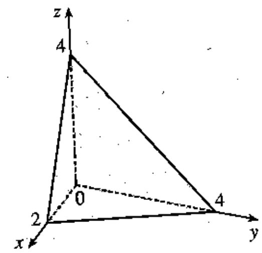


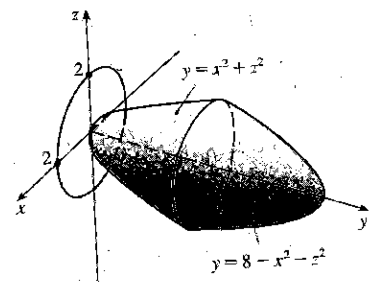
19. The plane $2x + y + z = 4$ intersects the xy-plane when

$2x + y + 0 = 4 \quad\Rightarrow\quad y = 4 - 2x$, so

$E = \{(x, y, z) \mid 0 \le x \le 2, 0 \le y \le 4 - 2x, 0 \le z \le 4 - 2x - y\}$ and

$$V = \int_0^2 \int_0^{4-2x} \int_0^{4-2x-y} dz\, dy\, dx = \int_0^2 \int_0^{4-2x} (4 - 2x - y)\, dy\, dx$$
$$= \int_0^2 \left[4y - 2xy - \tfrac{1}{2}y^2\right]_{y=0}^{y=4-2x} dx$$
$$= \int_0^2 \left[4(4 - 2x) - 2x(4 - 2x) - \tfrac{1}{2}(4 - 2x)^2\right] dx$$
$$= \int_0^2 (2x^2 - 8x + 8)\, dx = \left[\tfrac{2}{3}x^3 - 4x^2 + 8x\right]_0^2 = \tfrac{16}{3}$$

20. The paraboloids intersect when $x^2 + z^2 = 8 - x^2 - z^2 \quad\Leftrightarrow\quad x^2 + z^2 = 4$, thus the intersection is the circle $x^2 + z^2 = 4$, $y = 4$. The projection of E onto the xz-plane is the disk $x^2 + z^2 \le 4$, so $E = \{(x, y, z) \mid x^2 + z^2 \le y \le 8 - x^2 - z^2, x^2 + z^2 \le 4\}$. Let $D = \{(x, z) \mid x^2 + z^2 \le 4\}$. Then using polar coordinates $x = r\cos\theta$ and $z = r\sin\theta$, we have

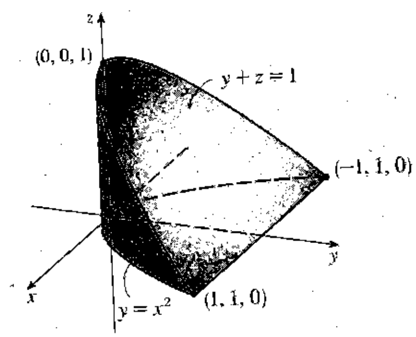


$$V = \iiint_E dV = \iint_D \left(\int_{x^2+z^2}^{8-x^2-z^2} dy\right) dA = \iint_D (8 - 2x^2 - 2z^2)\, dA$$
$$= \int_0^{2\pi} \int_0^2 (8 - 2r^2)\, r\, dr\, d\theta = \int_0^{2\pi} d\theta \int_0^2 (8r - 2r^3)\, dr$$
$$= \left[\theta\right]_0^{2\pi} \left[4r^2 - \tfrac{1}{2}r^4\right]_0^2 = 2\pi(16 - 8) = 16\pi$$

21. The plane $y + z = 1$ intersects the xy-plane in the line $y = 1$, so

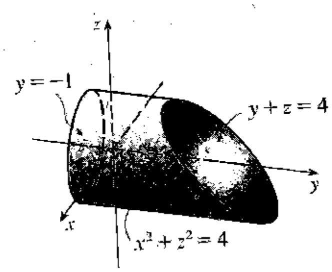
$E = \{(x, y, z) \mid -1 \le x \le 1, x^2 \le y \le 1, 0 \le z \le 1 - y\}$ and

$$V = \iiint_E dV = \int_{-1}^1 \int_{x^2}^1 \int_0^{1-y} dz\, dy\, dx = \int_{-1}^1 \int_{x^2}^1 (1 - y)\, dy\, dx$$
$$= \int_{-1}^1 \left[y - \tfrac{1}{2}y^2\right]_{y=x^2}^{y=1} dx = \int_{-1}^1 \left(\tfrac{1}{2} - x^2 + \tfrac{1}{2}x^4\right) dx$$
$$= \left[\tfrac{1}{2}x - \tfrac{1}{3}x^3 + \tfrac{1}{10}x^5\right]_{-1}^1 = \tfrac{1}{2} - \tfrac{1}{3} + \tfrac{1}{10} + \tfrac{1}{2} - \tfrac{1}{3} + \tfrac{1}{10} = \tfrac{8}{15}$$



