#3. Evaluate $\int_C F \cdot dr$, where C is parameterized by $r(t) = \langle t^3, 2, t^2 \rangle$, $(0 \le t \le 1)$ and $F(x,y,z) = \langle \frac{1}{y^3+1}, \frac{1}{z+1}, 1 \rangle$.

Solution: Use the definition of the line integral:

$$\int_C F \cdot dr = \int_0^1 \underbrace{\left(\frac{1}{9}\hat{\imath} + \frac{1}{t^2+1}\hat{\jmath} + 1\hat{k} \right)}_{F(r(t))} \cdot \underbrace{\left(3t^2\hat{\imath} + 0\hat{\jmath} + 2t\hat{k} \right)}_{r'(t)} dt$$

$$= \int_0^1 \left(\frac{1}{3}t^2 + 0 + 2t \right) dt$$

$$= \frac{1}{9}t^3 + t^2 \Big|_0^1$$

$$= \boxed{\frac{10}{9}}$$

#4. $F = \dfrac{y}{x^2+y^2}\,\hat{\imath} - \dfrac{x}{x^2+y^2}\,\hat{\jmath} = P\hat{\imath} + Q\hat{\jmath}$

(a) We have $\dfrac{\partial P}{\partial y} = \dfrac{\partial}{\partial y}\left[\dfrac{y}{x^2+y^2}\right] = \dfrac{(x^2+y^2)(1) - y(2y)}{(x^2+y^2)^2} = \dfrac{x^2-y^2}{(x^2+y^2)^2}$

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}\left[\frac{-x}{x^2+y^2}\right] = \frac{(x^2+y^2)(-1) - (-x)(2x)}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

So $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$.

(b) Does this imply F is conservative? No! The equality $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$ only implies F is conservative if F is defined on a simply connected region, and $\mathbb{R}^2 \setminus \{(0,0)\}$ is not simply connected!

#6 Let $f(x,y,z) = xy\sin(yz)$, $F = \nabla f$.

Evaluate $\int_C F \cdot dr$, where C is any path from $(0,0,0)$ to $(1,1,\pi)$.

Solution: Use the fundamental theorem for line integrals!

F is conservative, so the hypothesis is satisfied.

Thus, for any C from $(0,0,0)$ to $(1,1,\pi)$,

$$\int_C F \cdot dr = f(1,1,\pi) - f(0,0,0)$$

$$= (1)(1)\sin(\pi) - 0$$

$$= \boxed{0}.$$

#9 Use Green's thm to find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1$, where $a, b > 0$.

Solution: We can parameterize the boundary of the ellipse by $x = a\cos\theta$, $y = b\sin\theta$, for $0 \le \theta \le 2\pi$. By Green's theorem,

$$\text{area}(E) = \iint_E 1\, dA \overset{\text{(Green's thm)}}{=} \int_{\partial E} x\,dy = \int_0^{2\pi} (a\cos\theta)\, d(b\sin\theta)$$

$$= \int_0^{2\pi} ab\cos^2\theta\, d\theta$$

$$= ab \int_0^{2\pi} \tfrac{1}{2}(1 + \cos 2\theta)\, d\theta$$

$$= \boxed{\pi ab.}$$

Here E = the interior of the ellipse, ∂E = boundary of the ellipse, oriented cc-wise.