MATH 303 Practice Problems - Week 4

February 10, 2021

Problem 1

We consider a polygon with 5 vertices, labeled clockwise from 1 to 5, and we define a Markov chain $(X_n)_{n\geq 0}$ on the vertices as follows: at each turn n, one rolls an unbiased dice with 6 sides and move clockwise a number of steps equal to the outcome of the dice. For example, if X_n is at 1 and one rolls 1, 2, 3, 4, 5, or 6, then X_{n+1} is 2, 3, 4, 5, 1, 2, respectively.

1. Write the transition matrix of X_n.
2. Is the chain ergodic?
3. What is the mean number of turns it takes to re-visit a given state?

Problem 2

We consider the Markov chain (X_n) on the state space $\{1, 2, 3, 4\}$ with transition matrix

$$P = \begin{pmatrix} 0 & 1/2 & 1/4 & 1/4 \\ 1/4 & 0 & 1/2 & 1/4 \\ 1/4 & 1/4 & 0 & 1/2 \\ 1/2 & 1/4 & 1/4 & 0 \end{pmatrix}.$$

1. Check that the uniform distribution on $\{1, 2, 3, 4\}$ is stationary for this chain. In the other questions, we will assume that X_0 is picked according to this distribution.
2. Compute $\mathbb{P}(X_{N-1} = 1 | X_N = 2)$.
3. Compute $\mathbb{P}(X_{N-2} = 3 | X_N = 4)$.

Problem 3

We fix $p_A, p_B, p_C > 0$ such that $p_A + p_B + p_C = 1$. Alice owns 3 books entitled A, B and C, that she keeps arranged in a pile. Each day, she reads one of the books at random and put it back at the top of the pile, without to touch the other two. She chooses the book A with probability p_A, book B with probability p_B and book C with probability p_C.
We denote by X_n the order of the pile on the n-th day (for example $X_n = ABC$ if A is at the top and C at the bottom).
1. Give the transitions of the Markov chain (X_n).
2. Show that (X_n) is irreducible and ergodic.
3. What is the limiting probability that $X_n = ABC$? Is the chain reversible? *Hint*: first compute $\pi_{BAC} + \pi_{BCA}$.

Problem 4

Bob gambles in the following way: he starts with $i \geq 0$ dollars. At each step, he wins a dollar with probability $\frac{1}{3}$ and loses a dollar with probability $\frac{2}{3}$. However, if he has 0 dollar and loses, he stays at 0 dollar and can keep gambling (i.e. Bob cannot have debts). For example, if Bob has one dollar, loses twice and then wins, then he will have 1 dollar again. We are interested in the Markov chain (X_n) describing the fortune of Bob at time n.
1. Give the transitions of (X_n).
2. Find, with proof, the limiting probability that Bob owns i dollars at time n.

Problem 5 (Jupyter Notebooks)

1. Review the description of the Metropolis Hastings Algorithm in the Jupyter Notebook, and run the algorithm for any continuous target distribution of your choice (e.g. exponential, gamma, beta, Weibull, or a mixture of them etc.). Compare the histogram of the sampled trajectory to the theoretical distribution.

2. Edit the script to count the proportion of rejected sample (cf. step 6). Run the Metropolis Algorithm given in the Notebook for the mixture of standard normal distributions with pdf $f(x) \propto \exp\left(\frac{x^2}{2}\right)+\exp\left(\frac{(x-10)^2}{2}\right)^2$, where $n > 1$. Run the algorithm with different values of σ^2 (the variance associated with $q \sim \mathcal{N}(0, \sigma^2)$), or using $q \sim Unif([-a, a])$, with different values of $a > 0$. Can you observe differences in rejected samples?