$\mathbb{P}_2 - \mathbb{P}_1$ Mixed Finite Elements for the Steady Stokes Equations

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May 13, 2013

1 Introduction

The Stokes equations (or *Stokes flow*) is derived by making certain simplifying assumptions and applying them to the Navier-Stokes equations solution that the non-linear terms become negligably small. Then assuming that $\partial\mathbf{u}/\partial t = 0$, i.e. , that we have reached a steady state, we reach the steady Stokes equations. These equations describe the flow of an incompressible viscous fluid in an n-dimensional domain (with $n = 2$ or 3):

$$\begin{aligned} -\frac{1}{\mathrm{Re}}\nabla^2\mathbf{u} + \nabla p = \mathbf{f} \quad & \text{in } \Omega, \\ \nabla \cdot \mathbf{u} = 0 \quad & \text{in } \Omega, \\ \mathbf{u} = \mathbf{u}_0 \quad & \text{on } \partial\Omega \end{aligned} \tag{1}$$

with the vector-valued function $\mathbf{u} : \Omega \to \mathbb{R}^n$ denoting the velocity field and the scalar function $p : \Omega \to \mathbb{R}$ denoting the pressure[1]. The constant Re denotes the dimensionless *Reynolds number* which measures the effect of viscosity on the flow[2]. From here on I will refer to the steady Stokes equations as just the *Stokes equations*. Additionally, we will restrict ourselves to $n = 2$ dimensions.

1.1 Motivation

The Stokes equations describe fluid flow when the fluid velocities are very slow, the viscosities are very large, or the length-scales of the flow are very small. While the Navier-Stokes equations (being more general) would also model that phenomena, the increased difficulty and computational cost of solving a non-linear system outweighs any potential accuracy gains. Simply stated, solving Stokes involves one linear system, whereas Navier-Stokes would most likely involve an iterative method.

Nevertheless, the Stokes equation is not just an alternative to the Navier-Stokes equation. The solution may serve as an initial starting guess for those very iterative solvers for non-linear systems such as Newton's method[4].

1.2 Existing Methods

A common method to solve the Stokes equations is to utilize a *stream function* $\psi(x, y)$ with the property that $u = \partial\psi/\partial x$ and $v = -\partial\psi/\partial y$. If Ω is simply connected, then this fact reduces the Stokes problem down to the solution of a biharmonic problem $-(\nabla^2)^2\psi = f$ with $f = \partial f_2/\partial x - \partial f_1/\partial y$ [6]. This results in very high order finite elements being necessary to solve the problem. It also restricts domains to that which are simply connected, which hinders the finite element method's ability to elegantly handing irregular domains.

Another method of solving this equation involves approximating the incompressibility condition[6]. This involves seeking a discrete solution space of the form

$$V_h = \left\{ \mathbf{v}_h \in X_{0h} : \forall \phi_h \in \Phi_h, \sum_{\text{elements}} \int_{\text{element}} \phi_h \nabla \cdot \mathbf{v}_h \, dx = 0 \right\}.$$

This amounts to the pressure space appearing as an appropriate space of “Lagrange multipliers”[6]. According to Ciarlet, however, the “most promising” finite element approximations for the Stokes problem are of the mixed type[6]. These are what I consider in this report.

1.3 Taylor-Hood Element

For mixed finite element solutions to the Stokes equation, the function spaces defined on the element for velocity and pressure cannot be chosen at random, so to speak. The two function spaces must satisfy the LBB condition, which is stated later, in order to be properly suited as spaces in which to approximate the solution to the system of partial differential equations.

I chose the $\mathbb{P}_2 - \mathbb{P}_1$ Taylor-Hood element as my finite element space because it is a mixed method with low order, continuous polynomials as the approximating functions. As the notation would suggest, it uses quadratics to approximate the velocity mand linears to approximate the pressure. Lower order mixed elements exist but require the use of discontinuous constants to approximate the pressure. Since my original intention was to implement Newton's method with the Stokes equation solution as a starting point, the low-order continuous spaces were appealing.

