MATH 303 Midterm Exam 2B Solution

Problem 1

See Notebook report

Problem 2

At the grocery store, Khanh sees three cashiers serving Becca, Emmanuel and Merlin. Khanh assumes that the times they will spend at the cashier, T_B , T_M, and T_E are independent and exponentially distributed, with parameters λ_B, λ_M and λ_E for Becca, Emmanuel and Merlin, respectively. He also assumes that the variance of the time spent at the cashier is proportional to the number of items each person has.

The notebook will give the following possible outputs for Becca, Emmanuel and Merlin's number of items for question a, the probability to compute in question b and relation to use in question c

- Case (a):
  - [Becca, Emmanuel, Merlin] $= [1, 16, 4]$
  - P(Merlin leaves first and Becca leaves last)
  - $P(1 < \min(T_B, T_M) < 2) = \frac{1}{4}$
- Case (b):
  - [Becca, Emmanuel, Merlin] $= [4, 1, 16]$
  - P(Merlin leaves last and Emmanuel leaves first)
  - $\mathbb{E}[\min(T_B, T_E)] - Var(\min(T_B, T_E)) = -6$
- Case (c):
  - [Becca, Emmanuel, Merlin] $= [16, 4, 1]$
  - $T =$ Average time for the second person to leave given that Becca leaves first
  - $P(1 < \min(T_B, T_E) < 2) = \frac{1}{4}$
- Case (d):
  - [Becca, Emmanuel, Merlin] $= [16, 1, 4]$
  - $T =$ Average time for the first person to leave given that Becca leaves last
  - $\mathbb{E}[\min(T_B, T_M)] - Var(\min(T_B, T_M)) = -6$

a. Run the notebook for question 2a with your student ID to get everyone's number of items. Find the parameters λ_M and λ_E as a function of λ_B (justify your answer).

Solution: Let N_B, N_M and N_E denote the number of items of Becca, Merlin, and Emmanuel, respectively. Since $Var(Exp(\lambda)) = \frac{1}{\lambda^2}$, we have $\frac{Var(T_B)}{Var(T_E)} = \frac{\lambda_E^2}{\lambda_B^2} = \frac{N_B}{N_E}$, so $\lambda_E = \sqrt{\frac{N_B}{N_E}}\lambda_B$. Similarly $\lambda_M = \sqrt{\frac{N_B}{N_M}}\lambda_B$. Replacing N_B, N_M and N_E with their numerical values yields

- Case (a): $\lambda_E = \dfrac{\lambda_B}{4}$; $\lambda_M = \dfrac{\lambda_B}{2}$
- Case (b): $\lambda_E = 2\lambda_B$; $\lambda_M = \dfrac{\lambda_B}{2}$
- Case (c): $\lambda_E = 2\lambda_B$; $\lambda_M = 4\lambda_B$
- Case (d): $\lambda_E = 4\lambda_B$; $\lambda_M = 2\lambda_B$

b. Find the probability P or time T displayed in your notebook (justify your answers).

Solution:

- Case (a): The probability $P_{M,E,B}$ that Merlin, Emmanuel and Becca leave in this order is $P_{M,E,B} = P($ M leaves before E and B$) \times P($E leaves before B$)$, where we used the no-memory property of the exponential r.v.. From class, we also know that $P($ M leaves before E and B$) = \frac{\lambda_M}{\lambda_E+\lambda_M+\lambda_B} = \frac{2}{7}$and $P($E leaves before B$) = \frac{\lambda_E}{\lambda_E+\lambda_B} = \frac{1}{5}$, so
$$P_{M,E,B} = \frac{2}{35}.$$
- Case (b): Similarly as in (a), $P_{E,B,M} = \frac{8}{21}$.
- Case (c): By the no-memory property,
$$\begin{aligned} T &= \mathbb{E}[T_B] + \mathbb{E}[(\text{Time for the next person to leave after Becca leaves}] \\ &= \mathbb{E}[T_B] + \mathbb{E}[\min(Exp(\lambda_E), Exp(\lambda_M))] \\ &= \mathbb{E}[T_B] + \mathbb{E}[Exp(\lambda_E + \lambda_M)] \\ &= \frac{1}{\lambda_B} + \frac{1}{\lambda_M + \lambda_E} = \frac{7}{6\lambda_B}. \end{aligned}$$
- Case (d): We condition on the first person leaving (Emmanuel or Merlin), and the fact that $T_1 := Exp(\lambda_E + \lambda_M + \lambda_B)$ is independent from the ranking of T_B, T_E, T_M:
$$\begin{aligned} T &= \mathbb{E}[T_1 \mid \text{B leaves last}] \\ &= \mathbb{E}[T_1 \mid \text{E leaves before M, B leaves last}]P(\text{E leaves before M}) \\ &\quad + \mathbb{E}[T_1 \mid \text{M leaves before E, B leaves last}]P(\text{M leaves before E}) \\ &= \mathbb{E}[T_1]P(Exp(\lambda_E) < Exp(\lambda_M)) + \mathbb{E}[T_1]P(Exp(\lambda_M) < Exp(\lambda_E)) \\ &= \frac{1}{\lambda_E + \lambda_M + \lambda_B}. \end{aligned}$$
Remark: This result also shows that T_1 is independent from a partial ranking. Since the independence between T_1 and the ranking of T_B, T_E, T_M (Prop. 5.2, Ross) was not given, marks are awarded in the grading for conditioning and using T_E and T_B instead of T_1 in the third line.

c. Find the value of λ_B, given the equation obtained from running the notebook for question 2c (justify your answer). *Hint: this equation should lead to solve a second order polynomial equation*

Solution:

- Case (a): $\min(T_B, T_M) \sim Exp(\lambda_B + \lambda_M) \sim Exp\left(\frac{3}{2}\lambda_B\right)$. Using the cdf,

$$P(1 < \min(T_B, T_M) < 2) = \frac{1}{4} \Leftrightarrow -(e^{-\frac{3}{2}\lambda_B})^2 + e^{-\frac{3}{2}\lambda_B} = \frac{1}{4}.$$

  Solving $-X^2 + X = \frac{1}{4}$ (student's calculation to be shown) yields $X = \frac{1}{2}$, so $e^{-\frac{3}{2}\lambda_B} = \frac{1}{2}$ and $\lambda_B = \frac{2}{3}\ln(2)$.

