DIFFUSION-LIMITED ANNIHILATING-COALESCING SYSTEMS

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Abstract. We introduce a family of interacting particle systems in which particles diffuse as random walks on a transitive unimodular graph. Upon colliding, like particles coalesce and unlike particles annihilate. We describe a phase-transition as the initial particle density is varied and also the expected occupation time of the root.

1. Introduction

Coalescing and annihilating random walk were among the first interacting particle systems with infinitely many diffusive particles to be rigorously studied [Gri78, Arr83]. Subsequently, two-type annihilating systems were investigated by Bramson and Lebowitz [BL91]. These settings assumed particles diffuse at the same rate. There has been a recent resurgence in interest in systems with particles diffusing at different rates [DGJ$^+$19, PRS19, JJLS20, CRS18, BBJJ22, JL18]. Seeing as these dynamics arose as chemical reaction models, it is natural to allow for both coalescing and annihilating reactions.

Fix a simple, locally finite, and connected graph $G = (\mathcal{V}, \mathcal{E})$ with root $\mathbf{0}$. Each site $x \in \mathcal{V}$ initially contains $\xi_0(x)$ particles. If $\xi_0(x) > 0$, then the particles are all of type A_1, and if $\xi_0(x) < 0$, then the particles are B_1-particles. The site x additionally is assigned a tuple $(S^{x,j}, U^{x,j})_{j\in\mathbb{Z}}$. The $S^{x,j}$ are discrete simple random walk paths $S^{x,j} = (S^{x,j}_n)_{n\geq 0}$ started at x with Markov transition kernel K. So, K is a function $K \colon \mathcal{V}\times\mathcal{V} \to [0,1]$ such that $K(u,v) = 0$ if $(u,v) \notin \mathcal{E}$ and $\sum_{v\in N(u)} K(u,v) = 1$. The $U^{x,j}$ are uniform and independent *braveries* sampled from $(0,1)$. The jth particle started at x jumps along the vertices of $S^{x,j}$ according to a rate $\lambda_A > 0$ exponential clock if it is an A-particle, and a rate $\lambda_B \geq 0$ clock if it is a B-particle.

Fix *caps* $M, N \in [0,\infty]$. When multiple particles meet, the two particles with the highest bravery, depending on their type react. $A+B$ reactions result in mutual annihilation. When like particles meet they coalesce so long as both particles have not coalesced beyond the cap, otherwise, the particles do not interact and continue diffusing. These rules may be summarized as

$$
\begin{aligned}
&A_i + B_j \to \varnothing \quad \forall i,j \geq 1, \\
&A_i + A_j \to \begin{cases} A_{i+j}, & \max(i,j) \leq M \\ A_i\, A_j, & \text{else} \end{cases} \\
&B_i + B_j \to \begin{cases} B_{i+j}, & \max(i,j) \leq N \\ B_i\, B_j, & \text{else} \end{cases} .
\end{aligned} \tag{1}
$$

If multiple particles are present at the site, then the interactions are repeated pairwise giving priority to the highest bravery particles until no further interactions can occur at that instant. When a coalescing reaction occurs, the new particle inherits the path and bravery of whichever of the two most recently coalesced particles has the highest bravery. We refer to the index of coalesced particles as the *size*.

Call the interacting particle system with dynamics as at (1) a *diffusion-limited annihilating-coalescing system* (DLACS). We let $\xi_t^i = (\xi_t^i(x))_{x\in\mathbb{V}}$ denote the number of A_i particles at x at time t for $i \geq 1$. Similarly, we set $\xi_t^{-i} = (-\xi_t^{-i}(x))_{x\in\mathbb{V}}$ with $\xi_t^{-i}(x)$ the number of B_i-particles at x at time t. Set $\xi_t(x) = \sum_{i=1}^\infty \xi_t^i(x)$ so that positive entries of $\xi_t(x)$ correspond to that number of A-particles occupying x at time t, and negative entries the number of B-particles at x at time t. Note that A_i particles count as one particle in ξ_t regardless of the value of i.