2 Theoretical Results

2.1 Weak Formulation and Resulting System

For the context of this paper, I assume that $n = 2$ and write $\mathbf{u} = (u, v)^T$, take $\mathrm{Re} = 1$ unless noted otherwise, enforce homogenous Dirchlet boundary conditions $\mathbf{u}_0 = (u_0, v_0) = (0, 0)$ on $\partial\Omega$, and enforce a zero-mean pressure, i.e., $\int p(x, y)\, d\Omega = 0$. This gives us the set of equations

$$
\begin{aligned}
-\nabla^2 u + \frac{\partial}{\partial x} u &= -f_1 \quad \text{in } \Omega, \\
-\nabla^2 v + \frac{\partial}{\partial y} v &= -f_2 \quad \text{in } \Omega, \\
\frac{\partial}{\partial x} u + \frac{\partial}{\partial y} v &= 0 \quad \text{in } \Omega, \\
u = v &= 0 \quad \text{on } \partial\Omega, \\
\int p(x,y)\, d\Omega &= 0 \quad \text{on } \Omega.
\end{aligned}
\tag{2}
$$

We seek a weak formulation of the Stokes equations in order to motivate a finite element solution. One method of doing this is to choose separate and distinct function spaces X^2 and Q for the velocity and pressure functions, respectively. Motivated by the boundary conditions, we choose

$$
\begin{aligned}
V^2 = \left[H(\Omega)_0^1\right]^2 &:= \left\{(w_1, w_2) : w_i \in H_0^1(\Omega), i \in (1,2)\right\} \quad \text{and} \\
Q = L_0^2(\Omega) &:= \left\{q \in L_2(\Omega) : \int p\, d\Omega = 0\right\}.
\end{aligned}
\tag{3}
$$

Notice that the function space for velocity is not necessarily divergence-free. The zero divergence condition will be satisfied weakly when we "dump off" the derivatives of the pressure associated with the gradient ∇p onto the test functions via integration by parts as a result of our mean-zero pressure. This is known as *discrete incompressibility.* Multiplying by the test functions $(w_1, w_2) \in X^2$ and $q \in Q$ for the appropriate equations, integrating by parts, and boundary and other imposed conditions results in a variational formulation

$$
\begin{aligned}
\int \nabla u \cdot \nabla w_1 \, d\Omega - \int p \frac{\partial w_1}{\partial x} \, d\Omega &= \int f_1 w_1 \, d\Omega \qquad \forall w_1 \in X, \\
\int \nabla v \cdot \nabla w_2 \, d\Omega - \int p \frac{\partial w_2}{\partial y} \, d\Omega &= \int f_2 w_2 \, d\Omega \qquad \forall w_2 \in X, \\
\int \left(\frac{\partial u}{p x} + \frac{\partial v}{\partial y} \right) q \, d\Omega = 0 \forall q \in Q.
\end{aligned} \tag{4}
$$

Continuing as normal, we take discrete subsets $X_h^2 \subset V^2$ and $Q_h \subset Q$. Our variational form follows: find $(u_h, v_h) \in V_h^2$ and $p_h \in Q_h$ such that

$$
\begin{aligned}
\int \nabla u_h \cdot \nabla w_1 \, d\Omega - \int p_h \frac{\partial w_{1,h}}{\partial x} \, d\Omega &= \int f_1 w_{1,h} \, d\Omega \qquad \forall w_{1,h} \in X_h, \\
\int \nabla v_h \cdot \nabla w_{2,h} \, d\Omega - \int p_h \frac{\partial w_{2,h}}{\partial y} \, d\Omega &= \int f_2 w_{2,h} \, d\Omega \qquad \forall w_{2,h} \in X_h, \text{ and }, \\
\int \left(\frac{\partial u_h}{\partial x} + \frac{\partial v_h}{\partial y} \right) q \, d\Omega = 0 \qquad \forall q_h \in Q_h.
\end{aligned} \tag{5}
$$

Then expanding (u_h, v_h) and p_h as finite sums of basis functions, we write

$$
u_h = \sum_{j=1}^{n_v} a_j \varphi_j, \quad v_h = \sum_{j=1}^{n_v} b_j \varphi_j, \quad p_h = \sum_{j=1}^{n_p} c_j \psi_j. \tag{6}
$$

Finally, we get the block system of equations

$$
\begin{bmatrix} S & O & -G \\ O & S & -H \\ -G^T & -H^T & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ 0 \end{bmatrix} \tag{7}
$$

with

$$
A_{ij} = \int \nabla \varphi_j \cdot \nabla \varphi_i \, d\Omega, \tag{8}
$$

$$G_{ij} = \int \psi_j \frac{\partial \varphi_j}{\partial x} \, d\Omega, \tag{9}$$

$$H_{ij} = \int \psi_j \frac{\partial \varphi_j}{\partial y} \, d\Omega, \tag{10}$$

and I note that the bottom row of the block matrix is achieved by multiplying the incompressibility equation by -1 since the equation is equal to zero. This matrix is poorly conditioned but non-singular once the Dirchlet boundary conditions are imposed by removing the appropriate rows and columns.

