Anéis

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MAT-UnB

Exercício

Considere o conjunto $\mathbb{Z} \times \mathbb{Z}$ *com as operações*

$$(a, b) \oplus (c, d) = (a + c, b + d)$$
$$(a, b) \otimes (c, d) = (ac, ad + bc).$$

para todos $(a, b), (c, d) \in \mathbb{Z} \times \mathbb{Z}$. *Verifique se* $(\mathbb{Z} \times \mathbb{Z}, \oplus, \otimes)$ *é um anel. Caso afirmativo, esse anel é comutativo? Possui unidade?*

Solução: Sejam $\boxed{(x, y), (z, t)}$, $(a, b) \in \mathbb{Z} \times \mathbb{Z}$. Temos

$$\overline{[\underbrace{(x, y) \oplus (z, t)}] \oplus (a, \underline{b})} = (x+z, y+t) \oplus$$

$$(a, b) = \Big(\underbrace{(x+z) + a}_{\in \mathbb{Z}}, \underbrace{(y+t) + b}_{\in \mathbb{Z}}\Big) =$$

$$= \boxed{(x+z+a,\ y+t+b)}$$

$$(x,y) \oplus \left[(z,t) \oplus (a,b) \right] =$$

$$= (x,y) \oplus (z+a,\ t+b) = \big(\overbrace{x+(z+a)}^{\in \mathbb{Z}},\ \underbrace{y+(t+b)}_{\in \mathbb{Z}} \big) = \boxed{(x+z+a,\ y+t+b)}$$

Assim

$$[(x,y)\oplus(z,t)]\oplus(a,b)=(x,y)\oplus[(z,t)\oplus(a,b)].$$

Agora,

$$\boxed{(x,y)\oplus(z,t)} = (\overbrace{x+z}^{\in\mathbb{Z}},\ \overbrace{y+t}^{\in\mathbb{Z}})$$

$$= (z+x,\ t+y) = \boxed{(z,t)\oplus(x,y)}$$

Logo

$$(x,y)\oplus(z,t) = (z,t)\oplus(x,y).$$

$\underline{\underline{(\alpha, \beta)}} \in \mathbb{Z} \times \mathbb{Z}$

$$\underbrace{\underline{\underline{(x, y)}} \oplus (\overset{b}{\alpha}, \overset{b}{\beta})} \overset{?}{=} (x, y)$$

$$\overbrace{(x+\alpha}, \underbrace{y+\beta) = (x, y)}$$

$x + \alpha = x \Rightarrow$

$y + \beta = y \Rightarrow \beta = \alpha = 0$

AGORA TOMANDO $(0,0) \in \mathbb{Z} \times \mathbb{Z}$

TEMOS

$$(x,y) \oplus (0,0) = (x+0,\ y+0) = (x,y)$$

PARA TODO $(x,y) \in \mathbb{Z} \times \mathbb{Z}$. LOGO

$(0,0) \in \mathbb{Z} \times \mathbb{Z}$ É O ELEMENTO NEUTRO DA SOMA $\oplus$ EM $\mathbb{Z} \times \mathbb{Z}$.

$(x, y) \in \mathbb{Z} \times \mathbb{Z}$, Existe $(\alpha, \beta) \in \mathbb{Z} \times \mathbb{Z}$

TAL QUE

$$(x, y) \oplus (\alpha, \beta) = (0, 0)$$

$$(x + \alpha, y + \beta) = (0, 0)$$

$$\left.\begin{array}{l} x + \alpha = 0 \\ y + \beta = 0 \end{array}\right\} \Rightarrow \begin{array}{l} \alpha = \ominus x \in \mathbb{Z} \\ \beta = \ominus y \in \mathbb{Z} \end{array}$$

Dado $\boxed{(x, y) \in \mathbb{Z} \times \mathbb{Z}}$, tome o elemento $\underline{(-x, -y) \in \mathbb{Z} \times \mathbb{Z}}$. Assim

$$\underline{(x, y) \oplus (-x, -y)} = (x + (-x), y + (-y))$$

$$= \underline{\underline{(0, 0)}}.$$

LOGO O OPOSTO DE $(x, y) \in \mathbb{Z} \times \mathbb{Z}$ É O ELEMENTO $(-x, -y) \in \mathbb{Z} \times \mathbb{Z}$.