Since an A_i-particle is a mass of i-coalesced A_1-particles, define

$$W_T(x) = \int_0^T \sum_{i\geq 1} i\xi_t^i(x)\, dt$$

to be the *weighted occupation time of* x *by* A*-particles up to time* t. Also, let

$$V_T(x) = \int_0^T \sum_{i\geq 1} \xi_t^i(x)\, dt$$

be the *occupation time* of x by A-particles up to time t.

To simplify the presentation, we consider initial configurations with either a single A_1-particle or single B_1-particle independently at each site. That is $(\xi_0(x))_{x\in\mathcal{V}} \in \{-1,1\}^{\mathcal{V}}$ is assigned according to a product measure with $\mathbb{P}(\xi_0(x) = 1) = p \in [0,1]$.

To state our theorem we need a few quantities. We will denote the event that the vertex u initially contains an A_1-particle by $\mathfrak{a}_u$, and similarly for $\mathfrak{b}_u$. In a convenient abuse of notation we will refer to the generic particle started at u as $\bullet_u$, and as $\mathfrak{a}_u$ or $\mathfrak{b}_u$ if its type is specified. We will view A_i-particles (and B_i-particles) as clusters of i coalesced A-particles (or i B-particles) that are following the path of the bravest particle in the cluster. Define the events

$$\begin{aligned}
\mathfrak{a}_u \longleftrightarrow \mathfrak{b}_v &= \{\mathfrak{a}_u \text{ and } \mathfrak{b}_v \text{ mutually annihilate}\}\\
\mathfrak{a}_u \longleftrightarrow \mathfrak{b} &= \{\exists v \text{ such that } \mathfrak{a}_u \text{ mutually annihilates with } \mathfrak{b}_v\}\\
\mathfrak{b}_u \longleftrightarrow \mathfrak{a} &= \{\exists v \text{ such that } \mathfrak{b}_u \text{ mutually annihilates with } \mathfrak{a}_v\}.
\end{aligned}$$

We denote complements of these events with $\not\longleftrightarrow$.

Let $\tau(\mathfrak{a}_x)$ be the lifespan of an A-particle started at x. Set $\tau(\mathfrak{a}_x) = 0$ if x contains a B-particle, and $\tau(\mathfrak{a}_x) = \infty$ if the A-particle at x is never destroyed. Also, let $S(\mathfrak{b}_u)$ be the size of the particle that mutually annihilates with a B-particle at u. This implicitly is multiplied by $\mathbf{1}\{\mathfrak{b}_u\}$, so that $S(\mathfrak{b}_u) = 0$ if there is initially an A-particle at u, or if $\mathfrak{b}_u \not\longleftrightarrow \mathfrak{a}$. Define $S(\mathfrak{a}_u)$ similarly, but for the size of the B-particle that collides with an A-particle at $\mathbf{0}$. This lets us specify the sizes of particles at the time of mutual annihilation

$$\mathfrak{a}_u \overset{j,k}{\longleftrightarrow} \mathfrak{b}_v = \{\mathfrak{a}_u \text{ and } \mathfrak{b}_v \text{ mutually annihilate}\} \cap \{S(\mathfrak{a}_u) = k\} \cap \{S(\mathfrak{b}_v) = j\}.$$

We are interested in the phase transition at which some A-particles survive indefinitely. Since it is not immediately obvious that $\mathbb{P}(\mathfrak{a}_0 \not\longleftrightarrow \mathfrak{b} \mid \mathfrak{a}_0)$ is monotone in p, we define the upper and lower critical values

$$p_c^- = \inf\{p\colon \mathbb{P}(\mathfrak{a}_{\mathbf{0}} \not\longleftrightarrow \mathfrak{b} \mid \mathfrak{a}_{\mathbf{0}}) > 0\}$$
$$p_c^+ = \sup\{p\colon \mathbb{P}(\mathfrak{a}_{\mathbf{0}} \not\longleftrightarrow \mathfrak{b} \mid \mathfrak{a}_{\mathbf{0}}) = 0\}.$$