- Case (b): $\min(T_B, T_E) \sim \mathrm{Exp}(\lambda_B + \lambda_E) \sim \mathrm{Exp}(3\lambda_B)$, so

$$\mathbb{E}[\min(T_B, T_E)] - Var(\min(T_B, T_E)) = -6 \Leftrightarrow \frac{1}{3\lambda_B} - \left(\frac{1}{3\lambda_B}\right)^2 = -6.$$

  Solving $-X^2 + X = -6$ (student's calculation to be shown) yields $X = -2$ or $X = 3$, so $\frac{1}{3\lambda_B} = -2$ or $\frac{1}{3\lambda_B} = 3$. Since $\lambda_B > 0$, $\lambda_B = \frac{1}{9}$.

- Case (c): Similarly as in (a), $\lambda_B = \frac{1}{3}\ln(2)$.

- Case (d): Similarly as in (b), $\lambda_B = \frac{1}{9}$.

Problem 3

Let $N(t)$ be a Poisson process of rate λ, and let $Z \sim \mathrm{Exp}(\mu)$ be an independent exponential random variable.

a. Fix $n \in \{0, 1, 2, \ldots\}$. Compute $\mathbb{P}(N(Z) = n)$ as a function of n, λ and μ. [Hint: think of Z as the first arrival of a rate μ Poisson process.]

Solution: Consider the process $M(t) = N(t) + K(t)$, where $K(t)$ is an independent rate μ Poisson process, with first arrival time Z. By the superposition property, $M(t)$ is a rate $\lambda + \mu$ Poisson process. By a calculation from class, for any $t > 0$,

$$\mathbb{P}(N(Z) \geq n) = \mathbb{P}(K(t) = 0 | M(t) = n) = (\frac{\lambda}{\lambda + \mu})^n.$$

Thus

$$\mathbb{P}(N(Z) = n) = \mathbb{P}(N(Z) \geq n) - \mathbb{P}(N(Z) \geq n + 1) = \frac{\mu}{\lambda + \mu}(\frac{\lambda}{\lambda + \mu})^n.$$

b. Compute $\mathbb{E}[N(Z)/Z]$. For what value(s) of μ is $\mathbb{E}[N(Z)/Z] = \lambda$?

Solution: Just integrate:

$$
\begin{aligned}
\mathbb{E}[N(Z)/Z] &= \int_{-\infty}^{\infty} \mathbb{E}[N(Z)/Z | Z = z] \, d\mathbb{P}(Z = z) \\
&= \int_0^{\infty} \mu e^{-\mu z} \mathbb{E}\left[\frac{N(z)}{z}\right] dz \\
&= \int_0^{\infty} \mu e^{-\mu z} \frac{\lambda z}{z} \, dz \\
&= \int_0^{\infty} \lambda \mu e^{-\mu z} \, dz \\
&= \lambda.
\end{aligned}
$$

So $\mathbb{E}[N(Z)/Z] = \lambda$ for any $\mu \geq 0$!

Problem 4

Shooting stars appear in the sky at the times of a Poisson process with rate 1 per hour. Assume each star is visible to the human eye with probability 1/2 independently. Suppose that over the course of three hours, five total shooting stars arrived.

a. Find the probability displayed in your notebook.

P4 = [[2,4],[2,3],[3,4]] Question : What is the probability that P4[0] of the P4[1] stars were visible?

Solution: This is a direct application of the thinning property of a Poisson process. Given that m stars have appeared the probability that $k \in \{0, \cdots, m\}$ were visible is a Binomial distribution with parameters $(m, p = 1/2)$. So we get

- b = 0, [2,4] Proba = $6 \times (1/2)^4 = 0.375$
- b = 1, [2,3] Proba = $3 \times (1/2)^3 = 0.375$
- b = 2, [3,4] Proba = $4 \times (1/2)^4 = 0.25$

b. What's the probability that two visible shooting stars occurred in succession, with no invisible stars in between?

Solution: This can only occur if there are 2, 3, 4, or 5 visible stars. Representing V for visible star and I for invisible star, there are 4 possible orderings with 2 V's and 3 I's, 9 possible with 3 V's and 2 I's (all $10 = \binom{5}{3}$ are possible other than VIVIV), 5 possible with 4 V's and one I, and one with five V's. The probability of having exactly k visible stars for $k = 0, 1, 2, 3, 4, 5$ in a fixed ordering has probability 2^{-5}, since each star is V/I with probability 1/2 independently. Thus the total probability is

$$\frac{1}{32}(4 + 9 + 5 + 1) = 19/32.$$

c. Suppose four out of the five shooting stars were visible. What's the probability that exactly two of the visible ones occurred in the first hour, and exactly two visible ones occurred in the third hour?

Solution: Given that four were visible, the times at which the stars arrived are independent uniform random variables on $[0,3]$. Thus, the probability that two occurred in $[0,1]$ and two occurred in $[2,3]$ is $\binom{4}{2} \cdot \mathbb{P}(U \in [0,1])^4 = 2/27$. where U is a uniform random variable on $[0,3]$. (The binomial coefficient occurs because we must choose which two of the four arrivals are in $[0,1]$, and which two are in $[2,3]$.)