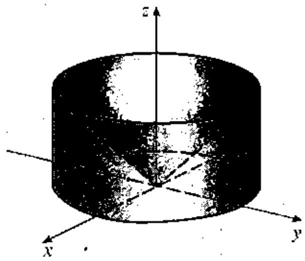
22. Here $E = \{(x, y, z) \mid -1 \le y \le 4 - z, x^2 + z^2 \le 4\}$, so

$$V = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{-1}^{4-z} dy\, dz\, dx = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4 - z + 1)\, dz\, dx$$
$$= \int_{-2}^{2} \left[5z - \tfrac{1}{2}z^2\right]_{z=-\sqrt{4-x^2}}^{z=\sqrt{4-x^2}} dx = \int_{-2}^{2} 10\sqrt{4 - x^2}\, dx$$
$$= 10\left[\tfrac{x}{2}\sqrt{4 - x^2} + 2\sin^{-1}\left(\tfrac{x}{2}\right)\right]_{-2}^{2} \qquad \left[\begin{array}{l}\text{using trigonometric substitution or} \\ \text{Formula 30 in the Table of Integrals}\end{array}\right]$$
$$= 10\left[2\sin^{-1}(1) - 2\sin^{-1}(-1)\right] = 20\left(\tfrac{\pi}{2} - \left(-\tfrac{\pi}{2}\right)\right) = 20\pi$$



[continued]

16.



The region of integration is given in cylindrical coordinates by $E = \{(r, \theta, z) \mid 0 \le \theta \le 2\pi, 0 \le r \le 2, 0 \le z \le r\}$. This represents the solid region enclosed by the circular cylinder $r = 2$, bounded above by the cone $z = r$, and bounded below by the xy-plane.

$$\int_0^2 \int_0^{2\pi} \int_0^r r\,dz\,d\theta\,dr = \int_0^2 \int_0^{2\pi} \left[rz\right]_{z=0}^{z=r} d\theta\,dr = \int_0^2 \int_0^{2\pi} r^2\,d\theta\,dr$$
$$= \int_0^2 r^2\,dr \int_0^{2\pi} d\theta = \left[\tfrac{1}{3}r^3\right]_0^2 \left[\,\theta\,\right]_0^{2\pi} = \tfrac{8}{3}\cdot 2\pi = \tfrac{16}{3}\pi$$

17. In cylindrical coordinates, E is given by $\{(r, \theta, z) \mid 0 \le \theta \le 2\pi, 0 \le r \le 4, -5 \le z \le 4\}$. So

$$\iiint_E \sqrt{x^2+y^2}\,dV = \int_0^{2\pi} \int_0^4 \int_{-5}^4 \sqrt{r^2}\,r\,dz\,dr\,d\theta = \int_0^{2\pi} d\theta \int_0^4 r^2\,dr \int_{-5}^4 dz$$
$$= \left[\,\theta\,\right]_0^{2\pi} \left[\tfrac{1}{3}r^3\right]_0^4 \left[\,z\,\right]_{-5}^4 = (2\pi)\left(\tfrac{64}{3}\right)(9) = 384\pi$$

