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# Newton's Method for the Navier-Stokes Equations with Finite-Element Initial Guess of Stokes Equations

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**Abstract**—It is shown that finite element solutions of Stokes equations may be chosen as the initial guess for the quadratic convergence of Newton's algorithm applied to Navier-Stokes equations provided there are sufficiently small mesh size $h$ and the moderate Reynold's number. We provide also a mixed convergence analysis in terms of iterations and finite-error estimates of the initial guess with a regularity estimate and error analysis for each Newton's step. 

**Keywords**—Navier-Stokes equations, Stokes equations, Convergence, Finite element method, Newton's method.

## 1. INTRODUCTION

In the course of solving nonlinear equations like Navier-Stokes equations, one may employ Newton and Newton-like methods combining with other methods (see, for example, [1–3], etc.). It is well known that if the initial guess is chosen nearby the exact solution of the given nonlinear differential equations the quadratic convergence is guaranteed (see [4]). It is known that the solution of Stokes equations may be chosen as the initial guess for Newton's method for Navier-Stokes equations

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which we consider now with *zero* boundary condition for the *velocity* $\mathbf{u} = (u_1, u_2)^t$ and the mean-zero condition for the *pressure* $p$ as follows:

$$\begin{aligned} -\mu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f}, &\quad \text{in } \Omega, \\ \nabla\cdot\mathbf{u} = 0, &\quad \text{in } \Omega, \\ \mathbf{u} = \mathbf{0}, &\quad \text{on } \partial\Omega, \\ \int_\Omega p\, d\Omega = 0, & \end{aligned} \tag{1.1}$$

where the symbols $\Delta$, $\nabla$, and $\nabla\cdot$ stand for the Laplacian, gradient, and divergence operators, respectively ($\Delta\mathbf{u}$ is the vector of components $\Delta u_i$); the number $\mu$ is a viscous constant; $\mathbf{f}$ is a vector function. The domain $\Omega$ is a convex polygonal bounded domain with boundary $\partial\Omega$.

For the quadratic convergence of Newton's method for Navier-Stokes equations (1.1), an initial guess should be chosen sufficiently close enough to the solution of (1.1). In the implementations of Newton's algorithm, one may use an initial guess as the finite-element approximations to Stokes equations. Hence it is better for us to know how far the finite-element solutions of Stokes equations are away from the solutions of Navier-Stokes equations (1.1). Therefore the purpose of this paper is to provide that the initial guess as the finite-element approximation solution to incompressible Stokes equations may be chosen if there are sufficiently small mesh size $h$ and a moderate Reynold's number $\lambda$ for the quadratic convergence of the sequence generated by Newton's method in the finite-element approaches. This is done by showing that if $\mathcal{U} = (\mathbf{u}, p) \in H^{k+1}(\Omega) \times H^m(\Omega)$ is the weak solution of (1.1) and the initial guess $\mathcal{U}_h^{(0)} = (\mathbf{u}_{0,h}, p_{0,h})$ is the finite element solution to Stokes equations, then the error can be estimated as

$$\|\mathbf{u} - \mathbf{u}_{0,h}\|_1 + \|p - p_{0,h}\| \le C\left(h^m\|p\|_m + h^k\|\mathbf{u}\|_{k+1} + \lambda\|\mathbf{u}\|_1^2\right), \tag{1.2}$$

where $\|\cdot\|_s$ is the standard norm for the Sobolev space $H^s(\Omega)$ where $s$ is a real number. Consequently, the convergence of Newton's sequence can be estimated in terms of mesh size and Reynold's number. As mentioned in earlier studies (see, for example, [4] and [5]), we assume that the incompressible Navier-Stokes equations (1.1) have a branch of nonsingular solutions.

Now interpreting Newton's method as an equivalent iterative method, for a sufficiently large viscosity $\mu$ (or sufficiently small $\lambda := 1/\mu$) regularity estimates and finite error analysis for the solution are also provided by each Newton step. For this purpose, the Newton's method for Navier-Stokes equations in [4] needs to be rewritten as an equivalent iterative method in a variational form. A similar analysis was done in [6] for compressible Navier-Stokes equations.

The rest of this paper is arranged in the following way. In Section 2, we review the quadratic convergence theory of Newton's method and rewrite the Newton's method in an equivalent iterative method. In Section 3, for the implementing of Newton's method by the finite-element method we choose the initial guess as the finite-element solution of Stokes equations. With estimation (1.2), we show that the Newton's sequence converges to the unique solution to the Navier-Stokes equations quadratically and finally provide an error analysis combining the quadratic convergence with the finite-element error of the initial guess and Reynold's number. In Section 4, a regularity and finite-error estimates are shown for each step of Newton's sequence.

## 2. NEWTON'S ALGORITHM IN A WEAK FORMULATION

Let us define a bilinear form $a(\cdot, \mathbf{z}, \mathbf{w}; \cdot)$ and a linear functional $(\mathbf{h}, \cdot)$ as follows:

$$a(\mathbf{u}, \mathbf{z}, \mathbf{w}\,; \mathbf{v}) = \int_\Omega \nabla\mathbf{u}\cdot\nabla\mathbf{v} + \lambda\left((\mathbf{z}\cdot\nabla)\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{w}\right)\mathbf{v}\, d\Omega, \qquad \text{for all } \mathbf{u}, \mathbf{v} \in \mathbf{V}, \tag{2.1}$$

and

$$(\mathbf{h}, \mathbf{v}) = \int_\Omega \mathbf{h} \cdot \mathbf{v} \, d\Omega, \qquad \text{for all } \mathbf{v} \in \mathbf{V}. \tag{2.2}$$

