Mathematics

Part I: Calculus

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Warsaw School of Economics

Organisation

Important contacts

- joint course by Maria Ekes and me
- name: Joanna Franaszek
- mail: jfrana@sgh.waw.pl,
- webpage (slides, notes, announcements etc.): `https://jfranaszek.github.io/`
- office hours: Monday 14:20, room TBA *(please e-mail me in advance)*

Grading

- 0-30 points: *average* of two written tests (mid-term and end-term, roughly 1/2 of the material)
- 0-30 points: final written exam (full material)
- 0-5 points: activity in exercise classes

score	grade
0-30	2
31-36	3
37-42	3.5
43-48	4
49-54	4.5
55+	5

Textbooks and other helpful resources

- official e-book
- WolframAlpha (desktop App or *https://www.wolframalpha.com/*
    - some tutorials are available
- Stewart James: Calculus Early Transcendentals, 2011, Brooks/Cole, Belmont CA,USA;
- Howard Anton, Chris Rorres: Elementary Linear Algebra with Suplemental Applications, 2010, Clarence Center Inc, Denver MA.

Sequences and limits

Key points

- definition of a sequence; arithmetic sequence; geometric sequence;
- bounded and monotone sequences;
- definition of a limit; simple arithmetic rules;
- squeeze theorem
- conditions for convergence;
- indeterminate forms

$$\frac{0}{0}, \frac{\infty}{\infty}, +\infty - \infty, 0 \cdot \infty, 1^{\infty}, 0^{0}, \infty^{0}$$

- the magical number $\mathbf{e}$

Sequence

Definition (Sequence)
A sequence is a function $a : \mathbb{N} \to \mathbb{R}$*, where* $\mathbb{N}$ *is the set of natural numbers, and* $\mathbb{R}$ *is the set of real numbers. The value* $a(n) = a_n$ *is called the* n*-th term of the sequence.*

Notation: a_n is a single number, while $(a_n)_{n=1}^{\infty}$ or $(a_n)_{n=1}^{\infty}$ or simply $(a_n)_n$ or $\{a_n\}_n$ denote a sequence.

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- A sequence is **bounded** if it is bounded from above and below

Limit of a sequence

Definition (Limit)
A number $g \in \mathbb{R}$is a limit of a sequence $(a_n)_n$ if:

$$\forall_{\epsilon>0}\exists_N\forall_{n>N}\ |a_n - g| < \epsilon.$$

If such a number exists, we say the sequence ***converges to g.***
Otherwise, the sequence is ***divergent****.*

Lemma
If $(a_n)_n$ has a limit, it is unique.

”Limits” in $+\infty$ or ∞

Definition (Improper limit)
A sequence $(a_n)_n$ has an improper limit in $+\infty$ ($-\infty$) if:

$$\forall_M \exists_{N_M} \forall_{n>N_M}\ a_n > M \quad (a_N < M).$$

We say the sequence ***diverges to infinity (minus infinity)*** *and denote it by $\lim_{n\to\infty} a_n = +\infty$ ($\lim_{n\to\infty} a_n = -\infty$) or, shorter $a_n \to \pm\infty$*

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- $\lim_{n\to\infty} a_n^{b_n} = a^b$, if only both sides well-defined (note: 0^0 **not** well-defined)

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- Note: rules for $b_n \to -\infty$ can be derived using last two slides:

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- important note: encountering indeterminate form does **not** necessarily mean the limit does not exist; it means we have to work harder to find it!
- important note 2: 'true' limits of statements in indeterminate form could be anything: a 'nice' number, zero, $-\infty$ etc.

”Classic” limits to remember

- exponential function diverges quicker than polynomial:

$$\lim_{n\to\infty} \frac{n^k}{2^n} = 0 \text{ for any } k > 0$$

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- but also (which is less obvious):

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Squeeze theorem

Very, very useful

Theorem (Squeeze theorem)
Let sequences $(a_n)_n, (b_n)_n, (c_n)_n$, satisfy:

$$a_n \leq b_n \leq c_n \quad \forall_n \quad (\text{or at least } \exists_N \forall_{n>N})$$

Then if $a_n \to g$ and $c_n \to g$, it must be that $b_n \to g$.

Theorem (Divergence)
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Then if $a_n \to \infty$, it must be that $b_n \to \infty$.

