Warm up Problems - Solutions

Recalling Some Useful Results from Real Analysis With Proof

Question 1:
For all $x, y \in \mathbb{R}$, show the following chain of inequalities.

$$|x| - |y| \leq ||x| - |y|| \leq |x - y| \leq |x| + |y|$$

Proof:
We first show that for all $x, z \in \mathbb{R}$, one has that $|x + z| \leq |x| + |z|$. Observe that

$$|x + z|^2 = (x + z)^2 = x^2 + 2xz + z^2 \leq |x|^2 + 2|x||z| + |z|^2 = (|x| + |z|)^2.$$

We thus have that $|x + z|^2 \leq (|x| + |z|)^2$. Taking the square root of both sides, noting that $f(x) = \sqrt{x}$ is monotonically increasing on $[0, \infty)$, proves the desired result. Letting $z = (-y)$ yields $|x - y| \leq |x| + |y|$. Also note that since $x \leq |x|$ for all $x \in \mathbb{R}$, it is immediate that $|x| - |y| \leq ||x| - |y||$. We lastly need to show that $||x| - |y|| \leq |x - y|$, a result known as the reverse triangle inequality. Observe that

$$|x| = |x - y + y| \leq |x - y| + |y| \implies |x| - |y| \leq |x - y|$$

and

$$|y| = |y - x + x| \leq |y - x| + |x| \implies -|x - y| \leq |x| - |y|.$$

Combining these two inequalities we have

$$-|x - y| \leq |x| - |y| \leq |x - y|$$

which implies the desired result. Putting everything together we have that

$$|x| - |y| \leq ||x| - |y|| \leq |x - y| \leq |x| + |y|,$$

as desired. $\square$

Question 2

Let x, y, p be real numbers satisfying $x, y \geq 0$ and $p \in (0, 1]$. Show that

$$(x+y)^p \leq (x)^p + (y)^p$$

Hint: Since $p \in (0, 1)$, there exists $m \in (0, 1)$ such that $p = 1 - m$.

Proof:

Note that the case of $x = 0$ or $y = 0$ is immediate and the case of $p = 1$ was shown above. We can thus assume that $x, y > 0$ and $p \in (0, 1)$. Using the hint we have that

$$\begin{aligned}
(x+y)^p &= (x+y)^{1-m} \\
&= x(x+y)^{-m} + y(x+y)^{-m} \\
&= \frac{x}{(x+y)^m} + \frac{y}{(x+y)^m} \\
&\leq \frac{x}{x^m} + \frac{y}{y^m} \\
&= xx^{-m} + yy^{-m} \\
&= x^{1-m} + y^{1-m} \\
&= x^p + y^p
\end{aligned}$$

proving the desired result. $\square$

Question 3:

(a) Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers such that $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$. Show that if there exists an $N \in \mathbb{N}$ such that $a_n \leq b_n$ for all $n \geq N$ then $a \leq b$.

Proof:

We first show that if a_n is a convergent sequence of real numbers with limit a and satisfying $a_n \geq 0$ for all $n \in \mathbb{N}$, then $a \geq 0$. Assume to the contrary that $a < 0$. Since $\lim_{n\to\infty} a_n = a$ and since $-a > 0$, we know that there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have that $|a_n - a| < -a$. This implies that $a_n < 0$, a contradiction. Now, consider the subsequences $A_{n_j} = a_N, a_{N+1}, a_{N+2}, \cdots$ and $B_{n_j} = B_N, B_{N+1}, B_{N+2}, \cdots$ of a_n and b_n respectively. Note that since $a_n \leq b_n$ for all $n \geq N$ we have that $B_{n_j} - A_{n_j} \geq 0$ for all $j \in \mathbb{N}$. Since every subsequence of a convergent sequence must converge to the same limit as the parent sequence we now know that

$$0 \leq \lim_{j\to\infty} B_{n_j} - \lim_{j\to\infty} A_{n_j} = \lim_{n\to\infty} (b_n - a_n) = b - a$$

This proves that $a \leq b$, which is what we wanted to show. $\square$

(b) Give an example to show that $a_n < b_n \quad \forall n$ and $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$, but $a \leq b$, as opposed to $a < b$.

Proof:

Take $a_n = \frac{1}{n}$ and $b_n = \frac{1}{2n}$. $\square$

Question 4 :
Show the following:

(a) Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable with $f(x) \geq 0$ for all $x \in [a, b]$. Show that $\int_a^b f(x)dx \geq 0$.

