4.4 Conditional expectation and variance revisited

Law of Iterated Expectations

Also know as the law of total expectation:
For two random variables X and Y,
$E[E(X \mid Y)] = E(X)$.
Note:

- $E(X \mid Y = y)$ is a function of y and non-random.
- $E(X \mid Y)$ is a function of Y and random.

Example 4.16

Let Y be a random variable producing a (probability) value in the interval $[0, 1]$.
Let $X \mid Y = y \sim \mathrm{Binomial}(n, y)$.

Example 4.16 (cont)

Determine:

- $E(X \mid Y = y)$
- $E(X \mid Y)$
- $E(X)$

Example 4.16 (cont)

Example 4.16 (cont)

Determine $E(X)$ if $Y \sim \mathrm{Uniform}(0, 1)$.

Example 4.17 Averaging Quiz Scores by Section

A class has n students and the quiz score of student i is x_i. The average quiz score is

$$m = \frac{1}{n} \sum_{i=1}^{n} x_i.$$

Example 4.17 (cont)

The students are divided into k disjoint subsets $A_1, A_2, \ldots, A_k$, with n_s denoting the number of students in subset A_s. The probability that a student is in section s is $1/n_s$.

The average score in section s is

$$m_s = \frac{1}{n_s} \sum_{i \in A_s} x_i.$$

Example 4.17 (cont)

Determine $E(X)$ using the law of iterated expectations.

Example 4.17 (cont)

The Conditional Expectation of an Estimator

Let an **estimator** $\hat{X} = E(X \mid Y)$.
The **estimation error** is $\tilde{X} = \hat{X} - X$.
Determine $E(\tilde{X})$

Mean Estimation Error (cont)

Uncorrelated Estimation Error

Show that $\cov\tilde{X}, \hat{X} = 0$, i.e., the estimation error and the estimator are uncorrelated.

Uncorrelated Estimation Error (cont)

The Conditional Variance

Law of Total Variance

The **law of total variance** states
$\mathrm{var}(X) = E[\mathrm{var}(X \mid Y)] + \mathrm{var}(E[X \mid Y])$.

Law of Total Variance (proof)

Law of Total Variance (proof)

Law of Total Variance (proof)

Example 4.16 (cont)

Let Y be a random variable producing a (probability) value in the interval $[0, 1]$. Let $X \mid Y = y \sim \text{Binomial}(n, y)$. Determine $\text{var}(X)$.

Example 4.16 (cont)

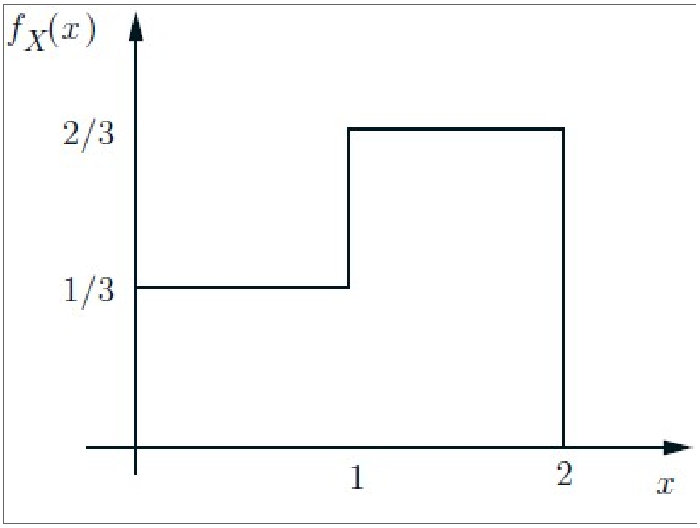
Example 4.21

Consider a continuous random variable X with the pdf given below.
Define

$$Y = \begin{cases} 1, & \text{if } x < 1, \\ 2, & \text{if } x \geq 1. \end{cases}$$

Determine $\text{var}(X)$.

Example 4.21 (cont)



The pdf in Example 4.21.

Example 4.21 (cont)

Example 4.21 (cont)