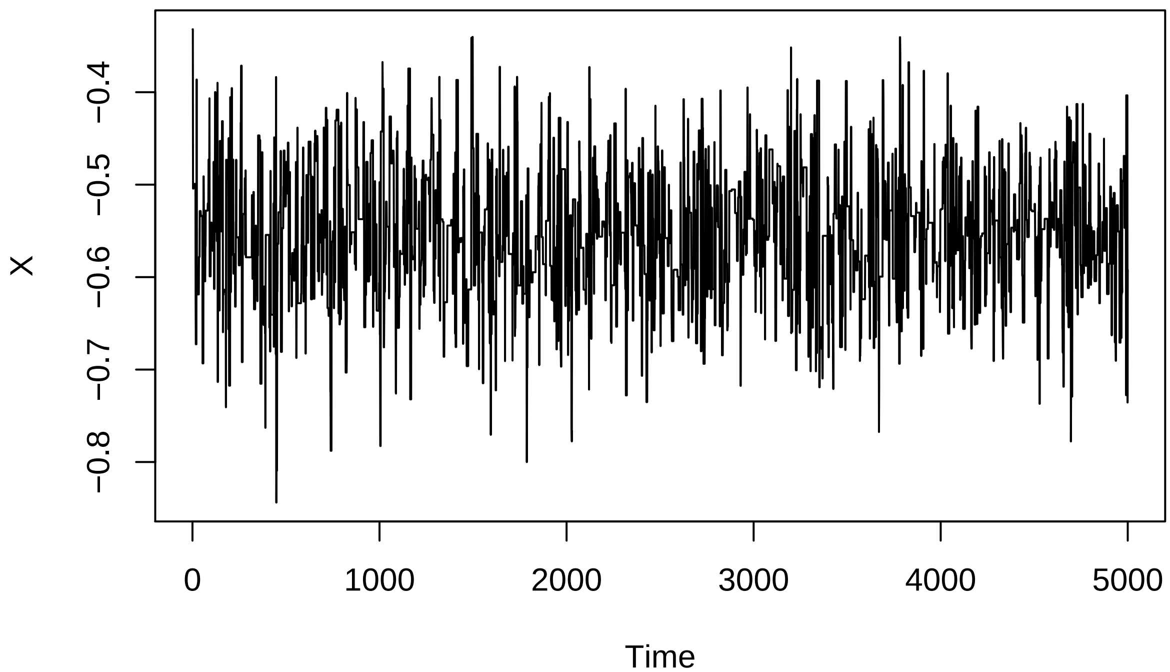
$$x_{t+1} = \begin{cases} Y_{t+1}, & \text{with probability } min\left\{\frac{p(Y_{t+1})}{p(x_t)}, 1\right\} \\ x_t, & \text{otherwise} \end{cases}$$

R code:

```
set.seed(3)
target = function(theta) (0.5 + 0.25 * theta)^125 * (0.25 * (1 - theta))^38 * (0.25 * theta)^34
NSim = 5000
X = Y = rep(0, NSim)

X[1] = runif(1,-0.5,0.5)

for (t in 2:NSim) {
  Y[t] = X[t-1] + runif(1,-0.5,0.5)
  Test = target(Y[t]) / target(X[t-1])
  X[t] = ifelse(runif(1) < min(1, Test), Y[t], X[t-1])
}

hist(X, nclass=75, freq=FALSE, xlab="",ylab="",col="wheat2", main="Random walk MH with U(-0.5, 0.5)")
```




```r
plot(ts(X), ylab = "X")
```



Commentary: The histogram appears to be centered around -0.5 and appears to have a multinomial shape.

(c)
```r
X.final = tail(X, 4000)
mean(X.final)
```

```
## [1] -0.5533081
```

```r
var(X.final)
```

```
## [1] 0.005062452
```

After discarding the burn-in samples from (b), we estimate that the posterior mean is -0.5533081 and the posterior variance is 0.005062452.

(d) We use R to find the normalizing constant

```r
integrate(target, 0,1)
```

```
## 5.84345e-91 with absolute error < 5.7e-93
```

Hence, the normalizing constant is 5.84345e-91.