08-29 Problem Set

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Question 1:
Let (X, ρ_1) and (Y, ρ_2) be metric spaces. Let f be an isometry between these two spaces. Show that f is continuous, uniformly continuous in fact!

Question 2:
(a) Let (X, ρ) be a metric space and let $M \subset X$. Prove that every contact point of M is either a limit point of M or an isolated point of M.
This is a significant result because it permits a classification all of the points of $\overline{M}$ as one of the following.

1. Limits points belonging to M
2. Limit points of M which do not belong to M
3. Isolated points of M

(b) Let L denote the set of all limit points of M. Show that $\overline{M} = M \cup L$.

Question 3:
For $x, y \in \mathbb{R}^n, n \geq 2$, define

$$d(x, y) = \left(\sum_{k=1}^{n} |x_k - y_k|^p \right)^{\frac{1}{p}}$$

Show that for every $p \in (0, 1)$, there exists $x, y \in \mathbb{R}^n$ such that

$$d(x + y, 0) > d(x, 0) + d(y, 0) = d(x + y, y) + d(y, 0)$$

and use this to conclude that for all such p, $(\mathbb{R}^n, d)$ is not a metric space.

Question 4:
For $x = (x_1, \cdots, x_n), y = (y_1, \cdots y_n)$, both in $\mathbb{R}^n$, define

$$\rho(x, y) = \underset{1 \leq k \leq n}{max} |x_k - y_k|.$$

Prove that $(\mathbb{R}^n, \rho)$ is a metric space.

Question 5:
As a subset of $\mathbb{R}$ with the absolute value metric, show that all of the points of $\mathbb{N}$ are isolated. Can you generalize this to find a subset of $\mathbb{R}^n$ endowed with the usual euclidean metric in $\mathbb{R}^n$ whose points are all isolated points. Note that this is the $p - metric$ we saw, with $p = 2$.