3.1 Continuous random variables and pdfs

Definining a continuous random variable

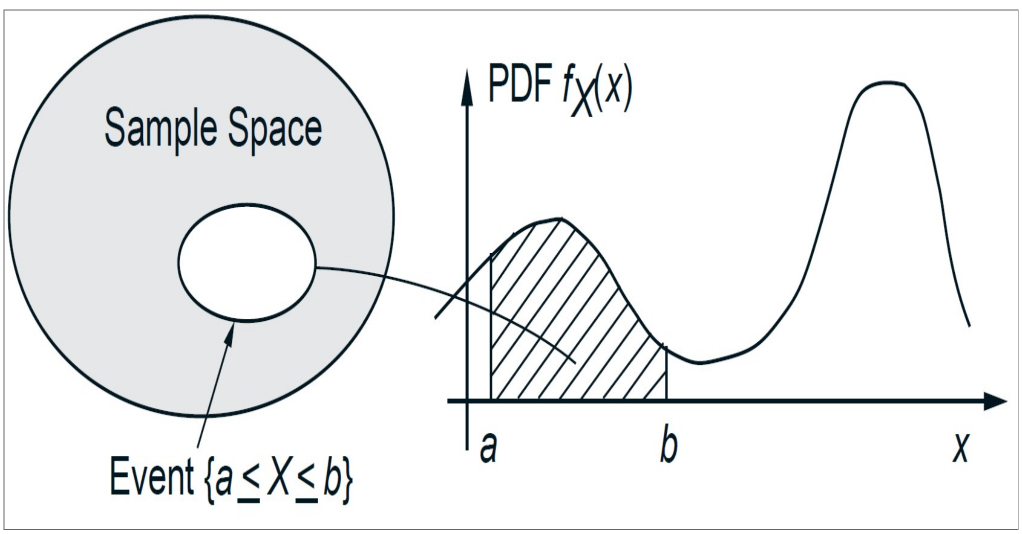
A random variable, X, is **continuous** if there exists a function, f_X, such that

$$P(a \leq X \leq b) = \int_a^b f_X(x)dx.$$

In that case, f_X is known as the probability density function (pdf).

pdf illustration

Visualizing the conditional pmf



The probability that X takes a value in an interval $[a, b]$ is $\int_a^b f_X(x)dx$.

pdf properties

A pdf must satisfy the following properties:

1. $f_X(x) \geq 0$ for all $x \in \mathbb{R}$.
2. $\int_{-\infty}^{\infty} f_X(x)dx = 1$.
3. $P(a \leq X \leq b) = \int_a^b f_X(x)dx$.

As a consequence of point 3,

$$P(X = a) = P(a \leq X \leq a) = \int_a^a f_X(x)dx = 0.$$

Example 3.1 Continuous Uniform Random Variable

A gambler spins a wheel of fortune, continuously calibrated betwen 0 and 1, and observes the resulting number. Assuming that any two subintevals of [0,1] of the same length have the same probability, this experiment can be modeled in terms of a random variable X with pdf

$$f_X(x) = \begin{cases} c & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

What should the constant c be?