1.1. **Graph Conditions.** We describe the graphs for which our results will hold. These include canonical graphs such as integer lattices and regular trees. Denote by $\mathrm{Aut}(G)$ the group of all automorphisms of G. Let $\mathrm{Aut}_K(G)$ be the subgroup of $\mathrm{Aut}(G)$ consisting of all K-preserving automorphisms; that is

$$\mathrm{Aut}_K(G) = \{\varphi \in \mathrm{Aut}(G)\colon K(u,v) = K(\varphi(u),\varphi(v)) \quad \forall u,v \in \mathcal{V}\}.$$

Given a subgroup $\Gamma_K \leq \mathrm{Aut}_K(G)$ of K-preserving automorphisms of G, for each $u,v \in \mathcal{V}$, denote $\Gamma_K(u,v) = \{\varphi \in \Gamma_K : \varphi(u) = v\}$. We define the following conditions on the triple (G,K,Γ_K).

1. (Transitivity) (G,K,Γ_K) is *transitive* if $\Gamma_K(u,v)$ is nonempty for each $u,v \in \mathcal{V}$.
2. (Unimodularity) (G,K,Γ_K) is *unimodular* if for each $u,v \in \mathcal{V}$,
$$|\Gamma_K(u,u)v| = |\Gamma_K(v,v)u| < \infty.$$
3. (Infinite Accessibility) For each $u,v \in \mathcal{V}$, we say that u is accessible from v if there exists a sequence $v = x_0, x_1, \dots, x_n = u$ of adjacent nodes such that $\prod_{i=0}^{n-1} K(x_i, x_i+1) > 0$. The triple (G,K,Γ_K) is *infinitely accessible* if there exist $\varphi \in \Gamma_K$ and $u \in \mathcal{V}$ such that $\{\varphi^n(u)\colon n \geq 0\}$ is infinite and u is accessible from $\varphi(u)$.

1.2. **Results.** Our first result establishes that the values p_c^- and p_c^+ coincide and are equal to the value of p for which the starting density of A-particles times the average number of B-particles destroyed per collision is smaller than the starting density of B-particles times the average number of A-particles destroyed per collision.

Theorem 1. *For DLACS on a transitive, infinitely accessible, unimodular graph, define*

$$p_c = p_c(\lambda_A,\lambda_B,M,N) := \sup\left\{p \in [0,1]\colon \frac{p}{1-p}\frac{\mathbb{E}[S(\mathfrak{a}_{\mathbf{0}}) \mid \mathfrak{a}_{\mathbf{0}} \longleftrightarrow \mathfrak{b}]}{\mathbb{E}[S(\mathfrak{b}_{\mathbf{0}}) \mid \mathfrak{b}_{\mathbf{0}} \longleftrightarrow \mathfrak{a}]} < 1\right\}.$$

It holds that $p_c^- = p_c = p_c^+$.

The formula for p_c in Theorem 1 lets us deduce that $p_c = 1/2$ for any symmetric DLACS.

Corollary 2. *If $\lambda_A = \lambda_B$ and $M = N$, then $p_c = 1/2$.*

Since $S(\mathfrak{a}_{\mathbf{0}}) \leq 2M$ and $S(\mathfrak{b}_{\mathbf{0}}) \leq 2N$, we obtain the following bounds on p_c.

Corollary 3. *For all $N,M \in [0,\infty]$*

$$\frac{1}{1+2M} \leq p_c \leq \frac{2N}{1+2N}.$$

We also characterize the weighted occupation time of $\mathbf{0}$ by A-particles.

Theorem 4. *For all $T \geq 0$*

$$\mathbb{E}[W_T] = \int_0^T \mathbb{P}(\tau(\mathfrak{a}_{\mathbf{0}}) > t)\,dt. \tag{2}$$

As $\mathbb{P}(\tau(\mathfrak{a}_\mathbf{0}) > t) \geq \mathbb{P}(\mathfrak{a}_\mathbf{0} \not\leftrightarrow \mathfrak{b})$, Theorem 4 and Theorem 1 imply linear growth of the weighted occupation time in the supercritical regime. The relationship $V_T \geq W_T/2M$ also gives a linear growth bound on V_T.

