Not Dense vs Nowhere Dense

Joseph Franks

In this note, we provide an example of a metric space setting where a set is not dense but fails to be nowhere dense.

Example:
Consider $\mathbb{R}$ endowed with the discrete metric, which we denote $(\mathbb{R}, d)$. We first show that the set $A = \{\{1\}, \{2\}\}$ is not dense in $\mathbb{R}$ with this metric. To see this, consider the ball of radius $\frac{1}{2}$ centered at 3, which we denote $B(\frac{1}{2}, 3)$. This ball is equal to the singleton set $\{3\}$ and thus does not intersect A, so every point of $\mathbb{R}$ is not a limit point of A, which implies that A is not dense in $\mathbb{R}$. However, we also claim that A fails to be nowhere dense in this space. Indeed, consider $B(\frac{1}{2}, 1) = \{1\}$. Then this ball is contained in $\overline{A} = A$, which implies that every point of this ball is a limit point of A, which is what it means for A to be dense in $B(\frac{1}{2}, 1)$. Since A is dense in some open ball, it fails to be nowhere dense.