MATH 303 Practice Problems Solutions - Week 7

March, 2021

Problem 1

See solution from Week 6 homework problems.

Problem 2

1. $\mathbb{P}(X_1 = \min(X_i, 1 \le i \le 3)) = \mathbb{P}(X_1 < \min(X_2, X_3))$.
From class, $\min(X_2, X_3) \sim Exp(\lambda_2 + \lambda_3)$ and is independent of X_1, so from class again, we obtain

$$\mathbb{P}(X_1 = \min(X_i, 1 \le i \le 3)) = \frac{\lambda_1}{\lambda_1 + \lambda_2 + \lambda_3}.$$

Similarly, since we know from class that the minimum of two independent Exp. r.v.'s of parameters λ_1 and λ_2 is $Exp(\lambda_1 + \lambda_2)$, we can generalize for n r.v.'s so $\sum_{i=1}^n Exp(\lambda_i) \sim Exp(\sum_{i=1}^n \lambda_i)$.
2. $\mathbb{P}(R_1 \text{ wins}) = \mathbb{P}(T_1 = \min(T_i, 1 \le i \le 8)) = \frac{1}{1+2+3+4+5+6+7+8} = \frac{1}{\frac{9\cdot 8}{2}} = \frac{1}{36}$ where T_i is the time of runner i, by the same reasoning as in 1. So $\mathbb{E}[R_1\text{'s gain}] = 10 \cdot \frac{1}{36} - 1\left(1 - \frac{1}{36}\right) = \frac{25}{36}[\$]$.
3. The winning time T is $Exp(36)$-distributed from 1., so conditioning on T, the expectation of the gain of X is

$$\begin{aligned}
\mathbb{E}[X] &= \int_0^\infty \mathbb{E}[X|T=t] f_T(t)\, dt = \int_0^\infty \mathbb{E}[X|T=t] 36 e^{-36t}\, dt \\
\mathbb{E}[X|T=t] &= e^{-\alpha t} \cdot \mathbb{P}(R_1 \text{ wins}) + 0 \cdot \mathbb{P}(\{R_1 \text{ wins}\}^C) = \frac{e^{-\alpha t}}{36}.
\end{aligned}$$

This means

$$\mathbb{E}[X] = \int_0^\infty \frac{e^{-(\alpha+36)t}}{36} 36\, dt = \frac{1}{\alpha + 36}.$$

Problem 3

1. The time where the first customer leaves is the minimum of 5 independent $Exp(\lambda)$ variables, so it is $Exp(5\lambda)$, so it has expectation $\frac{1}{5\lambda}$ and variance $\left(\frac{1}{5\lambda}\right)^2 = \frac{1}{25\lambda^2}$. Moreover, by the memoryless property, conditionally on this time, the time between when the first and the second customer leave is a minimum of 4 independent $Exp(\lambda)$ variables, so it is $Exp(4\lambda)$, so it has expectation $\frac{1}{4\lambda}$ and variance $\frac{1}{16\lambda^2}$.
Therefore, the mean time it takes for the first two customers to leave is

$$\frac{1}{5\lambda} + \frac{1}{4\lambda} = \frac{9}{20\lambda}.$$

Moreover, the conditional law of the second part does not depend on the first one, so both exponential variables are independent, so we can add the variances. Hence, the variance we are looking for is

$$\frac{1}{25\lambda^2} + \frac{1}{16\lambda^2} = \frac{41}{400\lambda^2}.$$

2. By the same reasoning as above once two customers have left, the time before the third one leaves is $Exp(3\lambda)$ and the time between when the third and the fourth one leave is $Exp(2\lambda)$, so we get

$$t + \frac{1}{3\lambda} + \frac{1}{2\lambda} = t + \frac{5}{6\lambda}.$$

Problem 4

1. The time where Merlin and Emmanuel leave are independent exponential variables with rates λ_1 and λ_2, so the probability that Merlin leaves first is $\frac{\lambda_1}{\lambda_1+\lambda_2}$. In this case, Becca goes to server 1 first. Moreover, in that case, by the memoryless property, the service times of Becca and Emmanuel are again independent exponential variables with parameters λ_1 and λ_2. Therefore, the conditional probability that Becca is done with server 1 before Emmanuel leaves is $\frac{\lambda_1}{\lambda_1+\lambda_2}$, so the answer is $\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^2$.

