3.4 Conditioning on an event

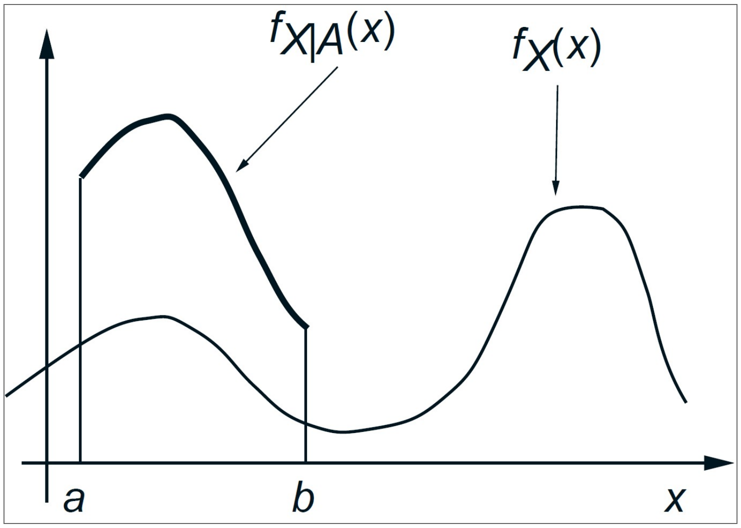
Conditional pdf

Let X be a continuous random variable with probability density function $f_X(x)$, and let $A \subseteq \mathbb{R}$ be an event such that $P(X \in A) > 0$.
The **conditional pdf** of X given the event $X \in A$ is defined as:

$$f_{X|A}(x) = \begin{cases} \frac{f_X(x)}{P(X \in A)} & \text{for } x \in A \\ 0 & \text{otherwise.} \end{cases}$$

This ensures that the conditional pdf integrates to 1 over the set A, and reflects the updated distribution of X under the condition that $X \in A$.

Conditional vs unconditional pdf



A comparison of a conditional and unconditional pdf.

Example 3.10 Memoryless property

Sammy Jankis goes to a bus stop where the time T between two successive buses has an exponential pdf with parameter λ. Suppose that Sammy arrives t secs after the preceding bus arrival and let us express this fact with the event $A = \{T > t\}$. Let X be the time that Sammy has to wait for the next bus to arrive. What is the conditional CDF $F_{X|A}(x|A)$?

Example 3.10 (cont)

Example 3.10 (cont)

Thus, the conditional cdf of X is exponential with parameter λ, regardless the time t that elapsed between the preceding bus arrival and Sammy's arrival.

This is known as the **memorylessness** property of the exponential.

Generally, if we model the time to complete a certain operation by an exponential random variable X, this property implies that as long as the operation has not been completed, the remaining time up to completion has the same exponential cdf, no matter when the operation started.

Conditional expectation

The conditional expectation of X given the event A is defined as:

$$E(X \mid A) = \int_{-\infty}^{\infty} x f_{X|A}(x)\, dx,$$

and

$$E[g(X) \mid A] = \int_{-\infty}^{\infty} g(x) f_{X|A}(x)\, dx.$$

More conditional expectation

If $A_1, A_2, \ldots, A_n$ partition the sample space, with $P(A_i) > 0$ for all i, then

$$f_X(x) = \sum_{i=1}^{n} P(A_i) f_{X|A_i}(x).$$

More conditional expectation

Additionally, in the same context,

$$E(X) = \sum_{i=1}^{n} P(A_i)E(X \mid A_i),$$

and

$$E[g(X)] = \sum_{i=1}^{n} P(A_i)E[g(X) \mid A_i.$$

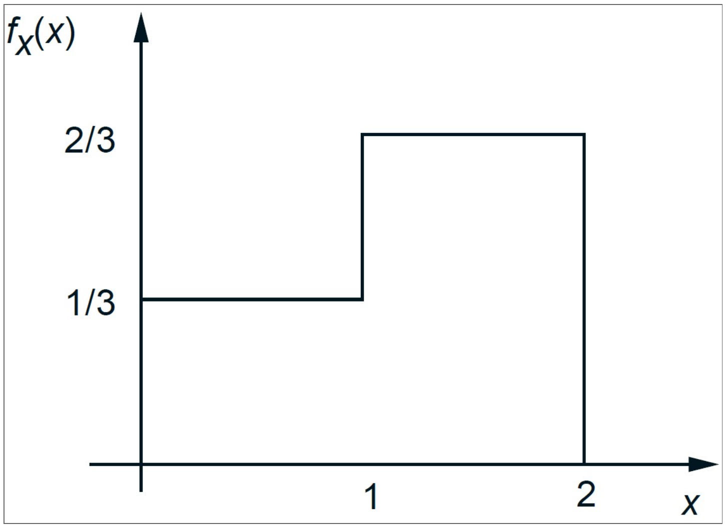
Example 3.11

Suppose that the random variable X has the piecewise constant pdf

$$f_X(x) = \begin{cases} 1/3 & \text{if } 0 \le x \le 1, \\ 2/3 & \text{if } 1 < x \le 2, \\ 0 & \text{otherwise.} \end{cases}$$

Determine the mean and variance of X.

Example 3.11 pdf



The piecewise constant pdf of Example 3.11.

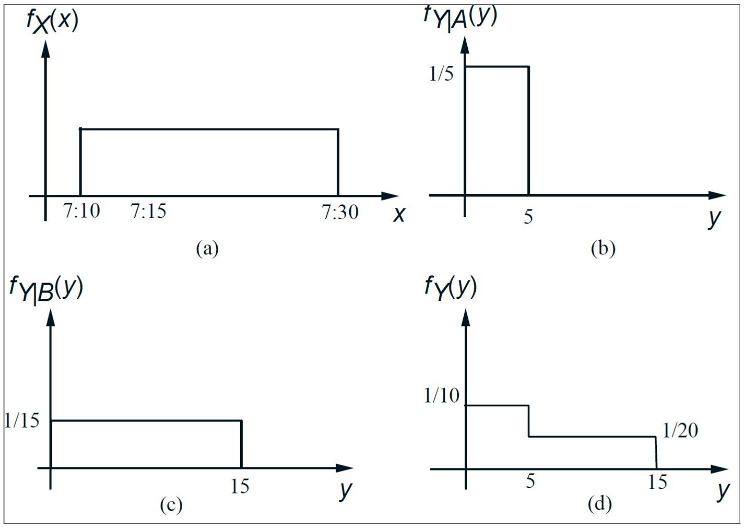
Example 3.11 (cont)

Example 3.11 (cont)

Example 3.12

The metro train arrives at the station near your home every quarter hour starting at 6:00 AM. You walk into the station every morning between 7:10 and 7:30 AM, with the time in this interval being a uniform random variable. What is the pdf of the time you have to wait for the first train to arrive?

Example 3.12 pdf



The relevant pdfs of Example 3.12.

Example 3.12 (cont)

Example 3.12 (cont)