3.5 Conditioning

Conditioning a Random Variable on an Event

Conditional pdf

Let X be a continuous random variable with probability density function $f_X(x)$, and let $A \subseteq \mathbb{R}$ be an event such that $P(X \in A) > 0$.
The **conditional pdf** of X given the event $X \in A$ is defined as:

$$f_{X|A}(x) = \begin{cases} \frac{f_X(x)}{P(X \in A)} & \text{for } x \in A \\ 0 & \text{otherwise.} \end{cases}$$

This ensures that the conditional pdf integrates to 1 over the set A, and reflects the updated distribution of X under the condition that $X \in A$.

Conditional vs unconditional pdf

A comparison of a conditional and unconditional pdf.

A comparison of a conditional and unconditional pdf.

Example 3.13 Memoryless property

Sammy Jankis goes to a bus stop where the time T between two successive buses has an exponential pdf with parameter λ. Suppose that Sammy arrives t secs after the preceding bus arrival and let us express this fact with the event $A = \{T > t\}$. Let X be the time that Sammy has to wait for the next bus to arrive. What is the conditional CDF $F_{X|A}(x|A)$?

Example 3.13 (cont)

Example 3.13 (cont)

Thus, the conditional cdf of X is exponential with parameter λ, regardless the time t that elapsed between the preceding bus arrival and Sammy's arrival.
This is known as the **memorylessness** property of the exponential.
Generally, if we model the time to complete a certain operation by an exponential random variable X, this property implies that as long as the operation has not been completed, the remaining time up to completion has the same exponential cdf, no matter when the operation started.

Conditional expectation

The conditional expectation of X given the event A is defined as:
$$E(X \mid A) = \int_{-\infty}^{\infty} x f_{X|A}(x)\, dx,$$
and
$$E[g(X) \mid A] = \int_{-\infty}^{\infty} g(x) f_{X|A}(x)\, dx.$$

More conditional expectation

If $A_1, A_2, \ldots, A_n$ partition the sample space, with $P(A_i) > 0$ for all i, then

$$f_X(x) = \sum_{i=1}^{n} P(A_i) f_{X|A_i}(x).$$

More conditional expectation

Additionally, in the same context,

$$E(X) = \sum_{i=1}^{n} P(A_i) E(X \mid A_i),$$

and

$$E[g(X)] = \sum_{i=1}^{n} P(A_i) E[g(X) \mid A_i].$$

Example

Suppose that the random variable X has the piecewise constant pdf

$$f_X(x) = \begin{cases} 1/3 & \text{if } 0 \le x \le 1, \\ 2/3 & \text{if } 1 < x \le 2, \\ 0 & \text{otherwise.} \end{cases}$$

Determine the mean and variance of X.

Example 3.17

The piecewise constant pdf of this Example

The piecewise constant pdf of this Example.

Example 3.17 (cont)

Example 3.17 (cont)

Example 3.14

The metro train arrives at the station near your home every quarter hour starting at 6:00 AM. You walk into the station every morning between 7:10 and 7:30 AM, with the time in this interval being a uniform random variable. What is the pdf of the time you have to wait for the first train to arrive?

Example 3.14 pdf

The relevant pdfs of Example 3.12.

The relevant pdfs of Example 3.14.

Example 3.14 (cont)

Example 3.14 (cont)

Conditioning one Random Variable on Another

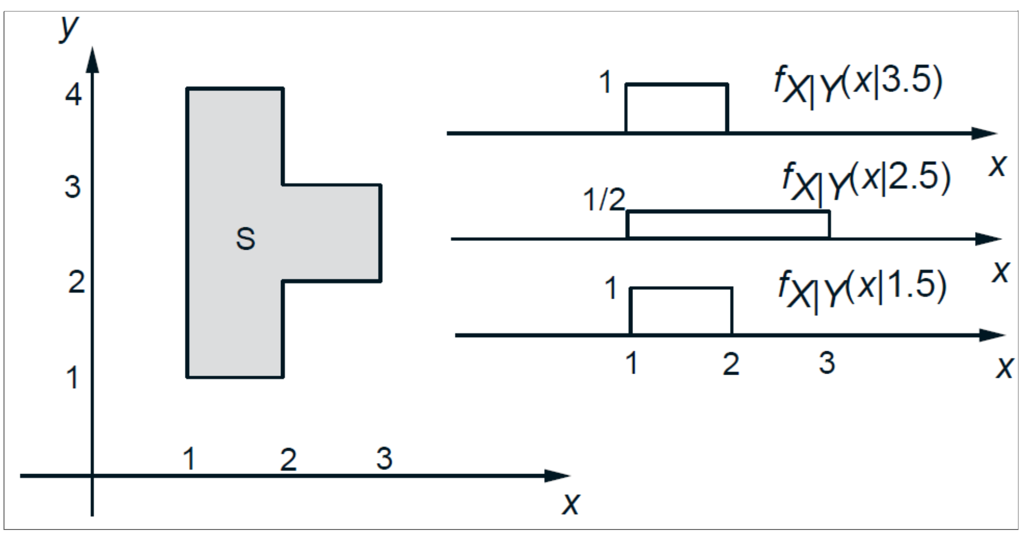
Conditional distribution (random variables)