2.2 The Ladyzenskaja-Babuska-Brezzi Condition

An import consideration before we proceed is that the discrete velocity and pressure function spaces, X_h and Q_h, satisfy a condition known as the Ladyzenskaja-Babuska-Brezzie (LBB) condition. This is an additional criterion that must be satisfied with the Lax-Milgram theorem. The use of mixed finite element spaces requires some attention be paid to saddle-point solutions. However, if a mixed element space satisfies the LBB condition, we are guarunteed a unique solution. I will simply state the condition as it pertains to my problem without much proof since Braess[1] states that the proofs are outside the scope of his text.

Definition 2.1 (LBB Condition[1])**.** A family of finite element spaces (X_h, Q_h) is said to satisffy the *LBB condition* provided there exists constants $\alpha > 0$ and $\beta > 0$ independent of h such that:

the bilinear form $a(\cdot, \cdot)$ is V_h-elliptic with ellipticity constant $\alpha > 0$ and

$$\sup_{v \in X_h} \frac{b(v, \lambda_h)}{||v||} \geq \beta \, ||\lambda_h|| \quad \text{for all } \lambda_h \in M_h.$$

The second condition is often call the *inf-sup condition*.

The Taylor-Hood $\mathbb{P}_2 - \mathbb{P}_1$ element satisfies this condition and thus is properly suited as a finite element space for which to solve the Stokes equation.

2.3 Error Estimate

Since our Taylor-Hood finite element satisfies the LBB conditions, we may prove the following error estimate.

Theorem 2.1. *Let $X_h \times Q_h \subset X \times Q$ satisfy the LBB conditions, Then it holds that*

$$||\nabla(v - v_h)|| + ||p - p_h|| \leq c \left(\min_{\phi_h \in V_h} ||\nabla(v - \phi_h)|| + \min_{\xi_h \in Q_h} ||p - \xi_h|| \right), \tag{11}$$

where the constant $c > 0$ depends on the inf-sup constant ψ_h. Further, on convex or smooth domains, it holds that

$$||v - v_h|| \leq ch \left(\min_{\phi_h \in V_h} ||\nabla(v - \phi_h)|| + \min_{\xi_h \in Q_h} ||p - \xi_h|| \right), \tag{12}$$

with constant $c = c(\gamma_h)$.

Proof. Define $e_v := v - v_h \in V$ and $e_p := p - p_h \in Q$. Then by Galerkin orthogonality

$$\begin{aligned} (\nabla e_v, \nabla \phi_h) &= (e_p, \nabla \cdot \phi_h) \quad \forall \phi_h \in V_h, \\ (\nabla \cdot e_v, \xi_h) &= 0 \quad \forall \xi_h \in Q_h. \end{aligned} \tag{13}$$

First start with an estimate of the velocity error:

$$||\nabla e_v||^2 = (\nabla e_v, \nabla e_v) - (e_v, \nabla \cdot e_v) + (e_v, \nabla \cdot e_v).$$

By Galerkin orthogonality, for any $\phi_h \in V_h$ and $\xi_h \in Q_h$,

$$\begin{aligned} ||\nabla e_v||^2 &= (\nabla e_v, \nabla(v - \phi_h)) - (e_p, \nabla \cdot (v - \phi_h)) + (\nabla \cdot e_v, p - \xi_h) \\ &\leq ||\nabla e_v||^2 \, ||\nabla(v - \phi_h)|| + ||e_p|| \, ||\nabla(v - \phi_h)|| + ||\nabla e_v|| \, ||p - \xi_h|| \, . \end{aligned}$$

By Young's inequality, for $\varepsilon > 0$:

$$||\nabla e_v|| \leq (2 + \varepsilon^{-1}) \, ||\nabla (v - \phi_h)|| + 2 \, ||p - \xi_h|| + \varepsilon \, ||e_p|| \, . \tag{14}$$

Next, we estimate the pressure error. Let $\xi_h \in Q_h$ be given. Then

$$||p - p_h|| \leq ||p - \xi_h|| + ||p_h - \xi_h|| \, . \tag{15}$$

For $p_h - \xi_h \in Q_h$, we use the discrete $\inf - \sup$ inequality to get

$$\begin{aligned} \gamma_h \, ||p_h - \xi_h|| &\leq \sup_{\phi_h \in V_h} \frac{(p_h - \xi_h, \nabla \cdot \phi_h)}{||\nabla \phi_h||} \\ &= \sup_{\phi_h \in V_h} \frac{(p - p_h, \nabla \cdot \phi_h)}{||\nabla \phi_h||} + \sup_{\phi_h \in V_h} \frac{(p - \xi_h, \nabla \cdot \phi_h)}{||\nabla \phi_h||} . \end{aligned} \tag{16}$$

