Math 324 Homework 5

For problems 1-3, use the definition of the line integral to evaluate $\int_C F \cdot dr$.

1. $F(x, y) = \langle x^2, xy \rangle$, $C =$ the line segment from $(0, 0)$ to $(2, 2)$.
2. $F(x, y) = \langle x^2, xy \rangle$, $C =$ part of the circle $x^2 + y^2 = 9$ in the second quadrant $x \leq 0, y \geq 0$, oriented clockwise.
3. $F(x, y, z) = \langle \frac{1}{y^3+1}, \frac{1}{z+1}, 1 \rangle$ over the curve $r(t) = \langle t^3, 2, t^2 \rangle$, for $0 \leq t \leq 1$.

4. In this problem, you will explore the vector field given by

$$\vec{F}(x, y) = \left\langle \frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right\rangle = P\hat{\imath} + Q\hat{\jmath}, \text{ where } P(x, y) = \frac{-y}{x^2 + y^2}, \; Q(x, y) = \frac{x}{x^2 + y^2},$$

which is defined on the plane $\mathbb{R}^2$ minus the origin $(0, 0)$.

   (a) Show that $\dfrac{dP}{dy} = \dfrac{dQ}{dx}$ where P and Q are defined.

   (b) Does this imply F is conservative?

   (c) Evaluate the integral

$$\int_C F \cdot dr = \int_C P dx + Q dy = \int_C \frac{x\,dy - y\,dx}{x^2 + y^2}$$

   over two different curves C: first, where C is the part of the unit circle in the first quadrant (that is, $0 \leq \theta \leq \pi/2$) oriented from $(1, 0)$ to $(0, 1)$; then, where C is the rest of the same circle − that is, the part of $x^2 + y^2 = 1$ not in the first quadrant $(\pi/2 \leq \theta \leq 2\pi)$ − oriented also from $(1, 0)$ to $(0, 1)$.

   (d) Does this example violate independence of path for conservative vector fields? Why or why not?

5. (Stewart 16.3.29) Show that

$$\int_C \frac{x\,dy - y\,dx}{x^2 + y^2} = 0$$

for any closed curve C that does not pass through or enclose the origin.

6. Let $f(x, y, z) = xy\sin(yz)$, and $F = \nabla f$. Evaluate $\int_C F \cdot dr$, where C is any path from $(0, 0, 0)$ to $(1, 1, \pi)$.

For problems 7 and 8, determine whether or not the vector field is conservative; if it is, give the corresponding potential function.

7. $F(x, y) = \langle 2xy + y^3, x^2 + 3xy^2 + 2y \rangle$.
8. $F(x, y, z) = \langle y, x, z^3 \rangle$.
9. Use Green's theorem to find the area of the ellipse given by $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where a and b are nonzero.

10. Use Green's theorem to evaluate the line integral

$$\int_C e^{2x+y}\,dx + e^{-y}\,dy,$$

where C is the triangle with vertices $(0,0), (1,0)$ and $(1,1)$ oriented counter clockwise.

11. (Stewart 16.4.13) Use Green's theorem to evaluate

$$\int_C F \cdot dr,$$

where $F(x,y) = \langle y - \cos(y), x\sin(y)\rangle$, and C is the circle $(x-3)^2 + (y+4)^2 = 4$ oriented clockwise.