Math 324 E - Fall 2017

Midterm exam 2

Wednesday, November 8, 2017

Name: Solutions

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Total	50	

- There are 4 problems on this exam. Make sure you have all four.
- You must show your work on all problems. The correct answer with no supporting work may result in no credit. **Put a box around your FINAL ANSWER for each problem and cross out any work that you don't want to be graded.**
- Give exact answers, and simplify as much as possible. For example, $\frac{\pi}{\sqrt{2}}$ is acceptable, but $\frac{1}{2}+\frac{3}{4}$ should be simplified to $\frac{5}{4}$.
- If you need more room, use the backs of the pages and indicate to the grader that you have done so.
- Raise your hand if you have a question.
- Any student found engaging in academic misconduct will receive a score of 0 on this exam.
- You have 50 minutes to complete the exam. Budget your time wisely!

GOOD LUCK!

1. (10 points) Let $f(x, y, z) = xz^2 - 2yz$.

a) Compute the directional derivative of f at the point $(-1, 0, 2)$ in the direction $u = \frac{1}{\sqrt{3}}(\hat{\imath}+\hat{\jmath}+\hat{k})$.

Use $\boxed{D_u f = \nabla f \cdot u.}$

$\nabla f = z^2\hat{\imath} - 2z\hat{\jmath} + (2xz - 2y)\hat{k}$, so $\nabla f(-1,0,2) = 4\hat{\imath} - 4\hat{\jmath} - 4\hat{k}$.

Thus $D_u f(-1,0,2) = (4\hat{\imath} - 4\hat{\jmath} - 4\hat{k}) \cdot \frac{1}{\sqrt{3}}(\hat{\imath} + \hat{\jmath} + \hat{k})$

$$= \frac{1}{\sqrt{3}}(4 - 4 - 4) = -\frac{4}{\sqrt{3}}.$$

b) What is the maximum value of directional derivative at $(-1, 0, 2)$, and which direction does it occur in?

$D_u f$ is maximized when $u = \frac{\nabla f}{\|\nabla f\|}$.

At $(-1,0,2)$, $\frac{\nabla f}{\|\nabla f\|} = \frac{4\hat{\imath} - 4\hat{\jmath} - 4\hat{k}}{\sqrt{4^2+4^2+4^2}} = \frac{1}{4\sqrt{3}}(4\hat{\imath} - 4\hat{\jmath} - 4\hat{k})$.

Moreover, the maximum value of $D_u f$ is $\|\nabla f\| = 4\sqrt{3}$.

So $D_u f$ is maximized in the direction $\frac{1}{\sqrt{3}}(\hat{\imath} - \hat{\jmath} - \hat{k})$, and the value of $D_u f$ in that direction is $4\sqrt{3}$.

2. (14 points) For each pair of conservative vector field F and curve C, first find a potential function for F, and then use the fundamental theorem of line integrals to evaluate $\int_C F \cdot dr$.

a) $F(x, y) = xy\hat{\imath} + \frac{1}{2}x^2\hat{\jmath}$, and C is the part of the hyperbola $y = \frac{3}{x}$ between the points $(1, 3)$ and $(3, 1)$, traversed from left to right.

Let $f(x,y) = \frac{1}{2}x^2 y$, so $F = \nabla f$. Thus

$$\int_C F \cdot dr = f(3,1) - f(1,3) = \frac{9}{2} - \frac{3}{2} = \boxed{3}.$$

b) $F(x, y) = x^{-2}y^{-1}\hat{\imath} + x^{-1}y^{-2}\hat{\jmath}$, and C is the infinite(!) ray along the line $x = 2y$ for $x \geq 1$, with initial point $(1, 2)$. (Hint: do the problem with the part of the ray out to the point $(n, 2n)$, and then let $n \to \infty$.)

Let $f(x,y) = -\frac{1}{xy}$, so $\nabla f = F$.

