Today: We conclude the section on time reversibility and we recall some important content with practice exercises.

⚠ → check comments we left on the files you submitted on Canvas!

Recall: We modeled a pbm with the following MC

$$0 \underset{1-p}{\overset{1}{\rightleftarrows}} 3 \underset{p}{\overset{p}{\rightleftarrows}} 1 \underset{1-p}{\overset{1-p}{\rightleftarrows}} 2 \circlearrowleft p$$

- The chain is ergodic (aperiodic, positive recurrent) and irreducible so we know the chain converges in distrib. to the stationary distrib. π.
- Assume that the MC is time reversible, let's find π.

detailed balance eq.
$$\begin{cases} \pi_i P_{ij} = \pi_j P_{ji} & \forall i \neq j \ (*) \\ \sum_i \pi_i = 1 \end{cases}$$

$$(*) \longrightarrow \begin{cases} \pi_0 \overset{1}{P_{03}} = \pi_3 \overset{1-p}{P_{30}} \\ \pi_3 \overset{p}{P_{31}} = \pi_1 \overset{p}{P_{13}} \\ \pi_1 \underset{1-p}{P_{12}} = \pi_2 \overset{1-p}{P_{21}} \end{cases}$$

②

$$\Longleftrightarrow \begin{cases} \pi_0 = \pi_3 \cdot (1-p) \\ \boxed{\pi_1 = \pi_2 = \pi_3} \end{cases}$$

$$\sum \pi_i = 1 \Rightarrow \pi_3(1-p) + 3\pi_3 = 1$$

$$\Rightarrow \boxed{\pi_3 = \frac{1}{4-p} \quad \text{and} \quad \pi_0 = \frac{1-p}{4-p}}$$

Reciprocally, one can easily verify that these values satisfy the detailed balance equations, so $\boxed{\pi \text{ is the stationary distribution}}$

Another example of time-reversible MC that is also a classical model is the Ehrenfest chain. (1907, Paul & Tatiana)

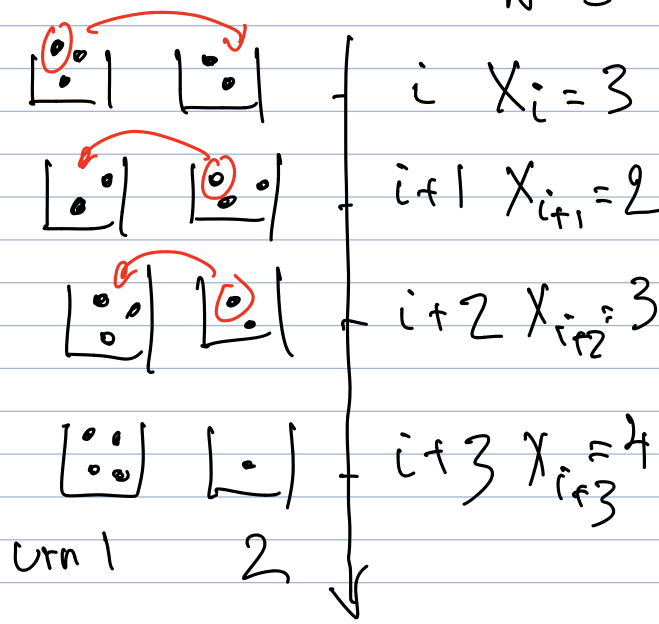
→ Toy model of gaz behaviour in 2 containers

→ We consider N balls distributed in 2 urns

→ At each step: select a ball at random and move it to the other urn.

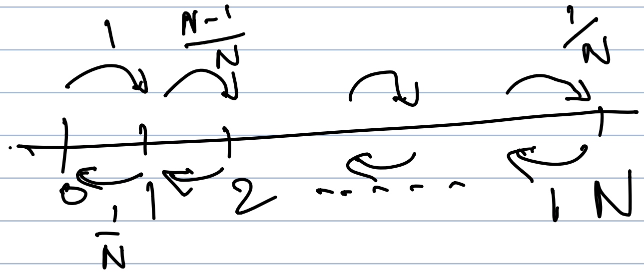
→ let's study $X_n := \#$ balls in urn 1 at time n

$N=5$



$S = \{0, \dots, N\}$

$P_{01} = 1$, $P_{N\,N-1} = 1$



$P_{i\,i+1} = \frac{N-i}{N}$ $i \neq 0$

$P_{i\,i-1} = \frac{i}{N}$ $i \neq N$

→ Period: 2 → Irreducibility: Y or N

⇒ the MC is positive recurrent and has a stationary distribution.