Corollary 5. *$\mathbb{E}[W_T] \geq \mathbb{P}(\mathfrak{a}_\mathbf{0} \not\leftrightarrow \mathfrak{b})T$ for all $T \geq 0$. And, if $M < \infty$, then*

$$\mathbb{E}[V_T] \geq \frac{\mathbb{P}(\mathfrak{a}_\mathbf{0} \not\leftrightarrow \mathfrak{b})}{2M} T.$$

When $M = \infty$ there is no obvious comparison between W_T and V_T. The case $p = 1$ and $M = \infty$ is classical coalescing random walk. It is proven in [BFGG+16, Theorem 1.2 (i)] that graphs with maximum degree D satisfy $\mathbb{E}[V_T] \geq \log(1 + DT)/D$. We prove an analogous logarithmic bound in the supercritical regime.

Theorem 6. *Suppose that $M = \infty$. Let D be the degree of the vertices in G. It holds that*

$$\mathbb{E}\sum_{k\geq 1} \xi_t^k \geq \frac{[\mathbb{P}(\tau(\mathfrak{a}_\mathbf{0}) > t)]^2}{1 + Dt} \text{ for all } t \geq 0, \tag{3}$$

and thus,

$$\mathbb{E}[V_T] \geq \int_0^T \frac{[\mathbb{P}(\tau(\mathfrak{a}_\mathbf{0}) > t)]^2}{1 + Dt} dt \geq \frac{[\mathbb{P}(\mathfrak{a}_\mathbf{0} \not\leftrightarrow \mathfrak{b})]^2}{D} \log(1 + DT)$$

for all $T \geq 0$.

As for an upper bound on $\mathbb{E}[V_T]$, the monotonicity result in Lemma 7 ensures that V_T is dominated by its behavior in the case $p = 1$ with coalescing random walk. The occupation time of the root for coalescing random walk is known to grow logarithmically for trees and high dimensional lattices and more rapidly for graphs on which random walk is recurrent [BFGG+16, FHJ18, Gri78].

1.3. **Further questions.** The characterization of p_c in Theorem 1 is implicit. We are interested in finding explicit formulas.

Question 1. *Find p_c for any infinite graph and non-symmetric choice of $\lambda_A, \lambda_B, M, N$.*

This is likely difficult for general G, but the graphs $\mathbb{Z}^d$ and the d-regular tree $\mathbb{T}_d$ are good places to start. An essential first step is proving that p_c is non-trivial when $M = \infty$ and $N < \infty$. Here is a concrete question.

Question 2. *Let $G = \mathbb{Z}^d$ and fix $\lambda_A = 1$, $\lambda_B \in [0, \infty)$, $M = \infty$, and $N = 0$. Is $p_c < 1$?*

Even showing that $p_c < 1$ on $\mathbb{Z}$ with $\lambda_B = 0$ would be interesting. This is similar to the setting from [DGJ+19, JJLS20], but with coalescence.

It would also be interesting to study the critical behavior at $p = p_c$ for DLACS. This was worked out by Bramson and Lebowitz in [BL91] for diffusion-limited annihilating systems in which $\lambda_A = \lambda_B$. They proved that the expected density of A-particles at the origin of $\mathbb{Z}^d$ at time t, call it $\rho_t(d)$, exhibits the following asymptotic behavior

$$\rho_t(d) \approx \begin{cases} t^{-d/4}, & d = 1, 2, 3 \\ t^{-1}, & d \geq 4 \end{cases}.$$

This confirmed non-mean-field behavior in low dimension and settled conflicting predictions from physicists. Followup work has focused on the critical behavior for diffusion-limited annihilating systems with asymmetric speeds [CRS18, JJLS20, DGJ+19]. All of these results would be worthwhile to extend to DLACS. Here is one of many such questions.

