Finding the source

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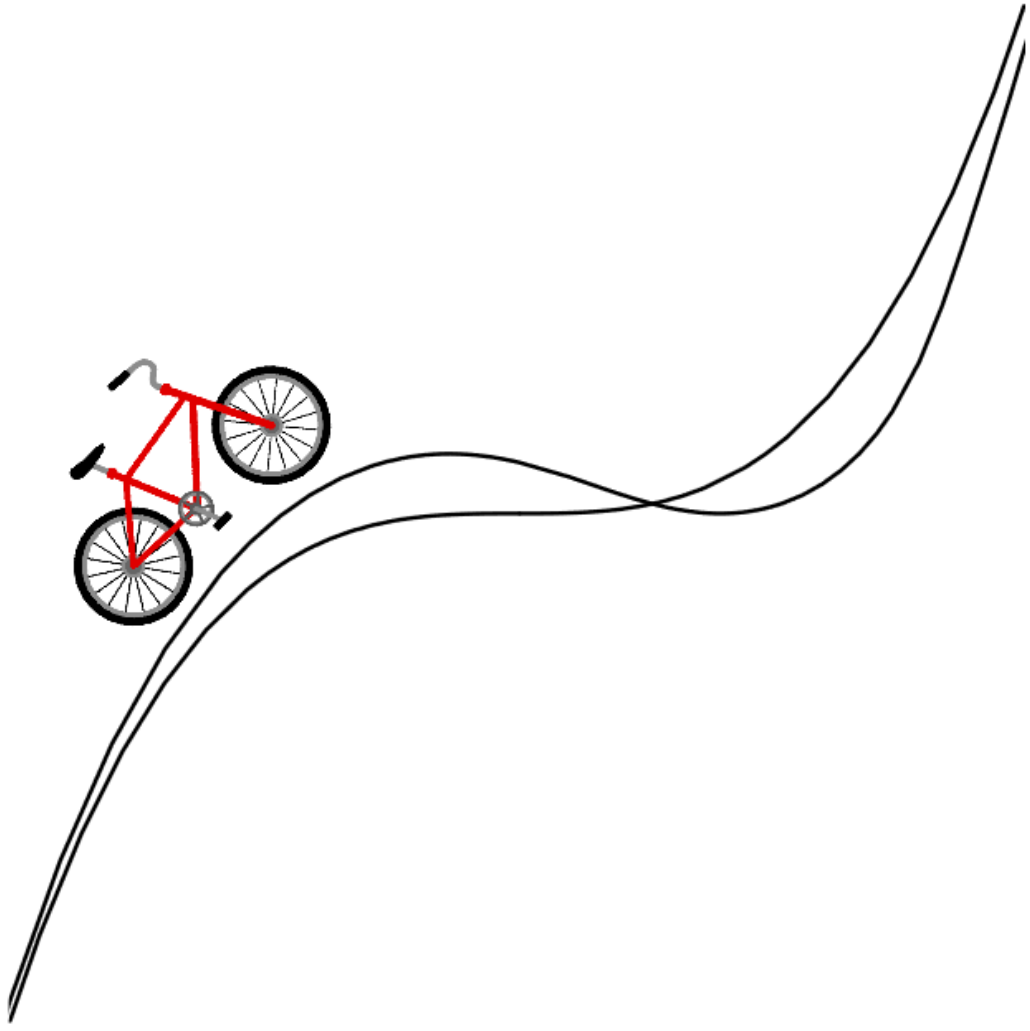
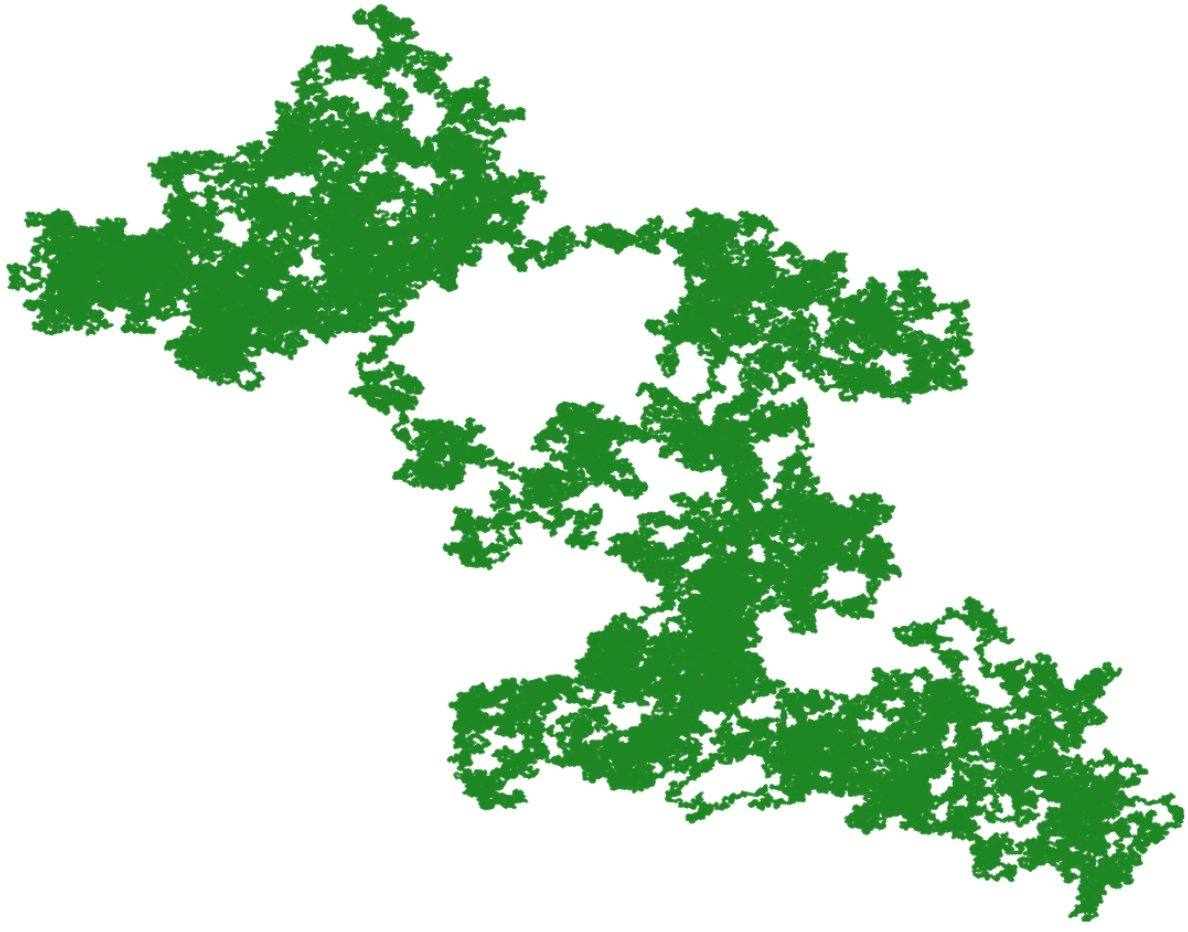
Dartmouth, November 2020