Now replace the pressure error e_p by the velocity error e_v:

$$\sup_{\phi_h \in V_h} \frac{(e_p, \nabla \cdot \phi_h)}{||\nabla \phi_h||} = \sup_{\phi_h \in V_h} \frac{(\nabla e_v, \nabla \phi_h)}{||\nabla \phi_h||} \leq ||\nabla e_v|| \, . \tag{17}$$

Then we have that

$$\gamma_h \, ||p_h - \xi_h|| \leq ||\nabla e_v|| + ||p - \xi_h|| \, , \tag{18}$$

and so

$$||e_p|| \leq (1 + \gamma_h^{-1}) \, ||p - \xi_h|| + \gamma_h^{-1} \, ||\nabla e_v|| \, . \tag{19}$$

Using $\varepsilon = \gamma_h / 2$, it follows that

$$||\nabla e_v|| \leq c(\gamma_h) \, (||\nabla (v - \phi_h)|| + ||p - \xi_h||) \, . \tag{20}$$

The previous two equations together give us the best-approximation property for the energy norm.

For the L^2-estimate, we define the adjoint Problems

$$(\nabla\phi, \nabla z) - (\xi, \nabla \cdot z) + (\nabla \cdot \phi, q) = ||e_v||^{-1} (e_v, \phi).$$

As $e_v / ||e_v|| \in L^2$ and the domain is convex or has a smooth enough boundary that

$$\left|\left|del^2 z\right|\right| + ||\nabla q|| \leq c_s \left|\left|\frac{e_v}{||e_v||}\right|\right| = c_s.$$

Now considering the interpolants $I_h z \in V_h$ and $I_h q \in Q_h$, it follows that

$$\begin{aligned}
||e_v|| &= (\nabla e_v, \nabla z) - (e_p, \nabla \cdot z) + (\nabla \cdot \phi, q) \\
&= (\nabla e_v, \nabla(z - \phi_h)) - (e_p, \nabla \cdot (z - \phi_h)) + (\nabla \cdot e_v, (q - \xi_h)) \\
&\leq ||\nabla e_v|| \, ||\nabla(z - I_h z)|| + ||\nabla e_p|| \, ||\nabla(z - I_h z)|| + ||\nabla e_v|| \, ||q - I_h q|| \, .
\end{aligned}$$

Finally, using the energy norm estimate and interpolation estimate, we get that

$$\begin{aligned}
||e_v|| &\leq c_I h(||\nabla e_v|| + ||e_p||)(\left|\left|\nabla^2 z\right|\right| + ||\nabla q||) \\
&\leq c(\gamma) c_S h \left(\min_{\phi_h \in V_h} ||\nabla(v - \phi_h)|| + \min_{\xi_h \in Q_h} ||p - \xi_h|| \right).
\end{aligned}$$

So the approximation order of the Stokes element depends on the degree of the polynomial chosen for the finite element pair. □

It can be shown that the optimal degree for velocity and pressure space differs by one.

3 Methodology

I employed a mixed element solution with the standard Galerkin finite element method. As has been stated numerous times already, I used the Taylor-Hood $\mathbb{P}_2 - \mathbb{P}_1$ finite element.

3.1 Meshing

Meshing for my implementation is fairly flexible. It requires a second order triangulation with the data structure such that it is easy to extract the "linear" nodes from each element in order to create the linear basis functions for the pressure space.

Since the mixed element method is not stricted to simply connected domains, I'll play with some novel geometries.

3.2 Implementation

Recalling the block matrix structure of the resulting system of equations:

$$\begin{bmatrix} S & O & -G \\ O & S & -H \\ -G^T & -H^T & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ 0 \end{bmatrix} \tag{21}$$

I considered a few options in dealing with the poor conditioning of the matrix.

It is suggested in Johnson[3] that you add a perturbation εI in the bottom-right most block and use that to create a semi-positive definite system

$$\left(\begin{bmatrix} S & O \\ O & S \end{bmatrix} + \frac{1}{\varepsilon} \begin{bmatrix} G \\ H \end{bmatrix} \begin{bmatrix} G^T & H^T \end{bmatrix} \right) \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \tag{22}$$

which is much better behaved. I can then solve for the pressure coefficients c from the velocity coefficients. The condition number of this system depends on ε, clearly, and as the perturbation gets smaller, the condition number increases.

Other options I considered was solving the matrix directly with the perturbation added across the lower-right block diagonal. This would yield the same system but seemed to have worse performance at the final linear system solve.

Lastly, it was suggested by Dr.Borggaard that I use the mass matrix for the linear basis functions, which is just the L^2 inner product of the basis functions scattered to their appropriate global index locations. I did not do any analysis on this suggestion since I felt that opting for a fundamentally smaller system to solve was the most efficient choice.

