Important Properties of Continuous Functions

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On the first day of class we were introduced to the set

$$C([a,b]) := \{f : [a.b] \to \mathbb{R} : f \text{ is continuous}\}$$

We further saw that endowed with the metric

$$\rho(f,g) = \max_{t\in[a,b]} |f(t) - g(t)|,$$

this set becomes a metric space. In my graduate experience so far, this space has proven to be one of the most important and avidly used/studied spaces in basic functional analysis. One of the reasons for this is because the elements of this space enjoy extremely rich properties, as one sees in Calculus and undergraduate Real Analysis. In this note, I want to recall some of the basic properties of the elements of this space because they are extremely common tools used in linear analysis proofs. I omit the proofs of these results and properties because either (a), we will prove them in a more general setting in this course, or (b) they are readily found in any Calculus or Analysis textbook.

Result 1: Linear Combinations of Continuous Functions:
For $f, g \in C[a,b]$, we first recall that $f \pm g$, $f \cdot g$, $f \circ g$, and $\frac{f}{g}$ are all continuous functions, where the last property requires g to be non-zero over $[a,b]$. Also note that for real scalars α (I only treat the real case here), the function αf is also continuous. Note that these properties together outline why $C[a,b]$ is a vector space, over the field $\mathbb{R}$ in this case.

Result 2: Interchange of Limits
Here we implicitly treat $\mathbb{R}$ as a metric space endowed with the usual absolute value metric. Let X be a subset of $\mathbb{R}$. Then f is continuous at $x_0 \in X$ if and only if for all sequence $\{x_n\}$ converging to x_0, with $x_n \in X \ \ \forall n$, one has

$$\lim_{n\to\infty} f(x_n) = f(x_0) = f\left(\lim_{n\to\infty} x_n\right)$$

So for continuous functions, the function and the sequential limit can be interchanged.

Similarly, recall that in calculus we saw that one way to define the continuity of $f : [a, b] \to \mathbb{R}$ is to require that

$$\lim_{x \to c} f(x) = f(c) = f\left(\lim_{x \to c} x\right)$$

So for continuous functions we also have that real variable limits and functions can be interchanged. I also want to note here that, independent of continuity we have for real valued functions that if $f(x) \leq g(x)$, $\lim_{x \to c} f(x) = M$, and $\lim_{x \to c} g(x) = N$, then $M \leq N$. In particular this says that non-strict ordering (in the usual sense) between two functions is preserved by limits. Note that the squeeze theorem for real valued functions is an immediate corollary to this result.

Result 3: Extreme Value Theorem
Let X be a compact subset of $\mathbb{R}$ and let $f : X \to \mathbb{R}$ be a continuous function. Then f attains both a maximum and a minimum on X. Note that by the Heine Borel theorem, this result holds for the elements of $C[a, b]$. Also note that this gives well defined meaning to the maximum metric we endowed $C[a, b]$ with in class.

Result 4: Mean Value Theorem
Let $f : [a.b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). The there exists $c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Result 5: Rolle's Theorem
Let $f : [a.b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Assume that $f(a) = f(b)$. Then for some $x \in (a, b)$ we have that $f'(x) = 0$. Note that this is an immediate corollary of the mean value theorem.