Problem Set 0

Recalling Some Basic Results from Real Analysis With Proof

The point of this problem set is to establish the proofs very common technical tools used to prove results in analysis. These results are often used without proof in graduate texts, which is why I choose to start here.

NOTE: Earlier problems may be useful in proving later problems

Question 1 - A vital chain of inequalities
For all $x, y \in \mathbb{R}$, show the following chain of inequalities.

$$|x| - |y| \leq ||x| - |y|| \leq |x - y| \leq |x| + |y|$$

Question 2- A Useulf Triangle Inequality Type Result:
Let x, y, p be real numbers satisfyinbg $x, y \geq 0$ and $p \in (0, 1]$. Show that

$$(x + y)^p \leq (x)^p + (y)^p$$

Hint: Since $p \in (0, 1)$, there exists $m \in (0, 1)$ such that $p = 1 - m$.

Question 3 - Limits Preserve Order
Let $\{a_n\}$ and $\{b_n\}$ be sequence of real numbers such that $\lim\limits_{n \to \infty} a_n = a$ and $\lim\limits_{n \to \infty} b_n = b$. Show that if there exists an $N \in \mathbb{N}$ such that $a_n \leq b_n$ for all $n \geq N$ then $a \leq b$.

Question 4 - Useful Properties of Integrals
Show the follwoing:

(a) Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable with $f(x) \geq 0$ for all $x \in [a, b]$. Show that $\int_a^b f(x)dx \geq 0$.

(b) Let $f, g : [a, b] \to \mathbb{R}$ be Riemann integrable with $f(x) \geq g(x)$ for all $x \in [a, b]$. Show that $\int_a^b f(x)dx \geq \int_a^b g(x)dx$.

(c) Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable. Let $d \in \mathbb{R}$ satisfy $a < b < d$. Show that $\int_a^b f(x)dx \leq \int_a^d f(x)dx$.

Question 5 - Series Small Tails Theorem

Let $a_n \in \mathbb{R}$ for all $n \in \mathbb{N}$ and assume that $\sum_{n=1}^{\infty} a_n < \infty$, that is assume that the series $\sum_{n=1}^{\infty} a_n$ converges. Show that $\lim_{n\to\infty} \sum_{k=n+1}^{\infty} a_k = 0$.

Question 6: The ϵ definition of the supremum

Let S be a non-empty set of real numbers that is bounded above. Show that a is the supremum of S if and only if for all $\epsilon > 0$ there exists $s \in S$ such that $a - \epsilon < s$.