18. The paraboloid $z = x^2 + y^2 = r^2$ intersects the plane $z = 4$ in the circle $x^2 + y^2 = 4$ or $r^2 = 4 \;\Rightarrow\; r = 2$, so in cylindrical coordinates, E is given by $\{(r, \theta, z) \mid 0 \le \theta \le 2\pi, 0 \le r \le 2, r^2 \le z \le 4\}$. Thus

$$\iiint_E z\,dV = \int_0^{2\pi} \int_0^2 \int_{r^2}^4 (z)\,r\,dz\,dr\,d\theta = \int_0^{2\pi} \int_0^2 \left[\tfrac{1}{2}rz^2\right]_{z=r^2}^{z=4} dr\,d\theta$$
$$= \int_0^{2\pi} \int_0^2 \left(8r - \tfrac{1}{2}r^5\right) dr\,d\theta = \int_0^{2\pi} d\theta \int_0^2 \left(8r - \tfrac{1}{2}r^5\right) dr = 2\pi \left[4r^2 - \tfrac{1}{12}r^6\right]_0^2$$
$$= 2\pi\left(16 - \tfrac{16}{3}\right) = \tfrac{64}{3}\pi$$

19. The paraboloid $z = 4 - x^2 - y^2 = 4 - r^2$ intersects the xy-plane in the circle $x^2 + y^2 = 4$ or $r^2 = 4 \;\Rightarrow\; r = 2$, so in cylindrical coordinates, E is given by $\{(r, \theta, z) \mid 0 \le \theta \le \pi/2, 0 \le r \le 2, 0 \le z \le 4 - r^2\}$. Thus

$$\iiint_E (x + y + z)\,dV = \int_0^{\pi/2} \int_0^2 \int_0^{4-r^2} (r\cos\theta + r\sin\theta + z)\,r\,dz\,dr\,d\theta = \int_0^{\pi/2} \int_0^2 \left[r^2(\cos\theta + \sin\theta)z + \tfrac{1}{2}rz^2\right]_{z=0}^{z=4-r^2} dr\,d\theta$$
$$= \int_0^{\pi/2} \int_0^2 \left[(4r^2 - r^4)(\cos\theta + \sin\theta) + \tfrac{1}{2}r(4 - r^2)^2\right] dr\,d\theta$$
$$= \int_0^{\pi/2} \left[\left(\tfrac{4}{3}r^3 - \tfrac{1}{5}r^5\right)(\cos\theta + \sin\theta) - \tfrac{1}{12}(4 - r^2)^3\right]_{r=0}^{r=2} d\theta$$
$$= \int_0^{\pi/2} \left[\tfrac{64}{15}(\cos\theta + \sin\theta) + \tfrac{16}{3}\right] d\theta = \left[\tfrac{64}{15}(\sin\theta - \cos\theta) + \tfrac{16}{3}\theta\right]_0^{\pi/2}$$
$$= \tfrac{64}{15}(1 - 0) + \tfrac{16}{3}\cdot\tfrac{\pi}{2} - \tfrac{64}{15}(0 - 1) - 0 = \tfrac{8}{3}\pi + \tfrac{128}{15}$$

20. In cylindrical coordinates E is bounded by the planes $z = 0$, $z = r\cos\theta + r\sin\theta + 5$ and the cylinders $r = 2$ and $r = 3$, so E is given by $\{(r, \theta, z) \mid 0 \le \theta \le 2\pi, 2 \le r \le 3, 0 \le z \le r\cos\theta + r\sin\theta + 5\}$. Thus

$$\iiint_E x\,dV = \int_0^{2\pi} \int_2^3 \int_0^{r\cos\theta + r\sin\theta + 5} (r\cos\theta)\,r\,dz\,dr\,d\theta = \int_0^{2\pi} \int_2^3 (r^2\cos\theta)\left[\,z\,\right]_{z=0}^{z=r\cos\theta + r\sin\theta + 5} dr\,d\theta$$
$$= \int_0^{2\pi} \int_2^3 (r^2\cos\theta)(r\cos\theta + r\sin\theta + 5)\,dr\,d\theta = \int_0^{2\pi} \int_2^3 (r^3(\cos^2\theta + \cos\theta\sin\theta) + 5r^2\cos\theta)\,dr\,d\theta$$
$$= \int_0^{2\pi} \left[\tfrac{1}{4}r^4(\cos^2\theta + \cos\theta\sin\theta) + \tfrac{5}{3}r^3\cos\theta\right]_{r=2}^{r=3} d\theta$$
$$= \int_0^{2\pi} \left[\left(\tfrac{81}{4} - \tfrac{16}{4}\right)(\cos^2\theta + \cos\theta\sin\theta) + \tfrac{5}{3}(27 - 8)\cos\theta\right] d\theta$$
$$= \int_0^{2\pi} \left(\tfrac{65}{4}\left(\tfrac{1}{2}(1 + \cos 2\theta) + \cos\theta\sin\theta\right) + \tfrac{95}{3}\cos\theta\right) d\theta = \left[\tfrac{65}{8}\theta + \tfrac{65}{16}\sin 2\theta + \tfrac{65}{8}\sin^2\theta + \tfrac{95}{3}\sin\theta\right]_0^{2\pi} =$$