Let $L_0^2(\Omega)$ be the subspace of $L^2(\Omega)$ which consists of the functions of mean zero. The $L^2$ inner product and norm are denoted by $(\cdot,\cdot)$ and $\|\cdot\|$, respectively. Define the following function spaces:

$$\mathbf{X} := \mathbf{V} \times \mathcal{Q}, \qquad \text{where } \mathbf{V} = H_0^1(\Omega)^2, \quad \mathcal{Q} = L_0^2(\Omega), \tag{2.3}$$

and

$$\mathbf{Y} := \mathbf{V}^*, \tag{2.4}$$

where $\mathbf{V}^*$ denotes the dual space of $\mathbf{V}$. With a little abuse of notation, let $\mathcal{U} := (\mathbf{u}, p)$ and $\mathcal{V} := (\mathbf{v}, q)$. Then the weak formulation corresponding to (1.1) can be written as: find $\mathcal{U} \in \mathbf{X}$ such that, for all $\mathbf{v} \in \mathbf{V}$ and $q \in \mathcal{Q}$,

$$\begin{aligned} a(\mathbf{u}, \mathbf{u}, \mathbf{0}; \mathbf{v}) - (p, \nabla \cdot \mathbf{v}) &= \lambda(\mathbf{f}, \mathbf{v}), && \text{for all } \mathbf{v} \in \mathbf{V}, \\ (\nabla \cdot \mathbf{u}, q) &= 0, && \text{for all } q \in \mathcal{Q}. \end{aligned} \tag{2.5}$$

For the derivation of Newton's algorithm, we cast the Navier-Stokes equations (2.5) in the canonical form

$$F(\lambda, \mathcal{U}) = \mathcal{U} + T \cdot G(\lambda, \mathcal{U}) = 0, \tag{2.6}$$

in which a linear solution operator $T$ for the Stokes equations is defined as $T : \mathbf{Y} \longrightarrow \mathbf{X}$ by $\mathcal{U} = T\mathbf{g}$ for $\mathbf{g} \in \mathbf{Y}$ if and only if

$$B(\mathcal{U}, \mathcal{V}) := a(\mathbf{u}, \mathbf{0}, \mathbf{0}; \mathbf{v}) - (p, \nabla \cdot \mathbf{v}) + (\nabla \cdot \mathbf{u}, q) = (\mathbf{g}, \mathbf{v}), \qquad \forall \mathcal{V} \in \mathbf{X}, \tag{2.7}$$

and a $\mathcal{C}^\infty$ operator $G$ for the incompressible Navier-Stokes equations (2.5) is defined as $G : \Lambda \times \mathbf{X} \longrightarrow \mathbf{Y}$, by $\mathbf{g} = G(\lambda, \mathcal{U})$ for $(\lambda, \mathcal{U}) \in \Lambda \times \mathbf{X}$ if and only if

$$(\mathbf{g}, \mathbf{v}) = (-\lambda(\mathbf{u} \cdot \nabla)\mathbf{u} + \lambda \mathbf{f}, \mathbf{v}), \qquad \forall \mathcal{V} \in \mathbf{X}, \tag{2.8}$$

where $\Lambda$ is a compact interval in $(0, \infty)$. Now assume that $\mathcal{U}$ is the solution of problem (2.6) with $T$ and $G$ given by (2.7) and (2.8), respectively. Then $\mathcal{U} = T\mathbf{g}$ if and only if $B(\mathcal{U}, \mathcal{V}) = (\mathbf{g}, \mathbf{v})$ for all $\mathcal{V} \in \mathbf{X}$, and $\mathbf{g} = G(\lambda, \mathcal{U})$ if and only if (2.8) holds. This argument shows that $\mathcal{U}$ is the solution of (2.5). Conversely, if $\mathcal{U}$ solves (2.5), let $\mathbf{g}$ be defined by (2.8). Then $B(\mathcal{U}, \mathcal{V}) = (\mathbf{g}, \mathbf{v})$ for all $\mathcal{V} \in \mathbf{X}$. Thus (2.5) and (2.6) are equivalent.

Throughout this paper, we will use the following assumption.

(A) There is $\lambda$ such that $\{(\lambda, \mathcal{U}(\lambda)); \lambda \in \Lambda\}$ is a branch of nonsingular solutions of equation (2.6).

Under this Assumption (A), the Navier-Stokes equations (1.1) have a unique solution $\mathcal{U} \in \mathbf{X}$ for a given $\mathbf{f} \in \mathbf{Y}$ (see [4, p. 297]).

Then the Newton's algorithm in a weak form for (1.1) reads as (see [4]): find $\mathcal{U}_{n+1} := (\mathbf{u}_{n+1}, p_{n+1}) \in \mathbf{X}$ such that, for all $\mathbf{v} \in \mathbf{V}$ and $q \in \mathcal{Q}$,

$$\begin{aligned} a(\mathbf{u}_{n+1}, \mathbf{u}_n, \mathbf{u}_n; \mathbf{v}) - (p_{n+1}, \nabla \cdot \mathbf{v}) &= \lambda(-(\mathbf{u}_n \cdot \nabla)\mathbf{u}_n + \mathbf{f}, \mathbf{v}), \\ (\nabla \cdot \mathbf{u}_{n+1}, q) &= 0. \end{aligned} \tag{2.9}$$

Let $\mathcal{U}$ be the unique solution to (1.1) and let an $\epsilon$-neighborhood of $\mathcal{U}$ be

$$S(\mathcal{U}; \epsilon) = \{\mathcal{V} \in \mathbf{X} : \|\mathbf{u} - \mathbf{v}\|_1 + \|p - q\| \le \epsilon\}. \tag{2.10}$$

The following convergence result for the sequence $\{\mathcal{U}_n\}$ by Newton's algorithm can be found in, for example, [4].

One can see that the weak second Frèchet-derivative $D^2_{\mathcal{U}}G(\lambda,\mathcal{U})(\hat{\mathcal{U}},\hat{\mathcal{U}})$ of $G$ is bounded on all bounded subsets of $\Lambda \times \mathbf{X}$, so that the first weak Frèchet derivative $D_{\mathcal{U}}G(\lambda,\mathcal{U})\hat{\mathcal{U}}$ is Lipschitz-continuous (see also [4, p.365]).

