Dense and Nowhere Dense - Example

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Definition 1:
Let (X,d) denote the parent metric space. Let A, B be subsets of X. A is said to be dense in B if $B \subset \overline{A}$, i.e. every point of B is a limit point of A.

Definition 2:
Let (X,d) denote the parent metric space. Then $A \subset X$ is everywhere dense if $\overline{A} = X$.

Definition 3:
Let (X,d) denote the parent metric space. Then A is said to be nowhere dense if it is dense in no open ball at all.

We use these definitions to prove the following result.

Proposition:
Let (X,d) be a discrete metric space. Then the only nowhere dense set in (X,d) is the empty set.

Proof:
We first show that it is in fact the case that the empty set is nowhere dense in this space. Let $B(x,\epsilon)$ be an open ball in (X,d). Then the result is clear, because the empty set is its own closure and cannot contain a non empty set. Now take any non empty subset E in (X,d). To show that this set is not nowhere dense, we must show that E is dense in some open ball. Take a point $y \in E$. Then $\overline{E} = E$ and

$$\{y\} = B(\frac{1}{2}, y) \subset E = \overline{E}$$

Thus, E is dense in $B(\frac{1}{2}, y)$, and is thus not nowhere dense in (X,d). $\square$