Isometries and Homeomorphisms

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Citation Note:
The following construction is not mine. I found it on MSE and thought it was cool. The link is below.

https://math.stackexchange.com/questions/914823/shift-numbers-into-a-different-range

Two closed Intervals are Homeomorphic :
Note that the parent space setting here is $(\mathbb{R}, |\ |)$.

Lets say we want to find a map that bijectively maps $[a, b]$ into $[c, d]$. If we first consider $f(t) = t - a$ then $f([a, b]) = [0, b - a]$. Now consider $g(t) = \frac{1}{b-a}t$. Observe that $h([0, b - a]) = [0, 1]$. From here, consider $h(t) = (d - c)t$. Then $h([0, 1]) = [0, d - c]$. Lastly, consider $j(t) = c + t$. Then $j([0, d - c]) = [c, d]$. Note that each one of these individual maps is a bijection so that

$$\begin{aligned}
S(t) &= j \circ h \circ g \circ f(t) \\
&= j \circ h \circ g(t - a) \\
&= j \circ h \left(\frac{t - a}{b - a} \right) \\
&= j \left(\frac{d - c}{b - a}(t - a) \right) \\
&= c + \frac{d - c}{b - a}(t - a)
\end{aligned}$$

Since $S(t)$ is a composition of bijections, we have that $S(t)$ is a bijection from $[a, b]$ to $[c, d]$. Also note that each one of these maps is continuous so that $S(t)$ is also continuous. Direct computation gives that $S^{-1}(t) = (t - c)\frac{b-a}{d-c} + a$, which is also a continuous function. We thus have that any two closed bounded intervals are homeomorphic.

Two Closed intervals Need Not be Isometric

As subspaces of $(\mathbb{R}, |\ |)$ the above gives that $[0,1]$ and $[0,2]$ are homeomorphic. However, they cannot be isometric. Assume to the contrary they were. Then there would have to a bijection between $f : [0,1] \to [0,2]$ such that $|f(x)-f(y)| = |x-y|$. Now, since f is bijective, there must exists $x, y \in [0,1]$ such that $f(x) = 2$ and $f(y) = 0$. Then

$$|f(x) - f(y)| = 2 \text{ and } \sup_{x,y\in[0,1]} |x-y| = 1$$

proving that such a map between these spaces is impossible. Note that it is easy to show that every isometry is a homeomorphism which follows immediately from the fact that f is bijective and the equality obtained from being an isometry. If this is not clear, it is a good exercise to prove it, and one that is not too hard to do.

Example Problem:

Lets show that $C[0,1]$ and $C[1,2]$ are isometric. We claim that the map $\phi : (C[0,1]) \mapsto C([1,2]) := \phi(f) = f(t-1)$ is an isometry between these spaces.

Direct computation yields that $\phi^{-1}(f) = f(t+1)$. Indeed

$$\phi^{-1}(\phi(f)) = \phi^{-1}(f(t-1)) = f(t-1+1) = f(t)$$

and

$$\phi(\phi^{-1}(f)) = \phi(f(t+1)) = f(t+1-1) = f(t)$$

so that ϕ is bijective. Now, to verify distance is preserved, observe that since the map $f : [1,2] \to [0,1]$ defined $f(t) = t-1$ is a bijection we have,

$$\sup_{t\in[1,2]} |f(t-1) - g(t-1)| = \sup_{t\in[0,1]} |f(t) - g(t)|$$

so that ϕ is an isometry. $\square$