3.2 Cumulative Distribution Functions

Definition of the cdf

The **cumulative distribution function** or cdf of a discrete random variable is defined as

$$F_X(x) = P(X \le x) = \sum_{x_i \le x} P(X = x_i).$$

The cdf of a continuous random variable is defined as

$$F_X(x) = P(X \le x) = \int_{-\infty}^{x} f_X(t)\, dt.$$

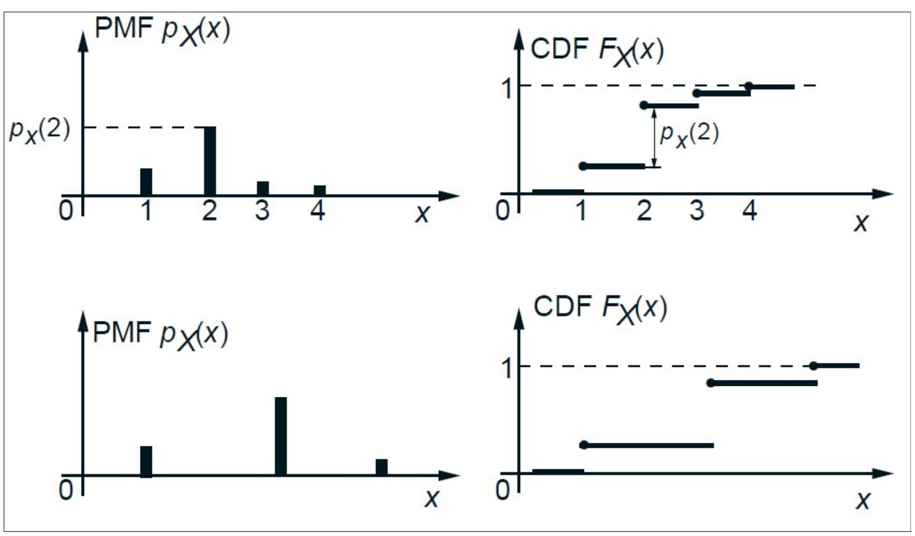
Comments

Informally, a cdf accumulates probabilities up to x.
The cdf, pmf, or pdf will be referred to as the **probability law** or **probability distribution** of the random variable X.
The pdf of a continuous random variable X can be obtained through the relationship

$$f_X(x) = \frac{dF_X(x)}{dx}.$$

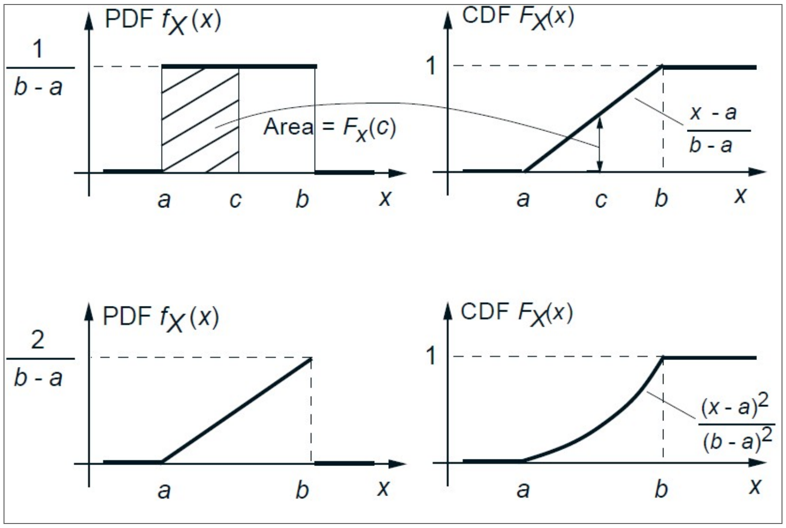
The cdf of a continuous random variable has no jumps.

Discrete cdf



cdfs of some discrete random variables.

Continuous cdf



cdfs of some continuous random variables.

cdf properties

The cdf of a random variable X is defined by
$F_X(x) = P(X \leq x), \quad \text{for all } x \in \mathbb{R}.$
A cdf has the following properties:

- $F_X(x)$ is monotonically increasing.
- $\lim x \to -\infty F_X(x) = 0.$
- $\lim x \to \infty F_X(x) = 1.$

More cdf properties

If X is discrete, then $F_X(x)$ is a piecewise constant function of x.
If X is continuous, then $F_X(x)$ is a continuous function of x.

Some discrete cdf properties

If X is discrete and takes integer values, then

$$F_X(k) = \sum_{i=-\infty}^{k} p_X(i),$$

and

$$\begin{aligned} p_X(k) &= P(X \leq k) - P(X \leq k - 10) \\ &= F_X(k) - F_X(k-1). \end{aligned}$$

Some continuous cdf properties

If X is continuous, then

$$F_X(x) = \sum_{-\infty}^{x} p_X(t)\, df$$

and

$$f_X(x) = \frac{dF_X(x)}{dx}.$$

Example 3.6 The maximum of several random variables

You are allowed to take a certain test three times, and your final score will be the maximum of the test scores.
Thus,
$X = \max\{X_1, X_2, X_3\}$,
where X_1, X_2, X_3 are the three test scores and X is the final score.
Assume that your score in each test takes one of the values from 1 to 10 with equal probability $1/10$.

Example 3.6 (cont)

What is the pmf, p_X, of the final score?

The Geometric and Exponential CDFs

The Geometric pmf and cdf

Let $X \sim \mathrm{Geometric}(p)$, with $p \in [0, 1]$.
Then the pmf, for $k = 1, 2, \ldots$ is given by

$$p_X(k) =$$

The cdf is given by

$$F_X(x) =$$

The Exponential pdf and cdf

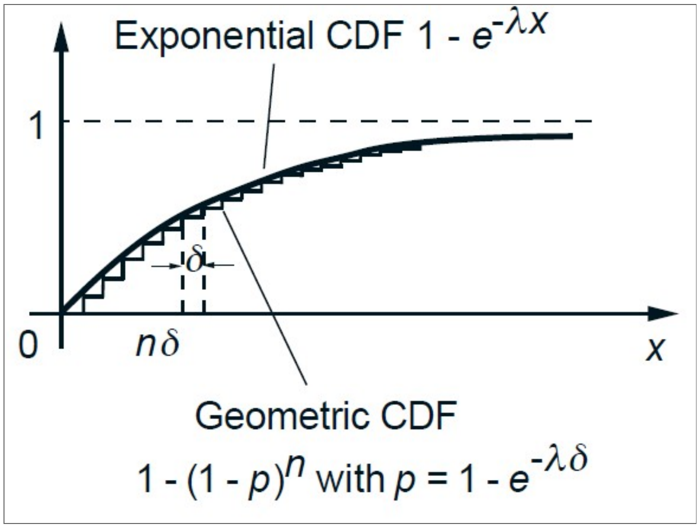
Let $X \sim \mathrm{Exponential}(\lambda)$ for $\lambda > 0$.
The pdf is given by

$$p_X(k) =$$

The cdf is given by

$$F_X(x) =$$

Geometric and exponential relationship



Relation of the geometric and exponential cdfs.

Geometric and exponential relationship

Define $\delta = -\ln(1-p)/\lambda$, so that
$e^{-\lambda\delta} = 1 - p$.
Then when $x = n\lambda$, with $n = 1, 2, \ldots$,
$F_{\exp}(n\delta) = F_{\text{geo}}(n)$.