2.6 Conditioning

Conditioning on events

The **conditional pmf of** X given an event A with $P(A) > 0$, is defined as

$$p_{X|A}(x) = P(X = x \mid A) = \frac{P(\{X = x\} \cap A)}{P(A)},$$

and satisfies

$$\sum_{x \in R_X} p_{X|A}(x) = 1.$$

Conditioning on events (cont)

The fact that

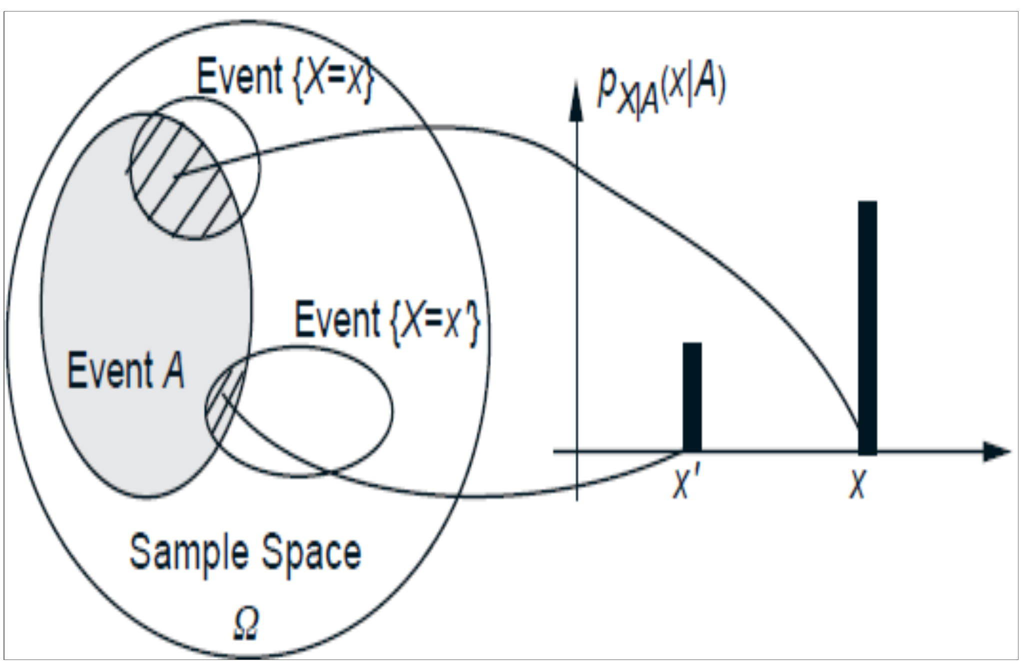
$$\sum_{x \in R_X} p_{X|A}(x) = 1$$

since

$$P(A) = \sum_{x \in R_X} P(\{X = x\} \cap A)$$

by the Total Probability Law.

Visualizing the conditional pmf



Visualization and calculation of the conditional pmf.

Conditioning and the Total Probability Law

If $A_1, \ldots, A_n$ are disjoint events that partition the sample space, with $P(A_i) > 0$ for $i = 1, 2, \ldots, n$, then

$$p_X(x) = \sum_{i=1}^{n} P(A_i) p_{X|A_i}(x).$$

Conditioning and the Total Probability Law (cont)

Additionally, for any event B, if $P(A_i \cap B) > 0$ for all i, we have

$$p_{X|B}(x) = \sum_{i=1}^{n} P(A_i \mid B) p_{X|A_i \cap B}(x).$$

Example 2.12

Let X be the roll of a fair-sided die and let A be the event that the roll is an even number. Determine the conditional pmf of X when $A = \{\text{the roll is an even number}\}$.

Example 2.13

A student can pass a test with probability p. The student will keep taking the test repeatedly, up to a maximum of n times, until they pass the test. Each test outcome is independent.
Let X be the number of attempts needed to pass the test if an unlimited number of events were allowed. Determine the pmf of X given the student passed.

Example 2.13 (cont)

Conditioning one Random Variable on Another

Conditional pmf definition

Let X and Y be discrete random variables. The **conditional pmf** of X given Y is defined as:

$$p_{X|Y}(x|y) = \frac{P(X = x, Y = y)}{P(Y = y)} = \frac{p_{X,Y}(x, y)}{p_Y(y)},$$

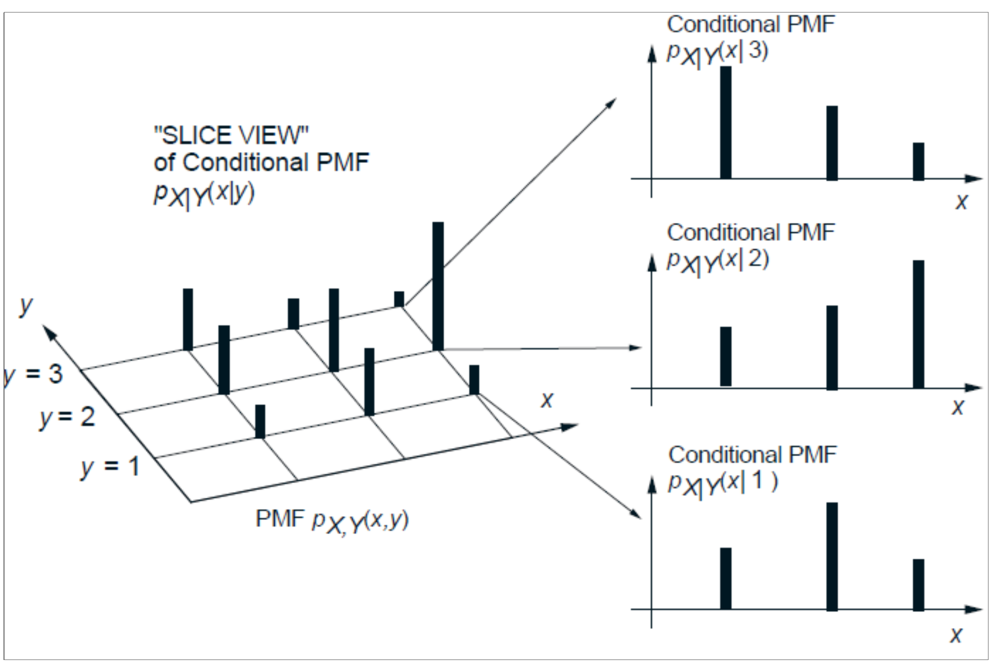
when $p_Y(y) > 0$.

