Homework 7

Thursday, July 13, 2023 5:23 PM

(Textbook exercise 2.49) Let X be a discrete random variable with possible values {0, 1, 2, . . . } and the following probability mass function: P(X = 0) = 4/5 and for k ∈ {1, 2, 3, . . . }, P(X = k) = 1/10 · (2/3)^k.
(a) Verify that the above is a probability mass function.
(b) For k ∈ {1, 2, . . . }, find P(X ≥ k |X ≥ 1).

a)

$$F_X(x) = \begin{cases} 4/5, & x=0 \\ \frac{1}{10}\cdot\left(\frac{2}{3}\right)^x, & x \geq 1 \quad (x \in \mathbb{N}) \\ 0, & \text{otherwise} \end{cases}$$

$$1 = \frac{4}{5} + \sum_{k=1}^{\infty} \frac{1}{10}\left(\frac{2}{3}\right)^k$$

$$= \frac{4}{5} + \frac{1}{10}\sum_{k=1}^{\infty}\left(\frac{2}{3}\right)^k$$

$$= \frac{4}{5} + \frac{(1/10)(2/3)}{1-2/3}$$ } simplifying geometric series to $\frac{a}{1-r}$ where $a = \frac{1}{10}\cdot\frac{2}{3}$ and $r = 2/3$:)

$$= 4/5 + \frac{\frac{1}{15}}{1/3}$$

$$= 4/5 + 1/5$$

$$1 = 1$$

LHS = RHS

∴ Since the LHS = RHS = 1, this means this is a valid probabilities since the sum of all the probabilities must equal 1. Therefore the PMF is valid

b)

$$P(X \geq k \mid X \geq 1) = \frac{P(X \geq k \cap X \geq 1)}{P(X \geq 1)}$$

$$= \frac{P(X \geq k)}{P(X \geq 1)}$$

$$= \frac{\sum_{x=k}^{\infty}\frac{1}{10}\left(\frac{2}{3}\right)^x}{\sum_{i=1}^{\infty}\frac{1}{10}\left(\frac{2}{3}\right)^i}$$ } Sum calculated above

$$= \frac{\frac{\frac{1}{10}(2/3)^k}{1-2/3}}{1/5}$$

$$= \frac{\frac{1}{10}\left(\frac{2}{3}\right)^k\cdot 3}{1/5}$$

$$= \frac{\frac{1}{10}\left(\frac{2}{3}\right)^k \cdot 3}{1/5}$$

$$= \frac{3}{\cancel{10}_{2}} \left(\frac{2}{3}\right)^k \cdot \cancel{5}$$

$$= \frac{3}{2}\left(\frac{2}{3}\right)^k$$

$$P(X \geq k \mid X \geq 1) = \left(\frac{2}{3}\right)^{k-1}$$

$\therefore$ The probability is $\left(2/3\right)^{k-1}$