THEOREM 2.1. *Assume that the first Frèchet-derivative $D_{\mathcal{U}}F(\lambda,\mathcal{V})$ of $F(\lambda,\mathcal{U})$ is Lipschitz-continuous with respect to $\mathcal{V}$ in the ball $S(\mathcal{U};\epsilon)$. Then there exists $\epsilon'$ with $0 < \epsilon' \le \epsilon$ such that for each initial guess $\mathcal{U}^{(0)}$ in $S(\mathcal{U};\epsilon')$ in Newton's algorithm (2.9) determines a unique sequence $\{\mathcal{U}_{n+1}\} \subset S(\mathcal{U};\epsilon')$ that converges to the unique solution $\mathcal{U}$ of (2.1). Furthermore the convergence is quadratic*

$$(\|\mathbf{u}_{n+1}-\mathbf{u}\|_1 + \|p_{n+1}-p\|) \le C(\|\mathbf{u}_n-\mathbf{u}\|_1 + \|p_n-p\|)^2. \tag{2.11}$$

We will choose the initial guess as the weak solution of (2.12) to rewrite an iterative scheme (2.13) and show that (2.13) is in fact Newton's algorithm.

First, solve for all $\mathbf{v}\in\mathbf{V}$ and $q\in\mathcal{Q}$

$$\begin{aligned} a\left(\mathbf{u}^{(0)},\mathbf{0},\mathbf{0};\mathbf{v}\right) - \left(p^{(0)},\nabla\cdot\mathbf{v}\right) &= \lambda(\mathbf{f},\mathbf{v}),\\ (\nabla\cdot\mathbf{u}^{(0)},q) &= 0. \end{aligned} \tag{2.12}$$

With the solution $\mathcal{U}^{(0)} = (\mathbf{u}^{(0)},p^{(0)})$ of the Stokes equation (2.12) as the initial guess, proceed with the following iterative scheme: find $\mathcal{U}^{(j)} = (\mathbf{u}^{(j)},p^{(j)}) \in \mathbf{X}$ where $j=1,2,\dots$ such that for all $\mathbf{v}\in\mathbf{V}$ and $q\in\mathcal{Q}$,

$$\begin{aligned} a\left(\mathbf{u}^{(j)},\mathbf{z}^{(j-1)},\mathbf{z}^{(j-1)};\mathbf{v}\right) - \left(p^{(j)},\nabla\cdot\mathbf{v}\right) &= -\lambda\left(\left(\mathbf{u}^{(j-1)}\cdot\nabla\right)\mathbf{u}^{(j-1)},\mathbf{v}\right),\\ \left(\nabla\cdot\mathbf{u}^{(j)},q\right) &= 0, \end{aligned} \tag{2.13}$$

where, with a nonnegative integer $k$,

$$\mathbf{z}^{(k)} = \sum_{i=0}^{k}\mathbf{u}^{(i)}.$$

Now let

$$\mathcal{U}_n = (\mathbf{u}_n,p_n) := \mathcal{U}^{(0)} + \mathcal{U}^{(1)} + \cdots + \mathcal{U}^{(n)}, \tag{2.14}$$

where

$$\mathbf{u}_n = \sum_{j=0}^{n}\mathbf{u}^{(j)} \quad\text{and}\quad p_n = \sum_{j=0}^{n}p^{(j)}, \tag{2.15}$$

which will be shown as the solution to (1.1) by Newton's method in the rest of this section.

THEOREM 2.2. *The solution $\mathcal{U}_n$ by the iterative scheme (2.13) is the solution generated by Newton's algorithm (2.9).*

PROOF. First summing (2.13) from $j=1$ to $n+1$, and then adding (2.12), we have

$$\begin{aligned} (\nabla\mathbf{u}_{n+1},\nabla\mathbf{v}) + \lambda\left(\sum_{j=1}^{n+1}\left((\mathbf{u}_{j-1}\cdot\nabla)\mathbf{u}^{(j)} + \left(\mathbf{u}^{(j)}\cdot\nabla\right)\mathbf{u}_{j-1}\right),\mathbf{v}\right) - (p_{n+1},\nabla\cdot\mathbf{v})&\\ = \lambda(\mathbf{f},\mathbf{v}) - \lambda\left(\sum_{j=1}^{n+1}\left(\mathbf{u}^{(j-1)}\cdot\nabla\right)\mathbf{u}^{(j-1)},\mathbf{v}\right)&,\\ (\nabla\cdot\mathbf{u}_{n+1},q) = 0&, \end{aligned} \tag{2.16}$$

which can be written as the Newton's iteration form

$$(\nabla \mathbf{u}_{n+1}, \nabla \mathbf{v}) + \lambda\left([(\mathbf{u}_n \cdot \nabla)\mathbf{u}_{n+1} + (\mathbf{u}_{n+1} \cdot \nabla)\mathbf{u}_n], \mathbf{v}\right) - (p_{n+1}, \mathbf{v})$$
$$= \lambda\left(-(\mathbf{u}_n \cdot \nabla)\mathbf{u}_n + \mathbf{f}, \mathbf{v}\right) + \lambda(\epsilon(n), \mathbf{v}), \tag{2.17}$$
$$(\nabla \cdot \mathbf{u}_{n+1}, q) = 0,$$

where

$$\epsilon(n) = (\mathbf{u}_n \cdot \nabla)\mathbf{u}_{n+1} + (\mathbf{u}_{n+1} \cdot \nabla)\mathbf{u}_n - \sum_{j=1}^{n+1}\left[\left(\mathbf{u}^{(j)} \cdot \nabla\right)\mathbf{u}_{j-1} + (\mathbf{u}_{j-1} \cdot \nabla)\,\mathbf{u}^{(j)}\right]$$
$$- \sum_{j=1}^{n+1}\left(\mathbf{u}^{(j-1)} \cdot \nabla\right)\mathbf{u}^{(j-1)} - (\mathbf{u}_n \cdot \nabla)\mathbf{u}_n. \tag{2.18}$$

Hence, it is enough to show that $\epsilon(n) = \mathbf{0}$, which can be shown by using the following identity:

$$\left(\sum_{i=0}^{n} a_i\right)\left(\sum_{i=0}^{n+1} b_i\right) = \sum_{i=1}^{n+1}\left(\sum_{j=0}^{i-1} a_j\right) b_i + \sum_{i=1}^{n} a_i\left(\sum_{j=0}^{i-1} b_j\right) + \sum_{i=1}^{n+1} a_{i-1}b_{i-1}.$$

This completes the assertion of theorem. ∎

In this paper, we will use a generic constant $C$, which does not depend on the Reynold's number $\lambda$, the mesh size $h$ of triangulations, and the number $n$ of Newton's iterations; otherwise we describe its dependence.

