Math 302, PSET 4

(1) Let U_1, U_2 be independent uniform random variables on $(0,1)$, and let $X = |U_1 - U_2|$.
  (a) Compute $\mathbb{E}[X]$ and $\mathbb{E}[X^2]$.
  (b) Compute $\mathrm{Corr}(U_1, X)$.
  (c) Determine a formula for the conditional probability density $f_{U_1|X}(u|x)$ of U_1 given X.

(2) Let X be a Poisson(1) random variable, and let Y be the random variable distributed as Uniform$(0, X)$.
  (a) Compute $\mathbb{E}Y$.
  (b) Compute $\mathrm{Corr}(X, Y)$.
  (c) Determine a formula for the conditional probability density $f_{X|Y}(x|y)$ of X given Y.
  (d) What is the distribution of the conditional expectation $\mathbb{E}[X|Y]$? What about $\mathbb{E}[Y|X]$?

(3) (Anderson, 4.16) Choose 500 numbers uniformly from the interval $[1.5, 4.8]$.
  (a) Approximate the probability that less than 65 of the numbers start with the digit 1.
  (b) Approximate the probability of the event that more than 160 of the numbers start with the number 3.

(4) Let $X_1, X_2, \ldots, X_n$ be independent Bernoulli(1/2) random variables, i.e. a sequence of n coin flips.
  (a) Let T_n be the number of indices $0 \leq i \leq n-2$ where X_i, X_{i+1} and X_{i+2} are all 1. Find $\mathbb{E}T$ and $\mathrm{Var}(T)$.
  (b) Fix $n = 5$. Describe the distribution of the conditional expectation $\mathbb{E}[T|X_3]$ in terms of the X_i's.

(5) Let Z_1, Z_2 be independent $\mathrm{N}(0,1)$ random variables. Identify the distribution of $Z_1 + Z_2$ (it should be familiar) in two different ways:
  (a) Using the convolution formula.
  (b) Using moment generating functions.

(6) Let Z_1, Z_2 be independent $\mathrm{N}(0,1)$ random variables. Identify the distribution of $Z_1^2 + Z_2^2$ (it should be familiar) in two different ways:
  (a) Using the convolution formula.
  (b) Using moment generating functions.

(7) Give an example of two jointly continuous random variables X, Y satisfying:
  - X and Y are not independent
  - X and Y do not have the same marginal distribution
  - $\mathrm{Cov}(X, Y) = 0$.

(8) Suppose X is a random variable with moment generating function $M_X(t) = \frac{1}{4}e^{-t} + \frac{1}{4} + \frac{1}{2}e^{4t}$.
  (a) Find the mean and variance of X by differentiating M.
  (b) Find the PMF of X, and use it to check your answers from part (a).

(9) (Anderson, 8.12) Let Z be Gamma$(2, \lambda)$ distributed for some $\lambda > 0$, i.e.

$$f_Z(z) = \begin{cases} \lambda^2 z e^{-\lambda z}, & z \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

  (a) Find the moment generating function of Z.

(b) Let X, Y be independent Exponential(λ) random variables. Show that $X + Y$ has the same distribution as Z.

(10) Let C be a Cauchy random variable, i.e. C has density function

$$f(x) = \frac{1}{\pi} \frac{1}{1 + x^2}.$$

(a) Show that the moment generating function $M_C(t)$ of C is infinite, except at $t = 0$.
(b) For which numbers $\alpha > 0$ is $\mathbb{E}[C^\alpha] < \infty$?
(c) Let Z_1, Z_2 be independent r.v.'s with $\mathrm{N}(0, 1)$ distribution. Show that Z_1/Z_2 has the same distribution as C.
(d) Let C_1, C_2 be independent r.v.'s with Cauchy distribution. Show that $\frac{1}{2}(C_1 + C_2)$ has the same distribution as C. [Challenging]

(11) Let $\theta_1, \theta_2, \theta_3$ be independent uniform random variables on $[0, 2\pi]$. Let T be the random triangle with vertices on the unit circle at angles $\theta_1, \theta_2, \theta_3$, and let X be the area of T.
(a) Let $\alpha = \min\{\theta_2, \theta_3\}, \beta = \max\{\theta_2, \theta_3\}, \gamma = \beta - \alpha = |\theta_3 - \theta_2|$. Find the PDF's of α, β, γ.
(b) Show that X is equal in distribution to

$$X = \frac{1}{2} (\sin \alpha - \sin \beta + \sin \gamma).$$

(Hint: assume WLOG that $\theta_1 = 0$.)

(c) Use the expression from part (b) to find $\mathbb{E}X$.

Additional exercises: Anderson 4.20, 6.28, 6.48, 6.58, 8.29, 8.64