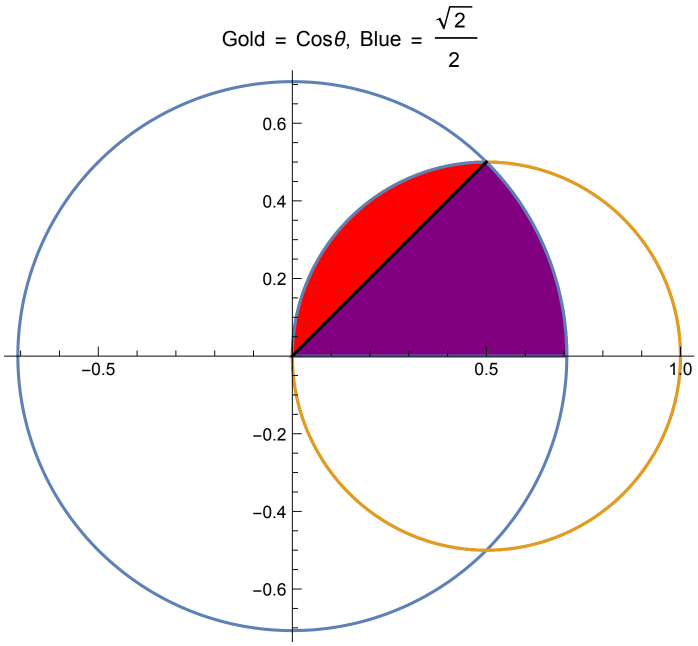
▪ 15.4 Polar Integration Example

We want to integrate the function $z = (x^2 + y^2)^{3/2}$ over the region D between the polar functions $r = \cos(\theta)$ and $r = \frac{\sqrt{2}}{2}$. There are many ways to do the integral. I think the simplest is to use the symmetry of the region, and of the function $(x^2 + y^2)^{3/2}$, and just integrate over the top half, i.e. the red and purple parts:

Out[33]=



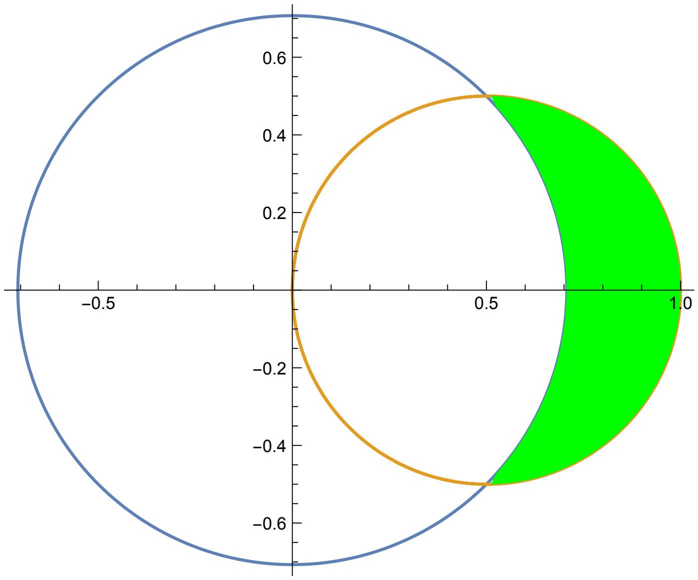
To do so, we split into two integrals in polar coordinates, to the left and right of the black line. The black line is at angle $\frac{\pi}{4}$. The purple region is bounded by the radial curves $r = 0$, $r = \frac{\sqrt{2}}{2}$, while the red region is bounded by the radial curves $r = 0$, $r = \cos\theta$. So we get

$$\iint_D (x^2 + y^2)^{3/2}\, dA = 2\left(\int_0^{\pi/4}\int_0^{\sqrt{2}/2} r^3\, r\, dr\, d\theta + \int_{\pi/4}^{\pi/2}\int_0^{\mathrm{Cos}\theta} r^3\, r\, dr\, d\theta\right) = \frac{16}{75} - \frac{43}{150\sqrt{2}} + \frac{\pi}{40\sqrt{2}} \approx .066165.$$

Evaluating this integral comes down to the substitution $u = \sin\theta$, as we saw in class.

Another way would be to integrate over the interior of gold circle $r = \cos\theta$, and subtract the integral over the green region (as below):

Out[83]=



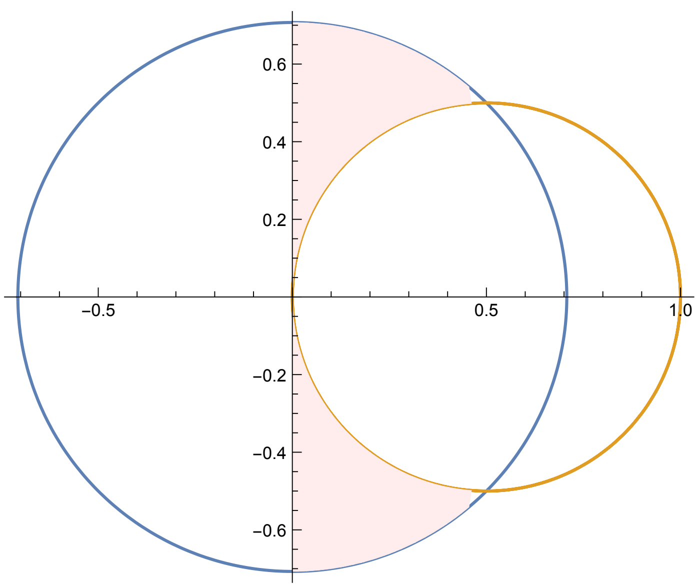
This integral looks like:

$$\iint_D (x^2+y^2)^{3/2}\,\mathrm{dA} = 2\int_0^{\pi/2}\int_0^{\mathrm{Cos}\theta} r^3\, r\,\mathrm{d}r\,\mathrm{d}\theta - \int_{-\pi/4}^{\pi/4}\int_{\sqrt{2}/2}^{\mathrm{Cos}\theta} r^3\, r\,\mathrm{d}r\,\mathrm{d}\theta \approx .066165.$$

Note that I've used the symmetry of the function $z = r^3$, and of the region, to do the first integral.

Yet another way would be to integrate over the right half of the blue circle $r = \frac{\sqrt{2}}{2}$, and subtract the integral over the pink part (as below):

Out[90]=



This would give the integral

$$\iint_D (x^2 + y^2)^{3/2}\, \mathrm{dA} = \int_{-\pi/2}^{\pi/2} \int_0^{\sqrt{2}/2} r^3\, r\, \mathrm{dr}\, \mathrm{d}\theta - 2 \int_{\pi/4}^{\pi/2} \int_{\mathrm{Cos}\theta}^{\sqrt{2}/2} r^3\, r\, \mathrm{dr}\, \mathrm{d}\theta \approx .066165.$$