Which way did the bicycle go?

Simple random walk on $\mathbb{Z}^2$, run for $5 \cdot 10^6$ steps.



Q: Given a snapshot of a (random) process, what can be determined?

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- Step distribution/generator?
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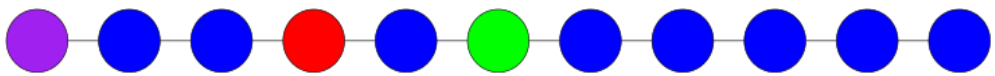

Given that the range is an interval of length N, what's the most likely starting point? Purple, red, or green?

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Proof sketch: think of the range as a 'coin switching' markov chain, compute transition probabilities recursively.

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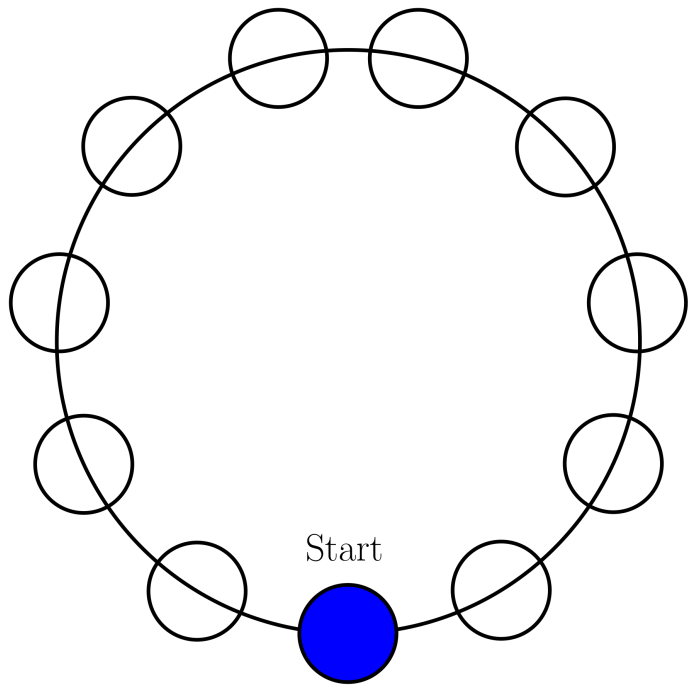
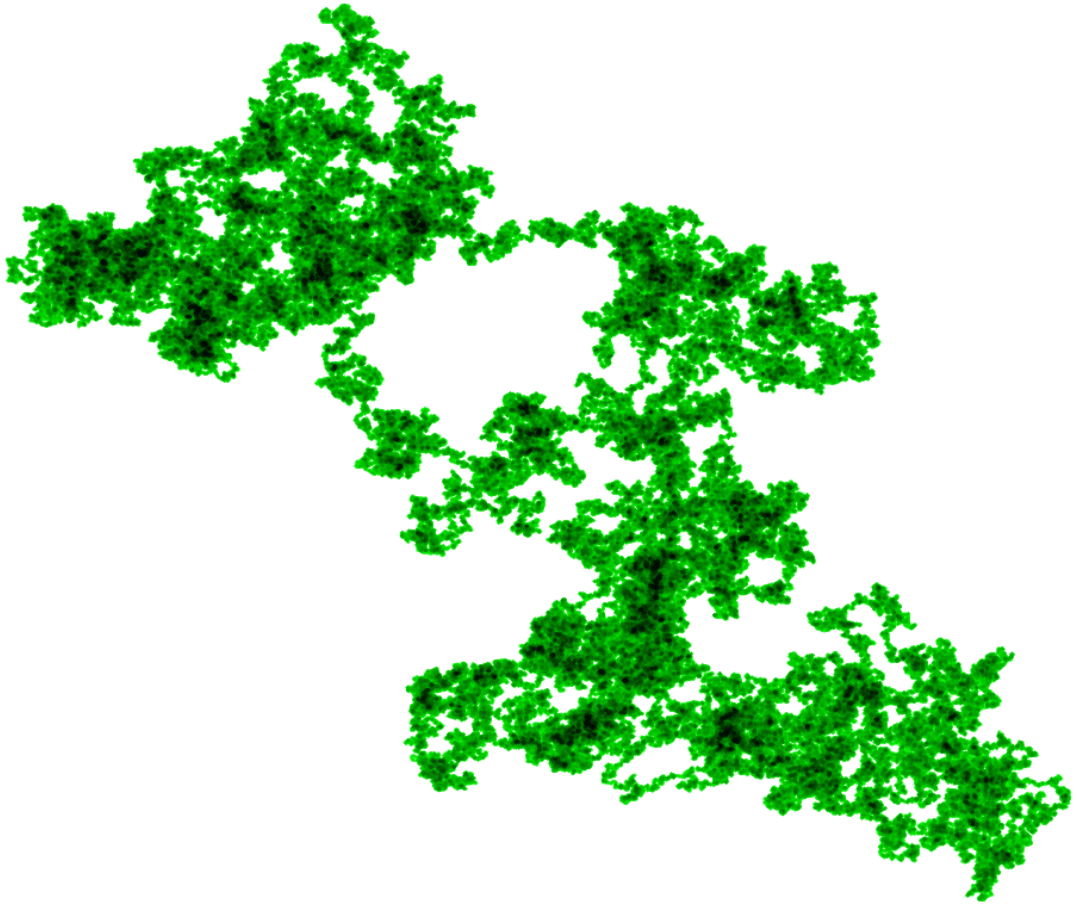
Alternatively: last vertex visited by SRW on the ring is uniform.