$=64/5\ \pi$

22. In spherical coordinates, H is represented by $\left\{(\rho,\theta,\phi) \mid 0 \le \rho \le 3, 0 \le \theta \le 2\pi, 0 \le \phi \le \frac{\pi}{2}\right\}$. Thus

$$\begin{aligned}
\iiint_H (9 - x^2 - y^2)\, dV &= \int_0^{\pi/2}\int_0^{2\pi}\int_0^3 \left[9 - (\rho^2 \sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta)\right]\, \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi \\
&= \int_0^{\pi/2}\int_0^{2\pi}\int_0^3 (9 - \rho^2\sin^2\phi)\, \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi \\
&= \int_0^{\pi/2}\int_0^{2\pi} \left[3\rho^3 - \tfrac{1}{5}\rho^5 \sin^2\phi\right]_{\rho=0}^{\rho=3} \sin\phi\, d\rho\, d\theta\, d\phi \\
&= \int_0^{\pi/2}\int_0^{2\pi} \left(81\sin\phi - \tfrac{243}{5}\sin^3\phi\right) d\theta\, d\phi \\
&= \int_0^{2\pi} d\theta \int_0^{\pi/2} \left[81\sin\phi - \tfrac{243}{5}(1-\cos^2\phi)\sin\phi\right] d\phi \\
&= 2\pi\left[-81\cos\phi - \tfrac{243}{5}\left(\tfrac{1}{3}\cos^3\phi - \cos\phi\right)\right]_0^{\pi/2} \\
&= 2\pi\left[0 + 81 + \tfrac{243}{5}\left(-\tfrac{2}{3}\right)\right] = \tfrac{486}{5}\pi
\end{aligned}$$

23. In spherical coordinates, E is represented by $\{(\rho,\theta,\phi) \mid 2 \le \rho \le 3, 0 \le \theta \le 2\pi, 0 \le \phi \le \pi\}$ and

$x^2 + y^2 = \rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta = \rho^2\sin^2\phi\left(\cos^2\theta + \sin^2\theta\right) = \rho^2\sin^2\phi$. Thus

$$\begin{aligned}
\iiint_E (x^2+y^2)\, dV &= \int_0^{\pi}\int_0^{2\pi}\int_2^3 (\rho^2\sin^2\phi)\,\rho^2\sin\phi\, d\rho\, d\theta\, d\phi = \int_0^{\pi}\sin^3\phi\, d\phi \int_0^{2\pi} d\theta \int_2^3 \rho^4\, d\rho \\
&= \int_0^{\pi}(1-\cos^2\phi)\,\sin\phi\, d\phi\, \left[\theta\right]_0^{2\pi} \left[\tfrac{1}{5}\rho^5\right]_2^3 = \left[-\cos\phi + \tfrac{1}{3}\cos^3\phi\right]_0^{\pi} (2\pi)\cdot\tfrac{1}{5}(243-32) \\
&= \left(1 - \tfrac{1}{3} + 1 - \tfrac{1}{3}\right)(2\pi)\left(\tfrac{211}{5}\right) = \tfrac{1688\pi}{15}
\end{aligned}$$

