Mixed Element Methods for Stokes Equations

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Overview

1. Theory
2. Numerical Results

Stokes with Homogeneous Boundary and Zero-Mean Pressure

$$
\begin{aligned}
-\mathrm{Re}^{-1}\,\nabla^2 u_1 + \frac{\partial}{\partial x} p &= f_1 \quad \text{in } \Omega, \\
-\mathrm{Re}^{-1}\,\nabla^2 u_2 + \frac{\partial}{\partial y} p &= f_2 \quad \text{in } \Omega, \\
\frac{\partial}{\partial x} u_1 + \frac{\partial}{\partial y} u_2 &= 0 \quad \text{in } \Omega, \\
u_1 &= 0 \quad \text{on } \partial\Omega, \\
u_2 &= 0 \quad \text{on } \partial\Omega, \\
\int p(x, y)\, d\Omega &= 0 \quad \text{on } \Omega.
\end{aligned} \tag{1}
$$

I take $\mathrm{Re} = 1$.

Mixed Elements

Framework

1. $\mathbf{u} \in H_0^1(\Omega)^2$
2. $H_0^1(\Omega)^2 = \{(u_1, u_2) : u_i \in H_0^1(\Omega)\}$
3. $p \in L_0^2(\Omega)$
4. $L_0^2(\Omega) = \{p \in L^2 : \int p \, d\Omega = 0\}$
5. Look for solution in $H_0^1(\Omega)^2 \times L_0^2(\Omega)$

We break up the function spaces for velocity $\mathbf{u}$ and pressure p into two distinct Hilbert spaces. We then choose appropriate finite dimensional subspaces to find our finite element solution. I chose the **Taylor-Hood** $\mathbb{P}_2 - \mathbb{P}_1$ finite element.

Taylor-Hood Finite Element

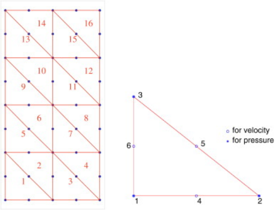

Figure: The $\mathbb{P}_2 - \mathbb{P}_1$ Element

Variatonal Weak Form

Variational Formulation

Find $\mathbf{u} \in H_0^1(\Omega)^2$ and $p \in L_0^2(\Omega)$ so that

$$\begin{aligned} (\nabla \mathbf{u}, \nabla \mathbf{v}) + (p, \nabla \cdot \mathbf{v}) = (\mathbf{f}, \mathbf{v}) \quad & \text{for all } \mathbf{v} \in H_0^1(\Omega)^2, \\ -(\nabla \cdot \mathbf{u}, q) = 0 \quad & \text{for all } q \in L_0^2(\Omega). \end{aligned} \tag{2}$$

Variational Weak Form Cont'd

Find $(u_{1,h}, u_{2,h}) \in V_h \subset H_0^1(\Omega)^2$ and $p_h \in Q_h \subset L_0^2(\Omega)$ such that

$$\begin{aligned}
\int \nabla u_{1,h} \cdot \nabla v_{1,h} \, d\Omega - \int p_h \frac{\partial v_{1,h}}{\partial x} \, d\Omega = \int f_1 v_{1,h} \, d\Omega \qquad \forall v_{1,h} \in V_h, \\
\int \nabla u_{2,h} \cdot \nabla v_{2,h} \, d\Omega - \int p_h \frac{\partial v_{2,h}}{\partial y} \, d\Omega = \int f_2 v_{2,h} \, d\Omega \qquad \forall v_{2,h} \in V_h \text{ and }, \\
\int \left(\frac{\partial u_{1,h}}{\partial x} + \frac{\partial u_{2,h}}{\partial y} \right) q_h \, d\Omega = 0 \qquad \forall q_h \in Q_h.
\end{aligned} \tag{3}$$

Conditions for Well-Posedness

We get a system of the form $\begin{bmatrix} A & B \\ B^T & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix}$.

Ladyzenskaya-Babuska-Breezi Condition (LBB)

A solution exists if and only if

$$\inf_{v \in V} \sup_{u \in U} \frac{a(u,v)}{||u|| \, ||v||} > \alpha_E > 0$$

and the solution is unique if and only if

$$\inf_{u \in U} \sup_{v \in V} \frac{(a(u,v))}{||u|| \, ||v||} > \alpha_U > 0$$

for a generic bilinear form $a(\cdot,\cdot) : U \times V \to \mathbb{R}$ defined on the Cartesian product of Hilbert spaces $U \times V$.

Error Estimate

Stokes Error Estimate

Let $V_h \times Q_h$ satisfy the LBB conditions with polynomials of order max order k in V_h and l in Q_h. Let $u \in H^{k+1}(\Omega)^2$ and $p \in L^2(\Omega)$ be the solution of the Stokes equation. Then

$$||\nabla(v - v_h)|| + ||p - p_h|| \leq Ch^{\min\{k,l+1\}}\left(\left|\left|\nabla^{k+1} v\right|\right| + \left|\left|\nabla^{l} p\right|\right|\right). \quad (4)$$

We see that the error is optimal for both spaces when $k = l + 1$. This is why $\mathbb{P}_2 - \mathbb{P}_1$ works.