2. We can decompose this time into five parts (which do not necessarily occur in the order we present them):

- The waiting time before either Merlin or Emmanuel leaves. This is an exponential variable with parameter $\lambda_1 + \lambda_2$, so it has expectation $\frac{1}{\lambda_1+\lambda_2}$.
- The service time for Becca at server 1, which has expectation $\frac{1}{\lambda_1}$.
- The service time for Becca at server 2, which has expectation $\frac{1}{\lambda_2}$.
- The waiting time between server 1 and server 2 if Emmanuel is not done yet. By the previous question, this time has probability $\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^2$ to exist. If it does, it is $Exp(\lambda_2)$, so it has conditional expectation $\frac{1}{\lambda_2}$, so the expectation is $\frac{1}{\lambda_2}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^2$.
- The waiting time between server 2 and server 1 if Merlin is not done yet. By the same computation as in the last item, this has expectation $\frac{1}{\lambda_1}\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^2$.

Therefore, the expected total spent by Becca is

$$\frac{1}{\lambda_1+\lambda_2} + \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_2}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^2 + \frac{1}{\lambda_1}\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^2 = \frac{2(\lambda_1^2+\lambda_1\lambda_2+\lambda_2^2)}{\lambda_1\lambda_2(\lambda_1+\lambda_2)}$$

Problem 5

1. The generating function of Z_n is

$$\begin{aligned}
G_{Z_n}(s) &= G_{I_n+\sum_{i=1}^{Z_{n-1}} X_{n-1,i}} \\
&= G_I(s) \cdot G_{\sum_{i=1}^{Z_{n-1}} X_{n-1,i}}(s) = G_I(s) \cdot G_{Z_{n-1}}(G_X(s)) \\
&= G_I(s) \cdot G_I(G_X(s)) \cdot G_{Z_{n-2}}(G_X(G_X(s)))
\end{aligned}$$

and, by induction,

$$G_{Z_n}(s) = \left(\prod_{i=0}^{n-1} G_I(G_X^{(i)}(s))\right) \cdot G_X^{(n)}(s).$$

2. If $X \sim Binomial(1,p)$, $G_X(s) = \mathbb{E}(s^X) = 1-p+ps$ and if $I \sim Poisson(\mu)$, $G_I(s) = e^{-\mu(1-s)}$. Further,

$$\begin{aligned}
G_X^{(0)}(s) &= s, \\
G_X^{(1)}(s) &= 1-p+ps, \\
G_X^{(2)}(s) &= 1-p+p(1-p+ps) = 1-p+p-p^2+p^2s = 1-p^2+p^2s,
\end{aligned}$$

and, by induction, $G_X^{(i)}(s) = 1 - p^i + p^i s, i \geq 0$.
Hence, using 1.,

$$
\begin{aligned}
G_{Z_n}(s) &= \left(\prod_{i=0}^{n-1} e^{-\mu(1-1+p^i-p^i s)}\right)(1-p^n+p^n s) \\
&= e^{-\mu(1-s)\sum_{i=0}^{n-1} p^i}(1-p^n+p^n s) \\
&= e^{-\mu(1-s)\frac{1-p^n}{1-p}}(1-p^n+p^n s).
\end{aligned}
$$

3. $\mathbb{E}(Z_n) = G'_{Z_n}(1)$ and $Var(Z_n) = G''_{Z_n}(1) + G'_{Z_n}(1) - (G'_{Z_n}(1))^2$. From 2.,

$$G'_{Z_n}(s) = \frac{\mu(1-p^n)}{1-p} e^{-\mu(1-s)\frac{1-p^n}{1-p}}(1-p^n+p^n s) + e^{-\mu(1-s)\frac{1-p^n}{1-p}} p^n$$

and

$$G''_{Z_n}(s) = \left(\frac{\mu(1-p^n)}{1-p}\right)^2 e^{-\mu(1-s)\frac{1-p^n}{1-p}}(1-p^n+p^n s) + 2\frac{\mu(1-p^n)}{1-p} e^{-\mu(1-s)\frac{1-p^n}{1-p}} p^n,$$

so $\mathbb{E}(Z_n) = \frac{\mu(1-p^n)}{1-p} + p^n$ and

$$Var(Z_n) = \left(\frac{\mu(1-p^n)}{1-p}\right)^2 + 2\frac{\mu(1-p^n)}{1-p}p^n + \frac{\mu(1-p^n)}{1-p} + p^n - \left(\frac{\mu(1-p^n)}{1-p}\right)^2 - 2\frac{\mu(1-p^n)}{1-p}p^n - p^{2n}.$$

Therefore, $Var(Z_n) = \frac{\mu(1-p^n)}{1-p} + p^n(1-p^n) = (\frac{\mu}{1-p} + p^n)(1-p^n)$.
4. When $n \to \infty$, $G_{Z_n}(s) \to e^{-\mu\frac{1-s}{1-p}}$ which is the generating function of $Poisson\left(\frac{\mu}{1-p}\right)$. Remark: Using the so-called "continuity theorem" that states that convergence for the generating function implies convergence in distribution, we can show that this is the limiting distribution of the process.