Problem Set 0

Recalling Some Basic Results from Real Analysis and Their Proofs

NOTE: Earlier problems may be useful in proving later problems.

Question 1:
For all $x, y \in \mathbb{R}$, show the following chain of inequalities.

$$|x| - |y| \leq ||x| - |y|| \leq |x - y| \leq |x| + |y|$$

Question 2:
Let x, y, p be real numbers satisfyinbg $x, y \geq 0$ and $p \in (0, 1]$. Show that

$$(x + y)^p \leq (x)^p + (y)^p$$

Hint: Since $p \in (0, 1)$, there exists $m \in (0, 1)$ such that $p = 1 - m$.

Question 3
(a) Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers such that $\lim\limits_{n\to\infty} a_n = a$ and $\lim\limits_{n\to\infty} b_n = b$. Show that if there exists an $N \in \mathbb{N}$ such that $a_n \leq b_n$ for all $n \geq N$ then $a \leq b$.

(b)Give an example to show that $a_n < b_n \quad \forall n$ and $\lim\limits_{n\to\infty} a_n = a$ and $\lim\limits_{n\to\infty} b_n = b$, but $a \leq b$, as opposed to $a < b$.

Question 4
Show the following:

(a) Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable with $f(x) \geq 0$ for all $x \in [a, b]$. Show that $\int_a^b f(x)dx \geq 0$.

(b) Let $f, g : [a, b] \to \mathbb{R}$ be Riemann integrable with $f(x) \geq g(x)$ for all $x \in [a, b]$. Show that $\int_a^b f(x)dx \geq \int_a^b g(x)dx$.

(c) Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable with $f(x) \geq 0$ for all $x \in [a, b]$. Let $d \in \mathbb{R}$ satisfy $a < d < b$. Show that $\int_a^d f(x)dx \leq \int_a^b f(x)dx$.

Question 5:
Let $a_n \in \mathbb{R}$ for all $n \in \mathbb{N}$ and assume that $\sum_{n=1}^\infty a_n < \infty$, that is assume that the series $\sum_{n=1}^\infty a_n$ converges. Show that $\displaystyle\lim_{n\to\infty} \sum_{k=n+1}^{\infty} a_k = 0$.

Question 6:
Let S be a non-empty set of real numbers that is bounded above. Show that a is the supremum of S if and only if for all $\epsilon > 0$ there exists $s \in S$ such that $a - \epsilon < s$.

Note that an analogous result holds for infimums, and it is a good idea to state and prove this result as well.