Recall: We established the Markov property:

$\forall n \in \mathbb{N}, \forall (x_0, \dots, x_n) \in S^{n+1}$

$$P(X_{n+1} = x_{n+1} \mid X_0 = x_0, X_1 = x_1, \dots, X_n = x_n)$$
$$= P(X_{n+1} = x_{n+1} \mid X_n = x_n).$$

Rmk: Given this property, we can simulate a realization of a M-C iteratively:

Pseudo-algorithm:

- Set up the initial state x_0
- for i from 1 to n
  randomly generate x_i from x_{i-1}
- Output: $(x_0, x_1, \dots, x_n)$

→ See Jupyter Notebook

Def: • A M-C is homogeneous if $\forall (x, y) \in S^2$, $P(X_{n+1} = x \mid X_n = y)$ is the same for all n.

• By indexing the states $S = \{s_1, \dots, s_i, \dots\}$, We can then define the TRANSITION MATRIX $\underset{\sim}{P}$ of $(X_n)_{n\in\mathbb{N}}$ as $(\underset{\sim}{P})_{ij} = p_{ij}$

$$= P(X_{n+1} = s_j \mid X_n = s_i)$$

- We can equivalently represent a M-C and its transition matrix as a directed graph, where each node represents a state, and s.t. we draw an arrow from x to y if $p_{xy} > 0$, with weight p_{xy}. This graph is called a TRANSITION DIAGRAM

Examples (see Ross textbook, examples 4.1-4.7)

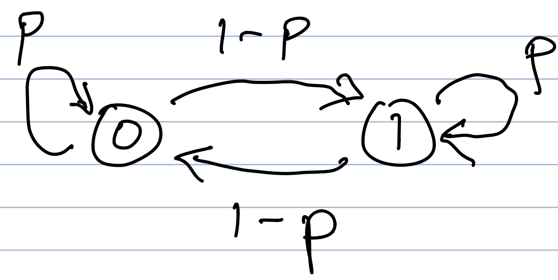
ex 4.2 : Communication system with 2 states 0 and 1; s.t. the message at the next step is $\begin{cases} \text{unchanged is } p & (0<p<1) \\ \text{changed is } 1-p \end{cases}$

$$P(X_{n+1} = 1 \mid X_n = 1) = P(X_{n+1} = 0 \mid X_n = 0) = p$$

$$P(X_{n+1} = 1 \mid X_n = 0) = P(X_{n+1} = 0 \mid X_n = 1) = 1-p$$

TRANSITION MATRIX $\qquad$ TRANSITION DIAGRAM

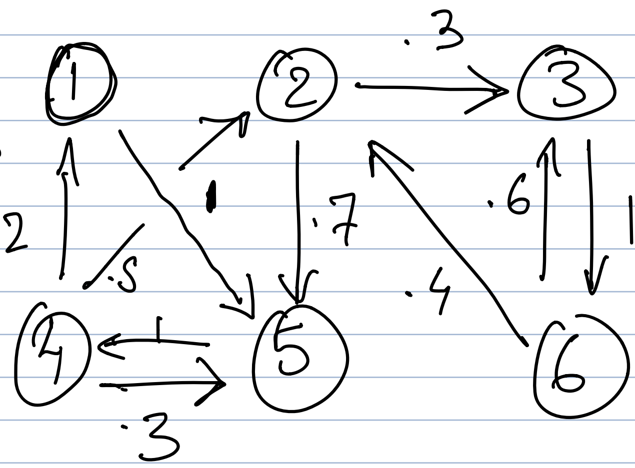
$$\underset{\sim}{P} = \begin{array}{c} \\ \\ \end{array}\begin{array}{cc} 0 & 1 \\ \begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix} \end{array}\begin{array}{c} \\ 0 \\ 1 \end{array} \longleftrightarrow$$



Rmk: As the number of states increases and with few transitions, it is more convenient to use the diagram.

ex:

$$\begin{array}{c|cccccc} & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & .3 & 0 & .7 & 0 \\ 3 & 0 & 0 & 0 & 0 & 0 & 1 \\ 4 & .2 & .5 & 0 & 0 & .3 & 0 \\ 5 & 0 & 0 & 0 & 1 & 0 & 0 \\ 6 & 0 & .4 & .6 & 0 & 0 & 0 \end{array} \longleftrightarrow$$



