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Asymptotics

1. Comparing functions Determine the asymptotic relationships between each pair of functions f and g in the following table (i.e. whether $f \in \mathcal{O}(g)$, $f \in \Omega(g)$ or $f = \Theta(g)$).

	$f(n)$	$g(n)$	$\mathcal{O}$	Ω	Θ
(1)	$n^2 - 30n + 5$	$0.7n^2 - 20n + 15$	○	○	○
(2)	$100n^3 + 40n^2 - n$	$0.5n^4 - 1000n^3$	○	○	○
(3)	$5n^2 - n$	$30n + 4$	○	○	○
(4)	n	$\sqrt{n}$	○	○	○
(5)	$n^{\frac{3}{4}}$	$\sqrt{n}$	○	○	○
(6)	$\log_2 n$	$\ln n$	○	○	○
(7)	$n(\log_2 n)^5$	$n\sqrt{n}$	○	○	○
(8)	2^n	2^{2n}	○	○	○
(9)	e^n	2^n	○	○	○
(10)	$n!$	n^n	○	○	○
(11)	$n!$	2^n	○	○	○
(12)	2^n	2^{n+1}	○	○	○

Lets start with a few different definitions of $\mathcal{O}$, Ω and Θ.

Intuitively, $f \in \mathcal{O}(g)$ if f grows at *most* as fast as g; $f \in \Omega(g)$ if f grows at most as *least* as g and $f \in \Theta(g)$ if f grows *about* as fast as g.

Another intuition could be $f \in \mathcal{O}(g)$ if $g(n)$ never becomes irrelevantly small when compared to $f(n)$; $f \in \Omega(g)$ if $f(n)$ never becomes irrelevantly small when compared to $g(n)$; grows at *most* as fast as g and $f \in \theta(g)$ if f and g never become irrelevant compared to each other.

Formally, $\mathcal{O}$, Ω and Θ are defined:

- $f \in \mathcal{O}(g)$ if there exist constants $c, m > 0$ such that $f(n) \leq cg(n)$ for all $n > m$.
- $f \in \Omega(g)$ if there exist constants $c, m > 0$ such that $f(n) \geq cg(n)$ for all $n > m$.
- $f \in \Theta(g)$ if $f \in \mathcal{O}(g)$ and $f \in \Omega(g)$. That is $\Theta = \mathcal{O} \cap \Omega$.

These are the definitions we expect you to know and use for this class. However, it is often useful to consider a slight alteration of these definitions:

- $f \in \mathcal{O}(g)$ if $\frac{f(n)}{g(n)}$ is bounded from above as n increases (as $n \to \infty$).
- $f \in \Omega(g)$ if $\frac{f(n)}{g(n)}$ remains strictly positive as n increases (as $n \to \infty$).
- $f \in \Theta(g)$ if $\frac{f(n)}{g(n)} \in (0, \infty)$ as n increases (as $n \to \infty$).

These are equivalent when you notice that c should be a real, finite, positive number.

2. Properties of asymptotic notation Let $f(n)$ and $g(n)$ be asymptotically positive functions (which means that $\lim_{n\to\infty} f(n) > 0$ and $\lim_{n\to\infty} g(n) > 0$). Prove or disprove each of the following conjectures

(1) $f(n) + g(n) \in \Theta(\min(f(n), g(n)))$.

(2) $f(n) + g(n) \in \Theta(\max(f(n), g(n)))$.

(3) $f(n) \in \mathcal{O}((f(n))^2)$.

(4) $f(n) \in \Omega((f(n))^2)$.

(5) $f(n) \in \mathcal{O}(g(n))$ implies $g(n) \in \Omega(f(n))$.

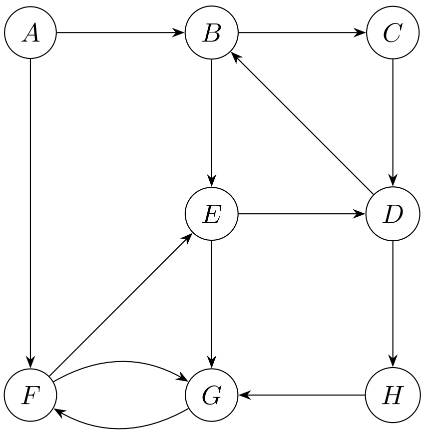
2nd tutorial (continued)

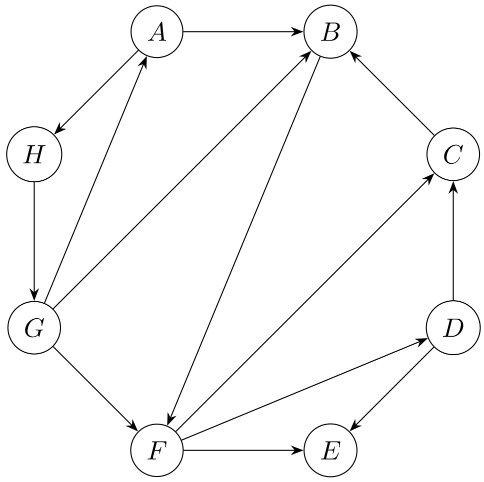
If the tree (and hence tree edges) represent a single path from the root to every other vertex, which edges constitute a second path?

Interlude: Bridges. Jamie will demo an algorithm for finding "bridges" in undirected graphs. See page 122 of https://iuuk.mff.cuni.cz/ koutecky/pruvodce-en-wip.pdf for a reference. Note that translation of this document into English is still a work in progress.

Depth First Search

3. DFS Runs Perform depth-first search on each of the following graphs; whenever there's a choice of vertices, pick the one that is alphabetically first. Classify each edge as a tree, forward, back or cross edge.





4. Singly Connected Graphs A directed graph G is *singly connected* if for every pair of vertices u and v there is at most one path from u to v in G. Give an efficient algorithm to determine whether or not a directed graph is singly connected.

If the tree (and hence tree edges) represent a single path from the root to every other vertex, which edges constitute a second path?

5. Longest Path In a DAG Given a directed, acyclic graph (DAG) $G = (V, E)$ and two of its vertices s and t, we want to compute the length of a longest path from s to t.

Proposition 1.1. *Running DFS on an acyclic graph will never produce a back edge.*

Back edges necessarily imply a cycle.

6. Count Paths In a DAG Given a DAG $G = (V, E)$ and two of its vertices s and t, design an algorithm that calculates the number of distinct paths from s to t.