1.4 Total Probability Theorem and Bayes' Rule

Total Probability Theorem

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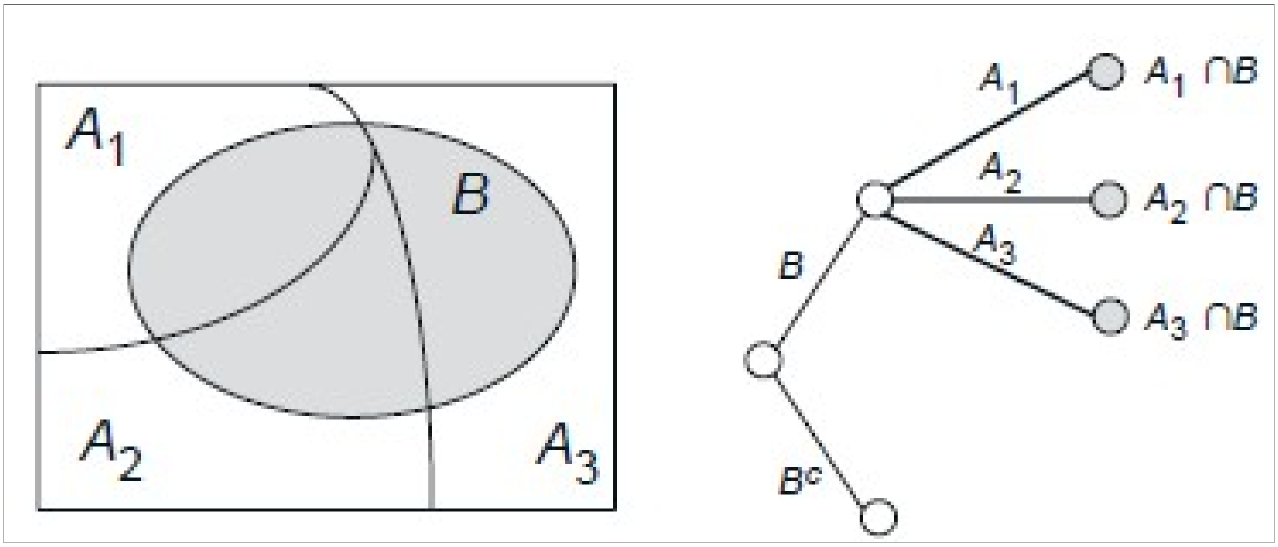
Let $A_1, \ldots, A_n$ be disjoint events that partition the sample space, Ω, and assume the $P(A_i) > 0$, for all i.
Then for any event $B \subseteq \Omega$,

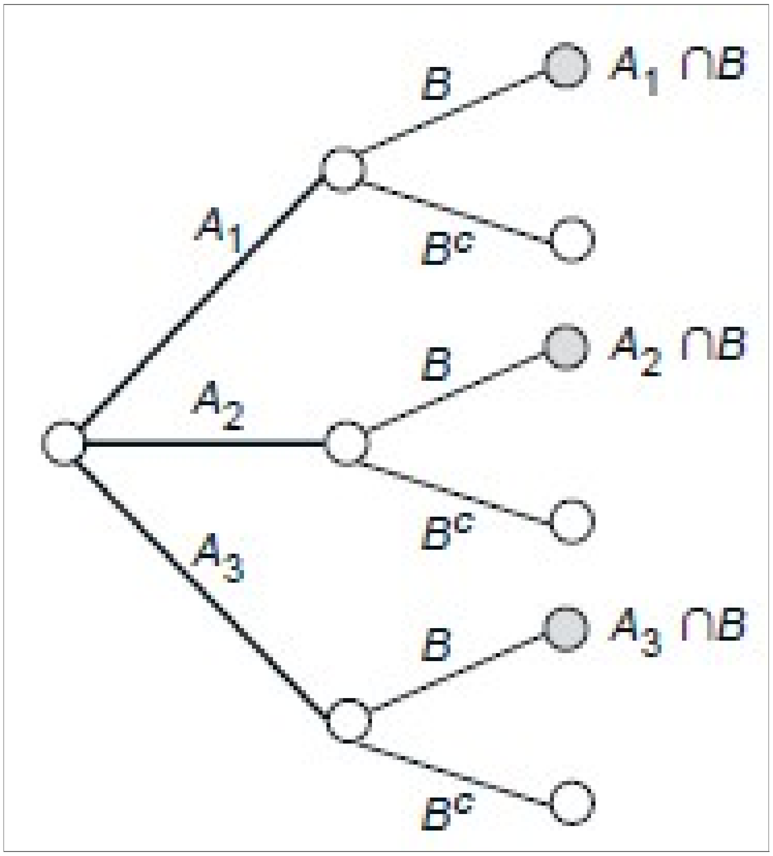
$$\begin{aligned} P(B) &= P(A_1 \cap B) + \cdots + P(A_n \cap B) \\ &= P(A_1)P(B \mid A_1) + \cdots + P(A_n)P(B \mid A_n). \end{aligned}$$

Total Probability Theorem Visualized



Visualization and verification of the total probability theorem

Total Probability Theorem Visualized (cont)



Alternative visualization of the total probability theorem

Example 1.13

You enter a chess tournament where your probability of wining a game is 0.3 against halp the players (type 1), 0.4 against a quarter of the players (type 2), and 0.5 against the remaining quarter of the players (type 3).
You play a game against a randomly chosen opponent. What is the probability of winning?