4 Numerical Results

I will preface this section by plainly admitting that I could not find a simple, closed form solution for my particular rendition of the Stokes equations. Satisfying the zero boundary conditions and zero divergence for an analytic solution was far from trivial. I only started looking into the matter toward the end of the project, again, as my primary focus for most of the time was to implement Newton's method using Stokes as an initial solution guess.

As a result of not being able to find a closed form solution, I was not sure how to numerically verify the convergence rates that I proved theoretically.

4.1 Unit Square

I chose the forcing function to be $\mathbf{f}(x, y) = 1\text{e}-2 \cdot (y, -x)$. The mesh is simple, just the unit square with three different meshes generated on it with characteristic lengths $h = 0.1, 0.05$, and 0.025.

The velocity functions *seem* to converge to nice, smooth solutions. Again, as I have nothing to compare it against, I cannot say for certain. The solution makes sense as it is forming a "bubble" to deal with the pressure in the system while still maintaining zero boundary conditions, similar to it having some factor of $\sin(\pi x)\sin(\pi y)$ in both $u(x,y)$ and $v(x,y)$.

The pressure coefficients are found by substituting back into the perturbed system. Since the system corresponding to those values was the culprit for making our system poorly conditioned, it makes sense that the coefficients seem oscillatory. By employing the very rigorous process of "eyeball integration" (a.k.a looking at the plot), it would appear from observing the symmetry that the mean-zero condition has been satisfied.

It should be noted that even though the velocity basis functions were quadratic and so are defined on 6 nodes of each triangle, for sake of plotting the data, the coefficients were downgraded to a linear representation. This was also at the suggestion of Dr. Borggaard when I inquired about plotting methods for higher order elements.