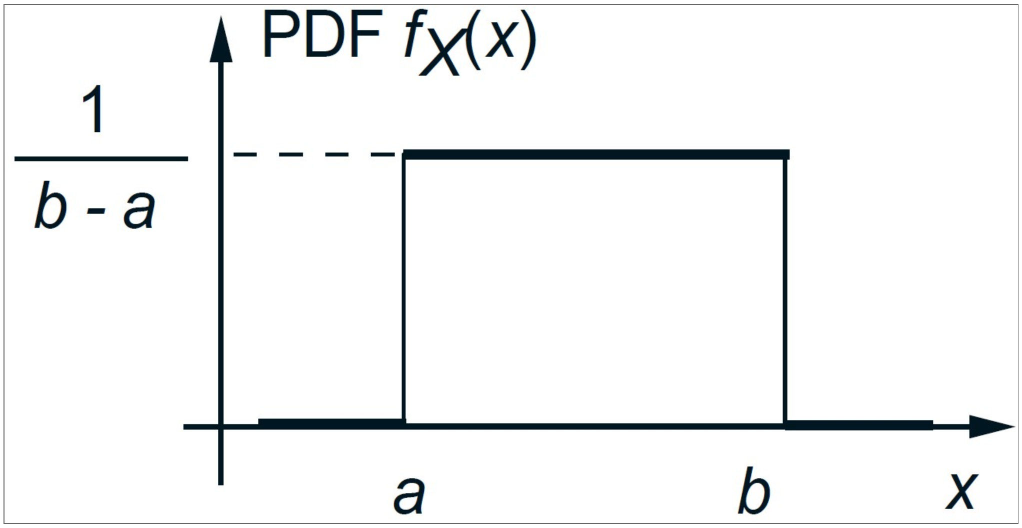
Example 3.1 (cont)

Uniform(a, b) random variable

A Uniform(a, b) random variable has the pdf

$$f_X(x) = \begin{cases} \frac{1}{b-a} & \text{if } a \leq x \leq b \\ 0 & \text{otherwise.} \end{cases}$$

Uniform(a, b) pdf



The pdf of a uniform random variable.

Uniform(a, b) continued

Verify that the Uniform(a, b) density is valid.

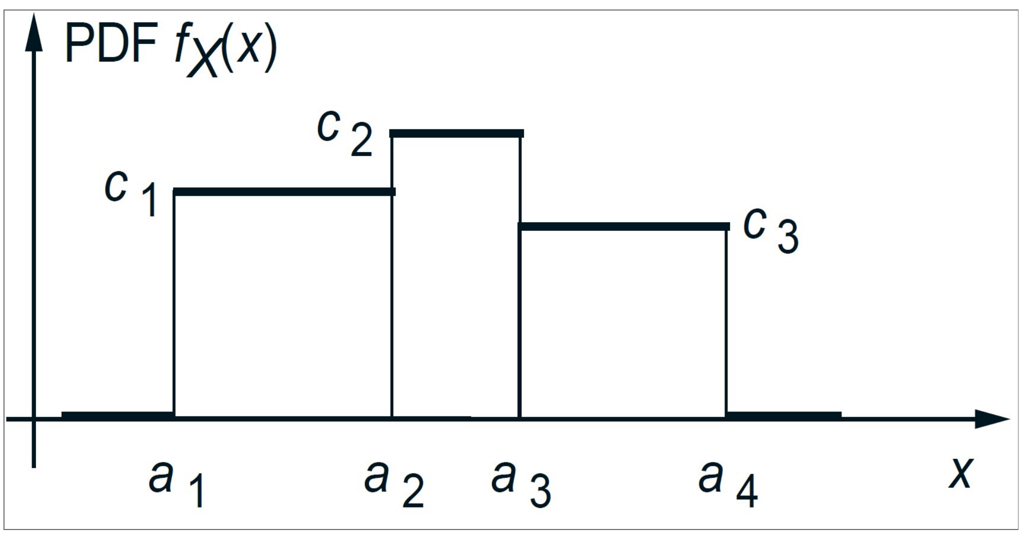
Example 3.2 Piecewise Constant pdf

Alvin's driving time to work is between 15 and 20 minutes if the day is sunny and between 20 and 25 minutes if the day is rainy, with all times being equally like in each case.
Assume that a day is sunny with probabilty 2/3 and rainy with probability 1/3. What is the pdf of the driving time, viewed as a random variable X?

Example 3.2 Piecewise Constant pdf (cont)

Example 3.2 Piecewise Constant pdf (cont)

A piecewise constant pdf



A piecewise constant pdf involving three intervals.