Previous work

Same SRW as before, with occupation times



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Theorem (Pemantle, Peres, Pitman, Yor '00)

Let $d \geq 3$, and consider Brownian motion in $\mathbb{R}^d$ run for time 1.

Given the occupation measure of the path projected onto the sphere, you can recover the range and the endpoint with probability 1.

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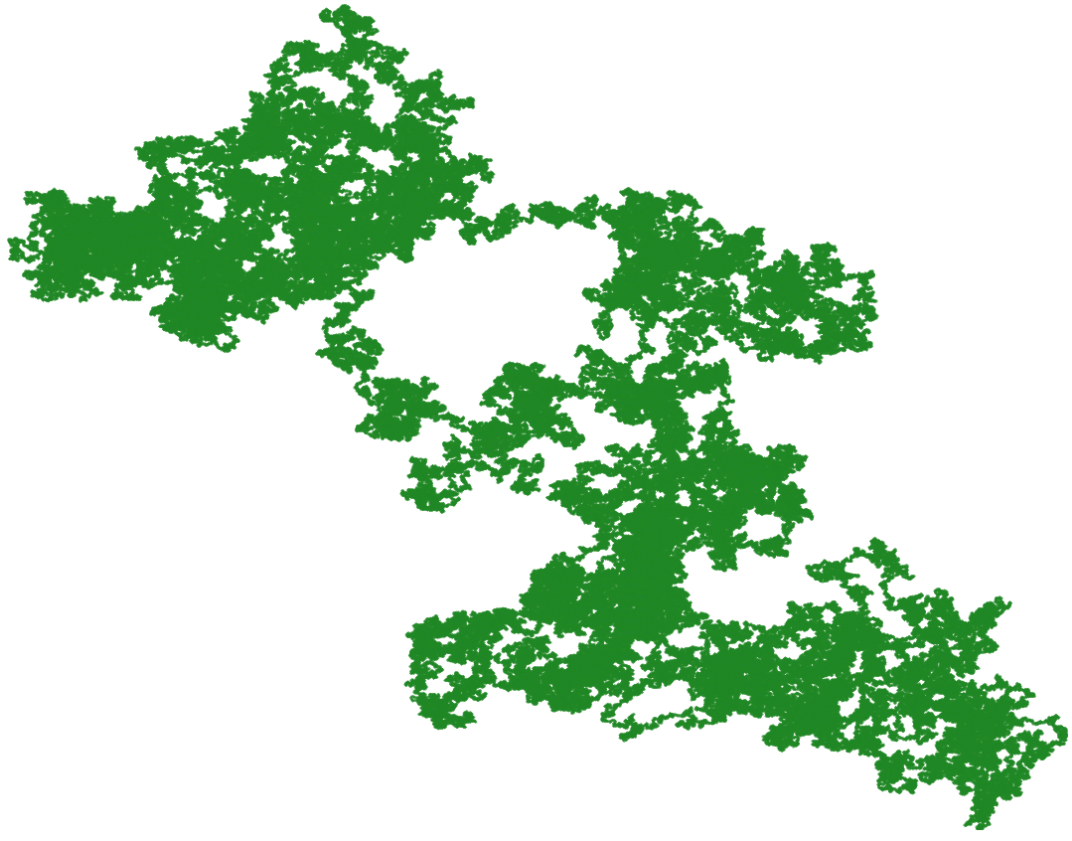
Conjecture (PPPY '00)

In dimension $d = 2$, the range cannot be recovered with probability 1.

