An Example of a Non-Separable Metric Space

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Notational Note: Throughout, $B(\epsilon, x)$ denotes the open ball, center x, radius ϵ.

Definition:
A sequence of real numbers is said to be bounded if there exists an $M \geq 0$ such that $|x_n| \leq M$ for all $n \in \mathbb{N}$. We define $\ell^\infty(\mathbb{R})$ to be the set of all bounded sequences of real numbers. Under the metric

$$\rho(x, y) = \sup_{k \in \mathbb{N}} |x_k - y_k|,$$

$\ell^\infty(\mathbb{R})$ becomes a metric space (If anyone needs help seeing this, please feel free to email).

Theorem:
$(\ell^\infty(\mathbb{R}), \rho)$ where ρ is defined above is not a separable metric space.

Proof:
We present the proof given in Kolmogorov and Fomin, specifically example 7 on pg. 49, as this construction makes the uncountability part of the argument simple.

Let E denote the set of all binary sequences, that is, all sequences whose terms consists entirely of zeros and ones. Then these sequences are all bounded (take $M = 1$ in the definition of bounded). Further, we know, via Cantor's diagonal argument, that E is uncountable. Now, to show that $\ell^\infty(\mathbb{R})$ is not separable, it suffices to show that any dense subset of $\ell^\infty(\mathbb{R})$ is uncountable. So let A be any dense subset of $\ell^\infty(\mathbb{R})$. Observe that for any $x, y \in E$ such that $x \neq y$ we have that $\rho(x, y) = 1$. This implies that for each $x, y \in E$, with $x \neq y$, $B(\frac{1}{2}, x) \cap B(\frac{1}{2}, y) = \emptyset$. Let's verify why. Lets assume to the contrary that there existed some $z \in \ell^\infty(\mathbb{R})$ such that $z \in B(\frac{1}{2}, x) \cap B(\frac{1}{2}, y)$. Then $d(x, z) < \frac{1}{2}$ and $d(y, z) < \frac{1}{2}$. But,

$$1 = d(x,y) \leq d(x,z) + d(y,z) < \frac{1}{2} + \frac{1}{2} = 1$$

a contradiction. We thus have uncountably many disjoint balls. Now, we assumed that A was dense in $(\ell^\infty(\mathbb{R}), \rho)$. Then for all $\epsilon > 0$ and for all $x \in \ell^\infty(\mathbb{R})$ we must have that $B(\epsilon, x) \cap A \neq \emptyset$. This implies that each of the uncountably many $B(\frac{1}{2}, x)$, must contain a point of A, and since these balls are disjoint, A must be uncountable. $\square$