1.6

General Order Statistics

1.6.2

The joint distribution of the minimum and maximum

Min/Max cdf

For an i.i.d. sample of continuous random variables $X_1, X_2, \ldots, X_n$ with cdf F, the joint cdf of $(X_{(1)}, X_{(n)})$ is
$F_{X_{(1)},X_{(n)}}(x, y) = [F(y)]^n - [F(y) - F(x)]^n$,
for $x < y$ and
$F_{X_{(1)},X_{(n)}}(x, y) = [F(y)]^n$
for $x \geq y$.

Proof of min/max cdf

Proof of min/max cdf (cont)

Proof of min/max cdf (cont)

Min/Max pdf

For an i.i.d. sample of continuous random variables $X_1, X_2, \ldots, X_n$ with cdf F, the joint pdf of $(X_{(1)}, X_{(n)})$ is
$f_{X_{(1)},X_{(n)}}(x, y) = n(n-1)[[F(y) - F(x)]^{n-2} f(x) f(y),$
for $x < y$.

Proof of min/max pdf

Example 1.6.1

Determine the joint pdf of $(X_{(1)}, X_{(15)})$ when $X_1, X_2, \ldots, X_{15} \overset{i.i.d.}{\sim} \mathrm{Uniform}(0, 1)$.

Example 1.6.1 (cont)

1.6.3

The distribution of all order statistics

PDF of all order statistics

For an i.i.d. sample of continuous random variables $X_1, X_2, \ldots, X_n$ with pdf f, the pdf of all n order statistics is

$$f_{X_{(1)}, X_{(2)}, \ldots, X_{(n)}}(x_1, x_2, \ldots, x_n)$$
$$= n! f(x_1) f(x_2) \cdots f(x_n) I(x_1 < x_2 < \cdots < x_{n-1} < x_n).$$

Proof

Proof

Proof

Proof

1.6.4

The distribution of $X_{(i)}$

CDF of $X_{(i)}$

For an i.i.d. sample of continuous random variables $X_1, X_2, \ldots, X_n$ with cdf F, the cdf of $X_{(i)}$ is

$$F_{X_{(i)}}(x) = P(X_{(i)} \leq x) = \sum_{j=i}^{n} \binom{n}{j} (1 - F(x))^{n-j} F(x)^j.$$

Proof

Proof (cont)

The PDF of $X_{(i)}$

For an i.i.d. sample of continuous random variables $X_1, X_2, \dots, X_n$ with cdf F, the pdf of $X_{(i)}$ is

$$f_{X_{(i)}}(x) = \frac{n!}{(i-1)!(n-i)!} F(x)^{i-1} (1 - F(x))^{n-k} f(x).$$

Proof

Proof (cont)

Proof (cont)

1.6.5

The joint distribution of $X_{(i)}$ and $X_{(j)}$

The joint pdf of $X_{(i)}$ and $X_{(j)}$

For an i.i.d. sample of continuous random variables $X_1, X_2, \ldots, X_n$ with cdf F, the joint pdf of $(X_{(i)}, X_{(i)})$ is

$$
\begin{aligned}
& f_{X_{(i)},X_{(j)}}(x_i, x_j) \\
& = \frac{n!}{(i-1)!(j-i-1)!(n-j)!} F(x_i)^{i-1} f(x_i) \\
& \quad \times [F(x_j) - F(x_i)]^{j-i-1} f(x_j) [1 - F(x_j)]^{n-j} I_{(-\infty, x_j)}(x_i).
\end{aligned}
$$