SRW in $\mathbb{Z}^d$



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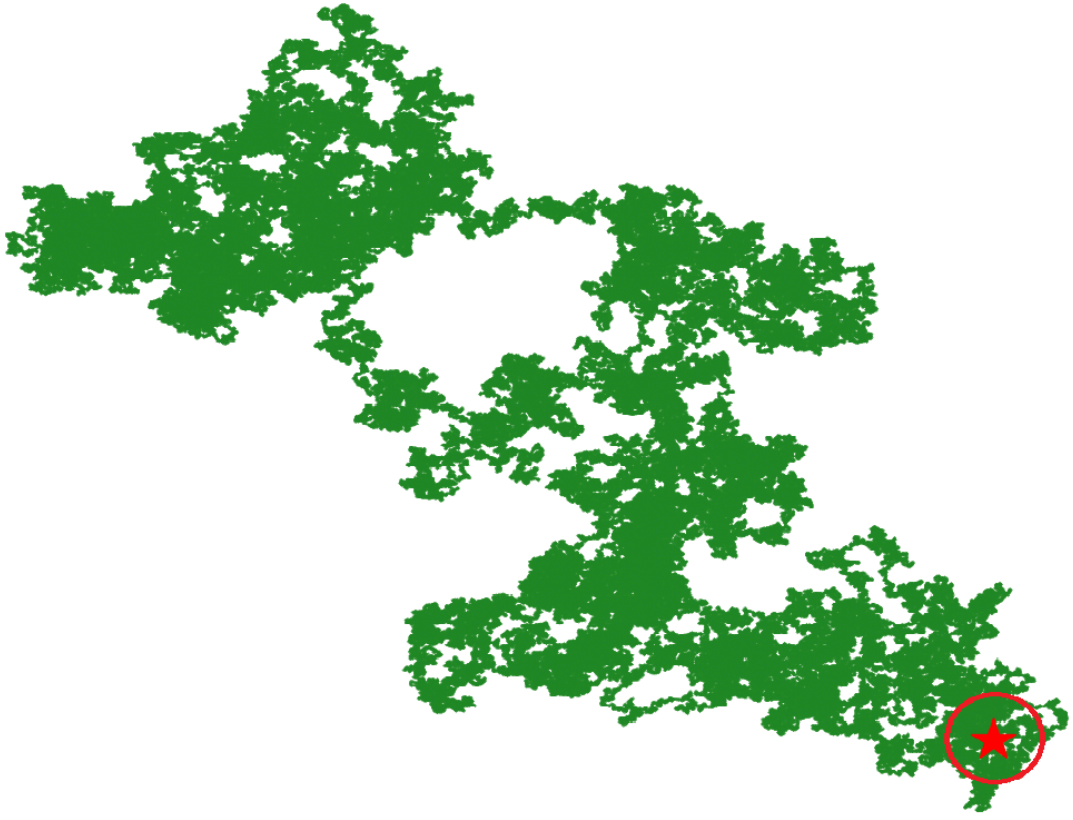
SRW in $\mathbb{Z}^d$



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Definition

For $x \in \mathbb{Z}^d$, let R_t^x be the range of an independent SRW started from x.
For any estimator $\widehat{v}$, the likelihood of $\widehat{v}$ is

$$L(\widehat{v}) = \mathbb{P}(R_t^{\widehat{v}} = R_t | R_t).$$

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Goal: Find a convenient estimator $\widehat{v}$ with

$$L(\widehat{v}) \approx L(\widehat{v}_{MLE})$$

Theorem (Hoffman, R.)

The following hold for SRW in $\mathbb{Z}^d$ as $t \to \infty$.

i. *If $d = 2$,*

$$\frac{L(\widehat{v}_{MLE})}{\sum_{w \in R_T} L(w)} \to_p 0$$

ii. *If $d \in \{3, 4, 5, 6\}$, there exists an estimator $\widehat{v}$ such that*

$$\mathbb{P}(\widehat{v} = 0) \geq \Theta(t^{-c_d})$$

for some constant $c_d \in (0, 1)$.

iii. *If $d \geq 7$, there exists an estimator $\widehat{u}$ such that*

$$\mathbb{P}(\widehat{u} = 0) = \Theta(1).$$

Conjecture

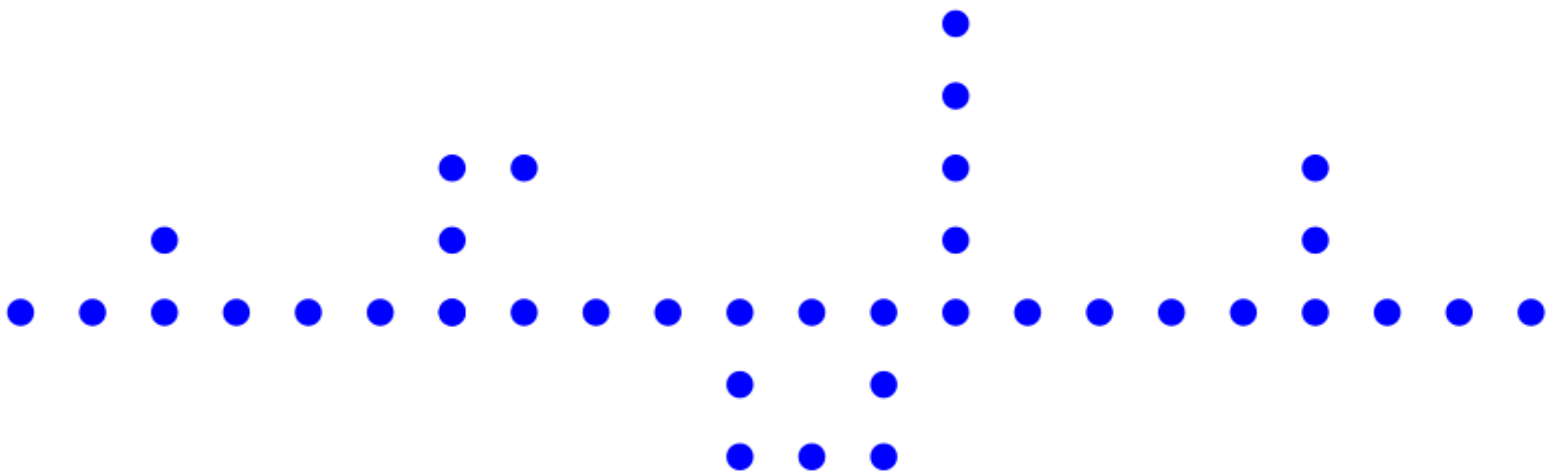
$$\mathbb{P}(\widehat{v}_{MLE} = 0) = \begin{cases} o(1), & d = 2 \\ \Theta(1), & d \geq 5 \end{cases}$$

Further Q's:

- SRW on d-regular tree, biased RW
- Performance of 'longest path' estimator for transient RW's?
- Good estimator for $\mathbb{Z}^2$?