Question 3. *Estimate the growth of* $\mathbb{E}[W_T]$ *or* $\mathbb{E}[V_T]$ *when* $p = p_c$ *for any DLACS with* $M > 0$ *on an infinite graph.*

1.4. **Overview of proofs.** In Lemma 7, we prove that the longevity of all A-particles is non-decreasing when additional A-particles are introduced to the system. The argument generalizes an idea from [CRS18] that uses tracers to track the difference between two systems. This monotonicity lets us deduce that $p_c^- = p_c^+$. The formula for p_c in Theorem 1 and bounds in Theorem 4 and Theorem 6 are derived from different applications of the mass transport principle stated in Lemma 8. The bound on $\mathbb{E}[V_T]$ is the most involved and makes use of an estimates on the particle size in coalescing random walk from [FHJ18].

2. Proofs

First we prove a monotonicity result for the lifespan of A-particles.

Lemma 7. *Consider two systems* ξ *and* ξ^+ *with identical underlying* $(S^{x,j}, U^{x,j})_{j\in\mathbb{Z}}$ *but* $\xi_0(x) \leq \xi_0^+(x)$ *for all* $x \in \mathcal{V}$*. Define the lifespans of* $\mathfrak{a}_\mathbf{0}$ *in the two systems as* $\tau(\mathfrak{a}_\mathbf{0})$ *and* $\tau^+(\mathfrak{a}_\mathbf{0})$*. There exists a coupling such that* $\tau(\mathfrak{a}_\mathbf{0}) \leq \tau^+(\mathfrak{a}_\mathbf{0})$*.*

Proof. Following [CRS18, Lemma 1] the quantity $\tau(\mathfrak{a}_\mathbf{0})$ can be approximated by systems with finitely many particles in a finite ball centered at $\mathbf{0}$. Thus, it suffices to exhibit a coupling with $\tau(\mathfrak{a}_\mathbf{0}) \leq \tau^+(\mathfrak{a}_\mathbf{0})$ in a system with finitely many particles $(\sum_{v\in\mathbb{V}} |\xi_0(v)| < \infty)$ and ξ^+ introducing an A_1-particle at a site x.

Suppose that $\xi^+(x) = j^*$. We modify the bravery U^{x,j^*} to be the minimum of all other braveries in the (finite) system divided by 2. Following [CRS18, Section 3.1], we will account for the differences introduced by an extra A_1-particle at x with a *tracer* that can be either *active, dormant,* or *dead.* We say that the tracer is tracking a given particle if it is following the tracked particles random walk path.

The tracer initially is *active* and tracks the extra A-particle at x following S^{x,j^*} when $j^* \geq 1$. The state of the tracer changes depending on the size/type of particle it is tracking.

Tracking an A-particle with size $\leq M$:

- Suppose that the tracked particle coalesces with another A-particle. If the resulting size is $\leq M+1$, then the tracer becomes/remains dormant and continues tracking the coalesced particle. If the resulting size is $> M+1$, then the tracer dies.
- Suppose that the tracked particle mutually annihilates with a B-particle. If the tracer is active, then it remains active and begins following the path of the B-particle. If the tracer is dormant, then it dies.

Tracking an A-particle with size $M+1$:

- Suppose that the tracked particle meets an A-particle with size $\leq M$. The particles do not coalesce. The tracer becomes/remains active and tracks whichever of the two non-coalesced particles has lower bravery.
- Suppose that the tracked particle mutually annihilates with a B-particle. If the tracer is active, the tracer remains active and begins following the path of the B-particle. If the tracer is dormant, then it dies.

Tracking a B-particle: If the tracer is active and meets another A-particle, then it begins tracking that particle and remains active. The size of the A-particle does not increase. If the tracer is dormant, then it dies.

The tracer and does not influence the evolution of ξ^+, rather, it accounts for the discrepancy between ξ and ξ^+. If the tracer is active and tracking an A-particle, then it accounts for an extra A-particle in ξ^+. If it is active and tracking a B-particle, this accounts for an extra B-particle in ξ. Otherwise, the positions of the particles in ξ and ξ^+ are identical. This is described for non-coalescing systems in [CRS18, Section 3.1]. Thus, if $\mathfrak{a}_\mathbf{0}$ never interacts with the tracer, then the extra particle has no effect on $\tau^+(\mathfrak{a}_\mathbf{0})$. So, $\tau(\mathfrak{a}_\mathbf{0}) = \tau^+(\mathfrak{a}_\mathbf{0})$. If $\mathfrak{a}_\mathbf{0}$ is at some point tracked by the tracer or interacts with a tracked particle, the tracer rules are such that A-particle paths are extended. Thus, $\tau(\mathfrak{a}_\mathbf{0}) \leq \tau^+(\mathfrak{a}_\mathbf{0})$. This gives a coupling with the claimed inequality. ☐

Our main tool is a mass-transport principle. The following is a minor modification of [LP16, Theorem 8.7].

