1.5 Independence

Definition

Independent Events Events A and B are **independent** if
$P(A \cap B) = P(A)P(B)$.

Equivalently, A and B are independent if
$P(A \mid B) = P(A)$,
assuming $P(B) > 0$.
The first definition is preferred because it can be used even when $P(A \mid B)$ is undefined.

Implications

Practically, two events are independent if knowledge about one of the events occurring has no impact the probability of the other event occurring.
If A is independent of B, then B is independent of A.
Shockingly, disjoint events cannot be independent!
Why? If $P(A) > 0$ and $P(B) > 0$ but A and B are disjoint, then
$0 = P(A \cap B) \neq P(A)P(B) > 0$.

Example 1.19

Consider an experiment involving two successive rolls of a 4-sided die in which all 16 possible outcomes are equally likely and have probability 1/16.

Example 1.19 (cont)

Are the events below independent?

$A_i = \{\text{1st roll results in } i\}, \quad B_j = \{\text{2nd roll results in } j\}$

Example 1.19 (cont)

Are the events below independent?

$A_i = \{\text{1st roll is a } 1\}, \quad B_j = \{\text{sum of the two rolls is } 5\}$

Example 1.19 (continued)

Are the events below independent?
$A_i = \{\text{maximum of the two rolls is } 2\}$,
$B_j = \{\text{minimum of the two rolls is } 2\}$

Conditional Independence

Definition

> **Conditionally Indpendent Events**
> Given an event C, the events A and B are **conditionally indpendent** if $P(A \cap B \mid C) = P(A \mid C)P(B \mid C)$.

The independence of A and B does not imply the conditional independence of the events.

Alternative Characterization

Example 1.20

Consider two independent fair coin tosses, in which all four possible outcomes are equally likely. Let
$\{H_1\} = \text{1st toss is a head,}$
$\{H_2\} = \text{2nd toss is a head,}$
$D = \{\text{the two tosses have different results}\}.$

Example 1.20 (cont)

Example 1.21

There are two coins, a blue and red one. We choose one of the two coins at random, each being chosen with probability $1/2$. We proceed with two independent oses. The coins are biased: with the blue coin, the probability of heads in any given toss is 0.99, whereas for the red coin it is 0.01.
Let B be the event that the blue coin was selected. Let H_i be teh event that the ith toss results in a head.
The events H_1 and H_2 are dependent. However, they are conditionally independent given the coin.

Example 1.21 (cont)

Summary

- Two event A and B are **independent** if $P(A \cap B) = P(A)P(B)$.
- If A and B are independent, then A and B^c are independent.
- Two events A and B are **conditionally independent** given another event C with $P(C) > 0$, if $P(A \cap B \mid C) = P(A \mid C)P(B \mid C)$.
- Independence does not imply conditional independence or vice versa.

Independence of a Collection of Events

Independence Generalization

Definition of Independence of Several Events

The events $A_1, A_2, \ldots, A_n$ are independent if

$$P\left(\cap_{i \in S} A_i\right) = \prod_{i \in S} P(A_i), \quad \text{for every subset} S \subseteq \{1, 2, \ldots, n\}.$$

Independence Generalization

If $A_1, A_2, \text{ and } A_3$ are events, what properties do they have to satisfy to be independent?

Example 1.22

Pairwise independence does not imply independence.
Consider two independent fair coin tosses, in which all four possible outcomes are equally likely. Let
$\{H_1\} = \text{1st toss is a head},$
$\{H_2\} = \text{2nd toss is a head},$
$D = \{\text{the two tosses have different results}\}.$

Example 1.22 (cont)

Example 1.23

The equality $P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2)P(A_3)$ does not guarantee independence.
Consider two rolls of a fair six-sided die, and the following events:
$A = \{\text{1st roll is} 1, 2, \text{ or } 3\}$,
$B = \{\text{1st roll is} 3, 4, \text{ or } 5\}$,
$C = \{\text{the sum of the two rolls is } 9\}$.

Example 1.23 (cont)

Intuition

Independence means that that the occurrence or non-occurrence of **any number** of the events from that collection carries no information on the remaining events or their complements.

