8 Nilpotent groups

Definition 8.1. A group G is *nilpotent* if it has a normal series

$$G = G_0 \leq G_1 \leq G_2 \leq \cdots \leq G_n = 1 \tag{1}$$

where

$$G_i/G_{i+1} \leq Z(G/G_{i+1}) \tag{2}$$

We call (1) a *central series* of G of *length* n. The minimal length of a central series is called the *nilpotency class* of G. For example, an abelian group is nilpotent with nilpotency class ≤ 1.

Some trivial observations: First, nilpotent groups are solvable since (2) implies that G_i/G_{i+1} is abelian. Also, nilpotent groups have nontrivial centers since $G_{n-1} \leq Z(G)$. Conversely, the fact that p-groups have nontrivial centers implies that they are nilpotent.

Theorem 8.2. *Every finite p-group is nilpotent.*

Proof. We will construct a central series of a p-group G from the bottom up. The numbering will be reversed: $G_{n-k} = Z^k(G)$ which are defined as follows.

0. $Z^0(G) = 1$ ($= G_n$ where n is undetermined).
1. $Z^1(G) = Z(G)$ (> 1 if G is a p-group).
2. Given $Z^k(G)$, let $Z^{k+1}(G)$ be the subgroup of G which contains $Z^k(G)$ and corresponds to the center of $G/Z^k(G)$, i.e., so that

$$Z^{k+1}(G)/Z^k(G) = Z(G/Z^k(G))$$

Since $G/Z^k(G)$ is a p-group it has a nontrivial center, making $Z^{k+1}(G) > Z^k(G)$ unless $Z^k(G) = G$. Since G is finite we must have $Z^n(G) = G$ for some n making G nilpotent of class $\leq n$. □

For an arbitrary group G the construction above gives a (possibly infinite) sequence of characteristic subgroups of G:

$$1 = Z^0(G) \leq Z(G) = Z^1(G) \leq Z^2(G) \leq \cdots$$

We call this the *upper central series* of G (because it has the property that $Z^{n-k}(G) \geq G_k$ for any central series $\{G_k\}$ of length n. However this series does not have good functorial properties since the $Z^k(G)$ are not fully invariant.

The lower central series is more useful. It comes from the following reformulation of condition (2) in Definition 8.1:

0. $G_i/G_{i+1} \leq Z(G/G_{i+1})$. In words this says:

1. G_i centralizes G modulo G_{i+1}, i.e.,
2. $xgx^{-1} \equiv g \mod G_{i+1}$ for $x \in G_i$ and $g \in G$. Equivalently,
3. $xgx^{-1}g^{-1} \in G_{i+1}$ for all $x \in G_i$ and $g \in G$. Equivalently,
4. $[G_i, G] \leq G_{i+1}$.

Thus the last inequality is equivalent to the first inequality. However, equality in the last condition is not equivalent to equality in the first condition. The equality $G_{i+1} = [G_i, G]$ defines G_{i+1} in terms of G_i.

Definition 8.3. The *lower central series* of any group G is defined to be the descending series of characteristic subgroups:

$$G \geq [G,G] \geq [[G,G],G] \geq [[[G,G],G],G] \geq \cdots$$

If for any $H \leq G$ we define $F_G(H) = [H,G]$ then these groups are $F_G(G), F_G(F_G(G)), F_G^3(G)$, etc.

Proposition 8.4. *The lower central series is functorial:*

1. *Any homomorphism $\phi : G \to H$ sends $F_G^k(G)$ into $F_H^k(H)$.*
2. *In particular each $F_G^k(G)$ is a fully invariant subgroup of G.*
3. *Any quotient map $\phi : G \twoheadrightarrow Q$ sends $F_G^k(G)$ onto $F_Q^k(Q)$.*

Proof. This is obvious. For example, $\phi[[G,G],G] = [[\phi(G), \phi(G)], \phi(G)]$. □

Theorem 8.5. *The nilpotency class of G is the smallest $c \geq 0$ so that $F_G^c(G) = 1$. [$c = \infty$ is G is not nilpotent.]*

Proof. Writing $F = F_G$, it suffices to show that $F^k(G) \leq G_k$ for any central series $\{G_k\}$ as in (1). This is true for $k = 0$ since $F^0(G) = G = G_0$. Suppose by induction that $F^k(G) \leq G_k$ where $k \geq 0$. Then

$$F^{k+1}(G) = [F^k(G), G] \leq [G_k, G] \leq G_{k+1}$$

□

Corollary 8.6. *If G is nilpotent of class c then every subgroup and quotient group is nilpotent of class $\leq c$.*

Proof. It is given that $F_G^c(G) = 1$. By the proposition this implies that $F_H^c(H) = 1$ for any $H \leq G$ and $F_Q^c(G) = 1$ for any quotient Q of G. □

Lemma 8.7. *If $H \leq G$ then $N_G(H) = \{x \in G \mid [x,H] \leq H\}$. Thus, $K \leq N(H)$ iff $[K,H] \leq H$.*