24. In spherical coordinates, E is represented by $\{(\rho,\theta,\phi) \mid 0 \le \rho \le 3, 0 \le \theta \le \pi, 0 \le \phi \le \pi\}$. Thus

$$\begin{aligned}
\iiint_E y^2\, dV &= \int_0^{\pi}\int_0^{\pi}\int_0^3 (\rho\sin\phi\sin\theta)^2\,\rho^2\sin\phi\, d\rho\, d\theta\, d\phi = \int_0^{\pi}\sin^3\phi\, d\phi \int_0^{\pi}\sin^2\theta\, d\theta \int_0^3 \rho^4\, d\rho \\
&= \int_0^{\pi}(1-\cos^2\phi)\,\sin\phi\, d\phi \int_0^{\pi}\tfrac{1}{2}(1-\cos 2\theta)\, d\theta \int_0^3 \rho^4\, d\rho \\
&= \left[-\cos\phi + \tfrac{1}{3}\cos^3\phi\right]_0^{\pi} \left[\tfrac{1}{2}\left(\theta - \tfrac{1}{2}\sin 2\theta\right)\right]_0^{\pi} \left[\tfrac{1}{5}\rho^5\right]_0^3 \\
&= \left(\tfrac{2}{3}+\tfrac{2}{3}\right)\left(\tfrac{1}{2}\pi\right)\left(\tfrac{1}{5}(243)\right) = \left(\tfrac{4}{3}\right)\left(\tfrac{\pi}{2}\right)\left(\tfrac{243}{5}\right) = \tfrac{162\pi}{5}
\end{aligned}$$

25. In spherical coordinates, E is represented by $\left\{(\rho,\theta,\phi) \mid 0 \le \rho \le 1, 0 \le \theta \le \frac{\pi}{2}, 0 \le \phi \le \frac{\pi}{2}\right\}$. Thus

$$\begin{aligned}
\iiint_E x e^{x^2+y^2+z^2}\, dV &= \int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 (\rho\sin\phi\cos\theta)e^{\rho^2}\,\rho^2\sin\phi\, d\rho\, d\theta\, d\phi = \int_0^{\pi/2}\sin^2\phi\, d\phi \int_0^{\pi/2}\cos\theta\, d\theta \int_0^1 \rho^3 e^{\rho^2}\, d\rho \\
&= \int_0^{\pi/2}\tfrac{1}{2}(1-\cos 2\phi)\, d\phi \int_0^{\pi/2}\cos\theta\, d\theta \left(\left[\tfrac{1}{2}\rho^2 e^{\rho^2}\right]_0^1 - \int_0^1 \rho e^{\rho^2}\, d\rho\right) \\
&\qquad \left[\text{integrate by parts with } u = \rho^2,\ dv = \rho e^{\rho^2}\, d\rho\right] \\
&= \left[\tfrac{1}{2}\phi - \tfrac{1}{4}\sin 2\phi\right]_0^{\pi/2} \left[\sin\theta\right]_0^{\pi/2} \left[\tfrac{1}{2}\rho^2 e^{\rho^2} - \tfrac{1}{2}e^{\rho^2}\right]_0^1 = \left(\tfrac{\pi}{4} - 0\right)(1-0)\left(0 + \tfrac{1}{2}\right) = \tfrac{\pi}{8}
\end{aligned}$$

26. $$\begin{aligned}
\iiint_E xyz\, dV &= \int_0^{\pi/3}\int_0^{2\pi}\int_2^4 (\rho\sin\phi\cos\theta)(\rho\sin\phi\sin\theta)(\rho\cos\phi)\,\rho^2\sin\phi\, d\rho\, d\theta\, d\phi \\
&= \int_0^{\pi/3}\sin^3\phi\cos\phi\, d\phi \int_0^{2\pi}\sin\theta\cos\theta\, d\theta \int_2^4 \rho^5\, d\rho = \left[\tfrac{1}{4}\sin^4\phi\right]_0^{\pi/3}\left[\tfrac{1}{2}\sin^2\theta\right]_0^{2\pi}\left[\tfrac{1}{6}\rho^6\right]_2^4 = 0
\end{aligned}$$

$\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1/4 & 1/4 \\ -3/4 & 1/4 \end{vmatrix} = \dfrac{1}{4}$, $4x + 8y = 4 \cdot \frac{1}{4}(u+v) + 8 \cdot \frac{1}{4}(v - 3u) = 3v - 5u$. R is a parallelogram bounded by the lines $x - y = -4$, $x - y = 4$, $3x + y = 0$, $3x + y = 8$. Since $u = x - y$ and $v = 3x + y$, R is the image of the rectangle enclosed by the lines $u = -4$, $u = 4$, $v = 0$, and $v = 8$. Thus

$$\begin{aligned}\iint_R (4x+8y)\,dA &= \int_{-4}^{4}\int_0^8 (3v - 5u)\left|\tfrac{1}{4}\right| dv\,du = \tfrac{1}{4}\int_{-4}^{4}\left[\tfrac{3}{2}v^2 - 5uv\right]_{v=0}^{v=8} du \\ &= \tfrac{1}{4}\int_{-4}^{4}(96 - 40u)\,du = \tfrac{1}{4}\left[96u - 20u^2\right]_{-4}^{4} = 192\end{aligned}$$