Euler's number

Theorem (Monotone convergence theoretm)
Every monotone and bounded sequence converges (to a proper limit).

Example
Let $a_n = \left(1 + \frac{1}{n}\right)^n$. We will show that the sequence a_n is strictly increasing and bounded and therefore has a limit.

Euler's number

Definition

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Note: this is a *definition* of e (one of few alternatives).

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Special case: $\left(1+\frac{c}{n}\right)^n = e^c$.

Functions of one variable

Function. Domain and image.

Definition

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- X is the **domain** of f (also denoted D_f, especially if we need to determine it!)
- Y is the **co-domain**; but usually we are interested in:
- $f(X) \subset Y$ is an **image** (sometimes: range) of f

Function one-to-one + ”onto” = bijection

- function is **one-to-one** (injective) if $x \neq y \Rightarrow f(x) \neq f(y)$;
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  - if $f : \mathbb{R} \to [0, +\infty)$ it is ’onto’
  - if $f : [0, +\infty) \to \mathbb{R}$ it is ’onto’ and one-to-one
    and has an inverse! $f^{-1}(y) = \sqrt{y}$

Image and preimage of a set

Let $f : X \to Y$ be a function

- image of $A \subset X$ is:

$$\{f(x) \in Y : x \in A\}$$

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- preimage of $B \subset Y$ is:

$$\{x \in X : f(x) \in B\}$$

  Notation: $f^{-1}(B)$ (not to be confused with an inverse function!) or $f^{-1}[B]$

Limit of a function

Definition (Limit points of a set)
*A point $x \in X$ is a **limit point** of X if there exists a seqence $(x_n)_n$ such that $x_n \in X \setminus x_0$ and $x_n \to x$. Otherwise, we call x an isolated point.*

Definition (Heine's limit)
Let x_0 be a limit point of X. A function $f(x) : X \to Y$ has a limit L in x_0 if for every sequence $(x_n)_n$ such that $x_n \in X \setminus x_0$ and $x_n \to x_0$ we have $f(x_n) \to L$.

Continuity

Definition
A function f is ***continuous*** *at $x_0 \in X$ if for every sequence $\lim_{n\to\infty} x_n \to x_0$ we have $\lim_{n\to\infty} f(x_n) = f(x_0)$.*

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- sum, difference, product of continuous functions is continuous
- quotient of continuous functions is continuous *if well-defined* (do not divide by 0!)
- polynomial, exponential, rational, logarithmic, trigonometric functions are continuous on their respective

Composition

Definition (Composition)
Let $f : X \to Y$ and $g : Y \to Z$. Then a composition $g \circ f =: h$ if a function $h : X \to Z$ defined by:

$$h(x) = g(f(x))$$

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Lemma
A composition of continuous functions is a continuous function.

Asymptotes

Definition (Vertical asymptote)
A function $f(x)$ has a vertical asymptote $x = c$ if f has at least one-sided improper limit in c:

$$\lim_{x \to c^-} f(x) = \pm\infty \text{ or } \lim_{x \to c^+} f(x) = \pm\infty$$

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- we do not require (and typically do not have) that $c \in D_f$.
- if f is continuous in x_0, there can't be an asymptote in x_0.
- $\Rightarrow$ vertical asymptotes may exist only in the limit points outside of the domain or in discontinuity points.

Asymptotes

Definition (Horizontal asymptote)
A function $f(x)$ has a horizontal asymptote $y = b$ if:

$$\lim_{x \to +\infty} f(x) = b \text{ or } \lim_{x \to -\infty} f(x) = b$$

Definition (Oblique asymptote)
A function $f(x)$ has an oblique asymptote $y = ax = b$ if:

$$\lim_{x \to +\infty} (f(x) - ax - b) = 0 \text{ or } \lim_{x \to -\infty} (f(x) - ax - b) = 0$$

Useful rules for oblique asymptotes:

$$a := \lim \frac{f(x)}{x}, \quad b := \lim f(x) - ax$$

Three improtant limits

It is useful to remember those three rules:

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

Click here for a beaufiful visual proof.

$$\lim_{x \to 0} \frac{\ln(1+x)}{x} = 1$$

Here is a proof with e, but I'll also show another one.

$$\lim_{x \to 0} \frac{e^x - 1}{x} = 1$$

Another visual proof.