Then if C_n is the segment $(1,2) \longrightarrow (n, 2n)$,

$$\int_{C_n} F \cdot dr = f(n,2n) - f(1,2) = -\frac{1}{n(2n)} - \left(-\frac{1}{1 \cdot 2}\right) = \frac{1}{2} - \frac{1}{2n^2}.$$

Thus $$\int_C F \cdot dr = \lim_{n \to \infty} \int_{C_n} F \cdot dr = \lim_{n \to \infty} \left(\frac{1}{2} - \frac{1}{2n^2}\right) = \boxed{\frac{1}{2}}.$$

3. (16 points) For each of the statements below, circle **True** if you think it is true, and circle **False** if you think it is false. You may use the space to do scratch work, but no partial credit will be awarded on this question. Note that for a vector field $F = P\hat{\imath} + Q\hat{\jmath}$ and a function g, we define gF as the vector field $gP\hat{\imath} + gQ\hat{\jmath}$. (4 points for each statement.)

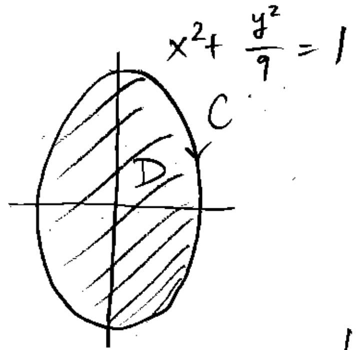
(a) **True** **False** The function $f(x, y) = e^{x^2-2y^2}$ satisfies $\nabla f = f(2x\hat{\imath} - 4y\hat{\jmath})$.

(b) **True** **False** The vector field $F = \frac{-y\hat{\imath}+x\hat{\jmath}}{x^2+y^2}$ satisfies $\int_C F \cdot dr = 0$ where C is the unit circle traversed counter clockwise.

(c) **True** **False** Let $G(x, y) = 6y\cos(2x)\hat{\imath} + 6\sin(x)\cos(x)\hat{\jmath}$. G is a conservative vector field.

(d) **True** **False** Let C denote the part of the parabola $y = 1 - x^2$ between the points $(0, 1)$ and $(4, -15)$, traversed from right to left. Then the vector $r(t) = 2t\hat{\imath} - 4t^3\hat{\jmath}$ is tangent to C for $0 \le t \le 2$.

4. (10 points) Let $F(x,y) = 2x^2y\hat{\imath} - 3x\hat{\jmath}$, and let C denote the curve defined by the ellipse $x^2 + \frac{y^2}{9} = 1$, traversed clockwise. Use Green's theorem to evaluate $\int_C F \cdot dr$.



By Green's theorem, since C is negatively oriented w.r.t. D,

$$\int_C F \cdot dr = -\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

$$= -\iint_D (-3 - 2x^2)\, dA$$

$$= \iint_D (2x^2 + 3)\, dA.$$

Use the transformation $x = u\cos(v)$, $y = 3u\sin(v)$, so

$x^2 + (y/3)^2 \le 1 \longrightarrow u^2 \le 1$, or $u \le 1$, $(v \in [0, 2\pi))$.

The Jacobian is $\begin{pmatrix} \cos v & -u\sin(v) \\ 3\sin(v) & 3u\cos(v) \end{pmatrix}$, so $|\det(J)| = 3u$.

Thus
$$\iint_D (2x^2+3)\, dA = \int_0^{2\pi} \int_0^1 (2u^2\cos^2(v) + 3) \cdot 3u \, du\, dv$$

$$= \int_0^{2\pi} \left(\frac{6}{4} u^4 \cos^2(v) + \frac{9}{2} u^2 \Big|_0^1 \right) dv$$

$$= \int_0^{2\pi} \left(\frac{3}{2} \cos^2(v) + \frac{9}{2} \right) dv = \frac{3\pi}{2} + 9\pi = \boxed{\frac{21\pi}{2}}.$$