Lemma 8. *Let $Z\colon \mathcal{V} \times \mathcal{V} \to [0,\infty)$ be a collection of random variables such that $\mathbb{E}[Z(x,y)] = \mathbb{E}[Z(\varphi(u),\varphi(v))]$ for all $\varphi \in \Gamma_K$ whenever v is accessible from u or u is accessible from v, and $\mathbb{E}[Z(u,v)] = 0$ otherwise. Then*

$$\mathbb{E}\sum_{v\in\mathcal{V}} Z(\mathbf{0},v) = \mathbb{E}\sum_{u\in\mathcal{V}} Z(u,\mathbf{0}).$$

Lemma 9. *For all $p \in [0,1]$ it holds that*

$$\mathbb{E}[S(\mathfrak{a}_\mathbf{0})] = \mathbb{E}[S(\mathfrak{b}_\mathbf{0})]. \tag{4}$$

Proof. For each $u, v \in \mathcal{V}$ and pair of integers $j,k \geq 1$, define an indicator variable

$$Z_{j,k}(u,v) = \mathbf{1}\{\mathfrak{a}_u \wedge \mathfrak{b}_v \wedge (\mathfrak{a}_u \xleftrightarrow{j,k} \mathfrak{b}_v)\}.$$

Lemma 8 ensures that

$$k\mathbb{P}(\mathfrak{a}_\mathbf{0} \xleftrightarrow{j,k} \mathfrak{b}) = \mathbb{E}\sum_{v\in\mathcal{V}} Z_{j,k}(\mathbf{0},v) = \mathbb{E}\sum_{u\in\mathcal{V}} Z_{j,k}(u,\mathbf{0}) = j\mathbb{P}(\mathfrak{a} \xleftrightarrow{j,k} \mathfrak{b}_\mathbf{0}).$$

We have used the fact that there are exactly k and j distinct sites corresponding to the k- and j-coalesced particles that are counted by the indicators. Summing over all $j,k \geq 1$ gives (4). ☐

Proof of Theorem 1. By conditioning, we may expand (4) as

$$\mathbb{E}[S(\mathfrak{a}_\mathbf{0}) \mid \mathfrak{a}_\mathbf{0} \longleftrightarrow \mathfrak{b}]\mathbb{P}(\mathfrak{a}_\mathbf{0} \longleftrightarrow \mathfrak{b} \mid \mathfrak{a}_\mathbf{0})p = \mathbb{E}[S(\mathfrak{b}_\mathbf{0}) \mid \mathfrak{b}_\mathbf{0} \longleftrightarrow \mathfrak{a}]\mathbb{P}(\mathfrak{b}_\mathbf{0} \longleftrightarrow a \mid \mathfrak{b}_\mathbf{0})(1-p).$$

Rearranging gives

$$\frac{p}{1-p}\frac{\mathbb{E}[S(\mathfrak{a}_\mathbf{0}) \mid \mathfrak{a}_\mathbf{0} \longleftrightarrow \mathfrak{b}]}{\mathbb{E}[S(\mathfrak{b}_\mathbf{0}) \mid \mathfrak{b}_\mathbf{0} \longleftrightarrow \mathfrak{a}]} = \frac{\mathbb{P}(\mathfrak{b}_\mathbf{0} \longleftrightarrow \mathfrak{a} \mid \mathfrak{b}_\mathbf{0})}{\mathbb{P}(\mathfrak{a}_\mathbf{0} \longleftrightarrow \mathfrak{b} \mid \mathfrak{a}_\mathbf{0})}.$$