(b) Let $f, g : [a, b] \to \mathbb{R}$ be Riemann integrable with $f(x) \leq g(x)$ for all $x \in [a, b]$. Show that $\int_a^b f(x)dx \leq \int_a^b g(x)dx$.

(c) Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable and assume that $f(x) \geq 0$ for all $x \in [a, b]$. Let $d \in \mathbb{R}$ satisfy $a < d < b$. Show that $\int_a^d f(x)dx \leq \int_a^b f(x)dx$.

Proof of (a)
Using the limit (of a sequence of partial sums) definition of the Riemann integral we have

$$0 = \sum_{i=1}^{n} 0\frac{b-a}{n} \leq \sum_{i=1}^{n} f(x_i)\frac{b-a}{n}$$

Using question 3 on the above chain of inequalities we have

$$0 \leq \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\frac{b-a}{n} = \int_a^b f(x)dx,$$

as desired. $\square$

Proof of (b)
Since $f(x) \leq g(x)$ for all $x \in [a, b]$ we have that $g(x) - f(x) \geq 0$ for all $x \in [a, b]$. By part (a) we have that $\int_a^b g(x) - f(x)dx \geq 0$. Using the linearity of the integral we have that $\int_a^b f(x)dx \leq \int_a^b g(x)dx$, which is what we wanted to show. $\square$

Proof of (c)
Note that:

$$\int_a^b f(x)dx = \int_a^d f(x)dx + \int_d^b f(x)dx$$

By part (a), this is all we need to conclude the desired result. $\square$

Question 5:
Let $a_n \in \mathbb{R}$ for all $n \in \mathbb{N}$ and assume that $\sum_{n=1}^{\infty} a_n < \infty$, that is assume that the series $\sum_{n=1}^{\infty} a_n$ converges. Show that $\lim\limits_{n\to\infty} \sum\limits_{k=n+1}^{\infty} a_k = 0$.

Proof:
Define the sequence $S_n := \sum_{i=1}^{n} a_i$. By the convergence of $\sum_{n=1}^{\infty} a_n$ we have that for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$

$$\left|\sum_{i=1}^{n} a_i - \sum_{i=1}^{\infty} a_i\right| = \left|\sum_{i=n+1}^{\infty} a_i - 0\right| < \epsilon,$$

proving the desired result. $\square$

Question 6:
Let S be a non-empty set of real numbers that is bounded above. Show that the upper bound a of the set S is the supremum of S if and only if for all $\epsilon > 0$ there exists $s \in S$ such that $a - \epsilon < s$.

Proof: First let a be the supremum of S. Assume to the contrary that for some $\epsilon > 0$ there does not exist such an $s \in S$. Then, for all $s \in S$ we have that $s \leq a - \epsilon < a$. But by definition this implies that $a - \epsilon$ is an upper bound of the set S, which contradicts that a is the supremum of S. Now assume that for all $\epsilon > 0$ there exists an $s \in S$ such that $a - \epsilon < s$. Let a' be the supremum of S and assume to the contrary that $a \neq a'$. Then there exists $r \in \mathbb{R}$ such that $a' < r < a$. Let $\epsilon = a - r$. Then by assumption, we have that there exists $s \in S$ such that

$$a' < r = a - (a - r) = a - \epsilon < s$$

But this is a contradiction, as we assumed $a' \geq s$ for all $s \in S$. It follows that $a = a'$, completing the proof. $\square$

Question 7:
Let X be a compact subset of $\mathbb{R}$, that is, let X be closed and bounded. Let $f : X \to \mathbb{R}$ be a continuous function. Show that $\underset{t \in X}{sup}|f(t)| = \underset{t \in X}{max}|f(t)|$.

Proof:
Let X and f be as above. Since the absolute value function is continuous on X and since the composition of continuous function is continuous, we know from the extreme value theorem that there exists $t^* \in X$ such that

$$|f(t)| \leq |f(t^*)| = \underset{t \in X}{max}|f(t)|$$

for all $t \in X$. If we consider the set

$$S = \{|f(t)| : t \in X\},$$

the above show that $|f(t^*)|$ is an upper bound for S. From here, we just need to observe that $|f(t^*)|$ is the least upper bound of S. To do so, let M be any upper bound of S. Then for all $s \in S$ we have that $s \leq M$. However, note that since t^* is in X, we have that $|f(t^*)| \in S$. It follows that $|f(t^*)| \leq M$. We can thus conclude that

$$|f(t^*)| = \underset{t \in X}{max}|f(t)| = \underset{t \in X}{sup}|f(t)|$$

which is what we wanted to show. $\square$