



Introdução à Álgebra Linear - Turma 11

Teste 1 - Módulo 2 - 11/10/2023

Nome: _____ Mat.: _____

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Questão 1: Seja

$$V = M_2(\mathbb{R}) = \left\{ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \mid a_{11}, a_{12}, a_{21}, a_{22} \in \mathbb{R} \right\}.$$

Considere em V as seguintes operações:

$$\begin{aligned} \bullet \quad A + B &= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{bmatrix} \\ \bullet \quad \alpha \cdot A &= \alpha \cdot \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} \alpha a_{11} & \alpha a_{12} \\ \alpha a_{21} & \alpha a_{22} \end{bmatrix} \end{aligned}$$

para todos $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \in V$ e todo $\alpha \in \mathbb{K} = \mathbb{R}$. Mostre que com essas operações $V = M_2(\mathbb{R})$ é um espaço vetorial sobre $\mathbb{K} = \mathbb{R}$.

Solução:

De fato,

A1) Sejam

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}.$$

Temos:

$$\begin{aligned} A + B &= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \\ &= \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{bmatrix} \\ &= \begin{bmatrix} b_{11} + a_{11} & b_{12} + a_{12} \\ b_{21} + a_{21} & b_{22} + a_{22} \end{bmatrix} \\ &= \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} + \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \\ &= B + A \end{aligned}$$

A2) Sejam

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}, C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}.$$

Temos:

$$\begin{aligned} (A + B) + C &= \left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \right) + \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \\ &= \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{bmatrix} + \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \\ &= \begin{bmatrix} (a_{11} + b_{11}) + c_{11} & (a_{12} + b_{12}) + c_{12} \\ (a_{21} + b_{21}) + c_{21} & (a_{22} + b_{22}) + c_{22} \end{bmatrix} \\ &= \begin{bmatrix} a_{11} + (b_{11} + c_{11}) & a_{12} + (b_{12} + c_{12}) \\ a_{21} + (b_{21} + c_{21}) & a_{22} + (b_{22} + c_{22}) \end{bmatrix} \\ &= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} b_{11} + c_{11} & b_{12} + c_{12} \\ b_{21} + c_{21} & b_{22} + c_{22} \end{bmatrix} \\ &= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \left(\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} + \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \right) \\ &= A + (B + C). \end{aligned}$$

A3) Tome

$$0_V = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Para toda matriz

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$

temos

$$\begin{aligned} A + 0_V &= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a_{11} + 0 & a_{12} + 0 \\ a_{21} + 0 & a_{22} + 0 \end{bmatrix} \\ &= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = A. \end{aligned}$$

Portanto a matriz

$$0_V = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

é o vetor nulo.

A4) Dada

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in \mathbb{M}_2(\mathbb{R}),$$

tome

$$-A = \begin{bmatrix} -a_{11} & -a_{12} \\ -a_{21} & -a_{22} \end{bmatrix} \in \mathbb{M}_2(\mathbb{R}).$$

Assim

$$\begin{aligned} A + (-A) &= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} -a_{11} & -a_{12} \\ -a_{21} & -a_{22} \end{bmatrix} \\ &= \begin{bmatrix} a_{11} - a_{11} & a_{12} - a_{12} \\ a_{21} - a_{21} & a_{22} - a_{22} \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0_V. \end{aligned}$$

Logo a matriz

$$-A = \begin{bmatrix} -a_{11} & -a_{12} \\ -a_{21} & -a_{22} \end{bmatrix}$$

é o vetor oposto.

M1) Sejam $\alpha, \beta \in \mathbb{R}$ e

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in \mathbb{M}_2(\mathbb{R}).$$

Então:

$$\begin{aligned} (\alpha\beta) \cdot A &= (\alpha\beta) \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \\ &= \begin{bmatrix} (\alpha\beta)a_{11} & (\alpha\beta)a_{12} \\ (\alpha\beta)a_{21} & (\alpha\beta)a_{22} \end{bmatrix} \\ &= \begin{bmatrix} \alpha(\beta a_{11}) & \alpha(\beta a_{12}) \\ \alpha(\beta a_{21}) & \alpha(\beta a_{22}) \end{bmatrix} \\ &= \alpha \cdot \begin{bmatrix} \beta a_{11} & \beta a_{12} \\ \beta a_{21} & \beta a_{22} \end{bmatrix} \\ &= \alpha \cdot \left(\beta \cdot \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \right) \\ &= \alpha \cdot (\beta \cdot A) \end{aligned}$$

M2) Seja $1 \in \mathbb{R}$ e

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in \mathbb{M}_2(\mathbb{R}).$$

Temos

$$\begin{aligned} 1 \cdot A &= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \\ &= \begin{bmatrix} 1 \cdot a_{11} & 1 \cdot a_{12} \\ 1 \cdot a_{21} & 1 \cdot a_{22} \end{bmatrix} \\ &= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = A. \end{aligned}$$

- Sejam $\alpha \in \mathbb{R}$ e

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in \mathbb{M}_2(\mathbb{R}), B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \in \mathbb{M}_2(\mathbb{R}).$$

Temos

$$\begin{aligned} \alpha \cdot (A + B) &= \alpha \cdot \left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \right) \\ &= \alpha \cdot \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{bmatrix} \\ &= \begin{bmatrix} \alpha(a_{11} + b_{11}) & \alpha(a_{12} + b_{12}) \\ \alpha(a_{21} + b_{21}) & \alpha(a_{22} + b_{22}) \end{bmatrix} \\ &= \begin{bmatrix} \alpha a_{11} + \alpha b_{11} & \alpha a_{12} + \alpha b_{12} \\ \alpha a_{21} + \alpha b_{21} & \alpha a_{22} + \alpha b_{22} \end{bmatrix} \\ &= \begin{bmatrix} \alpha a_{11} & \alpha a_{12} \\ \alpha a_{21} & \alpha a_{22} \end{bmatrix} + \begin{bmatrix} \alpha b_{11} & \alpha b_{12} \\ \alpha b_{21} & \alpha b_{22} \end{bmatrix} \\ &= \alpha \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \alpha \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \\ &= \alpha \cdot A + \alpha \cdot B \end{aligned}$$

D2) Sejam $\alpha, \beta \in \mathbb{R}$ e

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in \mathbb{M}_2(\mathbb{R}).$$

Temos

$$\begin{aligned} (\alpha + \beta) \cdot A &= (\alpha + \beta) \cdot \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \\ &= \begin{bmatrix} (\alpha + \beta)a_{11} & (\alpha + \beta)a_{12} \\ (\alpha + \beta)a_{21} & (\alpha + \beta)a_{22} \end{bmatrix} \\ &= \begin{bmatrix} \alpha a_{11} + \beta a_{11} & \alpha a_{12} + \beta a_{12} \\ \alpha a_{21} + \beta a_{21} & \alpha a_{22} + \beta a_{22} \end{bmatrix} \\ &= \begin{bmatrix} \alpha a_{11} & \alpha a_{12} \\ \alpha a_{21} & \alpha a_{22} \end{bmatrix} + \begin{bmatrix} \beta a_{11} & \beta a_{12} \\ \beta a_{21} & \beta a_{22} \end{bmatrix} \\ &= \alpha \cdot \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \beta \cdot \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \\ &= \alpha \cdot A + \beta \cdot A. \end{aligned}$$

Assim $V = \mathbb{M}_2(\mathbb{R})$ é um espaço vetorial sobre \mathbb{R} .