Math 324 Homework 6

For problems 1-2, use Green's theorem to evaluate $\int_C F \cdot dr$ for the given F and C.

1. $F(x,y) = \langle x-y, x+y \rangle$, C is the triangle with vertices $(0,1), (1,0)$, and $(-1,0)$, oriented counter-clockwise.

2. $F(x,y) = \langle x+y, x^2-y \rangle$, C is the boundary of the region enclosed by $y = x^2$ and $y = \sqrt{x}$ for $x \in (0,1)$, oriented counter-clockwise.

3. Let $f : [0,1] \to [0,1]$ be a differentiable function.

   a. Suppose $f(0) = f(1) = 0$. Let C be the graph of f oriented from $(1,0)$ to $(0,0)$. Use Green's theorem to show that

   $$\int_C x dy = \int_0^1 f(x) dx.$$

   (Hint: draw a picture.)

   b. Suppose instead that $f(0) = 0, f(1) = 1$, and let C be the graph of f, oriented from $(0,0)$ to $(1,1)$. Use Green's theorem to show that

   $$\int_C y dx = \int_0^1 f(x) dx.$$

4. Use Green's theorem to show that for any curve C enclosing the origin,

   $$\int_C \frac{x dy - y dx}{x^2+y^2} \cdot dr = 2\pi.$$

   [Hint: recall that we verified this in chapter 16.2 for $C =$ any circle centered at the origin. For an arbitrary C, apply Green's theorem to the region between C and a small circle centered at the origin.]

   For problems 4-6, compute the curl and divergence of the given vector field.

5. $F(x,y,z) = \langle x+yz, y+xz, z+xy \rangle$

6. $F(x,y,z) = \langle y/x, y/z, z/x \rangle$

7. $F(x,y,z) = \langle \frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2}, 0 \rangle$

8. Show that for a vector field $F = \langle P, Q, R \rangle$ and a function f, $\text{div}(fF) = f\ \text{div}(F) + F \cdot \nabla f$. Here fF is the vector field $\langle fP, fQ, fR \rangle$.

   For problems 6-7, determine whether or not the vector field is conservative, and if it is, find a potential function.

9. $F(x,y,z) = \langle xyz^2, x^2yz^2, x^2y^2z \rangle$.

10. $F(x,y,z) = \langle 1, \sin z, y\cos z \rangle$.

11. Is there a vector field G on $\mathbb{R}^3$ such that $\text{curl}(G) = \langle x\sin y, \cos y, z - xy \rangle$? Explain.