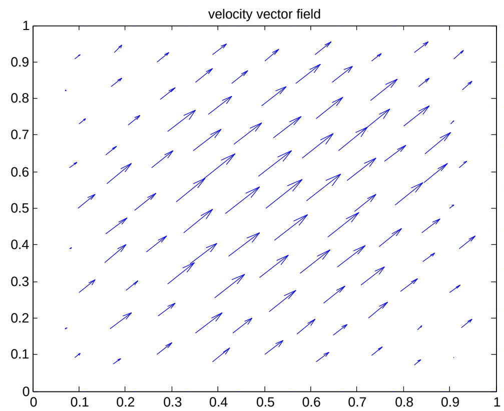

Figure 1: Velocity Field for $h = 0.1$

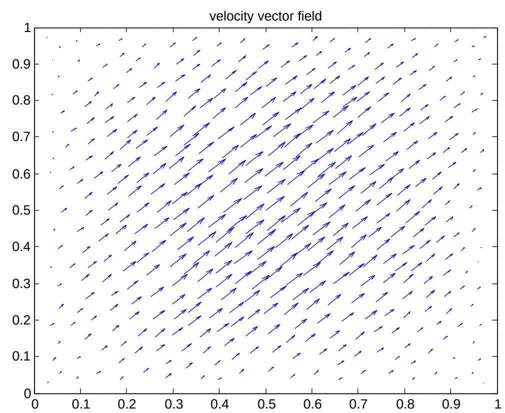

Figure 2: Velocity Field for $h = 0.05$

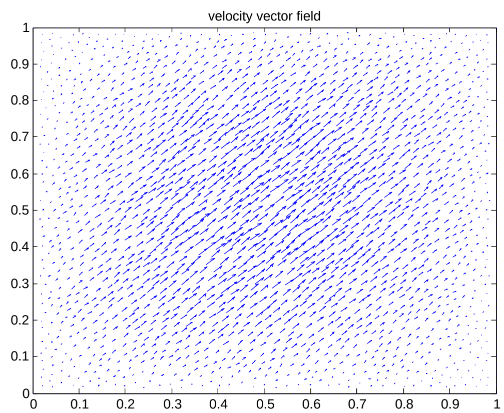

Figure 3: Velocity Field for $h = 0.025$

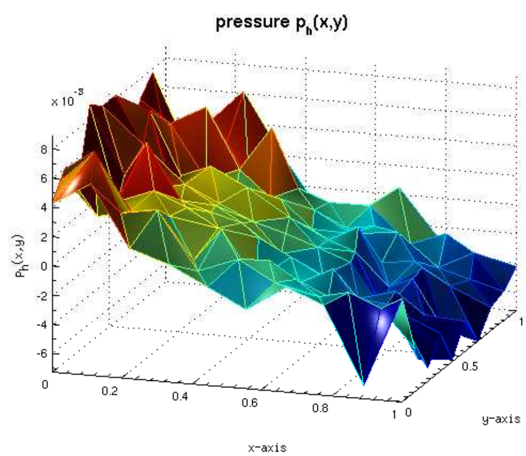

Figure 4: $p_h(x, y)$ for $h = 0.1$

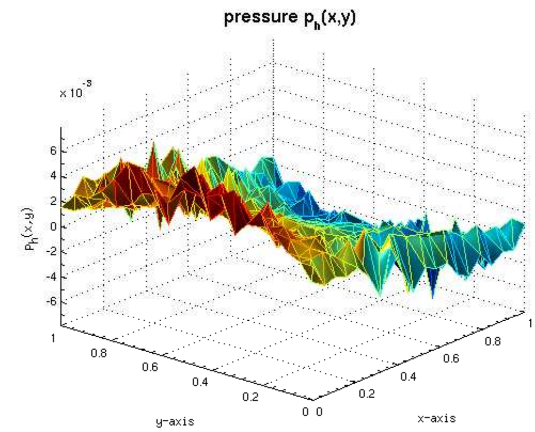

Figure 5: $p_h(x, y)$ for $h = 0.05$

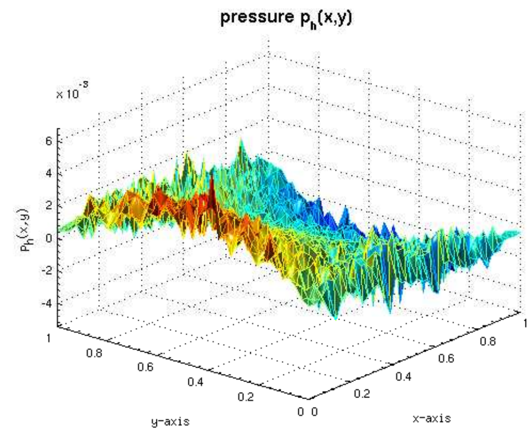

Figure 6: $p_h(x, y)$ for $h = 0.025$

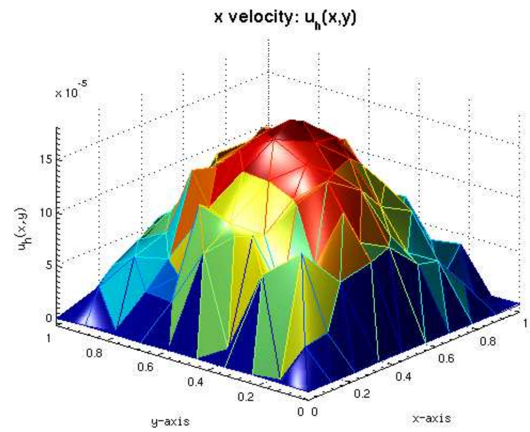

Figure 7: $u_h(x,y)$ for $h = 0.1$

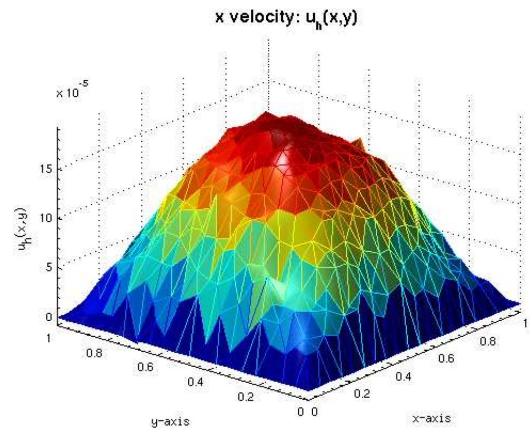

