Math 303 - Examples for Midterm Review

March 23

- Look at announcements
- Focus on:
  - Branching processes
  - Exp. random variables
  - Poisson processes

- Look at notebooks
- HW problems
- Exercises & Examples in lecture

Example ①: (Branching process)

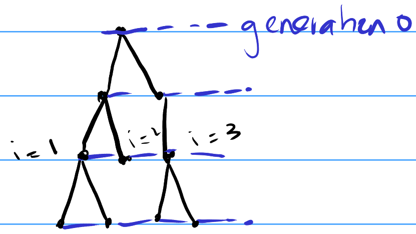
Assume we have a branching process with offspring distribution given by

$$p_0 = \frac{1}{6}$$
$$p_1 = \frac{1}{2}$$
$$p_3 = \frac{1}{3}$$

$p_i :=$ probability of an individual having i offspring

Z_n = size of the population at generation n

(a) $\mathbb{E}[Z_9]$



$$Z_n = \sum_{i=1}^{Z_{n-1}} X_{n-1,i}$$

where $X_{n-1,i}$ = number of offspring of individual i in generation $n-1$

iid with law of $X \sim$ offspring distribution

$$\mathbb{E}[Z_n] = \mathbb{E}\left[\sum_{i=1}^{Z_{n-1}} X_{n-1,i}\right]$$

$$= \mathbb{E}[Z_{n-1} X] \quad \text{by identically dist.}$$

$$= \mathbb{E}[Z_{n-1}]\,\mathbb{E}[X] \quad \text{by independence}$$

$$\vdots$$

$$= \mathbb{E}[X]^n \quad \text{by induction}$$

where: $\mathbb{E}[X] = 0\cdot P(X=0) + 1\cdot P(X=1) + 2\cdot \cancelto{0}{P(X=2)}$
$\qquad + 3\cdot P(X=3)$

$$= 0\cdot\tfrac{1}{6} + 1\cdot\tfrac{1}{2} + 3\cdot\tfrac{1}{3}$$

$$= \tfrac{3}{2}$$

$$\mathbb{E}[Z_9] = \mathbb{E}[X]^9$$

$$= \boxed{\left(\tfrac{3}{2}\right)^9}$$

In midterm: may use w/out proof: $\mathbb{E}[Z_n] = \mathbb{E}[X]^n$
$\mathrm{Var}(Z_n)$

(b) What is P(extinction)?

Key fact: $\eta = P(\text{extinction})$ is the smallest non-negative root of $G_X(s) = s$

$\uparrow$ generating function for X

Find $G_X(s)$:

$$G_X(s) = \mathbb{E}[s^X]$$

$$= s^0 P(X=0) + s^1 P(X=1) + s^3 P(X=3)$$

$$= 1\,\tfrac{1}{6} + \tfrac{1}{2}s + \tfrac{1}{3}s^3$$

$$= \tfrac{1}{6} + \tfrac{1}{2}s + \tfrac{1}{3}s^3$$

Solve the eq. $G_X(s) = s$ to find roots:

$$G_X(s) = s \iff \tfrac{1}{6} + \tfrac{1}{2}s + \tfrac{1}{3}s^3 = s$$

$$\iff \tfrac{1}{3}s^3 - \tfrac{1}{2}s + \tfrac{1}{6} = 0$$

$$\iff 2s^3 - 3s + 1 = 0$$

$$\iff (s-1)(2s^2 + 2s - 1) = 0$$

$$\iff \quad s=1 \quad \text{or} \quad s = \frac{-2 \pm \sqrt{2^2 - 4(2)(-1)}}{2(2)}$$

$$= 1\left(-\tfrac{1}{2} \pm \sqrt{3}\right)$$

$$\Rightarrow \eta = 1\left(-\tfrac{1}{2} + \sqrt{3}\right) \quad \leftarrow \text{may stop here}$$

$$\simeq 0.3660$$

(c) Assume we have 5 $\underline{\text{independent copies}}$ of this branching process (equivalently $Z_0 = 5$), then P(extinction) is ?

$\rightarrow BP_i$ indep BP_j, $i \neq j$

$$\begin{aligned}
P(\text{extinction}) &= P(\text{extinction of all 5 copies}) \\
&= P(\text{extinction of } BP_1, \text{ " " } BP_2, \dots, \text{ " " } BP_5) \\
&= P(\text{extinction of } BP_1)\, P(\text{extinction of } BP_2) \\
&\qquad \cdots P(\text{extinction } BP_5) \\
&= P(\text{extinction of } BP_1)^5 \\
&= \eta^5 \\
&\simeq 0.366^5
\end{aligned}$$

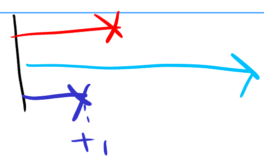
Remarks:

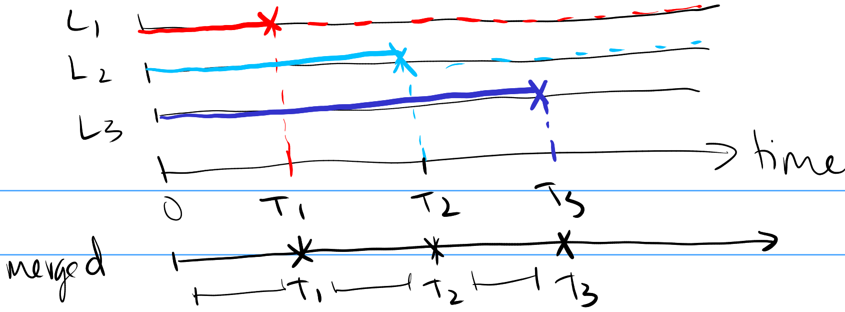
- Manipulate generating functions
- Use independence
- Variations in HW & Notebooks

Exercise: P(extinction at generation n)

Example ②:

Assume we have three lightbulbs with each having a lifetime which is exponentially dist with param. λ. Consider if we turn on all three lightbulbs at once, then what is the expected time at which the last lightbulb burns out?