## 3. A CONVERGENCE ANALYSIS

In this section, an error analysis will be provided in terms of mesh size $h$ and Reynold's number $\lambda$ for the sequence of Navier-Stokes equations generated by Newton's method if the finite element solution to Stokes equations is chosen as the initial guess.

Let $\mathcal{T}_h$ be a family of triangulations of $\Omega$ by a standard finite-element subdivisions of $\Omega$ into quasi-uniform triangles with $h = \max\{\operatorname{diam}(K) : K \in \mathcal{T}_h\}$. Assume that the finite-element subspaces $\mathbf{V}_h \subset \mathbf{V}$ and $\mathcal{Q}_h \subset \mathcal{Q}$ satisfy the following approximation properties:

$$\inf_{\mathbf{v}\in\mathbf{V}_h} \|\mathbf{u} - \mathbf{v}\|_1 \le Ch^k\|\mathbf{u}\|_{k+1}, \qquad \forall \mathbf{v} \in \mathbf{V} \cap H^{k+1}(\Omega)^2, \quad 1 \le k \le l, \tag{3.1}$$
$$\inf_{q\in\mathcal{Q}_h} \|p - q\|_0 \le Ch^m\|p\|_m, \qquad \forall p \in \mathcal{Q} \cap H^m(\Omega), \qquad 1 \le m \le l, \tag{3.2}$$

and further assume that the following uniform inf-sup condition holds on $\mathbf{V}_h \times \mathcal{Q}_h$: for each $q_h \in \mathcal{Q}_h$, there exists a $\mathbf{v}_h \in \mathbf{V}_h$ such that

$$(q_h, \nabla \cdot \mathbf{v}_h) = \|q_h\|^2, \qquad |\mathbf{v}_h|_1 \le C\|q_h\|, \tag{3.3}$$

with a positive constant $C$ independent of $h$.

The Newton's algorithm in the finite-element approach (2.9) can be stated as: find $(\mathbf{u}_{n+1,h}, p_{n+1,h}) \in \mathbf{V}_h \times \mathcal{Q}_h$ such that, for all $\mathbf{v} \in \mathbf{V}_h$ and $q \in \mathcal{Q}_h$,

$$a(\mathbf{u}_{n+1,h}, \mathbf{u}_{n,h}, \mathbf{u}_{n,h}; \mathbf{v}) - (p_{n+1,h}, \nabla \cdot \mathbf{v}) = \lambda((\mathbf{u}_{n,h} \cdot \nabla)\mathbf{u}_{n,h} + \mathbf{f}, \mathbf{v}),$$
$$(\nabla \cdot \mathbf{u}_{n+1,h}, q) = 0, \tag{3.4}$$

where $n = 0, 1, 2, \dots$, with the initial guess $\mathcal{U}_h^{(0)} := (\mathbf{u}_{0,h}, p_{0,h})$ which is the finite-element approximation solution of Stokes problem

$$\begin{aligned} a(\mathbf{u}_{0,h}, \mathbf{0}, \mathbf{0}; \mathbf{v}) - (p_{0,h}, \nabla \cdot \mathbf{v}) &= \lambda(\mathbf{f}, \mathbf{v}), && \text{for all } \mathbf{v} \in \mathbf{V}_h, \\ (\nabla \cdot \mathbf{u}_{0,h}, q) &= 0, && \text{for all } q \in Q_h. \end{aligned} \tag{3.5}$$

Equivalently, the discrete Newton's algorithm can be written as: find $\mathcal{U}_h^{(j)} = (\mathbf{u}_h^{(j)}, p_h^{(j)}) \in \mathbf{X}_h$ such that

$$\begin{aligned} a\left(\mathbf{u}_h^{(j)}, \mathbf{z}_h^{(j-1)}, \mathbf{z}_h^{(j-1)}; \mathbf{v}\right) - \left(p_h^{(j)}, \nabla \cdot \mathbf{v}\right) &= -\lambda\left(\left(\mathbf{u}_h^{(j-1)} \cdot \nabla\right) \mathbf{u}_h^{(j-1)}, \mathbf{v}\right), && \text{for all } \mathbf{v} \in \mathbf{V}_h, \\ \left(q, \nabla \cdot \mathbf{u}_h^{(j)}\right) &= 0, && \text{for all } q \in Q_h, \end{aligned} \tag{3.6}$$

where

$$\mathbf{z}_h^{(j-1)} = \sum_{i=0}^{j-1} \mathbf{u}_h^{(i)}.$$

Then, according to Theorem 2.2, the discrete solution to (3.4) can be written as

$$\mathbf{u}_{n,h} = \sum_{j=0}^{n} \mathbf{u}_h^{(j)} \quad \text{and} \quad p_{n,h} = \sum_{j=0}^{n} p_h^{(j)}. \tag{3.7}$$

The following technical known results are necessary for a future use.

LEMMA 3.1.

(i) *For every $p \in Q$, there is a $\mathbf{v} \in \mathbf{V}$ such that*

$$\nabla \cdot \mathbf{v} = p, \quad \textit{and} \quad \|\mathbf{v}\|_1 \le C_1 \|q\|_1, \tag{3.8}$$

*where $C_1$ is a positive constant.*

(ii) *The following Poincaré inequality holds:*

$$C_p \|\mathbf{u}\|_1^2 \le \|\nabla \mathbf{u}\|^2, \qquad \forall\, \mathbf{u} \in \mathbf{V}, \tag{3.9}$$

*where $C_p$ is a positive constant depending only on $\Omega$.*

(iii) *For all $u$, $v$, and $w \in H^1(\Omega)$*

$$\left| \int_\Omega D_i u v w \, d\Omega \right| \le C_2 \|u\|_1 \|v\|_1 \|w\|_1, \qquad i = 1, 2, \tag{3.10}$$

*where $D_i$ denotes the derivative with respect to the $i^{th}$ coordinate variable in $\Re^2$ and $C_2$ is a positive constant.*

