Some Brief Review

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In this note we review some of the basic results of set theory, function theory, and some small points of logic that will be needed in the material that follows.

Logic of The Empty Set
The logic accompanying the empty set is, while intuitive, often not explicitly outlined. This is done in what follows. If the negation of a statement is false, then the statement is true. A statement of the form:

$$\forall x \in \emptyset, x \text{ has some property}$$

is always true (vacuously). This is because its negation is always false, as a counter example to the above statement (an example that would render the negation true), would require there to be an element in the empty set, which is impossible by definition of the empty set. It immediately follows that the empty set is a subset of any set as for any set S, the statement $\forall x \in \emptyset, x \in S$ is vacuously true.

Definition: A family (or collection) of sets is said to be **pairwise disjoint** if any two sets in the family are disjoint.

Recall: The operations of union and intersection are commutative and associative.

"Or" is taken to be the inclusive or unless otherwise stated. That is, the statement, "either A or B is true" holds true if A is true or B is true or A and B are true, i.e. at least one is true. With this we have the distributive laws for the operations of unions and intersections.

$$(A \cup B) \cap C = (A \cap C) \cup (B \cap C) \qquad (1)$$

and

$$(A \cap B) \cup C = (A \cup C) \cap (B \cup C) \qquad (2)$$

Proof of (1): Let x be in $(A \cup B) \cap C$. Then x is in A or B, and x is in C. Thus, x is in A and C, or, x is in B and C, or x is in both A and C, and B and C, so that x is an element of $(A \cap C) \cup (B \cap C)$. Now let x be in $(A \cap C) \cup (B \cap C)$. Then, x is in A and C, or, x is in B and C, or x is in both A and C, and B and C. But this implies that x is in C and x is in A or B or both A and B, so that x is an element of $(A \cup B) \cap C$. $\square$

((2) is proven in an identical way.)

By a "universal" set we mean the largest possible set in the setting we are working in. That is, if we are performing analysis on the real line $\mathbb{R}$, then $\mathbb{R}$ would be the universal set and all other sets are subsets of $\mathbb{R}$. This is not a formal definition but rather intended to generate an image of the difference between the following two definitions.

Definition: The set difference $A - B$ is the set of all elements that are in A but are not in B.

Definition: The complement of a set A, denoted A^c, is the set of all elements that are not in A. Note that this set contains all elements of the universal set that are not in A.

Note the distinction between these two definitions: The first definition does not describe B^c. It is only the complement of B relative to A. The second definition describes the complement of A relative to the universal set. In set difference notation this would be denoted, $U - A$, where U denotes the universal set.

Definition: The symmetric difference of A and B denoted $A \Delta B$ is equal to $(A \cup B) - (A \cap B)$.

Proposition: Let U be a universal set. Let $A_1, A_2, \cdots$ be sets contained in U. Then the complement of the unions is the intersection of the complements and the complement of the intersections is the union of the complements. That is,

$$U - \bigcup_i A_i = \bigcap_i U - A_i \qquad (3)$$

and

$$U - \bigcap_i A_i = \bigcup_i U - A_i \qquad (4)$$

The above are known as DeMorgan's Laws.

Key Ideas

1) AN ELEMENT IS NOT IN THE INTERSECTION OF ANY NUMBER SETS IF AND ONLY IF IT IS NOT IN ONE OF THE SETS IN THE INTERSECTION, EQUIVALENTLY, IT MUST BE IN THE COMPLEMENT OF ONE OF THE ELEMENTS OF THE INTERSECTION.

2) AN ELEMENT IS NOT IN THE UNION OF ANY NUMBER OF SETS IF AND ONLY IF IT IS NOT IN ANY OF THE SETS.

The above two ideas are presented to serve as intuition for the proofs of (3) and (4).

Proof of (3) : Let x be in $U - \bigcup_i A_i$. Then x is not in any of the A_i. But then x is an element of $U - A_i$ for all indices i so that x is in the intersection of the complements of each of the A_i, which is precisely the right hand side. Now let x be in $\bigcap_i U - A_i$. Then x is in U and the complement of each of the A_i so that x is not in any of the A_i. Thus x is in the the universal set but not in $\bigcup_i A_i$. Thus x is in $U - \bigcup_i A_i$. $\square$

Proof of (4) : Let x be in the left-hand side of (4). Then x is in U and x is not in the intersection of the A_i. But then x is not in at least one of the A_i as if it were in all A_i it would be in there intersection. Thus x is in the complement of at least one of the A_i so that it is in the union of the complements of the A_i, which is precisely the right hand side of (4). Now let x be in the right-hand side of (4). Then, x is not in one of the sets A_i. This implies that x is not in the intersection of the A_i so that it is in the right hand side of (4). $\square$

FUNCTIONS:
The definition of a function and its domain, codomain, and range are assumed.

Definition: If $f : X \to Y$ is a function and $f(a) = b$ for $a \in X$ and b in Y, then b is said to be the image of a under f. All elements of X that are mapped to b make up the pre-image of b.