Properties: Let $\underset{\sim}{P}$ be a transition matrix

(i) $\forall\, i,j \qquad 0 \leq p_{ij} \leq 1$

(ii) $\forall\, i \in S \qquad \sum_j p_{ij} = 1$ ("sum of terms in a row $= 1$")

A matrix that satisfies (i) and (ii) is called a stochastic matrix

Proof: (i) trivial (p_{ij} is a probability)

(ii) $$\sum_j p_{ij} = \sum_j P(X_{n+1}=s_j \mid X_n = s_i)$$

$$\left(P(A\cap B) = P(A|B)\,P(B)\right) \quad = \sum_j \frac{P(X_{n+1}=s_j,\, X_n = s_i)}{P(X_n = s_i)}$$

$\{X_{n+1} = s_k\}_{k\in S}$ forms a complete set of mutually exclusive events, so

$$\sum_j P(X_{n+1}=s_j,\, X_n = s_i) = P(X_n = s_i)$$

$$\Rightarrow \sum_j p_{ij} = \frac{P(X_n = s_i)}{P(X_n=s_i)} = 1 \qquad \boxtimes$$

Rmk:
- When S is infinite, we generalize the concept of a finite matrix to a matrix with infinitely many rows and columns
- Given a transition matrix, one can simulate a realization of the process

Algorithm:
- set up the initial state x_0
- for i from 1 to n

randomly generate X_i from X_{i-1}
by sampling from the multinomial
distribution given by the
X_{i-1}-th row of the transition
matrix

- Output : $(X_0, \dots\dots, X_n)$

$\rightarrow$ see Jupyter Notebook

II. Chapman - Kolmogorov equation

Rmk : • For a M-C $(X_n)_{n\geq 0}$ with a transition matrix $\underset{\sim}{P}$, and if at time n, the distribution of X_n is $\mu = (\mu_1, \dots\dots, \mu_N)$, then the distribution of X_{n+1} is $\mu . \underset{\sim}{P}$,

by definition of the matrix product :

$$P(X_{n+1} = s_j) = \sum_{k \in S} P(X_{n+1} = s_j , X_n = s_k)$$

$$= \sum_{k \in S} \underbrace{P(X_{n+1} = s_j \mid X_n = s_k)}_{P_{kj}} . \underbrace{P(X_n = s_k)}_{\mu_k}$$

$$= \sum_{k \in S} \mu_k \cdot (\underset{\sim}{P})_{kj}$$

$$= (\mu \cdot \underset{\sim}{P})_j$$

- From this remark, if the distribution of $X_1 = \mu \cdot \underset{\sim}{P}$, then the distribution of

$$X_2 = \underbrace{(\mu \cdot \underset{\sim}{P})}_{\text{distrib. of } X_1} \cdot \underset{\sim}{P} = \mu \cdot \underset{\sim}{P} \cdot \underset{\sim}{P} = \mu \cdot \underset{\sim}{P}^2$$

and by recurrence, the distribution of X_n is $\mu \cdot \underset{\sim}{P}^n$

- We will generalize this result in this section

Def: For $n \in \mathbb{N}$, we define the n-step transition probability $P_{ij}^n = P(X_n = j \mid X_0 = i)$, which defines the n-step transition matrix

$$(\underset{\sim}{P}^{(n)})_{ij} = P_{ij}^n$$

Rmk : • $n = 0$: $P_{ij}^{0} = P(X_k = j \mid X_k = i)$

$$= \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

i.e. $\underset{\sim}{P}^{(0)} = Id_S$

• $n = 1$: $\underset{\sim}{P}^{(1)} = \underset{\sim}{P}$ by definition

• One can show (e.g. by induction on n, and because we consider a homogeneous M-C) that

$$\forall k \quad P(X_{n+k} = j \mid X_k = i) = P_{ij}^{n}$$

(stationarity of the process)

• If we know $\underset{\sim}{P}^{(n)}$ and $\underset{\sim}{P}^{(m)}$, one can get $\underset{\sim}{P}^{(n+m)}$:

Thm : (C-K. equation) : $P_{ij}^{n+m} = \sum_k P_{ik}^{n} \cdot P_{kj}^{m}$

we will prove this in the next lecture but an important consequence is

<u>Lemma</u>: (Matrix formulation):

$$\underset{\sim}{P}^{(n+m)} = \underset{\sim}{P}^{n+m} \leftarrow n+m\text{-th power of } \underset{\sim}{P}$$

$\uparrow$

$n+m$-step transition matrix