Proof ideas:

1. Get rid of the 'middle' of the range, using transience.
2. Infer chronological info using 'cut points.'



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Ingredients:

1. Long cycles: return probabilities / self-intersection exponents (Lawler)

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2. A cut time for X is a time $s \in [0, t]$ such that

$$X_{[0,s)} \cap X_{(s,t]} = \emptyset$$

If s is a cut time, X_s is called a cut point.

Theorem (James, Peres, '96)

In dimension $d \geq 3$, there are infinitely many cut times. In dimension $d \geq 5$, cut times have positive density.

Cutpoints are totally ordered (by their cut times).

Given all the cut points, find the ‘first’ and ‘last’ ones, pick uniformly from their small components.

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Problem: not all ‘divider’ points are cut points!

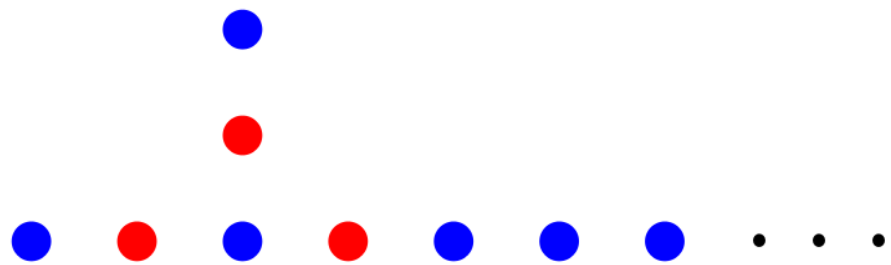

Figure: The three red ‘divider’ points can’t all be cut points.

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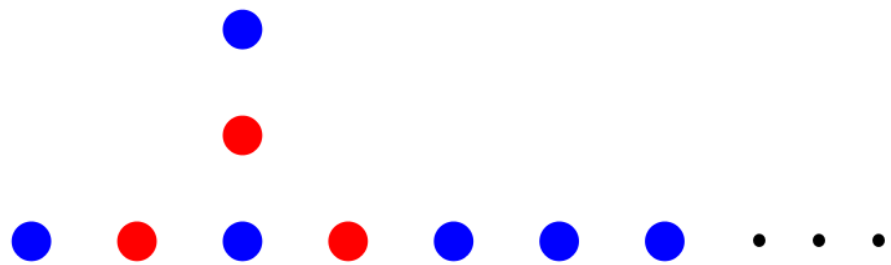

Figure: The three red 'divider' points can't all be cut points.

Need more information about how cutpoints are distributed.

Consider a rumor spreading through a network.

- The rumor starts from a ‘source’ vertex
- At integer times, nodes can pass the rumor to neighbors
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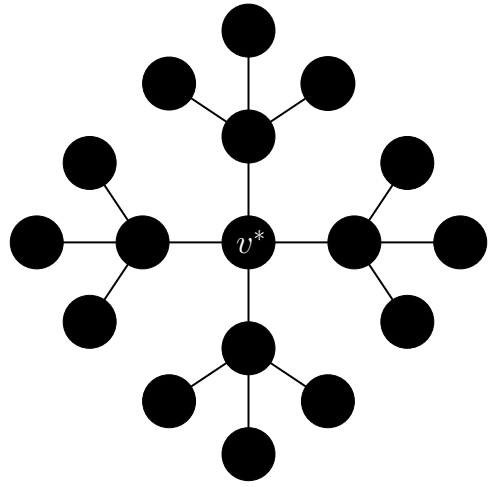
For today, focus on the d-regular tree.



The rumor is spread by a random algorithm known to the observer.

Goals for the rumor spreader:

- *Spreading*: spread the rumor to many nodes
- *Obfuscation*: minimize the probability that the observer guesses the source correctly
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- *Local spreading* (new): ensure that all vertices close to the source learn the rumor quickly

Anonymous messaging platforms, e.g. Secret, Yik Yak, Whisper

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Contact tracing / finding patient zero

Previous work: SI/SIR. MLE well understood. Rumor centrality

New algorithm: adaptive diffusion

- $G = d$-regular tree
- $G_t =$ set of nodes that know the rumor at time t
- $vs_t =$ virtual source at time t
- G_t **is a ball of radius** $t/2$ **centered at** vs_t **at even times** t
- Defined by transition probabilities $\alpha(t, h)$ for the virtual source

