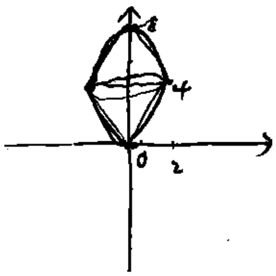
1. (a) (12 points) Evaluate the integral

$$\iiint_E x\,dV,$$

where E is the region between the paraboloids $x = 8 - y^2 - z^2$ and $x = y^2 + z^2$. You should include a picture of E or of any relevant cross section of E.



$$\iiint_E x\,dv$$

$$= \int_0^{2\pi}\int_0^2\int_{r^2}^{8-r^2} xr\,dx\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2 \frac{x^2}{2} r \Big|_{r^2}^{8-r^2} dr\,d\theta$$

$$\cancel{= \int_0^{2\pi}\int_0^2 \frac{r[(8-r^2)^2-r^2]}{2}\,dr\,d\theta}$$

$$= 2\pi\cdot\int_0^2 \frac{r[(8-r^2)^2-r^4]}{2}\,dr$$

$$\cancel{= 2\pi\cdot\int_0^2 \frac{r(r^4-17r^2+64r)}{2}\,dr}$$

$$\cancel{= 2\pi\cdot\int_0^2 \frac{r^5-17r^3+64r^2}{2}\,dr}$$

$$\cancel{= 2\pi\cdot \frac{\frac{r^6}{6}-\frac{17}{4}r^4+\frac{64r^3}{3}}{2}\Big|_0^2}$$

$$\cancel{= \pi\cdot\left(\frac{64}{6}-\frac{16\times17}{4}+\frac{64\times8}{3}\right)}$$

$$\cancel{= \frac{340}{3}\pi}$$

$$= 2\pi\cdot\int_0^2 \frac{r(64-16r^2)}{2}\,dr$$

$$= 2\pi\cdot\int_0^2 32r - 8r^3$$

$$= 2\pi\cdot 16r^2 - 2r^4\Big|_0^2$$

$$= 2\pi\cdot(64-32)$$

$$= 64\pi$$

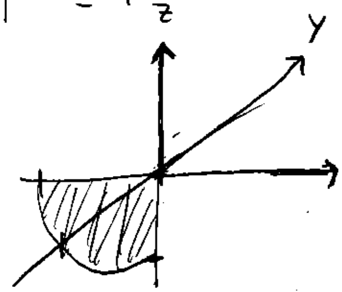
2. (12 points) Use spherical coordinates to evaluate the integral

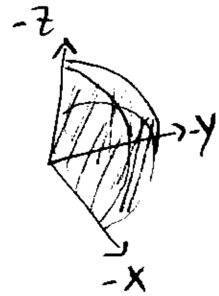
$$\iiint_E y\,dV, \qquad = \rho \sin\theta \sin\phi$$

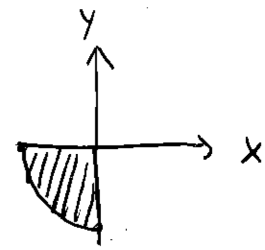
where E is the part of the unit sphere $x^2+y^2+z^2 \le 1$ lying inside the octant $x, y, z \le 0$.

$x^2+y^2+z^2 \le 1$

$\rho^2 \le 1$







$\pi \le \theta \le \frac{3\pi}{2}$

$\rho\cos\phi \le 0$

$0 \le \rho \le 1$ $\qquad \cos\phi \le 0$

$\frac{\pi}{2} \le \phi \le \pi$ $\qquad \sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$

$$\int_{\pi}^{\frac{3\pi}{2}} \int_{\frac{\pi}{2}}^{\pi} \int_0^1 \rho^3 \sin^2\phi \sin\theta \, d\rho \, d\phi \, d\theta$$

$$\int_{\pi}^{3\pi/2} \sin\theta \, d\theta \int_{\pi/2}^{\pi} \sin^2\phi \, d\phi \int_0^1 \rho^3 \, d\rho = \left(-\cos\theta \Big|_{\pi}^{\frac{3\pi}{2}}\right)\left(\tfrac{1}{2}\right)\left(\int_{\pi/2}^{\pi} (1-\cos 2\phi)\,d\phi\right)\left(\tfrac{1}{4}\right)$$

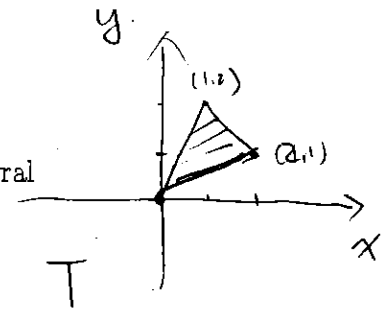
$$= (-1)\left(\tfrac{1}{2}\right)\left(\tfrac{1}{4}\right)\left(\theta - \tfrac{1}{2}\sin 2\phi \Big|_{\pi/2}^{\pi}\right)$$

$$= -\frac{1}{8}\left(\pi - \pi/2\right) = \boxed{-\frac{\pi}{16}}$$

$2y = 2u+4v$

$(y = u+2v)$

3. (12 points) Use the transformation $x = 2u+v, y = u+2v$ to evaluate the integral