PROOF. For (i), see Corollary 2.4 and the proof of Lemma 2.2 in [4]. (ii) is known as Poincare inequality. For (iii) see, for example, Lemma 7 in [7]. ∎

Consider the following problem: to find $(\mathbf{u}, p) \in \mathbf{X}$ such that

$$\begin{aligned} a(\mathbf{u}, \mathbf{z}, \mathbf{w}; \mathbf{v}) - (p, \nabla \cdot \mathbf{v}) &= (\mathbf{h}, \mathbf{v}), && \text{for all } \mathbf{v} \in \mathbf{V}, \\ (\nabla \cdot \mathbf{u}, q) &= 0, && \text{for all } q \in Q, \end{aligned} \tag{3.11}$$

for $\mathbf{z}, \mathbf{w} \in \mathbf{V}_\lambda$ where

$$\mathbf{V}_\lambda := \left\{ \mathbf{z} \in \mathbf{V} \mid \|\mathbf{z}\|_1 \le \frac{C_p}{2C_2} \frac{1}{\lambda},\ C_p \text{ is the Poincare constant} \right\}. \tag{3.12}$$

LEMMA 3.2. *For any* $\mathbf{u}, \mathbf{v} \in \mathbf{V}$ *and* $\mathbf{z}, \mathbf{w} \in \mathbf{V}_\lambda$, *the following holds:*

$$|((\mathbf{z} \cdot \nabla)\mathbf{u}, \mathbf{v})| \leq C_p \frac{1}{2\lambda} \|\mathbf{u}\|_1 \|\mathbf{v}\|_1 \quad \text{and} \quad |((\mathbf{u} \cdot \nabla)\mathbf{w}, \mathbf{v})| \leq C_p \frac{1}{2\lambda} \|\mathbf{u}\|_1 \|\mathbf{v}\|_1.$$

PROOF. A simple application of (3.10) yields the conclusion. ∎

LEMMA 3.3. *Let* $\mathbf{z}, \mathbf{w} \in \mathbf{V}_\lambda$. *Then we have*

$$|a(\mathbf{u}, \mathbf{z}, \mathbf{w}; \mathbf{v})| \leq C_3 \|\mathbf{u}\|_1 \|\mathbf{v}\|_1, \qquad \forall\, \mathbf{u}, \mathbf{v} \in \mathbf{V}, \tag{3.13}$$

*and*

$$\frac{C_p}{2} \|\mathbf{u}\|_1^2 \leq a(\mathbf{u}, \mathbf{z}, \mathbf{z}; \mathbf{u}), \qquad \forall\, \mathbf{u} \in \mathbf{V}, \tag{3.14}$$

*where* $C_3$ *is a constant dependent on the Poincare constant* $C_p$.

PROOF. The proof of (3.13) comes from Cauchy-Schwarz inequality, Poincare inequality, and Lemma 3.2. That is,

$$\begin{aligned} |a(\mathbf{u}, \mathbf{z}, \mathbf{w}; \mathbf{v})| &\leq |(\nabla \mathbf{u}, \nabla \mathbf{v})| + \lambda |((\mathbf{z} \cdot \nabla)\mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{w}, \mathbf{v})| \\ &\leq \|\nabla \mathbf{u}\| \|\nabla \mathbf{v}\| + C_p \|\mathbf{u}\|_1 \|\mathbf{v}\|_1 \\ &\leq C_4 \|\mathbf{u}\|_1 \|\mathbf{v}\|_1, \end{aligned}$$

where $C_3$ depends on $C_p$. From Lemma 3.2 and (3.9), we have

$$\begin{aligned} a(\mathbf{u}, \mathbf{z}, \mathbf{z}; \mathbf{u}) &= (\nabla \mathbf{u}, \nabla \mathbf{u}) + \lambda((\mathbf{z} \cdot \nabla)\mathbf{u}, \mathbf{u}) + ((\mathbf{u} \cdot \nabla)\mathbf{z}, \mathbf{u}) \\ &= (\nabla \mathbf{u}, \nabla \mathbf{u}) + \lambda((\mathbf{u} \cdot \nabla)\mathbf{z}, \mathbf{u}) \\ &\geq C_p \|\mathbf{u}\|_1^2 - \frac{C_P}{2} \|\mathbf{u}\|_1^2 \geq \frac{C_p}{2} \|\mathbf{u}\|_1^2. \end{aligned}$$

We have the conclusion. ∎

LEMMA 3.4. *Let* $\mathbf{h} \in \mathbf{V}^*$. *For any* $\mathbf{z}, \mathbf{w} \in \mathbf{V}_\lambda$, *the solution* $(\mathbf{u}, p) \in \mathbf{X}$ *of the weak form (3.11) satisfies a priori estimate*

$$\|\mathbf{u}\|_1 \leq C_6 \|\mathbf{h}\|_{-1} \quad \text{and} \quad \|p\| \leq C_6 \|\mathbf{h}\|_{-1}, \tag{3.15}$$

*where* $C_6 > 1$ *is a constant dependent on* $C_p$.

PROOF. Since $(\mathbf{u}, p)$ is the weak solution of (3.11), we have

$$a(\mathbf{u}, \mathbf{z}, \mathbf{w}; \mathbf{u}) = (\mathbf{h}, \mathbf{u}). \tag{3.16}$$

Then using Lemma 3.3, it follows that

$$\frac{C_p}{2} \|\mathbf{u}\|_1^2 \leq a(\mathbf{u}, \mathbf{z}, \mathbf{w}; \mathbf{u}) = (\mathbf{h}, \mathbf{u}) \leq \|\mathbf{h}\|_{-1} \|\mathbf{u}\|_1,$$

which implies

$$\|\mathbf{u}\|_1 \leq \frac{2}{C_p} \|\mathbf{h}\|_{-1}. \tag{3.17}$$

Since $p \in Q$, according to (3.8), there is a $\mathbf{v} \in \mathbf{V}$ such that

$$\nabla \cdot \mathbf{v} = p, \qquad \|\mathbf{v}\|_1 \leq C_1 \|p\|. \tag{3.18}$$