Definition: Let $f : X \to Y$ be a function. Let $A \subseteq X$. The set

$$f(A) = \{b \in Y : f(x) = b, x \in A\}$$

is said to be the image of A under f. The image of a subset of the domain is a subset of the codomain. (Set of all elements of codomain that get hit by elements of A.)

Definition: Let $f : X \to Y$ be a function. Let $B \subseteq Y$. The set

$$f^{-1}(B) = \{a \in X : f(a) = b, b \in B\}$$

is called the preimage of B under f. The preimage of a subset of the range of f is a subset of the domain of f.

Properties of Images vs. Properties of Pre-images: Let $A_1, A_2...$ be an arbitrary family of sets. Then,

$$f(\bigcup_i A_i) = \bigcup_i f(A_i) \qquad (5)$$

$$f^{-1}(\bigcup_i A_i) = \bigcup_i f^{-1}(A_i) \qquad (6)$$

$$f^{-1}(\bigcap_i A_i) = \bigcap_i f^{-1}(A_i) \qquad (7)$$

$$f^{-1}(A^c) = (f^{-1}(A))^c \qquad (8)$$

KEY IDEA: y is in $f(A)$ if and only if $y = f(x)$ for some x in A.
KEY IDEA: x is in $f^{-1}(A)$ if and only if $f(x)$ is in A.

In each proof, whether A or A_i is in the domain or codomain should be clear from the definition of image and pre-image and is thus not specified.

Proofs of (5): Let y be in the image of the union. Then $y = f(x)$ for some x in the union. This x is in one of the A_i so that y is also in $f(A_i)$ for that index i. Then y is in the union of the images of the A_i, which is the right hand side. Now let y be in the union of the images. Then, $y = f(x)$ for some x in some set A_i. But then x is in the union of the A_i so that $y = f(x)$ for x in the union of all the A_i. So y is in the image of the union. $\square$.

Proof of (6): Let x be in the left hand side. Then $f(x)$ is in the union of the A_i. So $f(x)$ is in some A_i. Then $x \in f^{-1}(A_i)$ for some A_i. So x is in the union of the pre-images. Now let x be in the right hand side. Then x is in $f^{-1}(A_i)$ for some i. So $f(x)$ is in A_i for some A_i. Then $f(x)$ is in the union of the A_i. Thus x is in the preimage of the union. $\square$

Proof of (7): Let x be in the left hand side of (7). Then $f(x)$ is in the intersection of the A_i. So $f(x)$ is in each of the A_i. Then x is in the pre-image of each of the A_i. So x is in the intersection of the preimages. Now let x be in the right hand side. Then $f(x)$ is in each of the A_i and is thus in their intersection. Then x is in the pre-image of the intersection $\square$.

Proof of (8): Let x be in the left hand side of 8. Then $f(x)$ is in A^c. Then $f(x)$ is not in A. Then x is not in the pre-image of A. So x is in $(f^{-1}(A))^c$. Now let x be in the right hand side. Then x is not in $f^{-1}(A)$ so that $f(x)$ is not in A. Then $f(x)$ is in A^c so that x is in $f^{-1}(A^c)$. $\square$

PRACTICE PROBLEMS: FROM KOLMOGOROV AND FOMIN - CHAPTER 1

(1) Claim: If $A \cup B = A$ and $A \cap B = A$, then $A = B$.

Proof: Let $x \in A$. Then x is in $A \cap B$ so that x is in B. Thus, $A \subseteq B$. Now let x be in B. Then x is in $A \cup B = A \cap B$, so that x is in A. $\square$

(2) Claim: If $x \in (A - B) \cup B$ then x need not be in A.
Proof: Consider an element x in $(A - B) \cup B$. Then it is possible that x is in B and not in A or $A - B$. As an example let $A = [0, 2]$ and $B = [1, 2]$ so that $A - B = [0, 1)$. Let $x = \frac{5}{4}$. Then x is in B so that x is in $(A - B) \cup B$. But x is both not in A or $A - B$ $\square$.

(3) If A is the set of all positive multiples of two and B is the set of all positive multiples of three, find $A \cap B$ and $A - B$. Soln: Let x be in $A \cap B$. Then x is a multiple of 2 and x is a multiple of 3. So three goes into x and 2 goes into x. By the fundamental theorem of arithmetic, 6 must go into x so that x is a multiple of 6. Thus, the intersection of $A \cap B$ is all multiples of six. $A - b$ is the set of all multiples of two that are not multiples of three. That is, $A - B$ is the set of all positive even numbers whose unique prime factorization does not contain three. $\square$

(4) Claim: $(A - B) \cap C = (A \cap C) - (B \cap C)$

Proof: Let x be in left hand side. Then x is in A, x is not in B, and x is in C. Then x is in A and C and not in the set of all elements that are in both B and C. So x is in the right hand side. Now let x be in the right hand side. Then x is in A and C and not in B. So x is in A but not B, and x is in C. Thus, x is in the left hand side. $\square$