Q: what is the stationary distribution

A: We will "guess and check"

→ Guess a solution:

* If we track a single ball in the long run how much fraction of time should it spend in the first urn? → $\frac{1}{2}$

* At stationarity → # balls in urn 1

$\shortparallel$

heads in N independent unbiased coin flips

$\shortparallel$

$$\text{Bin}\left(N, \frac{1}{2}\right)$$

Our guess $\pi_m = P(X = m) = \binom{N}{m} \times \frac{1}{2^N}$

→ Check : we'll check that π satisfies detailed balance.

Rmk: Our intuition that the process is time reversible comes from looking at the transition diagram and noticing that, at stationarity

0 only receives jump from 1 and gives jumps to 1

so fluxes between 0 and 1 are equal,

then the same applies for 1 and 2 etc.

→ $\pi_0 P_{01} = \frac{1}{2^N} \times 1$ $\qquad$ $\pi_1 P_{10} = \frac{\cancel{N}}{2^N} \times \frac{1}{\cancel{N}}$

$\Rightarrow \pi_0 P_{01} = \pi_1 P_{10}$

→ $\pi_N P_{N\,N-1} = \frac{1}{2^N} \times 1$ $\qquad$ $\pi_{N-1} \times P_{N-1\,N} = \binom{N}{N-1} \times \frac{1}{2^N} \times \frac{1}{N}$

$\Rightarrow \pi_N P_{N\,N-1} = \pi_{N-1} P_{N-1\,N} = \frac{1}{2^N}$

general case (exercise)

show that $\pi_i P_{i\,i+1} = \pi_{i+1} P_{i+1\,i} = \frac{N!}{(N-i-1)!\, i!}$

For the midterm

Check the announcement on Canvas

→ see learning outcomes.

In particular, make sure you practice on how

- to do a "1-step analysis"

  ↳ recursive equation

- to study limiting probabilities and stationary distributions

ex: You collect peaches. You start with 1 and At each step you get one more peach, but every peach that you collected has a probability p to get rotten (so you throw these away)

Let X_n = # peaches left at n^{th} step

Q: Find $E(X_n)$ and $\lim_{n\to+\infty} E(X_n)$ (if it exists)

↳ write a recursive eq. for $E(X_n)$

- $E(X_0) = 1$

- let $n \geq 0$

$$X_{n+1} = 1 + \sum_{i=1}^{X_n} \mathbb{1}\{\text{peach } i \text{ doesn't get rotten}\}$$

We apply expectation on both side

$$\mathbb{E}(X_{n+1}) = 1 + \mathbb{E}\Big(\sum_{i=1}^{X_n} \underbrace{\mathbb{1}\{\dots\}}_{=\begin{cases}1 & \text{w.p. } 1-p\\ 0 & \text{w.p. } p\end{cases}}\Big)$$

$$= 1 + \mathbb{E}\Big(\sum_{i=1}^{X_n} (1-p)\Big)$$

$$= 1 + (1-p)\,\mathbb{E}\Big(\sum_{i=1}^{X_n} 1\Big)$$

$$= 1 + (1-p)\,\mathbb{E}(X_n)$$

$$\boxed{\mathbb{E}(X_{n+1}) = 1 + q\,\mathbb{E}(X_n)}\text{, where } q = 1-p$$

$$\mathbb{E}(X_{n+1}) = 1 + q\big(1 + q\,\mathbb{E}(X_{n-1})\big)$$

$$= 1 + q + q^2\,\mathbb{E}(X_{n-1})$$

$$= 1 + q + q^2\big(1 + q\,\mathbb{E}(X_{n-2})\big)$$

$$= 1 + q + q^2 + q^3\big(\mathbb{E}(X_{n-2})\big)$$

$$(\dots) = 1 + q + q^2 + \dots + q^{n+1} = \sum_{k=0}^{n+1} q^k$$

So $$\mathbb{E}(X_n) = \sum_{k=0}^{n} q^k = \frac{1-q^{n+1}}{1-q}$$

so $\lim_{n\to+\infty} \mathbb{E}(X_n) = \frac{1}{1-q} = \boxed{\frac{1}{p}}$

Ex: We roll a fair dice repeatedly and add up all the numbers we get.

Let S_n = total sum after n rolls

Q: For n large, what is $P(S_n$ is divisible by $7)$? (i.e. what is $\lim_{n\to+\infty} P(7 \mid S_n)$)?

A: Consider the MC $X_n = S_n \mod 7$

State space: $\{0, 1, \dots, 6\}$

$$P = \frac{1}{6}\begin{pmatrix} 0 & & (1) \\ & \ddots & \\ (1) & & 0 \end{pmatrix}$$

→ The matrix is doubly stochastic!

$\Rightarrow \pi = \frac{1}{7}(1, \dots, 1)$ is stationary

so $P(7 \mid S_n) = P(X_n \equiv 0\ [7]) \xrightarrow[n\to+\infty]{} \boxed{\pi_0 = \frac{1}{7}}$