Thus, whenever

$$\frac{p}{1-p}\frac{\mathbb{E}[S(\mathfrak{a}_\mathbf{0}) \mid (\mathfrak{a}_\mathbf{0} \longleftrightarrow \mathfrak{b}) \wedge \mathfrak{a}_\mathbf{0}]}{\mathbb{E}[S(\mathfrak{b}_\mathbf{0}) \mid (\mathfrak{b}_\mathbf{0} \longleftrightarrow \mathfrak{a}) \wedge \mathfrak{b}_\mathbf{0}]} > 1,$$

we have $\mathbb{P}(\mathfrak{a}_\mathbf{0} \longleftrightarrow \mathfrak{b} \mid \mathfrak{a}_\mathbf{0}) < 1$.

On the other hand, when

$$\frac{p}{1-p}\frac{\mathbb{E}[S(\mathfrak{a}_\mathbf{0}) \mid (\mathfrak{a}_\mathbf{0} \longleftrightarrow \mathfrak{b}) \wedge \mathfrak{a}_\mathbf{0}]}{\mathbb{E}[S(\mathfrak{b}_\mathbf{0}) \mid (\mathfrak{b}_\mathbf{0} \longleftrightarrow \mathfrak{a}) \wedge \mathfrak{b}_\mathbf{0}]} < 1,$$

we have

$$\frac{\mathbb{P}(\mathfrak{b}_{\mathbf{0}} \longleftrightarrow \mathfrak{a} \mid \mathfrak{b}_{\mathbf{0}})}{\mathbb{P}(\mathfrak{a}_{\mathbf{0}} \longleftrightarrow \mathfrak{b} \mid \mathfrak{a}_{\mathbf{0}})} < 1.$$

This implies that $\mathbb{P}(\mathfrak{b}_{\mathbf{0}} \longleftrightarrow \mathfrak{a} \mid \mathfrak{b}_{\mathbf{0}}) < 1$. We claim that this further implies that $\mathbb{P}(\mathfrak{a}_0 \longleftrightarrow \mathfrak{b}_0 \mid \mathfrak{a}_0) = 1$. To see why, observe that by ergodicity and (3) (applied for B-particles), the set

$$\mathcal{B}_t = \{u\colon u \text{ contains a } B\text{-particle at time } t\}$$

satisfies

$$\lim_{r\to\infty} \frac{|\mathcal{B}_t \cap \mathbb{B}(\mathbf{0}, r)|}{|\mathbb{B}(\mathbf{0}, r)|} \geq \frac{[\mathbb{P}(\tau(\mathfrak{b}_{\mathbf{0}}) > t)]^2}{1 + Dt} \geq \frac{\delta}{1 + Dt}$$

for $\delta = [\mathbb{P}(\mathfrak{b}_0 \not\longleftrightarrow \mathfrak{a})]^2 > 0$ and D the maximum degree of vertices in G. By similar reasoning as [CRS18, Lemma 6] it suffices to show that an independent rate λ_A random walk (X_t) started at $\mathbf{0}$ will almost surely coincide with a site in $\mathcal{B}_t$.

It follows from Lemma 7 that $\mathbb{P}(\mathfrak{a}_0 \not\longleftrightarrow \mathfrak{b} \mid \mathfrak{a}_0) = \mathbb{P}(\tau(\mathfrak{a}_{\mathbf{0}}) = \infty \mid \mathfrak{a}_{\mathbf{0}})$ is increasing in p. Thus, for all $p < p_c$ we have $\mathbb{P}(\mathfrak{a}_0 \not\longleftrightarrow \mathfrak{b} \mid \mathfrak{a}_0) = 1$ and for all $p > p_c$ we have $\mathbb{P}(\mathfrak{a}_0 \not\longleftrightarrow \mathfrak{b} \mid \mathfrak{a}_0) < 1$. $\square$

