Recall: • We introduced $X \sim \mathrm{Exp}(\lambda)$ and basic properties ($\mathbb{E}(X)$, pdf, cdf)

• $\mathrm{Exp}(\lambda)$ is the only real continuous memoryless r.v.

$$\left(P(X > t+s \mid X > t) = P(X > s)\right)$$

→ This has many applications when we want to estimate some probabilities (cf. exs)

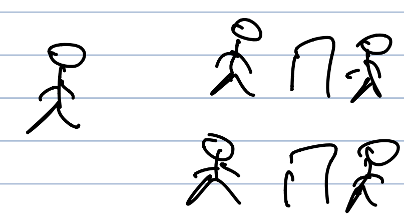
• Additional properties: for X_i independent $\mathrm{Exp}(\lambda_i)$

- $\mathrm{Min}(X_i) \sim \mathrm{Exp}(\sum \lambda_i)$ ✓
- $P(X_1 < X_2)$? → today
- $\sum X_i \sim$? → today

Rmk: • What happens for the memoryless property when the r.v. is discrete i.e. $\mathbb{N}$

(exercise) → The Geometric distribution is memoryless.

$$\left(P(X = k) = (1-p)^{k-1} p, \ 0 < p \leq 1\right)$$

• Example:  3 customers

Q: What is the expected time until all 3 customers have left?

- We assume that the service time at both counters is $Exp(\lambda)$

- Time until the 1^{st} customer leaves?

$$T \sim \min\left(Exp(\lambda), Exp(\lambda)\right) \sim Exp(2\lambda)$$

$$\Rightarrow E(T) = \frac{1}{2\lambda}$$

- By the memoryless property, the clock effectively restarts for the remaining customers, so the next customer to leave will do it with an average time $= \frac{1}{2\lambda}$
- The clock restarts, and the last customer leaves with time $\sim Exp(\lambda)$ so in average after $\frac{1}{\lambda}$

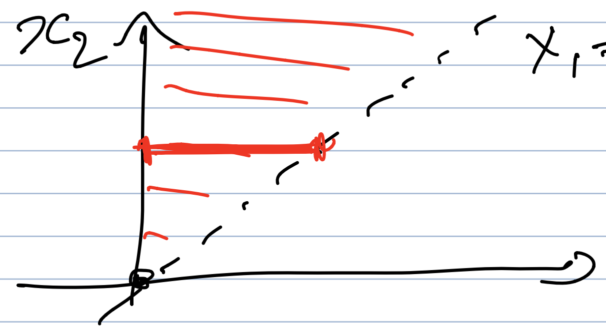
$$\Rightarrow \text{Total mean time} = \frac{1}{2\lambda} + \frac{1}{2\lambda} + \frac{1}{\lambda} = \boxed{\frac{2}{\lambda}}$$

For some similar exercise $\rightarrow$ see HW, pbm 4.

- Probability that $X_1 < X_2$ when $X_i \sim Exp(\lambda_i)$ (and independent)

Prop: If X_1 and X_2 are independent $Exp(\lambda_i)$ r.v.'s then $P(X_1 < X_2) = \dfrac{\lambda_1}{\lambda_1 + \lambda_2}$

Proof: $P(X_1 < X_2) = \iint_{x_1 < x_2} f_{X_1}(x_1) f_{X_2}(x_2)\, dx_1 dx_2$



$$= \int_0^{+\infty} dx_2\, f_{X_2}(x_2) \int_0^{x_2} f_{X_1}(x_1)\, dx_1$$

$$= (\ldots) = \frac{\lambda_1}{\lambda_1 + \lambda_2} \quad \boxtimes$$

(exercise)

- Sum of independent $Exp(\lambda)$

Rmk: If X and Y are independent continous real r.v.'s, then the pdf of $Z = X + Y$ is

$$f_Z(z) = \int_{-\infty}^{+\infty} f_X(z-y) f_Y(y)\, dy$$

$$= f_X * f_Y(z)$$

"convolution product of f_X and f_Y"

Exercise: If $X_1 \sim Exp(\lambda_1)$ and $X_2 \sim Exp(\lambda_2)$ and $\lambda_1 \neq \lambda_2$, then

$$(f_{X_1} * f_{X_2})(t) = \lambda_1 \lambda_2 e^{-\lambda_1 t} \int_0^t e^{-(\lambda_2 - \lambda_1)s} ds$$

(true for all λ_1, λ_2)

and for ($\lambda_1 \neq \lambda_2$)

$$f_{X_1+X_2}(t) = \frac{\lambda_1}{\lambda_1 - \lambda_2} \lambda_2 e^{-\lambda_2 t} + \frac{\lambda_2 \lambda_1}{\lambda_2 - \lambda_1} e^{-\lambda_1 t}$$

Rmk: This formula generalizes to n distinct $Exp(\lambda_i)$ (see Ross S. 2.4)

What happens when $\lambda_1 = \lambda_2 = \dots := \lambda$?

$$f_{X_1+X_2}(t) = (f_{X_1} * f_{X_2})(t) = \lambda^2 e^{-\lambda t} \int_0^t \cancel{e^{-(\lambda-\lambda)s}}^{\,1} ds$$

$(t \geq 0)$

so $f_{X_1+X_2}(t) = \lambda^2 e^{-\lambda t} \cdot t$

By induction, one can show $\boxed{f_{X_1+\dots+X_n} = \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!}}$

→ We recognize here the "Gamma" distribution $\Gamma(n, \lambda)$

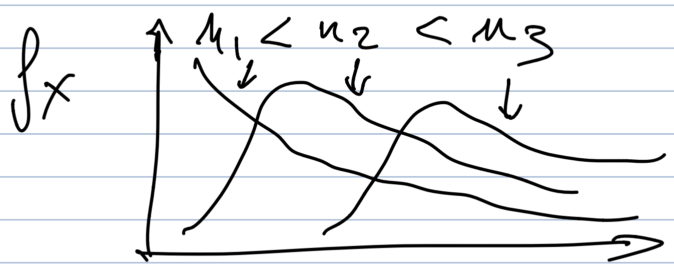
Prop: If $X_1, \dots, X_n$ are independent $\mathrm{Exp}(\lambda)$ r.v.'s, then $\sum_{i=1}^{n} X_i \sim \Gamma(n, \lambda)$,
i.e. the pdf of the sum is $f(t) = \lambda e^{-\lambda t} \dfrac{(\lambda t)^{n-1}}{(n-1)!}$

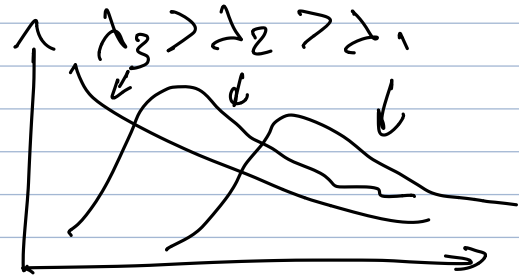
Rmk: Mean and variance of $\Gamma(n, \lambda)$

- $\mathbb{E}(X) = \sum_{i=1}^{n} \mathbb{E}(X_i) = \boxed{\dfrac{n}{\lambda}}$
- $\mathrm{Var}(X) \underset{\text{by independence}}{=} \sum_{i=1}^{n} \mathrm{Var}(X_i) = \boxed{\dfrac{n}{\lambda^2}}$

$\rightarrow$ when $n \nearrow$ mean and variance $\nearrow$

$\lambda \nearrow$ mean and variance $\searrow$





Upon considering such a sum of i.i.d $\mathrm{Exp}(\lambda)$ r.v.'s, we'll a first description of the Poisson process