Conditional pmf normalization

$$\sum_{x \in R_X} p_{X|Y}(x|y) = 1,$$

for each fixed y with $p_Y(y) > 0$.

Visualizing the conditional pmf



Visualization of the conditional pmf.

Joint and conditional pmfs

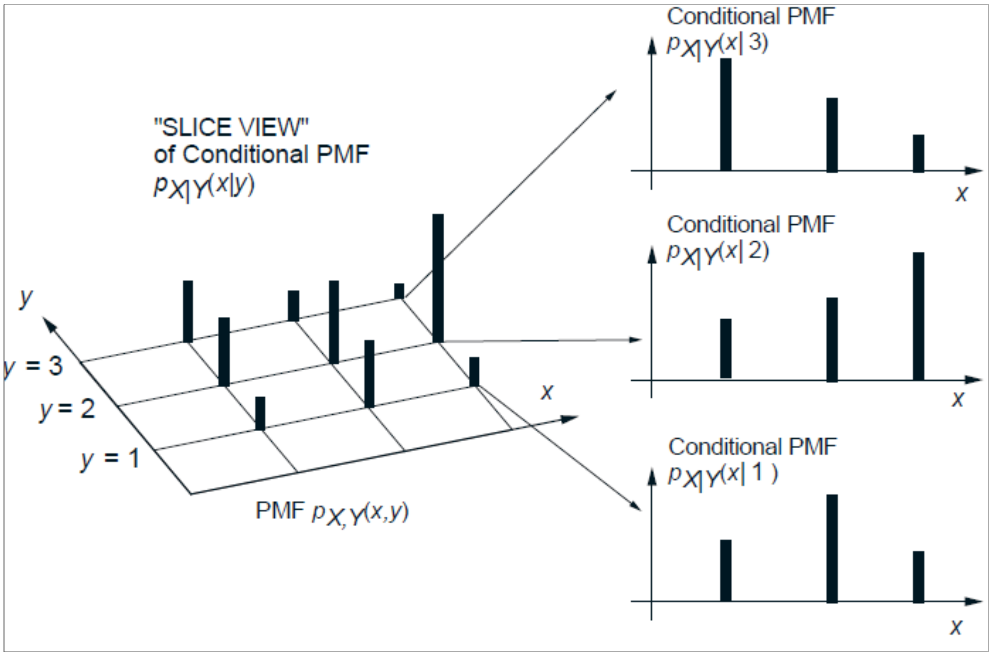
The joint PMF can be expressed in terms of the conditional and marginal PMFs:
$p_{X,Y}(x, y) = p_{X|Y}(x|y)p_Y(y)$
and
$p_{X,Y}(x, y) = p_{Y|X}(y|x)p_X(x).$

Example 2.14

Professor May B. Right often has his facts wrong, and answers each of his students' questions incorrectly with probability $1/4$, independent of other questions. In each lecture, May is asked 0, 1, or 2 questions with equal probability $1/3$. Let X and Y be the number of questions May is asked and the number of questions she answers wrong in a given lecture, respectively. Construct the joint pmf of X.

Example 2.14 (cont)

Example 2.14 (cont)



Example 2.14 joint pmf calculation.

Marginal and conditional pmfs

The conditional pmf of X given Y can be used to calculate the marginal pmf of X through the formula

$$p_X(x) = \sum_{y \in R_Y} p_Y(y) p_{X|Y}(x \mid y).$$

Example 2.15

Consider a transmitter that is sending messages over a computer network. Let us define the following two random variables:

X : the travel time of a given message,

Y : the length of the given message.

We know the pmf of the travel time of a message of a given length, and we know the mpf of the message length. Determine the pmf of the travel time of a message.

Example 2.15 (cont)

The length of a message is either 10^2 bytes with probability $5/6$ or 10^4 bytes with probability $1/6$.
The conditional distribution of the travel time when the message length $y = 10^2$ bytes is

$$p_{X|Y}(x \mid 10^2) = \begin{cases} \frac{1}{2} & \text{if } x = 10^{-2}, \\ \frac{1}{3} & \text{if } x = 10^{-1}, \\ \frac{1}{6} & \text{if } x = 1. \end{cases}$$

Example 2.15 (cont)

The conditional distribution of the travel time when the message length $y = 10^4$ bytes is

$$p_{X|Y}(x \mid 10^4) = \begin{cases} \frac{1}{2} & \text{if } x = 1, \\ \frac{1}{3} & \text{if } x = 10, \\ \frac{1}{6} & \text{if } x = 100. \end{cases}$$

Example 2.15 (cont)