Example 1.13 (continued)

Example 1.13 (continued)

Example 1.14

You roll a fair four-sided die. If the result is 1 or 2, you roll once more but otherwise, you stop.
What is the probability that the sum total of your rolls is at least 4?

Example 1.14 (continued)

Example 1.14 (continued)

Example 1.15

Alice is taking a probability class. At the end of each week, she can be either up-to-date or she may have fallen behind. If she is up-to-date in a given week, the probability that she will be up-to-date in the next week is 0.8. If she is behind in a given week, the probability that she will be up-to-date in the next week is 0.4. Alice is up-to-date when she start the class (by default). What is the probability that she is up-to-date after three weeks?

Example 1.15 (continued)

Example 1.15 (continued)

Inference and Bayes' Rule

Bayes' Rule

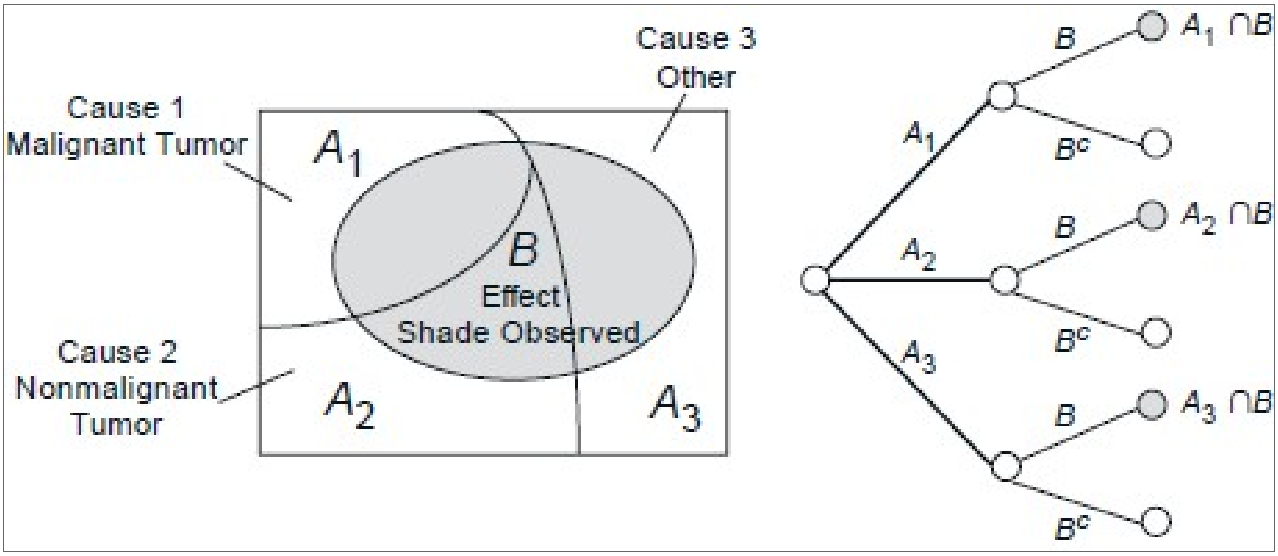
Bayes' Rule Let $A_1, \ldots, A_n$ be disjoint events that partition the sample space, Ω, and assume the $P(A_i) > 0$, for all i. Then, for any event B such that $P(B) > 0$, we have

$$\begin{aligned} P(A_i \mid B) &= \frac{P(A_i)P(B \mid A_i)}{P(B)} \\ &= \frac{P(A_i)P(B \mid A_i)}{P(A_1)P(B \mid A_1) + \cdots + P(A_n)P(B \mid A_n)}. \end{aligned}$$

Bayes' Rule Visualized

Suppose we observe a shade in a person's X-ray (event B).
We want to estimate the probability of three mutually exclusive and collectively exhaustic potential causes: 1. a malignant tumor (A_1), 2. a non-malignant tomor (A_2), 3. something other than a tumor (A_3).
Given that we see a shade (B has occurred), Bayes' rule gives us the conditional probabilities of the various causes.

Bayes' Rule Visualized



Visualization of Bayes' Rule

Bayes' Rule

Bayes' rule is often used for **inference**.
Given that effect B has been observed, we wish to evaluate the probability $P(A_i \mid B)$ that the cause A_i is present.

- $P(A_i \mid B)$ is the **posterior probability** of event A_i given the information B.
- $P(A_i)$ is the **prior probability** in this context.

Example 1.16

Let
$A = \{\text{An aircraft is present}\}.$
$B = \{\text{The radar signals an alarm}\}.$
We are given that
$P(A) = 0.05, \quad P(B \mid A) = 0.99, \quad P(B \mid A^c) = 0.1.$
Determine $P(\text{aircraft present} \mid \text{alarm}).$

Example 1.16 (cont)

Example 1.17 (Example 1.13 continued)

Let A_i be the event of getting a player of type i and B be the event of winning.
We have:
$P(A_1) = 0.5, \quad P(A_2 = 0.25), \quad P(A_3) = 0.25.$
$P(B \mid A_1) = 0.3, \quad P(B \mid A_2) = 0.4, \quad P(B \mid A_3) = 0.5.$
What is $P(A_1 \mid B)$, i.e., the probability you defeated opponent 1 given that you won the match?

Example 1.17 (cont)

Example 1.18 (The False-Positive Puzzle)

A test for a certain rare diesease correct with probability 0.95 given that the person has the disease.
If the person does not have the disease, the test results are negative with probability 0.95.
A random person drawn from a population has probability 0.001 of having the disease.
Given that a person tests positive for the disease, what is the probability the person has the disease?

Example 1.18 (cont)