Exercício

Considere $\mathbb{C}^*$ *um grupo multiplicativo. Verifique se* $f : \mathbb{C}^* \to \mathbb{C}^*$ *dada por*

$$f(z) = \sqrt{a^2 + b^2},$$

onde $z = a + bi$*, é um homomorfismo de grupos. Caso afirmativo, obtenha* $\ker(f)$.

$(G, *)$; (H, Δ)

$f : G \to H$ É HOMOMORFISMO SE

$f(x * y) = f(x) \Delta f(y)$

PARA TODOS $x, y \in G$.

SOLUÇÃO SEJAM $x, y \in \mathbb{C}^*$. DAÍ

$x = a + bi$

; $a, b, \alpha, \beta \in \mathbb{R}$.

$y = \alpha + \beta i$

$$\underbrace{f(xy)} = \underbrace{f(x)f(y)}$$

Đặt,

$$f(xy) = f[\overbrace{(a+bi)(\alpha+\beta i)}^{r+si}] =$$

$$= f\left(\overbrace{(a\alpha - b\beta)} + \overbrace{(a\beta + b\alpha)}i\right)$$

$$= \sqrt{(a\alpha - b\beta)^2 + (a\beta + b\alpha)^2}$$

$$f(xy) = \left(a^2\alpha^2 - \cancel{2a\alpha b}\beta + b^2\beta^2 + a^2\beta^2 + \cancel{2a\beta b}\alpha + b^2\alpha^2\right)^{\frac{1}{2}}$$

$$\underline{f(xy)} = \left(a^2\alpha^2 + \underline{b^2\beta^2} + \underline{a^2\beta^2} + \underline{b^2}\alpha^2\right)^{\frac{1}{2}}$$

AGORA

$$f(x)f(y) = f(a+bi)\,f(\alpha+\beta i)$$

$$= (a^2+b^2)^{\frac{1}{2}}(\alpha^2+\beta^2)^{\frac{1}{2}}$$

$$= \left[(a^2+b^2)(\alpha^2+\beta^2)\right]^{\frac{1}{2}}$$

$$\underline{f(x)f(y)} = \left(a^2\alpha^2 + a^2\beta^2 + b^2\alpha^2 + b^2\beta^2\right)^{\frac{1}{2}}$$

LOGO

$$f(xy) = f(x)f(y),$$

OU SEJA, f É UM HOMOMORFISMO DE GRUPOS.

MAS

$$\mathrm{nu}(f) = \{ z \in \mathbb{C}^{\times} \mid f(z) = 1 \}$$

é

$$f(z) = \sqrt{a^2 + b^2} \quad \text{se} \quad z = a + bi$$

Logo,

$$\sqrt{a^2+b^2} = 1$$

$$\Rightarrow a^2+b^2=1$$

PORTANTO

$$\ker(f) = \{ a+bi \in \mathbb{C}^* \mid \boxed{a^2+b^2=1} \}$$

ISTO É, $\ker(f) \neq \{\underline{0}\}$. LOGO

$\underbrace{f \text{ NÃO É INJETORA}}$. #