Figure 8: $u_h(x, y)$ for $h = 0.05$

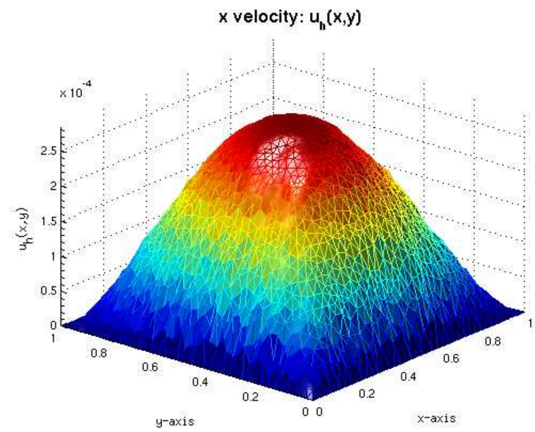

Figure 9: $u_h(x, y)$ for $h = 0.025$

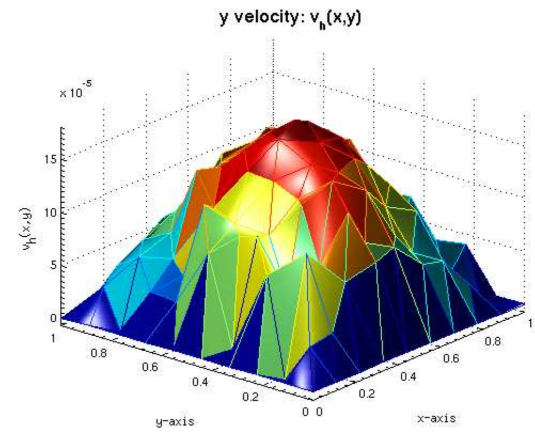

Figure 10: $v_h(x, y)$ for $h = 0.1$

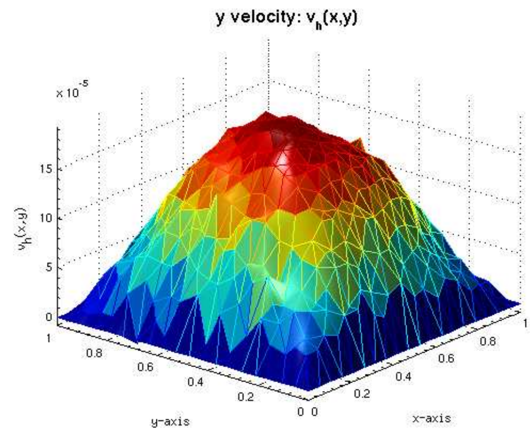

Figure 11: $v_h(x, y)$ for $h = 0.05$

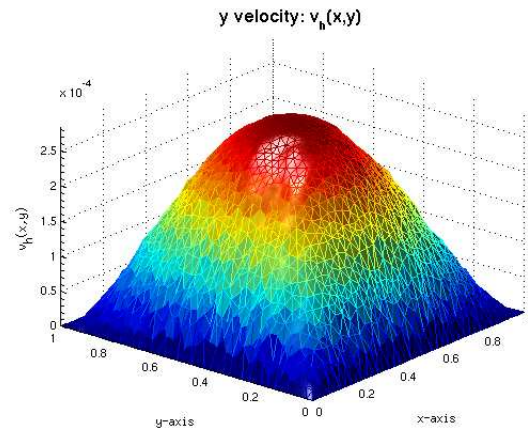

Figure 12: $v_h(x, y)$ for $h = 0.025$

4.2 Other Geometries

I will now present other plots with novel geometries. All meshes had characteristic length $h = 0.025$.

The forcing function for was $\mathbf{f}(x, y) = 1\mathrm{e}-2(y, -x)$ for this first set of plots.

