The University of British Columbia
Math 101-302 — Introduction to Probability
2018, October 24
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Exam solution

6 marks 1. Let X be a Poisson(1) random variable.

    (a) Show that the events $\{X \leq 1\}$ and $\{X \geq 1\}$ are not independent.

    (b) Let Y be the indicator of the event $\{X \leq 1\}$ (that is, $Y = 1$ if $X \leq 1$, and $Y = 0$ otherwise). Let Z be the indicator of $\{X \geq 1\}$. Compute the variance of $Y + Z$.

Solution:

(a). This can be done in a number of ways. The most straightforward way is to compute the probabilities:
$$\mathbb{P}(X \leq 1) = 2e^{-1}$$

and
$$\mathbb{P}(X \geq 1) = 1 - e^{-1}.$$

Also,
$$\mathbb{P}(\{X \leq 1\} \cap \{X \geq 1\}) = \mathbb{P}(X = 1) = e^{-1}.$$

Since
$$\mathbb{P}(\{X \leq 1\} \cap \{X \geq 1\}) = e^{-1} \neq 2e^{-1}(1 - e^{-1}) = \mathbb{P}(X \leq 1) \cdot \mathbb{P}(X \geq 1),$$

we see that the events are not independent.

(b). By the linearity of expectation, we have
$$\mathbb{E}[Y + Z] = \mathbb{E}Y + \mathbb{E}Z = \mathbb{P}(X \leq 1) + \mathbb{P}(X \geq 1) = 1 + e^{-1}.$$

The random variable $Y + Z$ takes values in $\{0, 1, 2\}$. In fact, $Y + Z$ cannot be 0, and moreover, $Y + Z = 2$ if and only if $X = 1$. Thus,
$$\mathbb{E}[(Y + Z)^2] = 1^2 \cdot \mathbb{P}(X \neq 1) + 2^2 \cdot \mathbb{P}(X = 1) = 1 - e^{-1} + 4e^{-1} = 1 + 3e^{-1}.$$

Therefore,
$$\mathrm{Var}(Y + Z) = \mathbb{E}[(Y + Z)^2] - (\mathbb{E}[Y + Z])^2 = 1 + 3e^{-1} - (1 + e^{-1})^2 = e^{-1} - e^{-2}.$$

Another, shorter solution is as follows: $Y + Z = 1 + W$, where W is the indicator of the event $\{X = 1\}$. Thus, $W \sim \mathrm{Ber}(e^{-1})$ so that
$$\mathrm{Var}(Y + Z) = \mathrm{Var}(1 + W) = \mathrm{Var}(W) = e^{-1}(1 - e^{-1}).$$

10 marks 2. Your friend has a bag with two types of dice: "boring" dice with 10 sides labeled 1 to 10, and "magic" dice with 10 sides labeled 1,2,2,3,3,3,4,4,4,4. Ten percent of the dice in the bag are magic and the rest are boring. You mix the bag, your friend takes out a die and rolls it.

(a) What is the probability that the die lands on 3?

(b) If the die lands on 3, what is the probability that it is a magic die?

(c) Your friend returns the die to the bag, takes out another die and rolls it twice. What is the probability that the die lands on the same number in both rolls?

Solution:

(a). Let M be the event that a magic die was chosen. Then, by the law of total probability,

$$\mathbb{P}(\text{die lands on } 3) = \mathbb{P}(3 \mid M) \cdot \mathbb{P}(M) + \mathbb{P}(3 \mid M^c) \cdot \mathbb{P}(M^c) = \frac{3}{10} \cdot \frac{1}{10} + \frac{1}{10} \cdot \frac{9}{10} = 12\%.$$

(b). By Baye's formula,

$$\mathbb{P}(M \mid \text{die lands on } 3) = \mathbb{P}(3 \mid M) \cdot \frac{\mathbb{P}(M)}{\mathbb{P}(3)} = \frac{3}{10} \cdot \frac{10\%}{12\%} = 25\%.$$

(c). Let D be the event that the die lands on the same number in both rolls. Then

$$\mathbb{P}(D \mid M^c) = 10\%$$

and

$$\mathbb{P}(D \mid M) = \mathbb{P}(1 \mid M)^2 + \mathbb{P}(2 \mid M)^2 + \cdots + \mathbb{P}(4 \mid M)^2 = \frac{1^2 + 2^2 + 3^2 + 4^2}{10^2} = 30\%.$$

Thus, by the law of total probability,

$$\mathbb{P}(D) = \mathbb{P}(D \mid M) \cdot \mathbb{P}(M) + \mathbb{P}(D \mid M^c) \cdot \mathbb{P}(M^c) = \frac{3}{10} \cdot \frac{1}{10} + \frac{1}{10} \cdot \frac{9}{10} = 12\%.$$

$\boxed{\text{10 marks}}$ 3. In a lottery, there is a ball machine containing 25 colored balls: 1 gold, 9 silver and 15 white. A gold ball is worth \$200, a silver \$15 and a white \$1.

