MATH 303 Midterm Exam B

February 11, 2021

INSTRUCTIONS: PLEASE READ ALL CAREFULLY BEFORE STARTING

After inputing your student ID, we recommend to write down the values set by the script. If you show results obtained with values different from those set by the Jupyter notebook script, your question **will not be graded.**

1. **Problems 1-3**: For each problem, you will need to separately assemble a pdf file of handwritten solutions. We recommend to not use more than 2 pages per problem. **Write your name** on top of each page and do not forget to indicate which subquestion (a, b...) you are solving.
2. **Problem 4**: Complete the Problem 4 set in `midtermB_students.ipynb` (there are three questions). Submit the completed notebook as `midtermB_complete.ipynb` (you don't have to finish all the questions to submit).
   Warning: Having "Success" displayed when running the cells does not necessarily mean that the solution is correct (this will be evaluated after you submit the notebook).

Grading

Problems 1-3 will count for $\sim 85\%$ of the grade. We recommend spending an equivalent amount of time on problems 1, 2 and 3.

Problem 1

Consider the transition matrix P obtained from running the notebook with your student ID.

a. Draw the transition diagram associated with P with the states corresponding to their row index in the matrix (i.e. first row correspond to state 1, second to state 2 etc.).
b. Determine all the communication classes (no need to justify).
c. Determine which states are recurrent and which are transient (briefly justify).
d. Determine the period of each state (briefly justify).
e. Consider the state j obtained from running the notebook and assuming $X_0 = j$, what is the probability that the chain gets absorbed to the class of i? (justify your calculation with key steps; answers directly written won't be accepted).

Problem 2

Consider the Markov Chain $(X_n)_{n\geq 0}$ defined on $\mathbb{N} = \{0, 1, \ldots\}$ with transition probabilities obtained from running the notebook with your student ID.

a. Draw the transition diagram.
b. For all $i > 0$, let p_i be the probability that given $X_0 = i$ the chain ever returns at state 0 (i.e. $p_i = P(X_n = 0 \text{ for some } n > 0 \mid X_0 = i)$). By conditioning on the outcome of the first step, find a relation between p_1 and p_2 (justify your answer).
c. We now also assume that $p_2 = p_1^2$ and $p_1 < 1$. Under these assumptions, replace p_2 by p_1^2 in the relation found in **b.** and find the value of p_1.
d. For $i > 1$, show that $p_k = p_{k-1}p_1$ (*hint: If the chain starts at $i > 1$, the chain has to visit state 1 before visiting 0*). In particular, this shows the previous assumption $p_2 = p_1^2$. Deduce the general formula for p_k as a function of k.

Problem 3

Your refrigerator has four cans of soda in it. You only buy two flavors of soda: purple berry and orange blast. At each step, you put your hand in the fridge, remove a soda (selecting one uniformly at random from those in the fridge), drink it, then replace it with a new can of the *opposite* flavor. Let X_n be the number of cans of purple berry in the fridge after you have drunk n cans.

a. Draw the transition diagram for X.

b. Consider the state i obtained from running the notebook. Find $\mathbb{P}(X_2 = i | X_0 = i)$. (justify your calculation; answers directly written won't be accepted).

c. Show that the distribution $\pi = \frac{1}{16}(1, 4, 6, 4, 1)$ is stationary for X_n, and that the process is reversible.

Consider a different Markov chain defined by slightly modifying the soda process: when all the soda in the fridge is the same flavor, you flip a fair coin, drink one can if it's heads and two cans if it's tails, and then replace any cans you drank with new cans of the opposite flavor. Let Y_n be the number of purple berry cans in the fridge after n steps in the modified process.

d. We assume that $\sigma = \frac{1}{17}(1, 4, 7, 4, 1)$ is stationary for Y. Use a result from class to argue that Y_n converges in distribution to σ as $n \to \infty$. Is Y reversible? Justify.

Problem 4

Complete the Problem 4 set in `midterm1B_students.ipynb` (there are three questions).
Submit the completed notebook as `midterm1B_complete.ipynb` (you don't have to finish all the questions to submit).

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