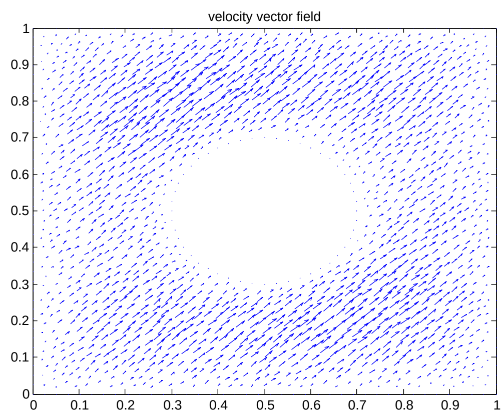

Figure 13: Velocity Field for $h = 0.025$

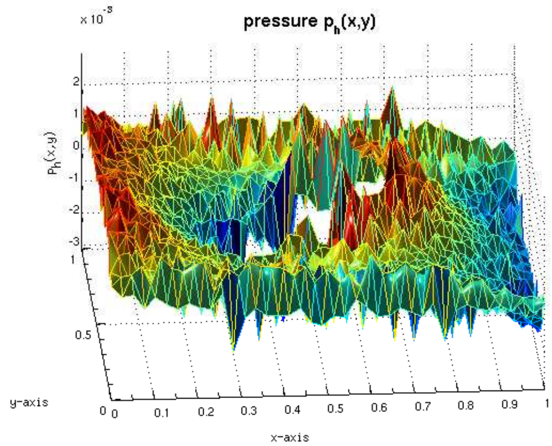

Figure 14: $p_h(x, y)$ for $h = 0.025$

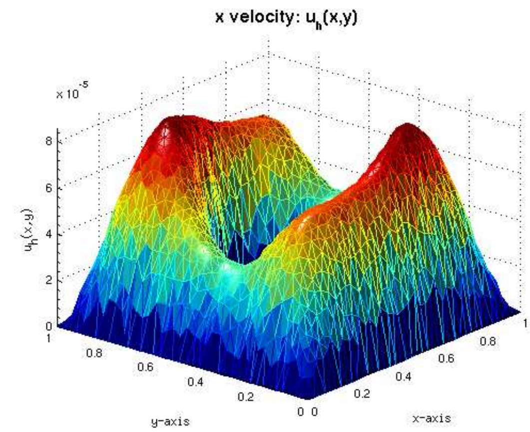

Figure 15: $u_h(x, y)$ for $h = 0.025$

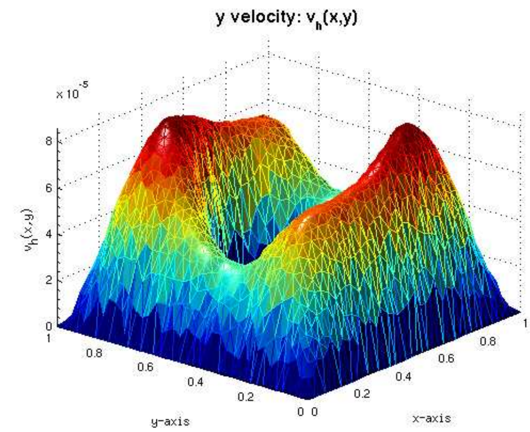

Figure 16: $v_h(x, y)$ for $h = 0.025$

And for this last set, the forcing function was $\mathbf{f}(x, y) = 10\cos(\pi x)y^3 - 4x, 0)$. This function was not chosen for any particular reason other than to be complicated.

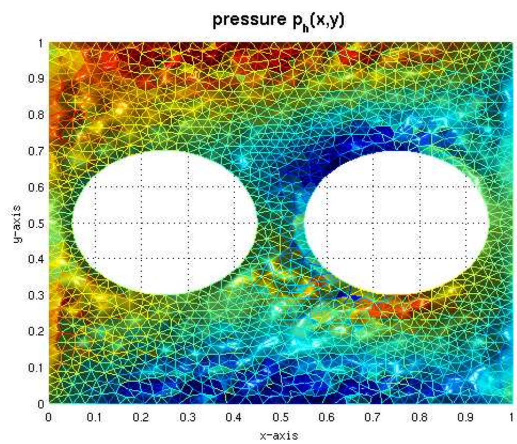

Figure 17: $p_h(x, y)$ for $h = 0.025$

5 Conclusions

To summarize, I was able to implement a mixed finite element solver for the Stokes equation with homogeneous boundary conditions and zero-mean pressure. The original intent was to use this to solve the steady-state Navier-Stokes equations with Newton's method. Unfortunately, calculating and correctly implementing the Jacobian of the non-linear Navier-Stokes system proved too much of a challenge.

I was able to prove that the error is optimal for the Taylor-Hood $\mathbb{P}_2-\mathbb{P}_1$ finite element. In terms of numerical work, I need to find a closed-form solution to test the actual convergence rates of my implementation against the theoretical results.

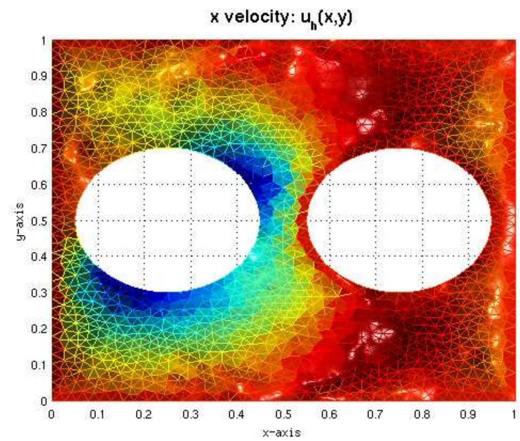

Figure 18: $u_h(x, y)$ for $h = 0.025$

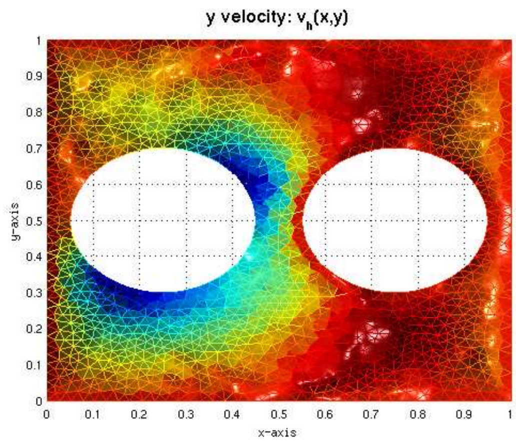

Figure 19: $v_h(x, y)$ for $h = 0.025$

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