$2u+v-2u-4v$

$$\iint_T (x-2y)\,dA,$$

where T is the triangle in the xy plane with vertices $(0,0),(2,1)$ and $(1,2)$. Draw pictures of T and the transformed version of T in the uv plane.

$$\left|\det\begin{vmatrix} dx/du & dx/dv \\ dy/du & dy/dv \end{vmatrix}\right| = \left|\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}\right| = |4-1| = 3.$$

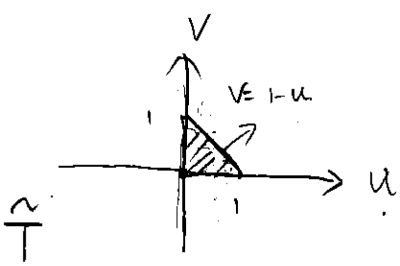
$$x-2y = 2u+v-2(u+2v) = 2u+v-2u-4v = -3v.$$

$$\iint_{\tilde{T}} (-3v)\cdot 3\,du\,dv.$$

$\begin{cases} 2y = 2u+4v \\ x = 2u+v \end{cases} \Rightarrow 3v = 2y-x \Rightarrow v = \dfrac{2y-x}{3}$

$\begin{cases} 2x = 4u+2v \\ y = u+2v \end{cases} \Rightarrow 3u = 2x-y \Rightarrow u = \dfrac{2x-y}{3}$

		coordinate,
$x-y$	$\Rightarrow$	$u \to v$
$(0,0)$		$(0,0)$
$(2,1)$		$(1,0)$
$(1,2)$		$(0,1)$



$$\int_0^1 \int_0^{1-u} (-3v)\cdot 3\,dv\,du$$

$$= -9\int_0^1\int_0^{1-u} v\,dv\,du$$

$$= -9\int_0^1 \frac{1}{2}(1-u)^2\,du$$

$$= -\frac{9}{2}\int_0^1 1-2u+u^2\,du$$

$$= -\frac{9}{2}\left(u-u^2+\frac{1}{3}u^3\Big|_0^1\right)$$

$$= -\frac{9}{2}\left(1-1+\frac{1}{3}\right)$$

$$= -\frac{3}{2}$$

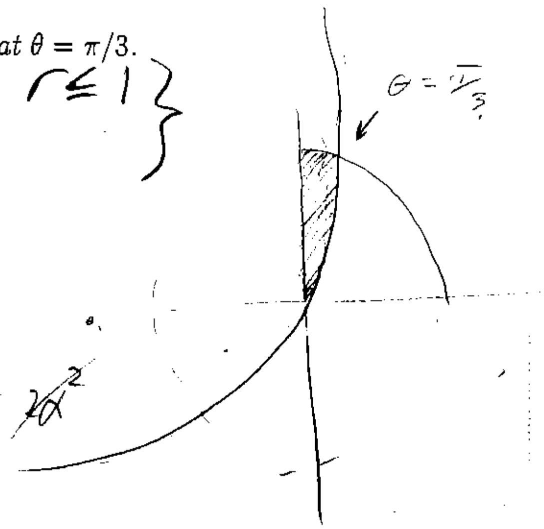
4. (14 points) Consider the region $D \subset \mathbb{R}^2$ bounded by the curves $x, y \geq 0$, $x^2 + y^2 \leq 1$, and

$$(x+\alpha)^2 + (y-\alpha)^2 \leq 2\alpha^2, \text{ where } \alpha = \frac{1+\sqrt{3}}{2}.$$

Draw a picture of D, and give a parameterization of D in polar coordinates (i.e. describe D as a set of r and θ values).

Hint: the intersection of the two circles in the first quadrant occurs at $\theta = \pi/3$.

$$D: \left\{ r, \theta \in \begin{matrix} 0 \leq r \leq 2\alpha(\sin\theta - \cos\theta) \ \& \ r \leq 1 \\ \frac{\pi}{4} \leq \theta \leq \frac{\pi}{2} \end{matrix} \right\}$$



$$(r\cos\theta + \alpha)^2 + (r\sin\theta - \alpha)^2 = 2\alpha^2$$

$$r^2\cos^2\theta + 2\alpha r\cos\theta + \cancel{\alpha^2} + r^2\sin^2\theta - 2\alpha r\sin\theta + \cancel{\alpha^2} = \cancel{2\alpha^2}$$

$$r^2\cancel{(\cos^2\theta + \sin^2\theta)} = 2\alpha r\sin\theta - 2\alpha r\cos\theta$$

$$r = 2\alpha(\sin\theta - \cos\theta)$$

0
$\cos\theta = \sin\theta$
$\theta = \pi/4$

All r values greater than zero, but less than both $2\alpha(\sin\theta - \cos\theta)$ and 1 for $\frac{\pi}{4} \leq \theta \leq \pi/2$

or

r values less than $2\alpha(\sin\theta - \cos\theta)$ for $\frac{\pi}{4} \leq \theta \leq \pi/3$
and r values less than 1 for $\frac{\pi}{3} \leq \theta \leq \pi/2$

Yes!