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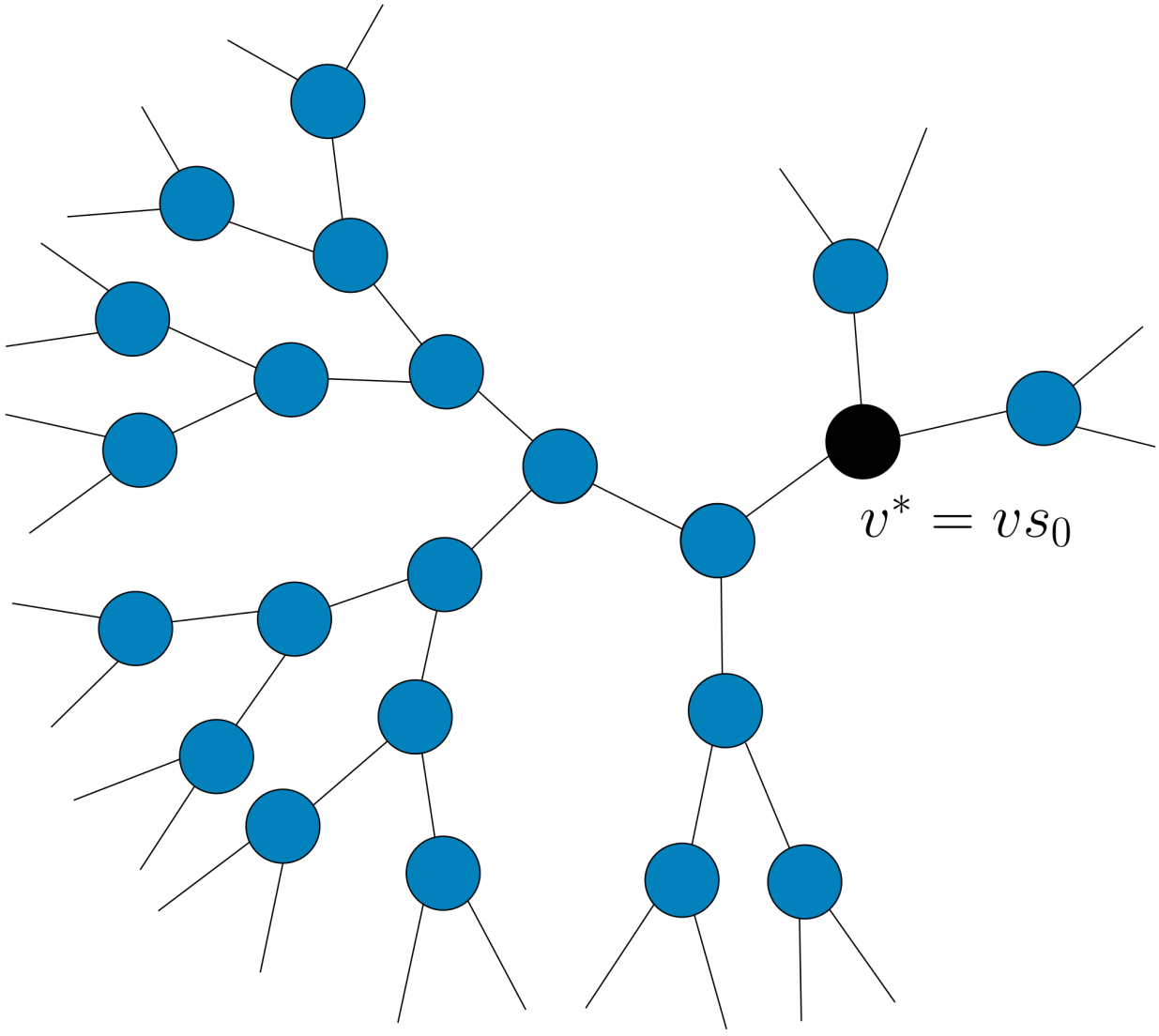
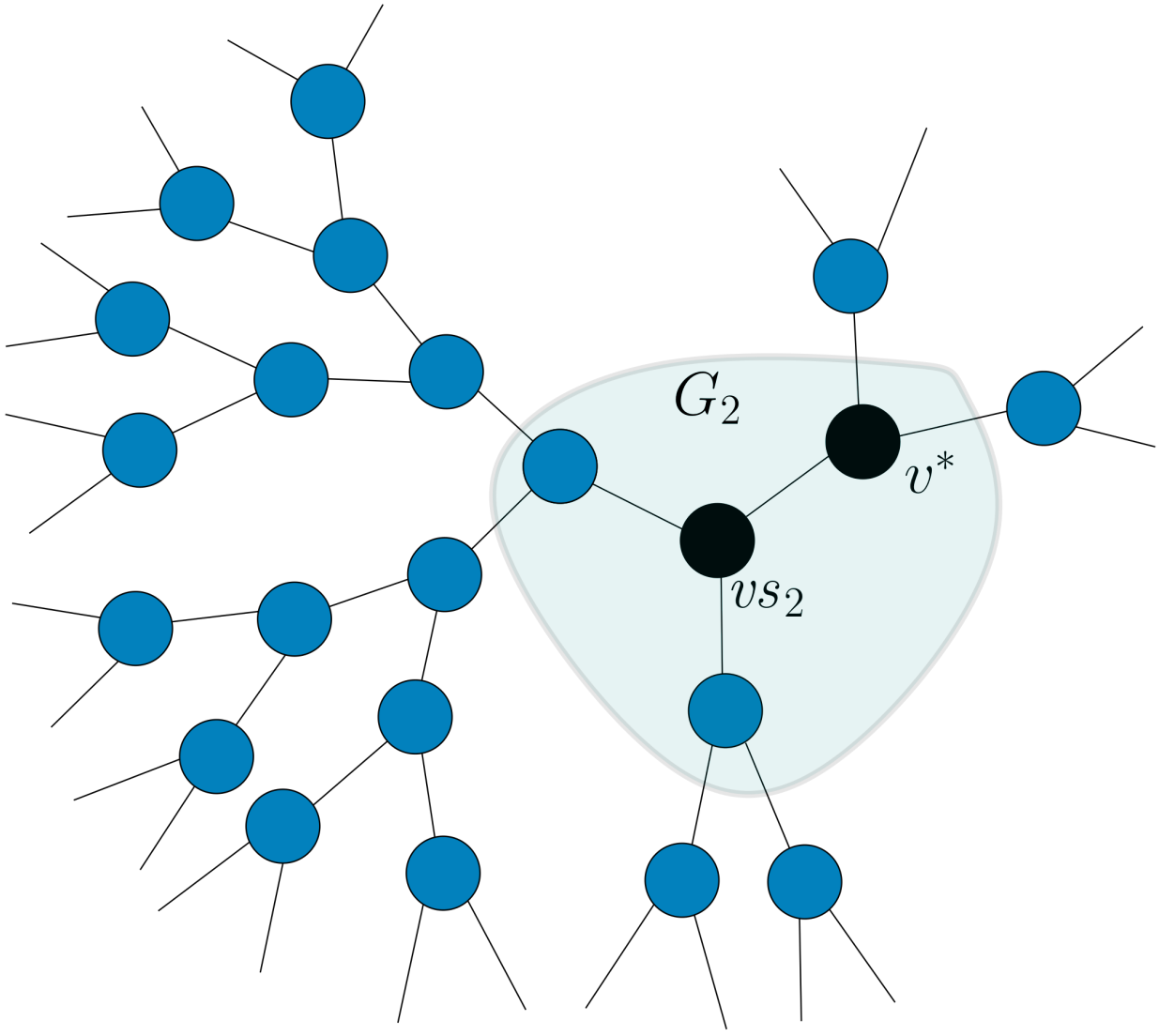