Let X and Y be continuous random variables with joint pdf $f_{X,Y}$. For any y with $f_Y(y) > 0$, the **conditional pdf** of X given that $Y = y$ is defined by $f_{X|Y}(x \mid y) = \dfrac{f_{X,Y}(x, y)}{f_Y(y)}$.
Note: - $Y = y$ is a fixed number in $f_{X|Y}$. - Only X is random.

Conditional pdf normalization

Demonstrate that a conditional pdf satisfies the normalization property.

Conditional pdf visualized



Visualization of the conditional pdf $f_{X|Y}(x \mid y)$.

Example 3.15 Circular Uniform pdf

Ben throws a dart at a circular target of radius r. We assume that he always hits the target, and that all points of impact (x, y) are equally likely, so that the joint pdf of the random variables X and Y is uniform. Compute $f_{X|Y}(x \mid y)$.

Example 3.15 (cont)

Example 3.15 (cont)

Example 3.15 (cont)

Example 3.16

The speed of a typical vehicle that drives past a police radar is modeled as an exponentially distributed random variable X with mean 50 miles per hour. The police's radar measurement of the vehicle's speed has an error that is modeled as a normal random variale with zero mean and standard deviation equal to one tenth of the vehicle's speed. What is the joint pdf of X and Y?

Example 3.16 (cont)

Example 3.16 (cont)

Example 3.16 (cont)

Conditional pdf for more random variables

$$f_{X,Y,Z}(x, y \mid z) = \frac{f_{X,Y,Z}(x, y, z)}{f_Z(z)}, \quad f_Z(z) > 0.$$

$$f_{X,Y,Z}(x, \mid y, z) = \frac{f_{X,Y,Z}(x, y, z)}{f_{Y,Z}(y, z)}, \quad f_{Y,Z}(y, z) > 0.$$

Conditional Expectation

Conditional Expectation Definitions

$$E(X \mid Y = y) = \int_{-\infty}^{\infty} x f_{X|Y}(x \mid y)\, dx.$$

$$E(g(X) \mid Y = y) = \int_{-\infty}^{\infty} g(x) f_{X|Y}(x \mid y)\, dx.$$

$$E(X) = \int_{-\infty}^{\infty} E(X \mid Y = y) f_Y(y)\, dy.$$

Conditional Expectation Definitions

$$E[g(X, Y) \mid Y = y] = \int_{-\infty}^{\infty} g(x, y) f_{X|Y}(x \mid y)\, dx.$$

$$E[g(X, Y)] = \int_{-\infty}^{\infty} E[g(X, Y) \mid Y = y] f_Y(y)\, dy.$$

Total Expectation Theorem Proof

Total Expectation Theorem Proof (cont)

Independence

Definition

Two continuous random variables X and Y are **independent** if their joint pdf is the product of their marginal pdfs, i.e.,
$f_{X,Y}(x, y) = f_X(x) f_Y(y), \quad \text{for all} x, y.$
Alternatively,
$f_{X|Y}(x \mid y) = f_X(x), \quad \text{for all } y \text{ with } f_Y(y) \geq 0 \text{ and all } x.$

Example 3.18 Independent Normal Random Variables

Let $X \sim N(\mu_x, \sigma_x^2)$ and $Y \sim N(\mu_y, \sigma_y^2)$ be independent normal random variables.

Example 3.18 (cont)

Determine the joint pdf.

Example 3.18 (cont)

Draw contours of the joint pdf.

Other facts about independence

Show that if X and Y are independent, then $P(X \in A, Y \in B) = P(X \in A)P(Y \in B)$.

Other facts about independence

Show that if X and Y are independent, then $F_{X,Y}(x, y) = F_X(x)F_Y(y)$.

Other facts about independence

Show that if X and Y are independent, then $E(XY) = E(X)E(Y)$.

Other facts about independence

If X and Y are independent, then
$E[g(X)h(Y)] = E[g(X)]E[h(Y)]$.