By taking such $\mathbf{v}$ into (3.11), Lemma 3.3 yields

$$\|p\|^2 = (p, \nabla \cdot \mathbf{v}) = a(\mathbf{u}, \mathbf{z}, \mathbf{w}; \mathbf{v}) - (\mathbf{h}, \mathbf{v}) \leq C_4(\|\mathbf{h}\|_{-1} + \|\mathbf{u}\|_1)\|\mathbf{v}\|_1,$$

where $C_4 = \max\{1, C_3\}$, which implies, using (3.18) and (3.17),

$$\|p\| \leq C_5(\|\mathbf{h}\|_{-1} + \|\mathbf{u}\|_1) \leq C_5 \left(1 + \frac{2}{C_p}\right) \|\mathbf{h}\|_{-1}.$$

These arguments yield conclusion (3.15) with a chosen $C_6 = \max\{2/C_p, C_5(1 + 2/C_p), 1\}$. ∎

Since $\mathbf{V}_h \subset H_0^1(\Omega)^2$ and $Q_h \subset L_0^2(\Omega)$ and Stokes equations are a particular type of (3.11) by choosing $\mathbf{z} = \mathbf{w} = \mathbf{0}$ and $\mathbf{h} = \lambda \mathbf{f}$, the following known result in [4] for the initial guess is immediate.

COROLLARY 3.1. *The solution $\mathcal{U}_{0,h} = (\mathbf{u}_{0,h}, p_{0,h}) \in \mathbf{V}_h \times Q_h$ of the finite-element solution to (3.5) satisfies*

$$\|\mathbf{u}_{0,h}\|_1 + \|p_{0,h}\| \leq C_6 \lambda \|\mathbf{f}\|_{-1}. \tag{3.19}$$

Now again consider problems (2.5) and (3.5) in order to estimate

$$\|\mathbf{u} - \mathbf{u}_{0,h}\|_1 + \|p - p_{0,h}\| \tag{3.20}$$

in the sense of the mesh size $h$ and the Reynold's number $\lambda$ where $(\mathbf{u}, p) \in \mathbf{V} \times \mathcal{Q}$ is the solution of the Navier-Stokes equations (2.5) and $(\mathbf{u}_{0,h}, p_{0,h}) \in \mathbf{V}_h \times \mathcal{Q}_h$ is the solution of Stokes equations (3.5). Note that $\mathbf{V}_h \subset \mathbf{V}$ and $\mathcal{Q}_h \subset \mathcal{Q}$. This estimate can be done by considering the exact solution $(\mathbf{u}^{(0)}, p^{(0)})$ of Stokes equations (2.12).

By triangle inequality, it follows that

$$\|\mathbf{u} - \mathbf{u}_{0,h}\|_1 + \|p - p_{0,h}\| \leq \left\|\mathbf{u} - \mathbf{u}^{(0)}\right\|_1 + \left\|p - p^{(0)}\right\| + \left\|\mathbf{u}^{(0)} - \mathbf{u}_{0,h}\right\|_1 + \left\|p^{(0)} - p_{0,h}\right\|. \tag{3.21}$$

Now subtracting (2.12) from (2.5), we have

$$\begin{aligned} a\left(\mathbf{u} - \mathbf{u}^{(0)}, \mathbf{0}, \mathbf{0}; \mathbf{v}\right) + \left(p - p^{(0)}, \nabla \cdot \mathbf{v}\right) &= -\lambda((\mathbf{u} \cdot \nabla)\mathbf{u}, \mathbf{v}), && \text{for all } \mathbf{v} \in \mathbf{V}, \\ \left(\nabla \cdot \left(\mathbf{u} - \mathbf{u}^{(0)}\right), q\right) &= 0, && \text{for all } q \in \mathcal{Q}. \end{aligned} \tag{3.22}$$

By choosing $\mathbf{z} = \mathbf{w} = \mathbf{0}$ and $\mathbf{h} = \lambda(\mathbf{u} \cdot \nabla)\mathbf{u}$ in (3.11), it can be shown from Lemma 3.4 that

$$\left\|\mathbf{u} - \mathbf{u}^{(0)}\right\|_1 + \left\|p - p^{(0)}\right\| \leq C\lambda \|\mathbf{u}\|_1^2, \tag{3.23}$$

and it can be also shown that the finite-error estimate for Stokes equations (see [4])

$$\left\|\mathbf{u}^{(0)} - \mathbf{u}_{0,h}\right\|_1 + \left\|p^{(0)} - p_{0,h}\right\| \leq C\left(h^k \|\mathbf{u}\|_{k+} + h^m \|p\|_m\right). \tag{3.24}$$

Combining (3.23) and (3.24), we have the following.

PROPOSITION 3.1. *Let $(\mathbf{u}, p) \in (\mathbf{V} \times Q) \cap (H^{k+1}(\Omega) \times H^m(\Omega))$ be the solution of (2.5) and $(\mathbf{u}_{0,h}, p_{0,h})$ be the solution of (3.5). Then*

$$\|\mathbf{u} - \mathbf{u}_{0,h}\|_1 + \|p - p_{0,h}\| \leq C\left(h^m \|p\|_m + h^k \|\mathbf{u}\|_{k+1} + \lambda \|\mathbf{u}\|_1^2\right). \tag{3.25}$$

The above Proposition 3.1 tells us that the finite-element approximate solutions $(\mathbf{u}_{0,h}, p_{0,h})$ can be chosen as an initial guess for the Newton's method (3.4) and (3.5) to solve Navier-Stokes equations if for an $\epsilon' < 1$, there are a mesh size $h$ and $\lambda$ such that

$$C\left(h^m\|p\|_m + h^k\|\mathbf{u}\|_{k+1} + \lambda\|\mathbf{u}\|_1^2\right) \leq \epsilon'. \tag{3.26}$$

In order to show the quadratic convergence of the sequence $\mathcal{U}_{n,h}$ generated by Newton's method for the chosen initial guess, we will provide the validity of the assumptions of Theorem 2.1 found in [4].

