Math 302, Assignment 2

(1) In a small town, there are three bakeries. Each of the bakeries bakes twelve cakes per day. Bakery 1 has two different types of cake, bakery 2 three different types, and bakery 3 four different types. Every bakery bakes equal amounts of cakes of each type.
You randomly walk into one of the bakeries, and then randomly buy two cakes.
  (a) What is the probability that you will buy two cakes of the same type?
  (b) Suppose you have bought two different types of cake. Given this, what is the probability that you went to bakery 2?

(2) An assembly line produces a large number of products, of which 1% are faulty in average. A quality control test correctly identifies 98% of the faulty products, and 95% of the flawless products. For every product that is identified as faulty, the test is run a second time, independently.
  (a) Suppose that a product was identified as faulty in both tests. What is the probability that it is, indeed, faulty?
  (b) What if the quality control test is only performed once?

(3) Let M be an integer chosen uniformly from $\{1, \ldots, 100\}$. Decide whether the following events are independent:
  (a) $E = \{M \text{ is even}\}$ and $F = \{M \text{ is divisible by } 5\}$
  (b) $E = \{M \text{ is prime}\}$ and $F = \{\text{at least one of the digits of } M \text{ is a } 2\}$
  (c) Can you replace the number 100 by a different number, in such a way that your answer to (a) changes? (E.g., if your answer was "dependent", try to change the number 100 in such a way your answer becomes "independent").
  (d) Show that if M is selected uniformly from $\{1, 2, \ldots, N\}$ for N very large, and p, q are fixed prime numbers, then the events $\{p|M\}$ and $\{q|M\}$ are roughly independent, i.e.

$$\mathbb{P}(pq|M) \approx \mathbb{P}(p|M)\mathbb{P}(q|M),$$

where $a|b$ means a divides b, and $\approx$ means the two sides of the above equation differ by something tending to 0 as $N \to \infty$. Does this still hold if p, q are allowed to be any integers?

(4) Let X be a discrete random variable with values in $\mathbb{N} = \{1, 2, \ldots\}$. Prove that X is geometric with parameter $p = \mathbb{P}(X = 1)$, i.e.

$$\mathbb{P}(X = k) = p(1-p)^{k-1} \text{ for } k \in \mathbb{N},$$

if and only if the *memoryless property*

$$\mathbb{P}(X = n + m \mid X > n) = \mathbb{P}(X = m)$$

holds.
Hint: Use $\mathbb{P}(X = k) = \mathbb{P}(X = k + 1 | X > 1)$ repeatedly.

(5) Let X take values $\{1, 2, 3, 4, 5\}$, and have p.m.f. given by

Table 1. The p.m.f. of X

k	1	2	3	4	5
$\mathbb{P}(X = k)$	$1/7$	$1/14$	$3/14$	$2/7$	$2/7$

(a) Calculate $\mathbb{P}(X \leq 3)$
(b) Calculate $\mathbb{P}(X < 3)$
(c) Calculate $\mathbb{P}(X < 4.12 | X > 1.6)$

(6) Consider the following lottery: There are a total of 10 tickets, of which 5 are "win" and 5 are "lose". You draw tickets until you draw the first "win". Drawing one ticket costs \$2, 2 tickets \$4, 3 tickets \$8, and so on. A winning ticket pays out \$8.

(a) Let X be the number of tickets you draw in the lottery (i.e. the number of tickets until the first win, including the winning ticket). Calculate the p.m.f. of X.
(b) Calculate the expectation $\mathbb{E}\,X$.
(c) Calculate the variance $\sigma^2(X)$.
(d) What are your expected winnings in this game?

(7) In a town, there are on average 2.3 children in a family and a randomly chosen child has on average 1.6 siblings. Determine the variance of the number of children in a randomly chosen family.

(8) Consider a random graph on $n \geq 3$ vertices labeled by $[n] := \{1, \ldots, n\}$ defined as follows: for each i, j draw an edge between i and j w.p. $1/2$ so that the events $A_{i,j} = \{i \text{ and } j \text{ are connected by an edge}\}$ are (mutually) independent for all pairs $\{i, j\}$ with $i \neq j \in [n]$. Let X be the number of triangles in the graph. That is, the number of $\{i, j, k\} \subset [n]$ so that $i \neq j$, $j \neq k$ and $i \neq k$ and each of the three pairs $\{i, j\}, \{i, k\}$ and $\{j, k\}$ has an edge between them.

(a) Calculate $\mathbb{E}\,X$
(b) Calculate $\sigma^2(X)$