Proof. $[x,H]$ has elements $xhx^{-1}h^{-1}$ which lie in H iff $xhx^{-1} \in H$, i.e., if x normalizes H. □

Theorem 8.8. *Nilpotent groups have no proper self-normalizing subgroups, i.e.,*

$$H < G \Rightarrow H < N(H).$$

Proof. Suppose that $H < G$. Then $F^0(G) = G \nleq H$. Let $k \geq 0$ be maximal so that $F^k(G) \nleq H$. Then

$$H \geq F^{k+1}(G) = [F^k(G), G] \geq [F^k(G), H]$$

so $F^k(G) \leq N(H)$ by the lemma. Since $F^k(G) \nleq H$, H is not self-normalizing. □

Corollary 8.9. *If G is a finite nilpotent group then every Sylow subgroup of G is normal. Thus G is the product of its Sylow subgroups.*

Proof. If a Sylow subgroup P of G is not normal then $N(P) < G$. But we know that $N(P)$ is self-normalizing [HW01.ex01] which contradicts the theorem above. □

The converse of this corollary is also true.

Lemma 8.10. *Any finite product of nilpotent groups is nilpotent.*

<u>HW4.ex01</u>: Prove this lemma (in your own words) and discuss the question: Is this true for infinite products?

Theorem 8.11. *A finite group is nilpotent if and only if all of its Sylow subgroups are normal (equivalently, it is the product of its Sylow subgroups).*

Frattini subgroup of a nilpotent group

Theorem 8.12 (Frattini). *The Frattini subgroup $\Phi(G)$ of any finite group G is nilpotent.*

Proof. It is enough to show that every Sylow subgroup of $\Phi(G)$ is normal. So suppose that P is a Sylow subgroup of $\Phi(G)$. Then $G = \Phi(G)N(P)$ by the Frattini argument [HW01.ex02]. If $N(P) \neq G$ then any maximal subgroup of G containing $N(P)$ will not containing $\Phi(G)$ contradicting the definition of $\Phi(G)$. □

<u>HW04.ex02</u>:(6.45 in Rotman 1st ed.)[Wielandt] A finite group G is nilpotent iff every maximal subgroup is normal.

Theorem 8.13 (Wielandt). *A finite group G is nilpotent iff $G' \subseteq \Phi(G)$, i.e., iff $G/\Phi(G)$ is abelian.*

Proof. We use Wielandt's lemma to prove his theorem. Suppose that $G/\Phi(G)$ is abelian. Then any maximal subgroup of G is normal since it corresponds to a maximal subgroup of $G/\Phi(G)$ all or whose subgroups are normal. Thus, by the exercise, G is nilpotent. Conversely, suppose that all maximal $M < G$ are normal. Then $G/M \cong \mathbb{Z}/p$ and $\Phi(G)$ is the kernel of the homomorphism:

$$G \to \prod G/M \cong \prod \mathbb{Z}/p_i$$

whose image ($\cong G/\Phi(G)$) is abelian. $\square$

The upper central series

[You may ignore this subsection.] The subgroups $Z^k(G)$ in the upper central series are natural only with respect to epimorphisms[1] in the following sense.

Lemma 8.14. *If $\phi : G \twoheadrightarrow H$ is an epimorphism then*

1. $\phi Z(G) \leq Z(H)$
2. $\phi Z^k(G) \leq Z^k(H)$ *for all* $k \geq 0$.

Proof. The first statement is clear since epimorphisms send central elements to central elements. [If z commutes with all $g \in G$ then $\phi(z)$ commutes with the elements $\phi(g)$ which make up all of H.]

The second statement follows from the first: By induction assume it is true for k. Then we get a new epimorphism:

$$\overline{\phi} : \frac{G}{Z^k(G)} \twoheadrightarrow \frac{H}{Z^k(H)}$$

The epimorphism $\overline{\phi}$ sends center to center. But the center of the first group is $Z^{k+1}(G)/Z^k(G)$ by definition and the center of the second group is $Z^{k+1}(H)/Z^k(H)$. In other words, $\phi Z^{k+1}(G) \leq Z^{k+1}(H)$. $\square$

Theorem 8.15. *The nilpotency class of G is the smallest $c \geq 0$ so that $Z^c(G) = G$.*

Proof. Given a central series $\{G_i\}$ for G it suffices to show that $G_{n-k} \leq Z^k(G)$ for all k. This is true for $k = 0, 1$. Suppose it is true for k then we get an epimorphism:

$$G/G_{n-k} \twoheadrightarrow G/Z^k(G)$$

which sends the central subgroup G_{n-k-1}/G_{n-k} to a subgroup of the center $Z^{k+1}(G)/Z^k(G)$ of $G/Z^k(G)$. Thus $G_{n-k-1} \leq Z^{k+1}(G)$ as required. $\square$

[1] We use *epimorphism* to mean *surjective homomorphism* and ignore the categorical definition.