THEOREM 3.1. *Let $\epsilon$ be in (2.10). Assume that (3.26) holds for $\epsilon' \leq \epsilon$, i.e., the solution $\mathcal{U}_h^{(0)} = (\mathbf{u}_{0,h}, p_{0,h})$ of Stokes equations (3.5) belongs to $S(\mathcal{U};\epsilon')$ where $\mathcal{U} = (\mathbf{u}, p)$ is the solution of (2.5). Then Newton's algorithm (3.4) determines a unique sequence $\{\mathcal{U}_{n+1,h}\} = \{(\mathbf{u}_{n+1,h}, p_{n+1,h})\} \subset S(\mathcal{U};\epsilon') \cap (\mathbf{V}_h \times Q_h)$ that converges to the unique solution $\mathcal{U} = (\mathbf{u}, p) \in H^{k+1}(\Omega) \times H^m(\Omega)$ of (2.5). Furthermore the convergence is quadratic with an initial guess $\mathcal{U}_h^{(0)}$*

$$\begin{aligned}(\|\mathbf{u}_{n+1,h} - \mathbf{u}\|_1 + \|p_{n+1,h} - p\|) &\leq C(\|\mathbf{u}_{n,h} - \mathbf{u}\|_1 + \|p_{n,h} - p\|)^2 \\ &\leq C_n\left(h^k\|\mathbf{u}\|_{k+1} + \lambda\|\mathbf{u}\|_1^2 + h^m\|p\|_m\right)^{2^{n+1}},\end{aligned} \tag{3.27}$$

*where $C_n$ is a positive constant dependent on the iteration number $n$.*

PROOF. Since the hypothesis holds the assumptions of Theorem 2.1, the Newton's sequence converges to the unique solution. Furthermore, the last inequality in (3.27) comes from (3.25). These arguments may complete the proof. ∎

REMARK. It can be noted from (3.26) that the Reynold's number $\lambda$ is more critical than the mesh size $h$ for using the finite-element solution of Stokes equations as the initial guess. This is because one may solve Stokes equations in finite-element sense for sufficiently small mesh size $h$. From this observation we infer that for large Reynold's numbers it may not be possible to choose the initial guess as the solution of Stokes equations.

## 4. A REGULARITY ESTIMATE

In this section, we discuss the $H^1$-regularity of the solution of the auxiliary problem (3.11) to show the $H^1$-regularity of the solution generated by Newton's method.

Assume that $\lambda$ satisfies

$$0 < \delta \leq \lambda < \frac{1}{C_6\left(\|\mathbf{f}\|_{-1} + 2C_2/C_p\right)}, \tag{4.1}$$

where $\delta$ is a constant. Let us consider the regularity of the finite-element solution (3.11) at each iteration step.

THEOREM 4.1. *Assume that (4.1) holds. Then the solution $(\mathbf{u}_{n,h}, p_{n,h})$ of (3.11) satisfies*

$$\|\mathbf{u}_{n,h}\|_1 \in \mathbf{V}_\lambda, \qquad \text{for all } n = 1, 2, \ldots, \quad \text{and} \quad \|\mathbf{u}_{n,h}\|_1 + \|p_{n,h}\| \leq \frac{C_p}{C_2}\frac{1}{\lambda}, \tag{4.2}$$

*where $C_p$ is the Poincare constant and $C_2$ is appeared in (3.10).*

PROOF. The proof will be done by mathematical induction for the solution of (3.6). Let

$$\gamma := C_6\lambda\|\mathbf{f}\|_{-1} < 1.$$

At first, note that the initial guess satisfies

$$\|\mathbf{u}_{0,h}\|_1 + \|p_{0,h}\| \leq C_6\lambda\|\mathbf{f}\|_{-1}, \tag{4.3}$$

from which

$$\left\|\mathbf{u}_h^{(0)}\right\| := \|\mathbf{u}_{0,h}\|_1 \leq \gamma \quad \text{and} \quad \left\|p_h^{(0)}\right\| := \|p_{0,h}\| \leq \gamma.$$

Hence, with $\mathbf{z} = \mathbf{w} = \mathbf{u}_{0,h}$ and $\mathbf{h} = -\lambda(\mathbf{u}_h^{(0)} \cdot \nabla)\mathbf{u}_h^{(0)}$ in (3.11), one may have from (3.15) and (4.1) that

$$\left\|\mathbf{u}_h^{(1)}\right\|_1 \le C_6 \left\|\left(\mathbf{u}_h^{(0)} \cdot \nabla\right) \mathbf{u}_h^{(0)}\right\|_{-1} \le C_6 \left\|\mathbf{u}_h^{(0)}\right\|_1^2 \le C_6\gamma^2$$

and

$$\left\|p_h^{(1)}\right\| \le C_6 \left\|\left(\mathbf{u}_h^{(0)} \cdot \nabla\right) \mathbf{u}_h^{(0)}\right\|_{-1} \le C_6 \left\|\mathbf{u}_h^{(0)}\right\|_1^2 \le C_6\gamma^2.$$

Note that from (4.1),

$$\frac{C_6}{1-\gamma} = \frac{C_6}{1 - C_6\lambda\|\mathbf{f}\|_{-1}} \le \frac{C_6}{1 - \|\mathbf{f}\|_{-1}/\left(\|\mathbf{f}\|_{-1} + 2C_2C_p^{-1}\right)} = \frac{C_p}{2C_2}\frac{1}{\lambda}. \tag{4.4}$$

Since $C_6 > 1$ and $\gamma < 1$ we have

$$\|\mathbf{u}_{1,h}\|_1 \le \left\|\mathbf{u}_h^{(1)}\right\|_1 + \left\|\mathbf{u}_h^{(0)}\right\|_1 \le C_6 \sum_{k=1}^{\infty} \gamma^k = \frac{C_p}{2C_2}\frac{1}{\lambda},$$

so that $\mathbf{u}_{1,h} \in \mathbf{V}_\lambda$ and

$$\|p_{1,h}\| \le \left\|p_h^{(1)}\right\| + \left\|p_h^{(0)}\right\| \le C_6 \sum_{k=1}^{\infty} \gamma^k = \frac{C_p}{2C_2}\frac{1}{\lambda}.$$

At the $j^{\text{th}}$ iteration process in (3.6), assume that we have

$$\left\|\mathbf{u}_h^{(j)}\right\|_1 \le C_6\gamma^{2^j} \quad \text{and} \quad \left\|p_h^{(j)}\right\| \le C_6\gamma^{2^j}. \tag{4.5}$$