Reliability

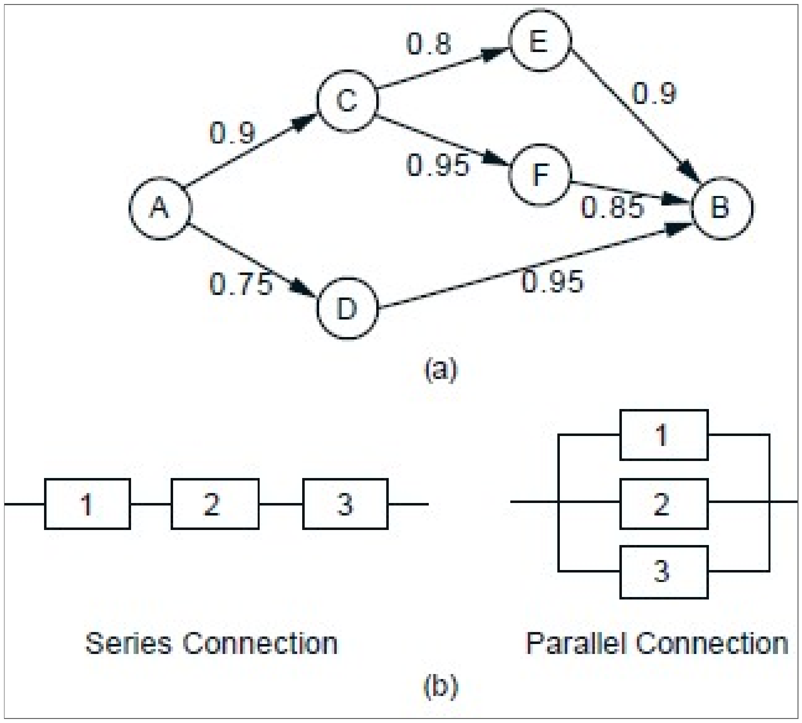
Context

In probabiliistic models of complex systems involving several components, it is often convenient (and intelligent) to assume that the behaviors of the components are uncoupled (independent).

Example 1.24 Network Connective

A computer network connects two nodes A and B through intermediate nodes C, D, E, F.
For every pair of directly connected nodes, say i and j, there is a given probability p_{ij} that the link from i to j is functioning properly. We assume that link failures are independent of each other.
What is the probability that there is a path connecting A and B in which all links are up?

Example 1.24 (cont)



Reliability example

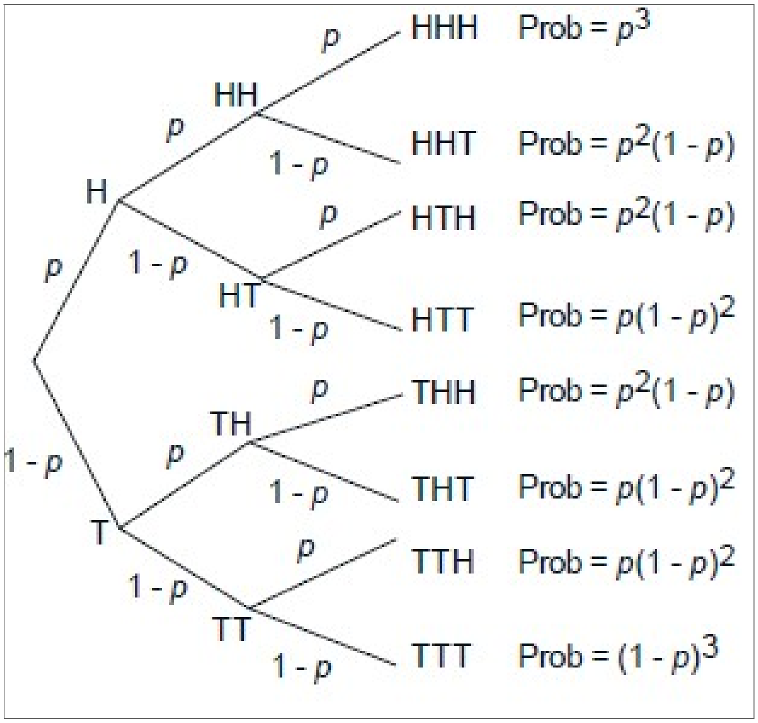
Example 1.24 (cont)

Independent Trials and the Binomial Probabilities

Context

If an experiment involves are sequence of identical smaller experiements, then we have a sequence of **independent trials**.
If each trial has only two possible outcomes, then we have a sequence of **Bernoulli trials**.
Assume we flip a coin three times, successively and independently. Let H_i denote flipping a head on trial i and $P(H_i) = p$.

Bernoulli trials visualized



Bernoulli trials tree diagram

Binomial probabilities

For a set of Bernoulli trials, consider the probability
$p(k) = P(k \text{ heads come up in an } n\text{-toss sequence}).$
The probability for a particular sequence (one branch of the tree) is
Thus,
$p(k) =$

Binomial probabilities

The numbers $\binom{n}{k}$ is read as "n choose k" and is the number of distinct n-toss sequences that contain k heads.
The numbers $\binom{n}{k}$ are known as the **binomial coefficients**.
The numbers $p(k)$ are known as the **binomial probabilities**.

Example 1.25 Grade of Service

An internet service provider has installed c modems to serve the needs of a population of n dial-up customers.
Each customer will need a connection with probability p independent of the other customers.
What is the probability that there are more customers needing a connection than modems?
Answer this problem generally, and then for $n = 100, p = 0.1$, and $c = 15$.

Example 1.25 (cont)