(a) You take out two balls. What is the probability that you win at least \$30?

(b) You take out one ball. What is the expected amount you win?

(c) The lottery owner replaces the ball machine. The lottery owner does not tell you the worth of the balls in the new machine, but promises that the expected value of a single ball has not changed, and also that the standard deviation is \$4. You take out one ball. Give a lower bound on the probability that you win at least \$6.

Solution:

(a). You win at least \$30 in any case except if you take out two white balls or one white ball and one silver ball. Let us compute the probabilities of these events.

$$\mathbb{P}(\text{two white}) = \frac{15}{25} \cdot \frac{14}{24} = \frac{7}{20} = 35\%$$

and

$$\mathbb{P}(\text{one white, one silver}) = 2 \cdot \frac{15}{25} \cdot \frac{9}{24} = \frac{9}{20} = 45\%.$$

Thus,

$$\mathbb{P}(\text{win at least }\$30) = 1 - 35\% - 45\% = 20\%.$$

(b). Let X denote the value in dollars of the chosen ball. Then

$$\mathbb{E}X = 200 \cdot \frac{1}{25} + 15 \cdot \frac{9}{25} + 1 \cdot \frac{15}{25} = 14.$$

(c). We no longer know the precise distribution of X, but rather only that $\mathbb{E}X = 14$ and $\text{Var}(X) = \sigma(X)^2 = 16$. Let us first give an upper bound on the probability that we win at most \$5. Since $X \leq 5$ implies that $|X - 14| \geq 9$, Chebyshev's inequality gives us that

$$\mathbb{P}(X \leq 5) \leq \mathbb{P}(|X - \mathbb{E}X| \geq 9) \leq \frac{\text{Var}(X)}{9^2} = \frac{16}{81}.$$

Therefore,

$$\mathbb{P}(\text{win at least }\$6) = \mathbb{P}(X \geq 6) = 1 - \mathbb{P}(X \leq 5) \geq \frac{65}{81}.$$

If we want a "nice" number, we could take say 75%, since $75\% \leq \frac{65}{81}$.

8 marks   4. You have a mini-deck of 16 cards numbered 1 to 8, with two cards of each number. You also have two 8-sided dice (whose sides are numbered 1 to 8). You draw three cards and roll the two dice. Find the probability that:

    (a) The two numbers on the dice are the same, whereas all three numbers on the cards are different.

    (b) The *set* of numbers on the two dice is the same as the *set* of numbers on the three cards.

Solution:

(a). Let A be the event that the dice are the same and let B be the event that the three cards are all different. Then

$$\mathbb{P}(A) = \frac{1}{8}$$

and

$$\mathbb{P}(B) = \frac{\binom{8}{3}\binom{2}{1}\binom{2}{1}\binom{2}{1}}{\binom{16}{3}} = \frac{4}{5}.$$

Since A and B are independent,

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B) = \frac{1}{10} = 10\%.$$

(b). The set of numbers on the two dice consists of either 1 or 2 distinct elements. The set of numbers on the three cards consists of either 2 or 3 distinct elements (since there are only two cards of each number). For these sets to be equal, both must consist of two distinct elements. That is, the dice must be different, and two of the three cards must have the same number. There are $\binom{8}{2} = 28$ ways to choose a set of two numbers from $\{1, \ldots, 8\}$. For any choice of two such numbers $\{a, b\}$, there is probability

$$\frac{2}{8^2} = \frac{1}{32}$$

that the set of numbers on the two dice is $\{a, b\}$, and there is probability

$$\frac{2 \cdot \binom{2}{2} \cdot \binom{2}{1}}{\binom{16}{3}} = \frac{1}{140}.$$

Thus, the probability of the required event is

$$28 \cdot \frac{1}{32} \cdot \frac{1}{140} = \frac{1}{160}.$$