1.6 Counting

Context

Probability calculations often involve counting how many outcomes are in an event of interest.
While counting *seems* straightforward, it is often difficult in practice and involves learning about the area of **combinatorics**.

The Counting Principle

Counting Strategy

The counting principle is based on a divide-and-conquer approach in which the process is broken down into stages through the use of a tree. E.g., an experiment might consist of two stages.

- The possible outcomes of stage 1 are $a_1, \ldots, a_m$.
- The possible outcomes of stage 2 are $b_1, \ldots, b_n$.

The possible result of the experiment is the set of all possible **ordered** pairs (a_i, b_j), $i = 1, \ldots, m$, and $j = 1, \ldots, n$.
In this example, there are mn total results.

Counting strategy visualized

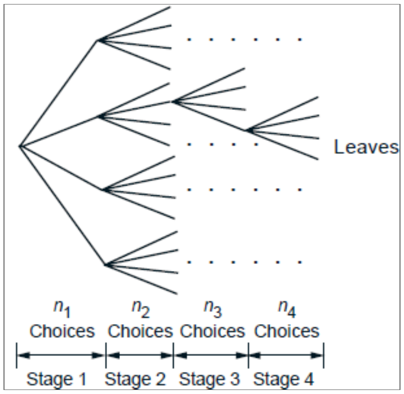

Illustration of the basic counting principle

The Counting Principle

> Consider a process that consists of r stages. Suppose that
>
> a. There are n_1 possible results at the first stage.
>
> b. For every possible result of the first stage, there are n_2 possible results at the second stage.
>
> c. More generally, for any sequence of possible results at the first $i-1$ stages, there are n_i possible results at the ith stage.
>
> Then the total number of possible results of the r-stage process is $n_1 n_2 \ldots n_r$.

Example 1.26 The Number of Telephone Numbers

A local telephone number is a 7-digit sequence, but the first digit must be different from 0 or 1. How many distinct telephone numbers are there?

Example 1.27 The Number of Subsets of an n-Element Set

Consider an n-element set $\{s_1, s_2, \ldots, s_n\}$. How many subsets does it have?

Focus

We will focus on counting when we want to select k objects out of a collection of n objects.

- **Permutation** is when the selection order matters.
- **Combination** is when the selection order doesn't matter, only the set we end up with.

k-permutations

k-permutations

We have n distinct objects.
How many ways can we pick k objects out of the n objects and arrange them in a sequence? ($k \leq n$)

- We want the number of distinct k-object sequences.

k-permutation counting

We have n choices for the first selection, $n-1$ for the second selection, etc.
The number of possible sequences, called k-**permutations**, is

$$
\begin{aligned}
&n(n-1)\cdots(n-k+1) \\
&\quad = \frac{n(n-1)\dots(n-k+1)(n-k)\dots(2)(1)}{(n-k)\dots(2)(1)} \\
&\quad = \frac{n!}{(n-k)!}.
\end{aligned}
$$

Example 1.28

How many unique “words” can we create from four unique (English alphabet) letters?

Example 1.29

You have n_1 classical music albums, n_2 rock music albums, and n_3 country music albums.
How many different ways can you arrange the albums so that the albums of the same type are contiguous?

Example 1.29 (cont)

We offer to give a friend k_i out of the n_i albums for each type of music, where $k_i < n_i, i = 1, 2, 3.$
How many different ways can we arrange the remaining albums so that the albums of the same type are contiguous?

Combinations

Combination context

There are n people and we are interested in forming a committee of size k. How many different committees are possible?

- Note that the ordering of the people on the committee isn't important.
- We are interested in combinations, not permutations.

Combinations versus permutations

The 2-permutations of the letters A, B, C, D are:

The combinations of size 2 from these 4 letters are:

Combination intuition

Each combination is associated with $k!$ “duplicate” k-permutations (e.g., AB, BA).
The number of possible combinations of size k from n objects is the number of k-permutations divided by $k!$, i.e.,

$$\frac{n!}{k!(n-k)!}.$$

Note that this is the same definition we used in the binomial coefficient, $\binom{n}{k}$.

Example 1.31

We have a group of n persons. Consider clubs that consist of a special person from the group (the club leader) and a number (possibly zero) of additional club members. How many possible clubs of this type are there?

Partitions

Partitions context

A combination is a choice of k elements out of an n-element set without regard to order.

- A combination partitions the set into two subsets: one with k elements and one with $n - k$ elements.

We can generalize this notion to more than 2 subsets.

Partitions intuition

We want to partition n elements into r subsets, where the first group has n_1 elements, the second has n_2 elements, and so on, such that
$n_1 + n_2 + \cdots + n_r = n$.
We have $\binom{n}{n_1}$ ways of forming the first subset.
We have $\binom{n-n_1}{n_2}$ ways of forming the second subset, and so on.

Partition combinations

Using the Counting Principle for this r-stage process, the total number of choices is

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\cdots\binom{n-n_1-\cdots-n_{r-1}}{n_r},$$

which simplifies to

$$\frac{n!}{n_1!n_2!\ldots n_r!}.$$

Multinomial coefficient

The **multinomial coefficient**

$$\binom{n}{n_1, n_2, \ldots, n_r} = \frac{n!}{n_1! n_2! \ldots n_r!}.$$

Example 1.32 Anagrams

How many different words can be obtained by rearranging the letters in the word TATTOO?

Example 1.32 (cont)

Example 1.33

A class consisting of 4 graduate and 12 undergraduate students is randomly divided into 4 groups of 4. What is the probability that each group includes a graduate student?

Summary of Counting Results

- **Permutations** of n objects: $n!$.
- k-**permutations** of n objects: $n!/(n-k)!$
- **Combinations** of k out of n objects: $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.
- **Partitions** of n objects into r groups, with the ith group having n_i objects:
$$\binom{n}{n_1, n_2, \ldots, n_r} = \frac{n!}{n_1! n_2! \ldots n_r!}.$$