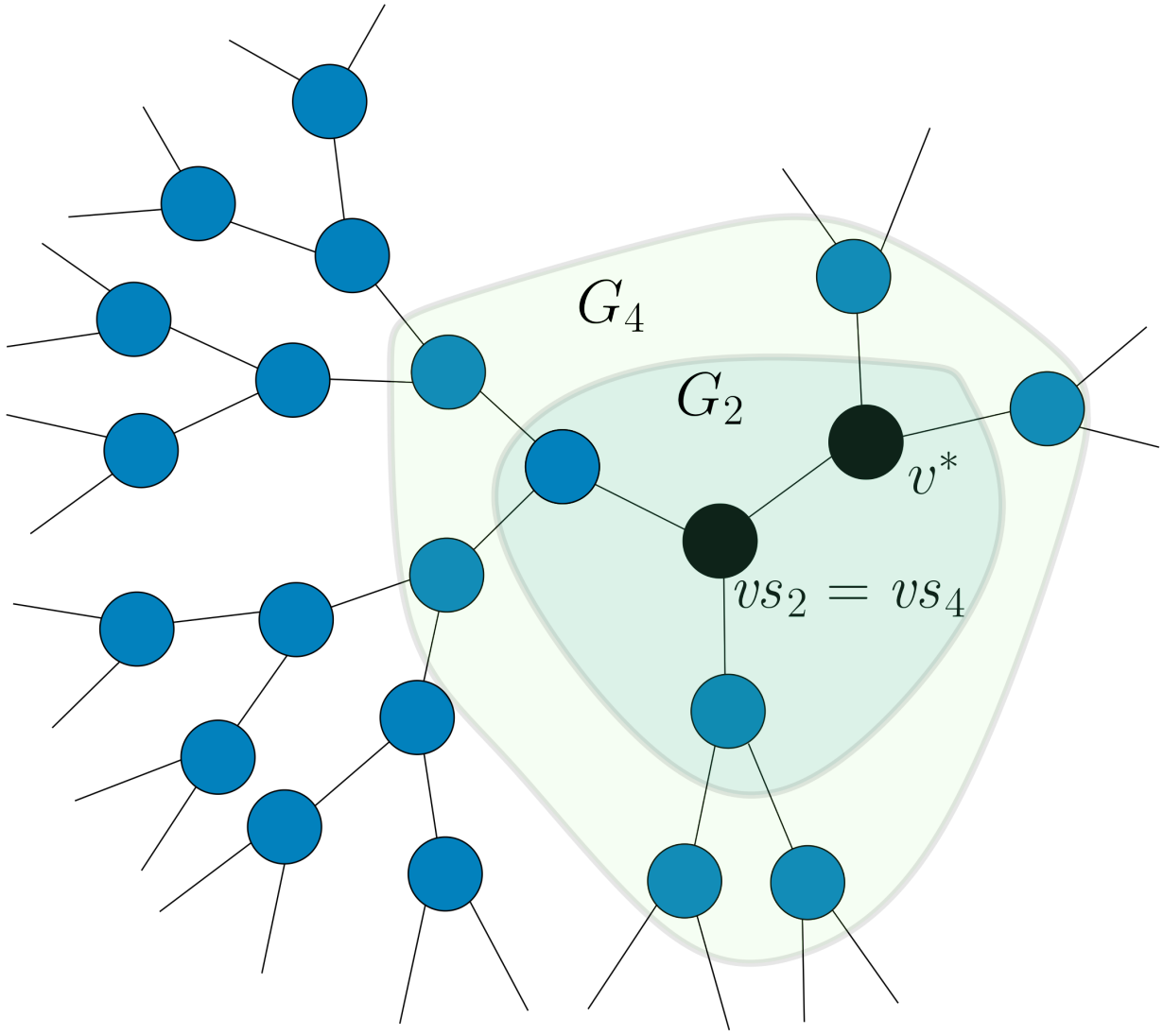
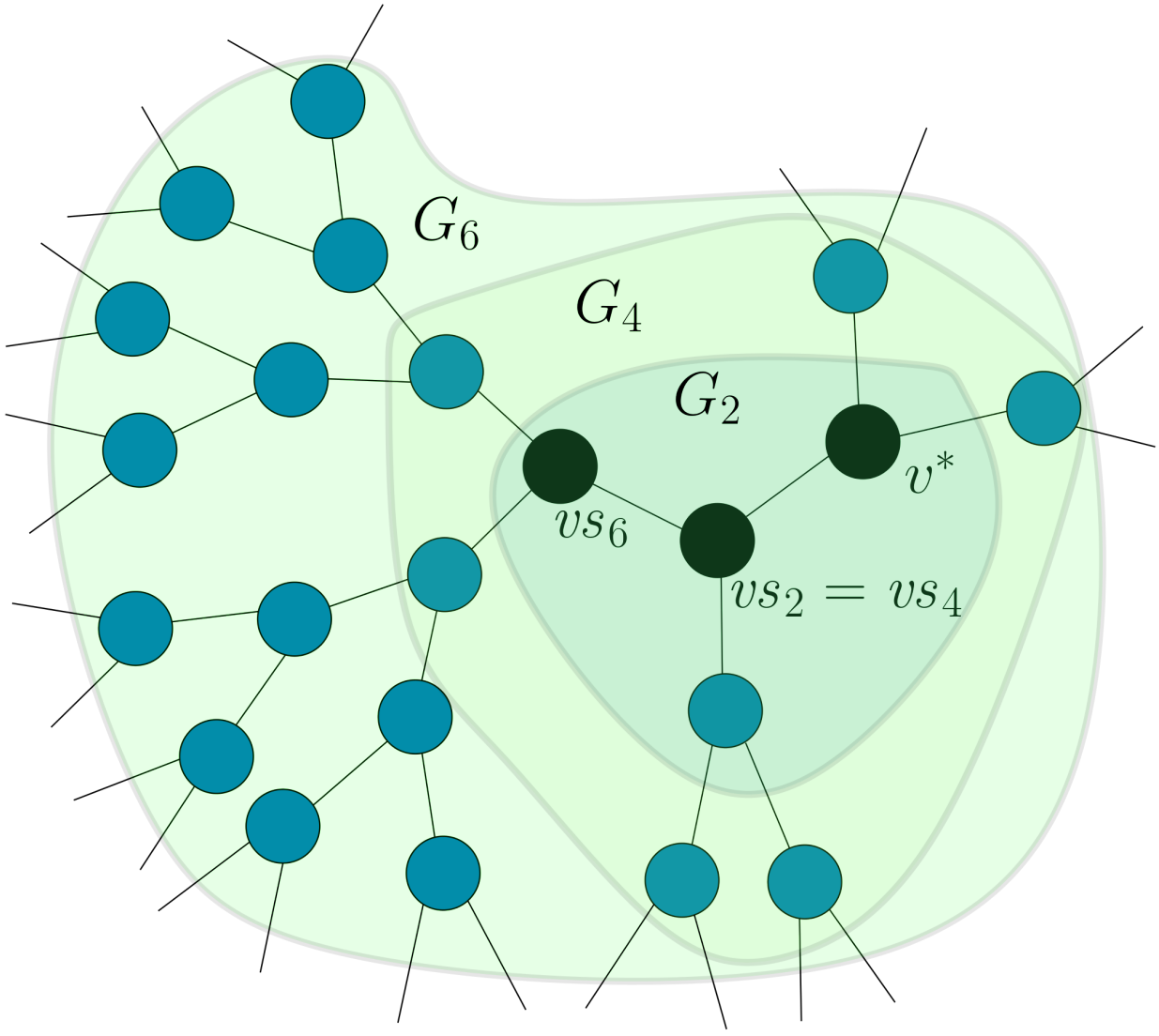
When the virtual source moves, it always moves in a uniform direction away from v^*.