Proof of Theorem 4. To obtain (2), let $L_t(\mathfrak{a}_u)$ denote the location of $\mathfrak{a}_u$ at time t if $\mathfrak{a}_u$ is still in the system at time t. Define the family of indicator random variables $W(u, v, t) = \mathbf{1}\{L_t(\mathfrak{a}_u) = v\}$ for $u, v \in \mathcal{V}$ and $t \geq 0$. Lemma 8 gives

$$\mathbb{P}(\tau(\mathfrak{a}_{\mathbf{0}}) > t) = \mathbb{E}\sum_{v\in\mathcal{V}} W(\mathbf{0}, v, t) = \mathbb{E}\sum_{u\in\mathcal{V}} W(u, \mathbf{0}, t) = \mathbb{E}\sum_{i\geq 1} i\xi_t^i(\mathbf{0}). \tag{5}$$

Integrating gives the claimed formula. $\square$

Proof of Theorem 6. Let $N_{t,k} = \xi_t^k(\mathbf{0})$ be the number of size k A-particles at $\mathbf{0}$ at time t, $N_t = \sum_{k=1}^\infty N_{t,k}$ be the total number of A-particles at $\mathbf{0}$ at time t, and $n_t = \sum_{k=1}^\infty k\xi_t^k(\mathbf{0})$ be the weighted number of A-particles at $\mathbf{0}$ at time t. Let $\mathtt{size}_t(\mathfrak{a}_u)$ denote the size of the cluster that $\mathfrak{a}_u$ belongs to at time t with the convention that it is zero if $\mathfrak{a}_u$ is no longer in the system. For $u, v \in \mathbb{V}$, $t \in [0, \infty]$ and $k \geq 1$ define the indicators

$$R(u, v, t, k) = \mathbf{1}\{L_t(\mathfrak{a}_u) = v, \mathtt{size}_t(\mathfrak{a}_u) = k\}.$$

By Lemma 8

$$\mathbb{P}(\tau(\mathfrak{a}_{\mathbf{0}}) > t, \mathtt{size}_t(\mathfrak{a}_{\mathbf{0}}) = k) = \mathbb{E}\sum_{v\in\mathcal{V}} R(\mathbf{0}, v, t, k) = \mathbb{E}\sum_{u\in\mathcal{V}} R(u, \mathbf{0}, t, k) = \mathbb{E}[kN_{t,k}].$$

Multiplying both sides by k and summing gives

$$\mathbb{E}[\mathtt{size}_t(\mathfrak{a}_{\mathbf{0}})] = \sum_{k=1}^\infty k^2 \mathbb{E}[N_{t,k}]. \tag{6}$$

Let D be the degree of vertices in G. If follows from [FHJ18, Proposition 2.4] and Lemma 7 that $\mathbb{E}[\mathtt{size}_t(\mathfrak{a}_{\mathbf{0}})] \leq 1 + 2Dt$. Applying this to (6) gives

$$\mathbb{E}\sum_{k=1}^\infty k^2 N_{t,k} \leq 1 + 2Dt. \tag{7}$$

Writing $\sum_{k=1}^\infty kN_{t,k} = \sum_{k=1}^\infty k\sqrt{N_{t,k}}\sqrt{N_{t,k}}$ and applying the Cauchy-Schwartz inequality yields

$$n_t^2 \leq \left(\sum_{k=1}^\infty k^2 N_{t,k}\right)\left(\sum_{k=1}^\infty N_{t,k}\right) = \left(\sum_{k=1}^\infty k^2 N_{t,k}\right) N_t.$$

Taking expectation and then applying the bound at (7) gives

$$\mathbb{E}[n_t^2] \leq \mathbb{E}[N_t]\left(\sum_{k=1}^\infty k^2 \mathbb{E}[N_{t,k}]\right) \leq \mathbb{E}[N_t](1+Dt).$$

Since $(\mathbb{E}[n_t])^2 \leq \mathbb{E}[n_t^2]$, we may rearrange the above inequality to obtain $\mathbb{E}[N_t] \geq (\mathbb{E}[n_t])^2/(1+Dt)$. It is proven at (5) that $\mathbb{E}[n_t] = \mathbb{P}(\tau(\mathfrak{a}_{\mathbf{0}}) > t)$. This gives the claimed inequality for $\mathbb{E}[N_t]$. $\square$

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