Example 3.3 A pdf can take arbitrarily large values

Consider a random variable X with pdf

$$f_X(x) = \begin{cases} \frac{1}{2\sqrt{x}} & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Verify that this is a pdf.

Example 3.3 (cont)

Expected value

The **expected value** of a continuous random variable X is
$E(X) = \int_{-\infty}^{\infty} x f_X(x)\, dx.$
Similarly,
$E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx.$

More expected value

The **variance** of a continuous random variable X is
$\mathrm{var}(X) = \int_{-\infty}^{\infty} (x - E[X])^2 f_X(x)\, dx.$
Oftentimes, it is more convenient to use the relationship
$\mathrm{var}(X) = E(X^2) - (E[X])^2.$

Expectation of a linear function

If $Y = aX + b$, then
$E(Y) = aE(X) + b$
and
$\mathrm{var}(Y) = a^2\mathrm{var}(X)$.

Example 3.4 Mean and variance of a uniform random variable

Let $X \sim \mathrm{Uniform}(a, b)$.
Write the pdf of X.

Example 3.4 (cont)

Determine $E(X)$.

Example 3.4 (cont)

Determine $\mathrm{var}(X)$.

The Exponential Random Variable

Let $X \sim \text{Exponential}(\lambda)$, where $\lambda > 0$ is the rate parameter. The pdf of X is:

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

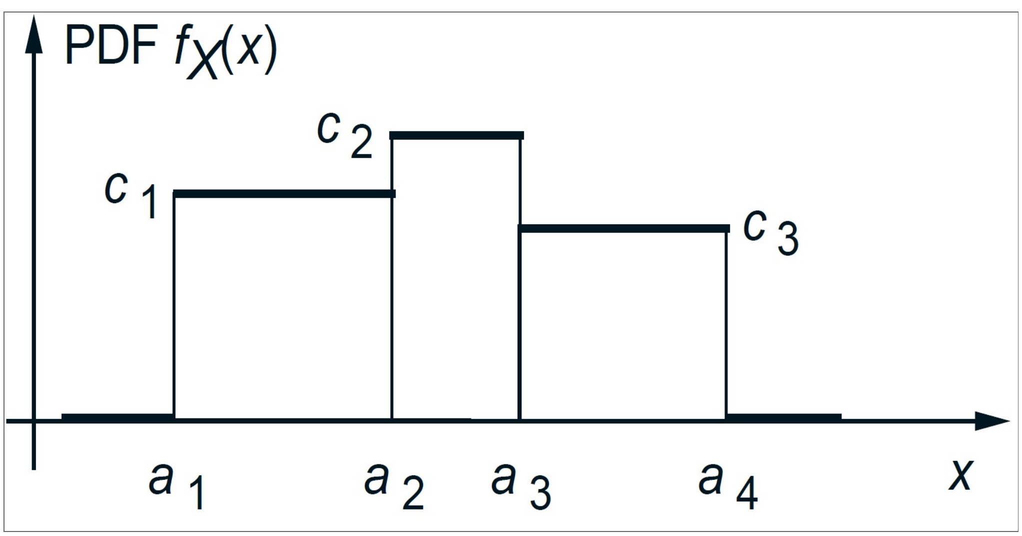
Exponential pdf validy

Verify that the pdf is valid.

Exponential probabilities

Determine $P(X \geq a)$ for $a \geq 0$.

Exponential mean visualized



Line plot $\lambda e^{-\lambda x}$.

Exponential variance

Determine $\mathrm{var}(X)$.

Example 3.5

The time until a small metorite first lands anywhere in the Sahara desert is modeled as an exponential random variable with a mean of 10 days. The time is currently midnight. What is the probability that a meteorite first lands sometime between 6 a.m. and 6 p.m. of the first day?
Let X be the time elapsed until the event of interest, measured in days. Then X is Exponential with mean $1/\lambda = 10$, which yields $\lambda = 10$.

Example 3.5 (cont)

The desired probability is