Since $\gamma < 1$. Then, for the $(j+1)^{\text{th}}$ iteration process, since $\mathbf{z} := \mathbf{u}_{j,h} = \sum_{i=0}^{j} \mathbf{u}_h^{(i)}$ it follows that

$$\|\mathbf{z}\|_1 = \|\mathbf{u}_{j,h}\| \le \sum_{i=0}^{j} \left\|\mathbf{u}_h^{(i)}\right\|_1 \le C_6 \sum_{i=0}^{j} \gamma^{2^j} < \frac{C_6}{1-\gamma} \tag{4.6}$$

and by the same way

$$\|p_{j,h}\| < \frac{C_6}{1-\gamma}. \tag{4.7}$$

Note that from (4.4),

$$\mathbf{z} := \mathbf{u}_{j,h} = \sum_{i=0}^{j} \mathbf{u}_h^{(i)} \in \mathbf{V}_\lambda.$$

For the last step of mathematical induction, consider (3.6) and (3.11) with $\mathbf{h} = -\lambda(\mathbf{u}_h^{(j)} \cdot \nabla)\mathbf{u}_h^{(j)}$. Then (3.15) implies

$$\left\|\mathbf{u}_h^{(j+1)}\right\|_1 \le C_6 \left\|-\left(\mathbf{u}_h^{(j)} \cdot \nabla\right) \mathbf{u}_h^{(j)}\right\|_{-1}^2 \le C_6 \left\|\mathbf{u}_h^{(j)}\right\|_1^2 \le C_6 \left(2^{2^j}\right)^2 = C_6 2^{2^{j+1}},$$

similarly we have

$$\left\|p_h^{(j+1)}\right\| \le C_6 2^{2^{j+1}}.$$

Thus, from (3.7) we conclude that

$$\|\mathbf{u}_{j+1,h}\|_1 \le \sum_{i=0}^{j+1} \left\|\mathbf{u}_h^{(i)}\right\|_1 \le C_6 \sum_{i=0}^{\infty} \gamma^{2^i} = C_6 \frac{\gamma}{1-\gamma} \le \frac{C_6}{1-\gamma}$$

and

$$\|p_{j+1,h}\| \le \sum_{i=0}^{j+1} \left\|p_h^{(i)}\right\| \le C_6 \sum_{i=0}^{\infty} \gamma^{2^i} = C_6 \frac{\gamma}{1-\gamma} \le \frac{C_6}{1-\gamma}.$$

In particular, from (4.4) is implied

$$\mathbf{u}_{j+1,h} \in \mathbf{V}_\lambda, \quad \text{and} \quad \|\mathbf{u}_{j+1,h}\|_1 + \|p_{j+1,h}\| \le \frac{2C_6}{1-\gamma} = \frac{C_p}{C_2}\frac{1}{\lambda}. \qquad \blacksquare$$

REMARK. The above theorem tells that the solution $(\mathbf{u}_{n,h}, p_{n,h})$ generated by Newton's method is always bounded for $n = 1, 2, \dots$ for a certain range of Reynold's number $\lambda$ provided the initial guess $(\mathbf{u}_h^{(0)}, p_h^{(0)})$ satisfies (4.3). In this case $\{\mathbf{u}_{n,h}\}$ stay always in the set $\mathbf{V}_\lambda$ and the upper bound of Reynold's number does depend on $\mathbf{f}$, the Poincare constant $C_p$, and constants which depend on the domain $\Omega$.

We can now show the finite-element discretization error for the solution generated by Newton's method. We note that if (3.3) and (4.1) are satisfied, then the finite-element approximation problem (3.11) has at least one solution. Now one can easily verify that the following error estimate for the finite-element approximate solution generated by Newton's method (3.11) can be derived by using the orthogonality property of the error

$$\mathbf{u}_{n+1} - \mathbf{u}_{n+1,h}, \qquad p_{n+1} - p_{n+1,h},$$

the uniform inf-sup condition (3.3), and the coercivity of $a(\cdot,\cdot,\cdot;\cdot)$ because of Theorem 4.1. Hence, we omit its proof.

THEOREM 4.2. *Under the assumption of Theorem 4.1, the weak solution $(\mathbf{u}_{n+1}, p_{n+1}) \in \mathbf{V} \times Q$ by Newton's method of (2.9) and its approximate solution $(\mathbf{u}_{n+1,h}, p_{n+1,h}) \in \mathbf{V}_h \times Q_h$ of (3.4) by finite-element methods satisfy the following error estimate:*

$$\|\mathbf{u}_{n+1} - \mathbf{u}_{n+1,h}\|_1 + \|p_{n+1} - p_{n+1,h}\| \le C \inf \left\{ \|\mathbf{u}_{n+1} - \mathbf{v}_h\|_1 + \|p_{n+1} - q_h\| \right\}, \tag{4.8}$$

*where the infimum is taken over all $\mathbf{v}_h \in \mathbf{V}_h$ and $q_h \in Q_h$ and $C$ is a positive constant independent of $\mathbf{v}_h$ and $q_h$.*

Hence an error estimate can be summarized as follows.

COROLLARY 4.1. *Assume that the hypotheses of Theorem 4.1 hold and suppose that the solution of the weak form (3.4) by Newton's method is in $(\mathbf{u}_{n+1}, p_{n+1}) \in (\mathbf{V} \times Q) \cap (H^{k+1}(\Omega) \times H^m(\Omega))$. Then the finite-element approximate solution $(\mathbf{u}_{n+1,h}, p_{n+1,h}) \in \mathbf{V}_h \times Q_h$ of (3.11) satisfies the following:*

$$\|\mathbf{u}_{n+1} - \mathbf{u}_{n+1,h}\|_1 + \|p_{n+1} - p_{n+1,h}\| \le C \left( h^k \|\mathbf{u}_{n+1}\|_{k+1} + h^m \|p_{n+1}\|_m \right),$$

*where $C$ is a positive constant.*

PROOF. It comes from (3.1) and (3.2) with Theorem 4.2. $\blacksquare$

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