Math 308 E - Spring 2018
Final exam
Thursday, June 7th, 2018

Name: ______________________________

Problem 1	15	
Problem 2	15	
Problem 3	20	
Problem 4	15	
Problem 5	15	
Problem 6	20	
Total	100	

- There are 6 questions on this exam. Make sure you have all six.
- Always explain your reasoning clearly and concisely.
- Any student found engaging in academic misconduct will receive a score of 0 on this exam.
- You have 110 minutes to complete the exam.

GOOD LUCK!

1. (15 points) Let $\mathcal{B} = \{u_1, u_2, u_3, u_4\} \subset \mathbb{R}^4$ be a basis for $\mathbb{R}^4$, and let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be a linear transformation satisfying

$$T(u_1) = T(u_3) = T(u_4) = 0, T(u_2) \neq 0.$$

(a) Find a basis for $\mathrm{Ker}(T)$.

(b) Is $S = \{v \in \mathbb{R}^4 : T(v) \neq \vec{0}\} \cup \{\vec{0}\}$ a subspace of $\mathbb{R}^4$? (In words, S is the set of four-dimensional vectors v such that either $T(v) \neq 0$ or $v = 0$.) Justify your answer.

(c) Is T one-to-one/onto/invertible? Justify your answer.

2. (15 points) Let A be any square matrix, and let n be any positive integer.

(a) Simplify the matrix product

$$(A - I)(A^{n-1} + A^{n-2} + \cdots + A + I).$$

(b) Assuming that $A^{n-1} + A^{n-2} + \cdots + A + I$ is not invertible, use your result from part (a) to show that 1 is an eigenvalue of A^n.

3. (20 points) Consider the subspace of $\mathbb{R}^3$ given by

$$S = \operatorname{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}.$$

(a) Find a basis for $\mathbb{R}^3$ that contains the two given vectors above.

(b) Find the change of basis matrix U from your basis to the standard basis on $\mathbb{R}^3$.

(c) Use your results from part (a) and (b) to construct a 3×3 matrix A such that $\mathrm{Null}(A) = S$. (Hint: A can be written as $B \cdot U$ for some matrix B.)

4. (15 points) For each matrix below, diagonalize it or explain why it is not diagonalizable.

(a) $A = \begin{bmatrix} 0 & 0 & 1 \\ 2 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

(b) $B = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

5. (15 points) Suppose A is a 3×3 matrix with eigenvalues -1 and 4, and corresponding eigenvectors

$$v_{-1} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, v_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, w_4 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

(The vector v_1 is an eigenvector for the eigenvalue $\lambda = -1$, and v_4, w_4 are both eigenvectors for the eigenvalue $\lambda = 4$.)

(a) Find $A^7 \cdot (-2v_{-1})$, $A^3 \cdot (w_4 + v_4)$, and $A \cdot (v_{-1} - 2w_4)$.

(b) Show that A is invertible, and find $A^{-1} \cdot w_4$.

(c) What is the rank of $A - I_3$?

6. (20 points) Consider two 'probability vectors' $p = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$ and $q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$, where the entries satisfy

$$p_1 + p_2 = q_1 + q_2 = 1, \text{ and } 0 \le p_1, p_2, q_1, q_2 \le 1.$$

Suppose that we have two biased coins, where one coin is heads with probability p_1 and tails with probability p_2, and the second coin is heads with probability q_1 and tails with probability q_2.

(a) Assume that the probability that both coins have the same result (both heads or both tails) is 1/2, and so is the probability of the coins having different results (one heads and the other tails). This is the same as saying

$$p_1 q_1 + p_2 q_2 = \frac{1}{2}$$

and

$$p_2 q_1 + p_1 q_2 = \frac{1}{2}$$

Now suppose p_1 and p_2 are fixed numbers, and q_1, q_2 are variables. Find a matrix A made up of p_1's and p_2's so that the above equations are equivalent to the matrix equation $A \cdot q = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$.

(b) Show that your matrix A is invertible if $p_1 \neq p_2$, and find the inverse matrix.

(c) Solve the equation $A \cdot q = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$ for q by multiplying both sides by A^{-1}.

(d) What can you conclude about the second coin?