Claim: $(A \cup B) - (A \cap B) = (A - B) \cup (B - A)$.
Proof: If x is in the left hand side, then x is in A or B but not both A and B. So x is in either A and not B or B and not A. Thus x is in the right hand side. Let x be in the right hand side. First note that the two sets on the right hand side are disjoint. Then, x must either be in A and not B or it must be in B and not A. Then x is in A or B but not both A and B. So x is in the left hand side. (Note that this proof really requires that $A \cup B$ be the universal set.) $\square$

(5) Claim: $\bigcup\limits_i A_i - \bigcup\limits_j B_j \subset \bigcup\limits_i \bigcup\limits_j A_i - B_j$.
Proof: Let x be in the left hand side. Then x is not in any of the B_j and x is at least one of the A_i. Immediately we have that x is in the right hand side. $\square$.

(6) Let A_n be the set of all positive integers divisible by n. What is $\bigcup_{n=2}^{\infty} A_n$ and $\bigcap_{n=2}^{\infty} A_n$.

The union is the set of all natural numbers greater than or equal to 2 as n divides n for all $n \in \mathbb{N}$. Due to the existence of prime numbers bigger than or equal to 2, the intersection is empty. $\square$

(7) Very Important: Calculate $\bigcup_{n=1}^{\infty}[a+\frac{1}{n}, b-\frac{1}{n}]$ and $\bigcap_{n=1}^{\infty}(a-\frac{1}{n}, b+\frac{1}{n})$.

The following proofs are very informal and give an idea of the rigorous argument. The Archimedean Property of the real line is typically needed to rigorously prove such statements.

For $n \in \mathbb{N}$ note that $a+\frac{1}{n} > a$ and $b-\frac{1}{n} < b$. As n increases we approach a from the right but never attain the value of a (but do attain all values bigger than a) as an end point. Similarly, we approach the value of b from the left but never attain the value of b (but do attain all smaller values] as an endpoint. So the above union is equal to (a, b).

For the case of the intersection let $n \in \mathbb{N}$.Then, $a-\frac{1}{n} < a$ and $b+\frac{1}{n} > b$. As n increases we approach a from the left, never attaining the value of $a-\frac{1}{n}$ (but attaining all larger values) as an end point. Similarly, we approach the value of b from the right never attaining the value of $b+\frac{1}{n}$. Thus the intersection is equal to $[a, b]$.

(8)Let A be the set of points lying on the curve $y = \frac{1}{x^{\alpha}}, (0 < x < \infty)$. Find the intersection of all such sets for $\alpha \geq 1$.
We simply find the point of intersection for all such curves. Setting $\frac{1}{x^{\alpha_1}} = \frac{1}{x^{\alpha_2}}$, we see that $x = 1$ is the only possible solution over all positive α_i. When $x = 1$, $y = 1$, so that $\cap_{\alpha} = (1, 1)$. $\square$

(9) Let $[a, b]$ denote a closed interval on the real line $\mathbb{R}$. Let $f(x) = frac(x)$ take an element in $[a, b]$ to its fractional part. As an example, on the interval $[1, 2], f(1.234000....) = 234000....$ Answer the following:

(a) Prove the image of f is the same for any closed interval of length 1.
(b) Is f injective or surjective or both?
(c)Calculate the preimage of f for the interval $[\frac{1}{4}, \frac{3}{4}]$.

(a) We first compute the image of f for an arbitrary closed interval of length one:

$$f([a, b]) = \{.x_1x_2... : x_i \in \mathbb{N} \cup \{0\} \forall n \in \mathbb{N}\}$$

Let $f(x) \in f([a, b])$. If $x = a$ or $x = b$, then $f(x) = .00000 = f(c)$, so that $f(x)$ is in $f([c, d])$. If $x \in (a, b)$ then $x = a.x_1x_2....$ Then $f(x) = .x_1x_2...$ which is precisely equal to $f(c.x_1x_2...)$ for some $y \in (c, d)$. Thus $f(x) \in$

$f([c,d])$. We thus have that $f([a,b]) \subseteq f([c,d])$. An identical argument using an arbitrary element of $f([c,d])$ gives the set equality we want. $\square$

(b) For any given interval $[a,b]$, with $|a-b|<1$ we immediately see that f is not injective as $f(a)=f(b)$ but a cannot equal b since $|a-b|=1$. For surjectivity, assume to the contrary that there existed y in $f([a,b])$ such that $y \neq f(x)$ for some $x \in [a,b]$. This would imply that $a.x_1x_2..., x_i \in \mathbb{N} \ \ \forall i$ is not in $[a,b]$. But $a < a.x_1x_2... < b$ so that $a.x_1x_2..$ is clearly in $[a,b]$, a contradiction. We thus have that no such y can exist so that f is surjective. $\square$

(c) $f^{-1}([\frac{1}{4},\frac{3}{4}])$=$[a.25x_1x_2, ..., a.75y_1y_2...]$ with x_i, y_i non-negative integers for all i. $\square$