1.4 Minimums and Maximums

Order statistics notation

Let $X_1, X_2, \ldots, X_n$ denote a sample of observations. The order statistics are denoted as:

- $X_{(1)} = \min(X_1, X_2, \ldots, X_n)$.
- $X_{(2)} = \text{2nd smallest of } X_1, X_2, \ldots, X_n$.
- $X_{(n)} = \max(X_1, X_2, \ldots X_n)$.

1.4.2 The distribution of a minimum by example

Example 1.4.1

Let $X_1, X_2, \ldots, X_n \overset{i.i.d.}{\sim} \text{Exponential}(\lambda)$. Determine the distribution of $X_{(1)}$.

Example 1.4.1 (cont)

Example 1.4.1 (cont)

Example 1.4.1 (cont)

Example 1.4.1 (cont)

1.4.3 The general distribution of minimums and maximums

Distribution of the minimum

Let $X_1, X_2, \ldots, X_n$ be a random sample from a continuous distribution. Determine the distribution of $X_{(1)}$.

Distribution of the minimum (cont)

Distribution of the minimum (cont)

Distribution of the minimum (cont)

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PDF for a minimum

Let $X_1, X_2, \ldots, X_n$ be a random sample from a continuous distribution with pdf f and cdf X.
The pdf for $X_{(1)}$ is
$f_{X_{(1)}} = n[1 - F(x)]^{n-1} f(x)$.

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Distribution of the maximum

Let $X_1, X_2, \ldots, X_n$ be a random sample from a continuous distribution. Determine the distribution of $X_{(n)}$.

Distribution of the maximum (cont)

Distribution of the maximum (cont)

Distribution of the maximum (cont)

PDF for a maximum

Let $X_1, X_2, \ldots, X_n$ be a random sample from a continuous distribution with pdf f and cdf X.
The pdf for $X_{(n)}$ is
$f_{X_{(n)}} = n[F(x)]^{n-1} f(x)$.

1.5 Moment Generating Functions (mgfs)

Definition of the mgf

The moment generating function (mgf) of a random variable X is $M_X(t) = E[e^{tX}]$.

Uses of mgfs

Mgfs are often useful in two settings:

1. Determining the distribution of a transformation of 1 or more random variables (particularly the sum of n i.i.d. random variables.)
2. Determine the "moments" of a random variable, i.e., $E(X), E(X^2), E(X^3), \ldots$.

1.5.1 The expectation of a function X

Example 1.5.1

Suppose the random variable X has the following distribution.

x	-1	0	1
$f(x)$	4/12	3/12	5/12

Determine $E(X^2)$.

Example 1.5.1 (cont)

Example 1.5.1 (cont)

Law of the Unconscious Statistician

If X is a discrete random variable, then

$$E[g(X)] = \sum_x g(x) f_X(x),$$

where $f_X(x)$ is the pmf of X.

If X is a continuous random variable, then

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx,$$

where $f_X(x)$ is the pdf of X.

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Proof (specific case)

☰

Proof (specific case)

Proof (specific case)

Proof (specific case)

1.5.2 Finding mgfs and the Poisson distribution

The Poisson distribution

A discrete random variable $X \sim \mathrm{Poisson}(\lambda)$ if it has the pmf
$f_X(x) = \dfrac{e^{-\lambda}\lambda^x}{x!} I_{\{0,1,2,...\}}(x)$.

Poisson interpretation

A Poisson random variable can be thought of as the number of arrivals observed in a fixed amount of time where:

1. The mean time between arrivals is constant.
2. The number of arrivals in non-overlapping time periods are independent.

Example 1.5.2

Determine the mgf for the Poisson distribution

Example 1.5.2 (cont)

Example 1.5.2 (cont)

Example 1.5.3

Suppose that $X \sim \text{Exponential}(\lambda)$. Determine the mgf of X.

Example 1.5.3 (cont)

Example 1.5.3 (cont)

Example 1.5.4

Suppose that $X \sim N(\mu, \sigma^2)$ with pdf

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2\sigma^2}(x-\mu)^2\right].$$

Determine the mgf of X.

Example 1.5.4 (cont)

Example 1.5.4 (cont)

Example 1.5.4 (cont)

1.5.3 Finding moments

Defining moments

The **moments** of a random variable X are expectations of the form $E(X^k)$ for $k = 1, 2, 3, \ldots$.

Computing moments

The kth moment of a random variable X can be computed as $M_X^{(k)}(0)$, where $M_X^{(k)}(0)$ is the kth derivative of the mgf of X evaluated at zero.

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Mgf moment proof

≡

Mgf moment proof (cont)

Mgf moment proof (cont)

1.5.4 The mgf uniquely determines a distribution

The DNA of the mgf

The mgf of a random variable uniquely determines its distribution.
Example: If we find the mgf of a random variable X is $\exp[15(e^t - 1)]$, then $X \sim \text{Poisson}(15)$ since it matches the mgf of a Poisson random variable with $\lambda = 15$.

1.5.6 Sums of iid random variables

Mgf of sum of independent r.v.s

Let $X_1, X_2, \ldots, X_n$ be independent random variables. The mgf of

$$Y := \sum_{i=1}^{n} X_i$$

is

$$M_Y(t) = \prod_{i=1}^{n} M_{X_i}(t).$$

Proof

Proof (cont)

Proof (cont)

≡

Mgf of sum of i.i.d. r.v.s

Let $X_1, X_2, \ldots, X_n \overset{i.i.d.}{\sim} F$. The mgf of

$$Y := \sum_{i=1}^{n} X_i$$

is

$$M_Y(t) = [M_X(t)]^n.$$

≡

Example 1.5.6

Suppose that $X_1, X_2, \ldots, X_n \overset{i.i.d.}{\sim} \text{Poisson}(\lambda)$. Determine the distribution of $Y := \sum_{i=1}^{n} X_i$.

Example 1.5.6 (cont)

Example 1.5.6 (cont)

Example 1.5.7

Suppose that $X_1, X_2, \ldots, X_n \overset{i.i.d.}{\sim} \text{Exponential}(\lambda)$. Determine the distribution of $Y := \sum_{i=1}^{n} X_i$.

Example 1.5.7 (cont)

Example 1.5.7 (cont)

Example 1.5.8

Suppose that $X_1, X_2, \ldots, X_n$ be independent random variables with $X_i \sim \mathrm{Poisson}(\lambda_i)$. Determine the distribution of $Y := \sum_{i=1}^n X_i$.

Example 1.5.8 (cont)

Example 1.5.8 (cont)

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Example 1.5.8 (cont)