Chapter 4 - Further Topics on Random Variables

4.1 Derived Distributions

Context

A **derived distribution** is the distribution of a random variable $Y = g(X)$ based on the distribution of X.

- We will consider continuous distributions.

Main approach

To compute the pdf of Y:

1. Calculate the cdf of Y using the formula $F_Y(y) = P[g(X) \leq y] = \int_{\{x|g(x)\leq y\}} f_X(x)\,dx.$
2. Differentiate $F_Y(y)$ to get the pdf of Y: $f_Y(y) = \frac{d}{dy} F_Y(y).$

Example 4.1

Let $X \sim \mathrm{Uniform}(0, 1)$ and $Y = \sqrt{X}$. Determine the pdf of Y.

Example 4.1 (cont)

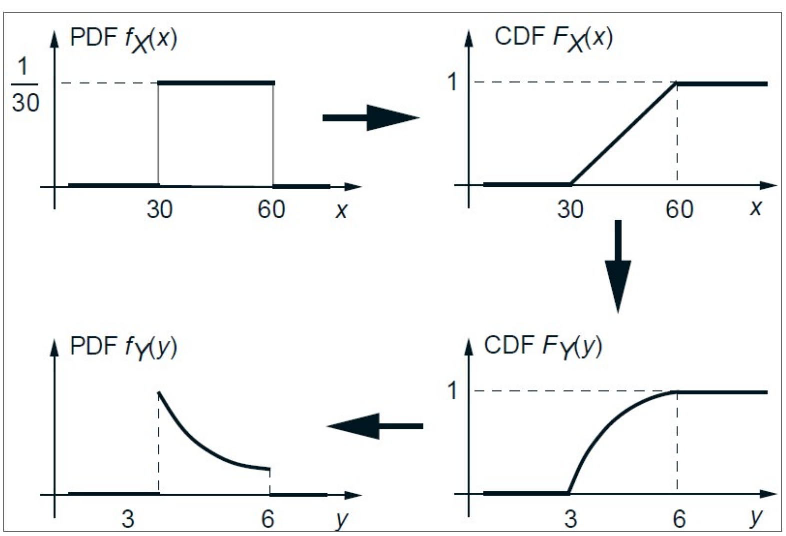
Example 4.2

John Slow is deriving from Boston to the New York, a distance of 180 miles at a constant speed, whose values is uniformly distributed between 30 and 60 miles per hour. What is the pdf of the duration of the trip?

Example 4.2 (cont)

Example 4.2 (cont)

Example 4.2 visualized



Calculation of the pdf of Y in Example 4.2.

Example 4.3

Let $Y = g(X) = X^2$, where X is a random variable with known pdf. Determine the pdf of Y.

Example 4.3 (cont)

Example 4.3 (cont)

The Linear Case

The pdf of a linear function

Let X be a continuous random variable with pdf f_X.
Let
$Y = aX + b,$
where $a, b \in \mathbb{R}$ and $a \neq 0$. Then
$f_Y(y) = \dfrac{1}{|a|} f_x\left(\dfrac{y-b}{a}\right).$

Formula verification

Example 4.4

Suppose that $X \sim \mathrm{Exponential}(\lambda)$ and $Y = aX + b$, where $a, b \in \mathbb{R}$ and $a \neq 0$.
Determine the pdf of Y.

Example 4.4 (cont)

Example 4.5

Suppose that $X \sim N(\mu, \sigma^2)$ and $Y = aX + b$, where $a, b \in \mathbb{R}$ and $a \neq 0$. Determine the pdf of Y.

Example 4.5 (cont)

The Monotonic Case

Strictly monotonic

g is **strictly monotonically increasing** if $x < y$ means the $g(x) < g(y)$.
g is **strictly monotonically decreasing** if $x < y$ means the $g(x) > g(y)$.

Invertibility

If $y = g(x)$ and g is monotonically increasing or decreasing, then g is invertible and $x = g^{-1}(y) = h(y)$.

Monotonicity and invertibility

The relationship between monotonicity and invertibility.

