2.5 Joint pmfs of multiple random variables

Definition of the joint pmf

Let X and Y be two discrete random variables.
The **joint probability mass function (pmf)** is defined as:
$p_{X,Y}(x, y) = P(X = x, Y = y)$,
for all $x, y \in \mathbb{R}$.
Note:

$$\begin{aligned} P(X = x, Y = y) &\equiv P(\{X = x\} \cap \{Y = y\}) \\ &\equiv P(X = x \text{ and } Y = y). \end{aligned}$$

Properties of a joint pmf

Non-negativity

$p_{X,Y}(x, y) \geq 0, \quad x, y \in \mathbb{R}.$

Normalization

$$\sum_{x \in R_X} \sum_{y \in R_Y} p_{X,Y}(x, y) = 1.$$

Additivity

Additivity

Let A be an event of interest contained in $\mathrm{Range}(X, Y)$.

$$\begin{aligned} P(A) &= P((X, Y) \in A) \\ &= \sum_{(x,y) \in A} p_{X,Y}(x, y). \end{aligned}$$

Marginal pmf computation

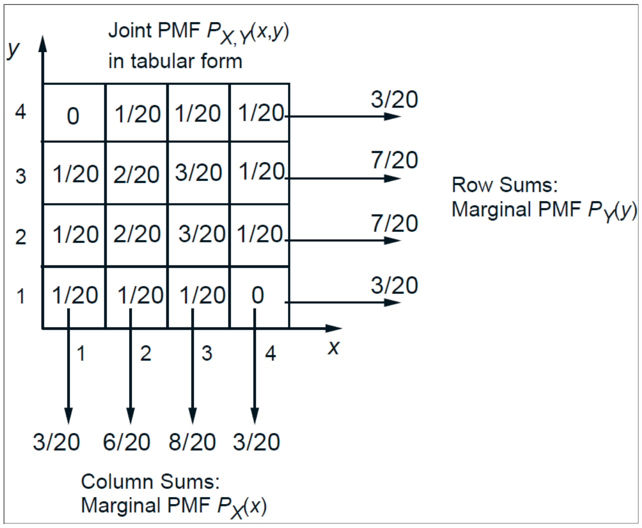
The marginal pmfs are obtained by summing over the other variable:
$$p_X(x) = \sum_{y \in R_Y} p_{X,Y}(x, y),$$
$$p_Y(y) = \sum_{x \in R_X} p_{X,Y}(x, y).$$

Transformations of multiple random variables

A function $g(X, Y)$ defines another random varible, and
$$E[g(X, Y)] = \sum_{x \in R_X} \sum_{y \in R_Y} g(x, y) p_{X,Y}(x, y).$$
If g is linear and has the form $Z = aX + bY + c$ for $a, b \in \mathbb{R}$, then
$$E(aX + bY + c) = aE(X) + bE(Y) + c.$$

Example 2.9

Consider two random variables, X and Y, described by the joint pmf in the graphic below. Obtain the marginal pmfs of X and Y.



The joint pmf of X and Y.

Example 2.9 (cont)

Example 2.9 (cont)

Let $Z = X + 2Y$. Determine the pmf of Z.

Example 2.9 (cont)

Example 2.9 (cont)

Determine $E(Z)$ and $\mathrm{var}(Z)$.

More than Two Random Variables

Multivariate pmf definition

The **joint probability mass function (pmf)** of discrete random variables X, Y, and Z is defined as:

$$p_{X,Y,Z}(x, y, z) = P(X = x, Y = y, Z = z), \quad x, y, z \in \mathbb{R}.$$

Valid multivariate pmf properties

Non-negativity

$$p_{X,Y,Z}(x, y, z) \geq 0 \quad \text{for all } x, y, z \in \mathbb{R}.$$

Normalization

$$\sum_{x \in R_X} \sum_{y \in R_Y} \sum_{z \in R_Z} p_{X,Y,Z}(x, y, z) = 1.$$

Marginal pmfs

Given the joint probability mass function $p_{X,Y,Z}(x, y, z) = P(X = x, Y = y, Z = z)$, the marginal pmf of X is obtained by summing over all possible values of Y and Z:

$$p_X(x) = \sum_{y \in R_Y} \sum_{z \in R_Z} p_{X,Y,Z}(x, y, z), \quad x \in R_X.$$

Similar definitions hold for Y and Z.

Functions of multiple random variables

$$E[g(X, Y, Z)] = \sum_{x \in R_X} \sum_{y \in R_Y} \sum_{z \in R_Z} g(x, y, z) p_{X,Y,Z}(x, y, z).$$

If g is linear and has the form $aX + bY + cZ + d$, then $E(aX + bY + cZ + d) = aE(X) + bE(Y) + cE(Z) + d$.

Generalization

For any random variables $X_1, X_2, \ldots, X_n$ and scalars $a_1, a_2, \ldots, a_n$,
$E(a_1X_1 + a_2X_2 + \cdots + a_nX_n)$
$= a_1E(X_1) + a_2E(X_2) + \cdots + a_nE(X_n)$.

Example 2.11 (The Hat Problem)

Suppose that n people throw their hats in a box and then each picks one hat at random (without replacement). What is the expected value of X, the number of people that get back their own hat?

Example 2.11 (cont)