$f_1(x, y) = y, f_2(x, y) = -x$

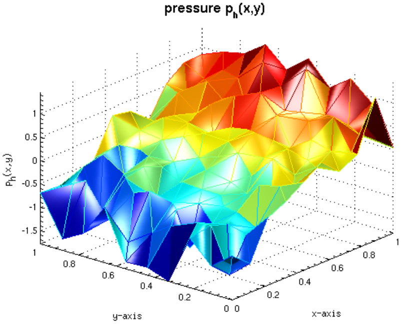

Figure: p_h for $h = 0.1$

$f_1(x, y) = y, f_2(x, y) = -x$

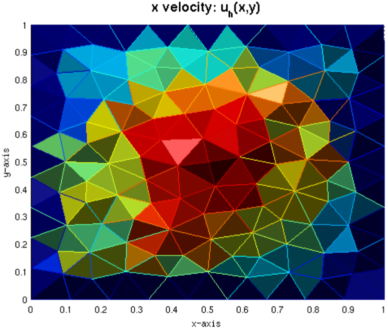

Figure: $|u_h|$ for $h = 0.1$

$f_1(x, y) = y, f_2(x, y) = -x$

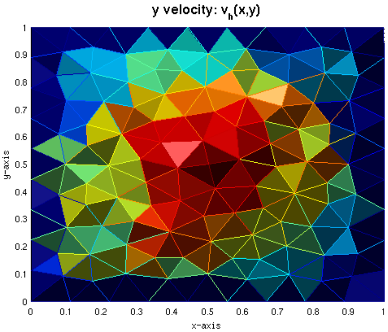

Figure: $|v_h|$ for $h = 0.1$

$f_1(x, y) = y, f_2(x, y) = -x$

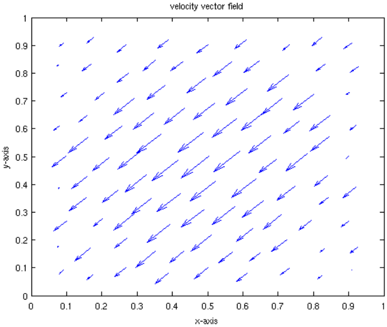

Figure: Quiver Plot for $h = 0.1$

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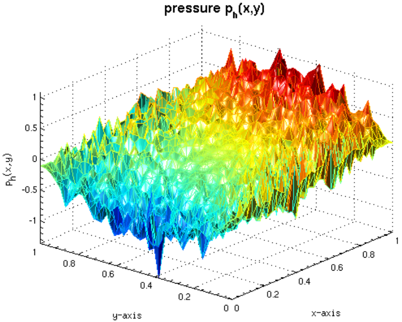
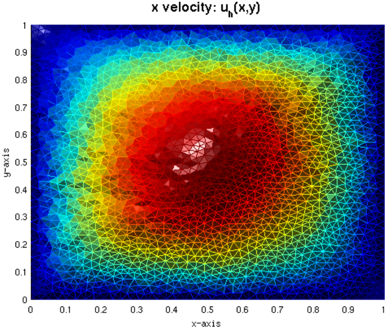


Figure: p_h for $h = 0.025$

$f_1(x, y) = y, f_2(x, y) = -x$



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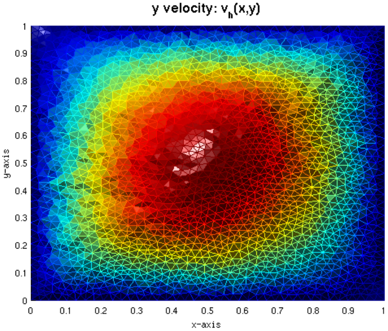

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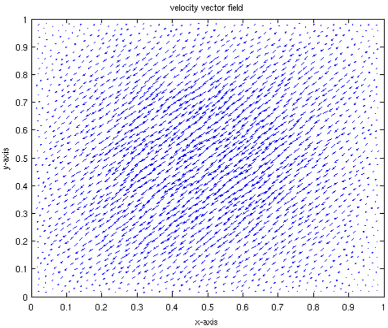
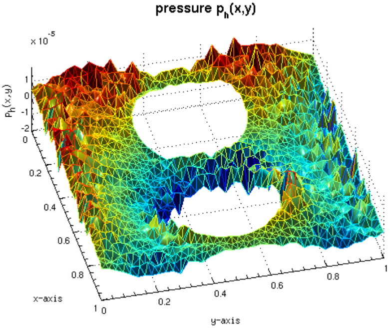


Figure: Quiver Plot for $h = 0.025$

$f_1(x,y) = x^2y^3\cos(\pi y), f_2(x,y) = 1e - 4 \cdot |y - 0.5|$



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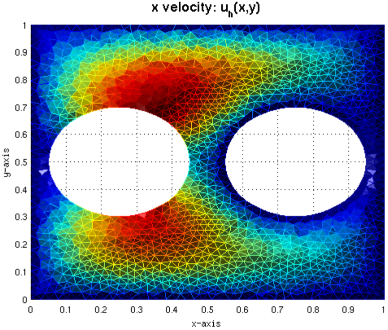

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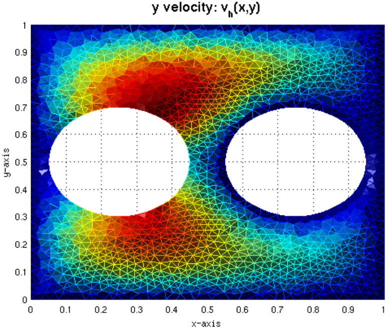

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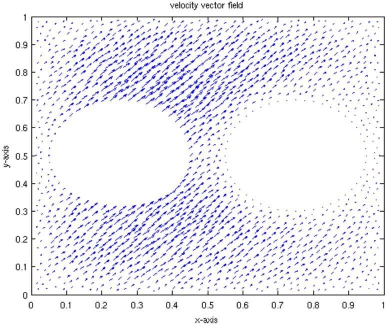

Figure: Quiver Plot for $h = 0.025$

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