AGORA DADOS $\underline{(x, y), (z, t), (a, b)} \in \mathbb{Z} \times \mathbb{Z}$ TEMOS:

$$\big[\overbrace{(x,y)\otimes(z,t)}\big]\otimes(a,b) = (\overline{xz},\ xt+yz)$$

$$\otimes(a,b) = \big(xza,\ xzb + (xt+yz)a\big)$$

$$= \boxed{\big(xza,\ \overline{xz}b + x\overline{\overline{t}}a + \overline{\overline{y}}za\big)}$$

$$(x,y)\otimes\big[\underbrace{(z,t)\otimes(a,b)}\big] =$$

$$= (x, y) \otimes (za, zb + ta) =$$

$$= (xza, \; x(zb + ta) + yza)$$

$$\boxed{= (xza, \; xzb + xta + yza)}$$

DA;

$$[(x,y)\otimes(z,t)]\otimes(a,b)=(x,y)\otimes[(z,t)\otimes(a,b)].$$

AG⊃NA,

$$\overbrace{\left[(x,y)\oplus(z,t)\right]}\otimes(a,b) = (x+z,\ y+t)\otimes(a,b)$$

$$= \big((x+z)a,\ (x+z)b+(y+t)a\big)$$

$$= \boxed{(xa+za,\ xb+zb+ya+ta)}$$

$$\underbrace{(x,y)\otimes(a,b)}\oplus\underbrace{(z,t)\otimes(a,b)} =$$

$$= (xa,\ xb + ya) \oplus (za,\ zb + ta)$$

$$\boxed{= (\underbrace{xa + za}_{\in \mathbb{Z}},\ \underbrace{xb + ya + zb + ta}_{\in \mathbb{Z}})}$$

Com isso,

$$[(x,y) \oplus (z,t)] \otimes (a,b) = (x,y) \otimes (a,b) \oplus (z,t) \otimes (a,b).$$

FINALMENTE

$$(x,y)\otimes[(z,t)\otimes(a,b)] = (x,y)\otimes(z+a,\,t+b)$$

$$= \big(x(z+a),\ x(t+b)+y(z+a)\big)$$

$$= \boxed{(xz+xa,\ xt+xb+yz+ya)}$$

$$\overbrace{(x,y)\otimes(z,t)} \oplus \overbrace{(x,y)\otimes(a,b)} =$$

$$=(xz,\ xt+yz)\oplus(xa,\ xb+ya)$$

$$\boxed{=(xz+xa,\ xt+yz+xb+ya)}$$

DAÍ:

$$(x,y) \otimes [(z,t) \oplus (a,b)] = (x,y) \otimes (z,t)$$

$$\oplus\, (x,y) \otimes (a,b).$$

PORTANTO $(\mathbb{Z} \times \mathbb{Z}, \oplus, \otimes)$ É UM ANEL.

DADO $(x,y),(z,t) \in \mathbb{Z} \times \mathbb{Z}$ TEMOS

$$(x,y) \otimes (z,t) = (xz, xt + yz)$$

$$(z,t) \otimes (x,y) = (zx, zy + tx)$$

LOGO,

$$(x,y)\otimes(z,t) = (z,t)\otimes(x,y).$$

OU SEJA, $(\mathbb{Z}\times\mathbb{Z}, \oplus, \otimes)$ É UM

ANEL COMUTATIVO.

$(x,y) \in \mathbb{Z} \times \mathbb{Z}$; existe $\boxed{(\alpha, \beta) \in \mathbb{Z} \times \mathbb{Z}}$

$(x,y) \otimes (\alpha, \beta) = (x,y)$

$(x\alpha, x\beta + y\alpha) = (x,y)$

$$\begin{cases} x\alpha = x \Rightarrow \alpha = 1 \\ x\beta + y\alpha = y \end{cases}$$

$x\beta + y = y \Rightarrow x\beta = 0$

$\Downarrow$

$\beta = 0$

TOMANDO O PAR $(1,0) \in \mathbb{Z} \times \mathbb{Z}$

TEMOS

$$(x, y) \otimes (1, 0) = (x, y)$$

PARA TODO $(x, y) \in \mathbb{Z} \times \mathbb{Z}$.

PORTANTO $(\mathbb{Z}\times\mathbb{Z}, \oplus, \otimes)$ POSSUI

UNIDADE E A UNIDADE DESSE

ANEL É O PAR $(1,0)$.

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