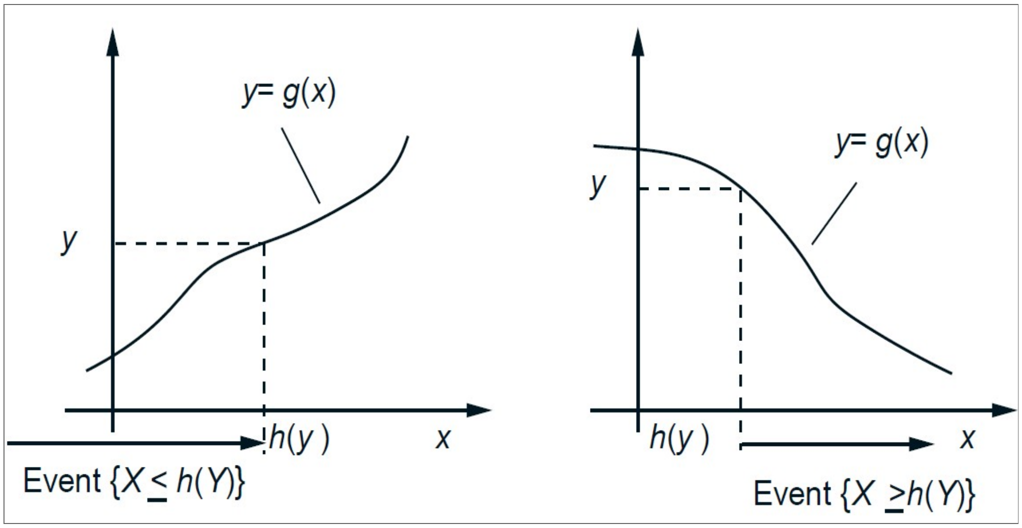
The relationship between monotonicity and invertibility.

pdf of a strictly monotonic function

Let X be a continuous random variable and $Y = g(X)$, where g is strictly monotonic and differentiable. Then
$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dg^{-1}(y)}{dy} \right|.$

Formula verification

Formula visualized



Monotonic transformation probability.

Example 4.2 (cont)

John Slow is deriving from Boston to the New York, a distance of 180 miles at a constant speed, whose values is uniformly distributed between 30 and 60 miles per hour. What is the pdf of the duration of the trip?

Example 4.2 (cont)

Examle 4.2 (cont)

Example 4.6

Let $X \sim \mathrm{Uniform}(0,1)$ and $Y = X^2$. Determine the pdf of Y.

Example 4.6 (cont)

Functions of Two Random Variables

Example 4.7

Two archers shoot at a target. The distance of each shot from the center of the target is uniformly distributed from 0 to 1, indpependent of the other other shot. What is the pdf of the distance of the losing shot from the center?

Example 4.7 (cont)

Example 4.7 (cont)

Example 4.8

Let X and Y be independent random variables that are uniformly distributed on the interval $[0, 1]$. What os the pdf of the random variable $Z = Y/X$?

Example 4.8 (cont)

Example 4.8 (cont)

CDF calculation of $Z = Y/X$

The calculation of the cdf of Z = Y/X.

The calculation of the cdf of $Z = Y/X$.

Example 4.9

Romeo and Juliet have a date at a given time, and each, independently, will be late by an amount of time that is exponentially distributed with paramater λ. What is the pdf of the difference between their arrival times?

Example 4.9 (cont)

Example 4.9 (cont)

Romeo and Juliet cdf calculation

Romeo and Juliette cdf calculation.

Romeo and Juliet cdf calculation.

Sums of Independent Random Variables - Convolution

Discrete convolution

Suppose that $Z = X + Y$ for independent random variables X and Y.
For discrete X and Y,

$$\begin{aligned}
p_Z(z) &= P(X + Y = z) \\
&= \sum_{\{(x,y) \mid x+y=z\}} P(X = x, Y = y) \\
&= \sum_x P(X = x, Y = z - x) \\
&= \sum_x p_X(x) p_Y(z - x).
\end{aligned}$$

This pmf p_Z is called the convolution of the pmfs of X and Y.

Continuous convolution

For continuous X and Y,

$$\begin{aligned} P(Z \leq z \mid X = x) &= P(X + Y \leq z \mid X = x) \\ &= P(x + Y \leq z) \\ &= P(Y \leq z - x) \end{aligned}$$

Taking the derivative of both sides, $f_{Z|X}(z \mid x) = f_Y(z - x)$.

Continuous convolution continued

Thus,

$f_{X,Z}(x, z) = f_X(x) f_{Z|X}(z \mid x) = f_X(x) f_Y(z - x)$.

and

$$f_Z(z) = \int_{-\infty}^{\infty} f_{X,Z}(dx) = \int_{-\infty}^{\infty} f_X(x) f_Y(z - x) dx.$$

Example 4.10

The random variables X and Y are independent and uniformly distributed in the interval $[0, 1]$. Determine the pdf of $Z = X + Y$.

Example 4.10 (cont)

Example 4.10 (cont)

Example 4.11 Sum of Two Indep. Normals

Let $X \sim N(\mu_x, \sigma_x^2), Y \sim N(\mu_y, \sigma_y^2)$, with X and Y independent.
Determine the pdf of $Z = X + Y$.

Example 4.12

Determine the convolution formula for $Z = X - Y$.

Example 4.12 (cont)

Example 4.12 (cont)

Example 4.12 (cont)