3.4 Joint pdfs of multiple random variables

Jointly continuous definition

Two random variables are **jointly continuous** with **joint pdf** $f_{X,Y}$ if there exists a nonnegative function that satisfies

$$P[(X,Y) \in B] = \iint_{(x,y)\in B} f_{X,Y}(x,y)\, dx\, dy,$$

for any $B \subseteq \mathbb{R}^2$.

Additional properties

Additionally, a joint pdf must satisfy the normalization property of probability, i.e.,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy = 1.$$

Marginal pdf

The **marginal** pdf f_X of X is computed from the joint pdf by

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy.$$

Similarly, the marginal pdf of Y is

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx.$$

Example 3.9 Two-dimensional uniform pdf

Romeo and Juliet have a data at a given time, and each will arrive at the meeting place with a delay between 0 and 1 hour. Let X and Y denote the delays of Romeo and Julien, respectively, Assuming that no pairs (x, y) in the unit square are more likely than others, a natural model involves a joint pdf of the form

$f_{X,Y}(x, y) = cI_{[0,1]}(x)I_{[0,1]}(y)$.

Determine c.

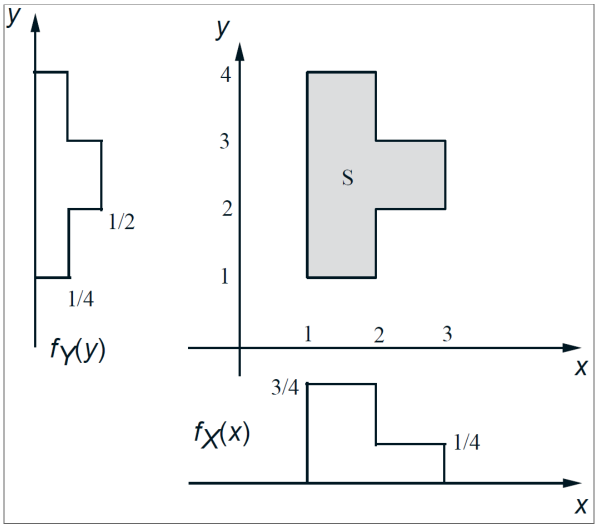
Example 3.9 (cont)

Example 3.10

The joint pdf of random variables X and Y is a constant c on the set S shown in the Figure below and 0 otherwise.
Determine c. Determine the marginal pdf of X and the marginal pdf of Y.

Example 3.10 (cont)

Conditional vs unconditional pdf



The joint pdf of X and Y.

Example 3.10 (cont)

Example 3.10 (cont)

Example 3.10 (cont)

Hydration example

Josh refills his water bottle every 3 hours. Let Y denote the proportion of the water bottle filled with water at the beginning of the 3-hour window. Let X denote the amount of water the author consumes in the 3-hour window (measured in the the proportion of total water bottle capacity). We know that $0 \leq X \leq Y \leq 1$. The joint density of the random variables is
$f(x, y) = 4y^2, \quad 0 \leq x \leq y \leq 1,$
and 0 otherwise.

Hydration example (cont)

$P(0.5 \leq X \leq 1, 0.75 \leq Y)$

Hydration example (cont)

Determine the marginal distributions of X and Y.

Hydration example (cont)

Joint cdfs

The joint cdf of jointly continuous random variables X and Y is
$F_{X,Y}(x,y) = P(X \le x, Y \le y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(s,t)\, dt\, ds.$
Converely,
$f_{X,Y}(x,y) = \frac{\partial^2}{\partial x \partial y} F_{X,Y}(x,y).$

Example 3.12

Determine the cdf of the following jointly continuous pdf
$f_{X,Y}(x,y) = I_{[0,1]}(x) I_{[0,1]}(y).$

Example 3.12 (cont)

Verify the pdf from the cdf.

Expectation

Expected value

The expectation of a function of jointly continuous random variables is
$E[g(X,Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y)\, dx\, dy.$
Additionally, for scalars a, b, c,
$E[aX + bY + c] = aE(X) + bE(Y) + c.$

More than two random variables

The ideas presented above extend directly to multiple dimensions.

Extended jointly continuous definition

Three random variables are **jointly continuous** with **joint pdf** $f_{X,Y,Z}$ if there exists a nonnegative function that satisfies

$$P[(X,Y,Z) \in B] = \iiint_{(x,y,z)\in B} f_{X,Y,Z}(x,y,z)\,dx\,dy\,dz,$$

for any $B \subseteq \mathbb{R}^3$.

Extended marginal pdf

The **marginal** pdf f_X of X is computed from the joint pdf by

$$f_X(x) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y,Z}(x,y,z)\,dy\,dz.$$

Expected value

The expectation of a function of jointly continuous random variables is

$$E[g(X,Y,Z)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y,z) f_{X,Y,Z}(x,y,z)\,dx\,dy\,dz.$$

Additionally, for scalars $a_1, a_2, \ldots, a_n$,

$$E\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} E(X_i).$$