17. $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} = 6$, $x^2 = 4u^2$ and the planar ellipse $9x^2 + 4y^2 \le 36$ is the image of the disk $u^2 + v^2 \le 1$. Thus

$$\begin{aligned}\iint_R x^2\,dA &= \iint_{u^2+v^2\le 1} (4u^2)(6)\,du\,dv = \int_0^{2\pi}\int_0^1 (24r^2\cos^2\theta)\,r\,dr\,d\theta = 24\int_0^{2\pi}\cos^2\theta\,d\theta \int_0^1 r^3\,dr \\ &= 24\left[\tfrac{1}{2}x + \tfrac{1}{4}\sin 2x\right]_0^{2\pi}\left[\tfrac{1}{4}r^4\right]_0^1 = 24(\pi)\left(\tfrac{1}{4}\right) = 6\pi\end{aligned}$$

18. $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \sqrt{2} & -\sqrt{2/3} \\ \sqrt{2} & \sqrt{2/3} \end{vmatrix} = \dfrac{4}{\sqrt{3}}$, $x^2 - xy + y^2 = 2u^2 + 2v^2$ and the planar ellipse $x^2 - xy + y^2 \le 2$ is the image of the disk $u^2 + v^2 \le 1$. Thus

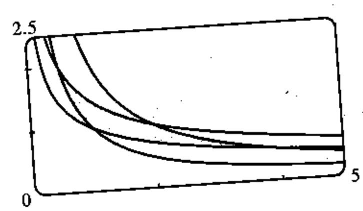
$$\iint_R (x^2 - xy + y^2)\,dA = \iint_{u^2+v^2\le 1} (2u^2 + 2v^2)\left(\tfrac{4}{\sqrt{3}}\,du\,dv\right) = \int_0^{2\pi}\int_0^1 \tfrac{8}{\sqrt{3}}r^3\,dr\,d\theta = \tfrac{4\pi}{\sqrt{3}}$$

19. $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1/v & -u/v^2 \\ 0 & 1 \end{vmatrix} = \dfrac{1}{v}$, $xy = u$, $y = x$ is the image of the parabola $v^2 = u$, $y = 3x$ is the image of the parabola $v^2 = 3u$, and the hyperbolas $xy = 1$, $xy = 3$ are the images of the lines $u = 1$ and $u = 3$ respectively. Thus

$$\iint_R xy\,dA = \int_1^3\int_{\sqrt{u}}^{\sqrt{3u}} u\left(\frac{1}{v}\right) dv\,du = \int_1^3 u\left(\ln\sqrt{3u} - \ln\sqrt{u}\right) du = \int_1^3 u\ln\sqrt{3}\,du = 4\ln\sqrt{3} = 2\ln 3.$$

20. Here $y = \dfrac{v}{u}$, $x = \dfrac{u^2}{v}$ so $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 2u/v & -u^2/v^2 \\ -v/u^2 & 1/u \end{vmatrix} = \dfrac{1}{v}$ and R is the image of the square with vertices $(1,1)$, $(2,1)$, $(2,2)$, and $(1,2)$. So

$$\iint_R y^2\,dA = \int_1^2\int_1^2 \frac{v^2}{u^2}\left(\frac{1}{v}\right) du\,dv = \int_1^2 \frac{v}{2}\,dv = \frac{3}{4}$$



21. (a) $\dfrac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc$ and since $u = \dfrac{x}{a}$, $v = \dfrac{y}{b}$, $w = \dfrac{z}{c}$ the solid enclosed by the ellipsoid is the image of the ball $u^2 + v^2 + w^2 \le 1$. So

$$\iiint_E dV = \iiint_{u^2+v^2+w^2\le 1} abc\,du\,dv\,dw = (abc)(\text{volume of the ball}) = \tfrac{4}{3}\pi abc$$