Math 324 C - Spring 2019

Final exam

Monday, June 10, 2019

Name: ____

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- There are 5 questions on this exam. Make sure you have all five.
- **You must show your work on all problems.** The correct answer with no supporting work may result in no credit. Put a box around your FINAL ANSWER for each problem and cross out any work that you don't want to be graded.
- Give exact answers, and simplify as much as possible.
- Use the backs of pages *for scratch work only.*
- Raise your hand if you have a question.
- Any student found engaging in academic misconduct will receive a score of 0 on this exam.
- You have 110 minutes to complete the exam. Budget your time wisely!

GOOD LUCK!

1. Let S be the boundary surface of the cone region $x^2 + y^2 \leq z^2 \leq 1$, so S consists of the cone $x^2 + y^2 = z^2$ for $0 \leq z \leq 1$ and the disk $x^2 + y^2 \leq 1$ in the plane $z = 1$. Equip S with the outward pointing normal (normal points away from the cone). Use the divergence theorem to evaluate the surface integral

$$\iint_S \langle 5, xy^2 z, -xyz^2 \rangle \cdot d\mathbf{S}.$$

2a. Show that $\nabla \times \langle 3y, -2yz, \log z \rangle = \langle 2y, 0, -3 \rangle$.

2b. Let S be the part of the ellipsoid $x^2 + \frac{y^2}{9} + \frac{z^2}{4} = 5$ lying above the plane $z = 2$ oriented downward. Use Stokes' theorem and the first part of this problem to evaluate

$$\iint_S \langle 2y, 0, -3 \rangle \cdot d\mathbf{S}.$$

3. Let $F = \langle xy^2, x+y\rangle$ be a vector field in the xy-plane, and let C denote the upper half of the unit circle $x^2+y^2=1, y\geq 0$ oriented counter-clockwise. Also, let D be the upper half of the unit disk $x^2+y^2\leq 1, y\geq 0$.

   a. Draw a picture of D and C, and evaluate $\int_C F\cdot d\mathbf{r}$.

   b. Henry, Jacob's evil twin brother, claims he is able to use Green's theorem to solve part a. He reasons as follows: "The curve C bounds the region D with the positive orientation, so by Green's theorem,

$$\int_C F\cdot d\mathbf{r} = \iint_D \left(\frac{d}{dx}(x+y) - \frac{d}{dy}(xy^2)\right) dA = \int_0^{\pi}\int_0^1 (1-2r^2\sin\theta\cos\theta) r\,dr\,d\theta = \frac{\pi}{2}."$$

   What is wrong with Henry's argument?

   c. Use Green's theorem correctly to relate a double integral over D to a line integral. Explain why Henry got the right answer, even though his reasoning is flawed.

4a. Show that $\nabla \cdot (\nabla \times F) = 0$ for any smooth vector field $F = \langle P, Q, R \rangle$ on $\mathbb{R}^3$.

4b. Use the result from part a to show that

$$\iint_S \text{curl}(F) \cdot d\mathbf{S} = 0$$

where S is the surface $x^2 + (y-1)^2 + (z+1)^2 = 3$, and $F = \langle 9y^2x^3 + 3z^4, 2xz^2, z^5 \rangle$. [If you don't know how to use the result from part a, but have some other way of showing this fact, you can receive partial credit on this problem.]

5. For each of the statements below, circle **True** if you think it is true, and circle **False** if you think it is false. You may use the space to do scratch work, but no partial credit will be awarded on this question.

(a) **True** **False** The vector field $G(x, y, z) = \langle 3x - 2ye^x, ye^y + z, 3x\rangle$ satisfies

$$\nabla \cdot (\nabla \times (\nabla \times \nabla(\nabla \cdot G))) = -2ye^x + e^y.$$

(b) **True** **False** Any surface $S \subset \mathbb{R}^3$ can be given an orientation (i.e. continuous choice of normal vector $\hat{n}$) so that the integral

$$\iint_S F \cdot d\mathbf{S} = \iint_S F \cdot \hat{n}\, dS$$

exists.

(c) **True** **False** There exists a function $h(x, y, z)$ and a closed loop C in $\mathbb{R}^3$ (same starting and ending point) so that

$$\int_C h \cdot ds \neq 0.$$

6. Let C be the cylinder $x^2 + y^2 \leq 1$ for $0 \leq z \leq 1$, and let S be the boundary surface of C, oriented outward, so S consists of the cylinder for $0 < z < 1$ and the disks of radius 1 in the planes $z = 0$ and $z = 1$. Without using the divergence theorem or evaluating any integrals, explain the following two (true) equalities:

$$\iint_S z\hat{k} \cdot d\mathbf{S} = \pi,$$

and

$$\iint_S |x|\hat{\imath} \cdot d\mathbf{S} = 0.$$