T_i := time the ith lightbulb burnt out

L_i := lifetime of lightbulb i

$\mathbb{E}[\max\{L_1, L_2, L_3\}]$

we have three independent poisson process where $L_i \sim Exp(\lambda)$

Use ideas from superposition:

$$\mathbb{E}[T_1] = \frac{1}{3\lambda} = \frac{1}{(\lambda+\lambda+\lambda)}$$

$T_1 = \min\{L_1, L_2, L_3\}$
$\Rightarrow T_1 \sim Exp(3\lambda)$

$$\mathbb{E}[T_2 - T_1] = \frac{1}{2\lambda}$$ $\leftarrow$ $\mathbb{E}$[interarrival time of Poisson process $\bar{w}$ intensity 2λ] $\sim Exp(2\lambda)$

$$\mathbb{E}[T_3 - T_2] = \frac{1}{\lambda}$$

$$\mathbb{E}[T_3] = \mathbb{E}[T_1 + (T_2 - T_1) + (T_3 - T_2)]$$
$$= \mathbb{E}[T_1] + \mathbb{E}[T_2 - T_1] + \mathbb{E}[T_3 - T_2]$$
$$= \frac{1}{3\lambda} + \frac{1}{2\lambda} + \frac{1}{\lambda}$$

Example (3):

- Shocks occur according to a Poisson process $\bar{w}$ rate λ.
- Each shock causes the system to fail with prob p.
- Let T := time the system fails
  N := number of shocks taken by the system (between $[0, T]$)

(a) What is the conditional distribution of T given $N = n$, $n \geq 0$.

$T \mid \{N = n\}$ has the same distribution as the nth arrival time of the Poisson process.

$\Rightarrow T \mid \{N = n\} \sim Gamma(n, \lambda)$

(b) Calculate the conditional distribution of N given $T = t$, $t \geq 0$

Let $n \in \mathbb{N}$

$$P(N = n \mid T = t) = \frac{P(N = n, T = t)}{P(T = t)}$$

$f_{T \mid N=n}(t)$

$$= \frac{P(T=t \mid N=n)\, P(N=n)}{P(T=t)} \leftarrow f_T(t)$$

$$= \frac{\lambda^n t^{n-1} e^{-\lambda t}}{(n-1)!} \frac{(1-p)^{n-1} p}{f_T(t)}$$

(Remark: common trick to find normalization constant)

$$= C \frac{(\lambda t)^{n-1}}{(n-1)!} (1-p)^{n-1}$$ where C is constant wrt n

Know. $\sum_{n=1}^{\infty} P(N=n \mid T=t) = 1$

$$\cdot \sum_{n=1}^{\infty} \frac{(\lambda t)^{n-1}}{(n-1)!} (1-p)^{n-1} = \sum_{n=1}^{\infty} \frac{(\lambda t(1-p))^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \frac{(\lambda t(1-p))^n}{n!}$$

$$= e^{\lambda t(1-p)}$$

$$\Rightarrow \quad C \cdot e^{\lambda t(1-p)} = 1$$

$$\Rightarrow \quad C = e^{-\lambda t(1-p)}$$

$$P(N=n \mid T=t) = e^{-\lambda t(1-p)} \cdot \left(\frac{(\lambda t)^{n-1}}{(n-1)!} (1-p)^{n-1} \right)$$

Remark: Should know by heart:

- Exponential distribution
- Gamma (sum of indep. exp. r.v.s) distribution.
- Poisson distribution

Observe $P(N=n \mid T=t) = P(M(t) = n-1)$

where $M(t) \sim \text{Poisson}(\lambda(1-p)t)$

(c) Find the probability dist. in (b) with no calculations;

Remember properties of Poisson process:

- memoryless
- stationarity
- superposition
- thinning
- conditional arrival times
- more corollaries of these
- alternative definition

Consider that shocks arriving are 1 of 2 types:

- Type 1 - causes failure $N_1(t) \sim \text{Poisson}(\lambda p t)$
- Type 2 - does not cause failure $N_2(t) \sim \text{Poisson}(\lambda(1-p)t)$

Thinning

$$
\begin{aligned}
P(N=n \mid T=t) &= P(N_1(t)+N_2(t)=n \mid N_1(t)=1) \\
&= P(N_2(t)=n-1 \mid N_1(t)=1) \\
&= P(N_2(t)=n-1) \\
&= e^{-\lambda(1-p)t}\,\frac{(\lambda(1-p)t)^{n-1}}{(n-1)!}
\end{aligned}
$$

Will post one more example on Queing processes

- uses conditioning on what next event will be (analogy to discrete time Markov chain arguments)