$v^* = vs_0$


G_2
v^*
vs_2


G_4
G_2
v^*
$vs_2 = vs_4$


G_6
G_4
G_2
v^*
vs_6
$vs_2 = vs_4$

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MLE for the source vertex:

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Fact

$\widehat{v}_{MLE}$ is uniform over all vertices at distance h^* from vs_t, where

$$h^* = \arg\max_{h \in \{1,2,\ldots,t/2\}} \frac{p(t,h)}{(d-1)^h}.$$

Spreading

For adaptive diffusion,

$$|G_t| = N_t = \frac{1}{d-2}(d-1)^{t/2}.$$

deterministically at even times t. (Order-optimal spreading)

Obfuscation

$$\mathbb{P}(\hat{v}_{MLE} = v^*) = \begin{cases} \Theta(N_t^{-1}) & \text{(perfect obfuscation)} \\ \Theta(N_t^{-\gamma}) & \text{(polynomial obfuscation)} \\ o(1) & \text{(weak obfuscation)} \end{cases}$$

SI/SIR: good spreading, weak obfuscation; not even weak obfuscation under multiple observations. [Shah, Zaman, Dong, Tan, Wang, Zhang]

Adaptive diffusion (Fanti, Kairouz, Oh, Viswanath '15)

Let $G = d$-regular tree. There exists an adaptive diffusion algorithm that achieves perfect obfuscation:

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Proof sketch: choose $p(t, h) \sim (d-1)^h$, so the MLE picks a uniform random vertex in G_t. Show this is realizable for some values $\alpha(t, h)$.

Q: Does it have good local spreading?

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Definition

The *local spread* R_t is the radius of the largest ball centered at v^* and contained in G_t.

For adaptive diffusion, $R_t = \#$ of times the virtual source has *not* moved:

$$R_t = t/2 - \text{dist}(v^*, vs_t).$$

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The algorithm from the theorem doesn't even achieve weak local spread!

$$p(t,h) \sim (d-1)^h \implies \text{dist}(vs_t, v^*) \approx t/2 - O(1).$$

Spreading/obfuscation trade-off [Racz, Richey '18]

Consider any adaptive diffusion with polynomial obfuscation of order $\gamma \in (0, 1)$, i.e.

$$\mathbb{P}(\widehat{v}_{MLE} = v^*) = O(N_t^{-\gamma}).$$

Then the local spreading is bounded from above:

$$\mathbb{E}[R_t] \leq (1 - \gamma)\frac{t}{2} + O(\log t).$$

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Obfuscation and local spreading are **inversely linked** in this case.

The trade-off is essentially tight:

Spreading/obfuscation trade-off [Racz, Richey '18]

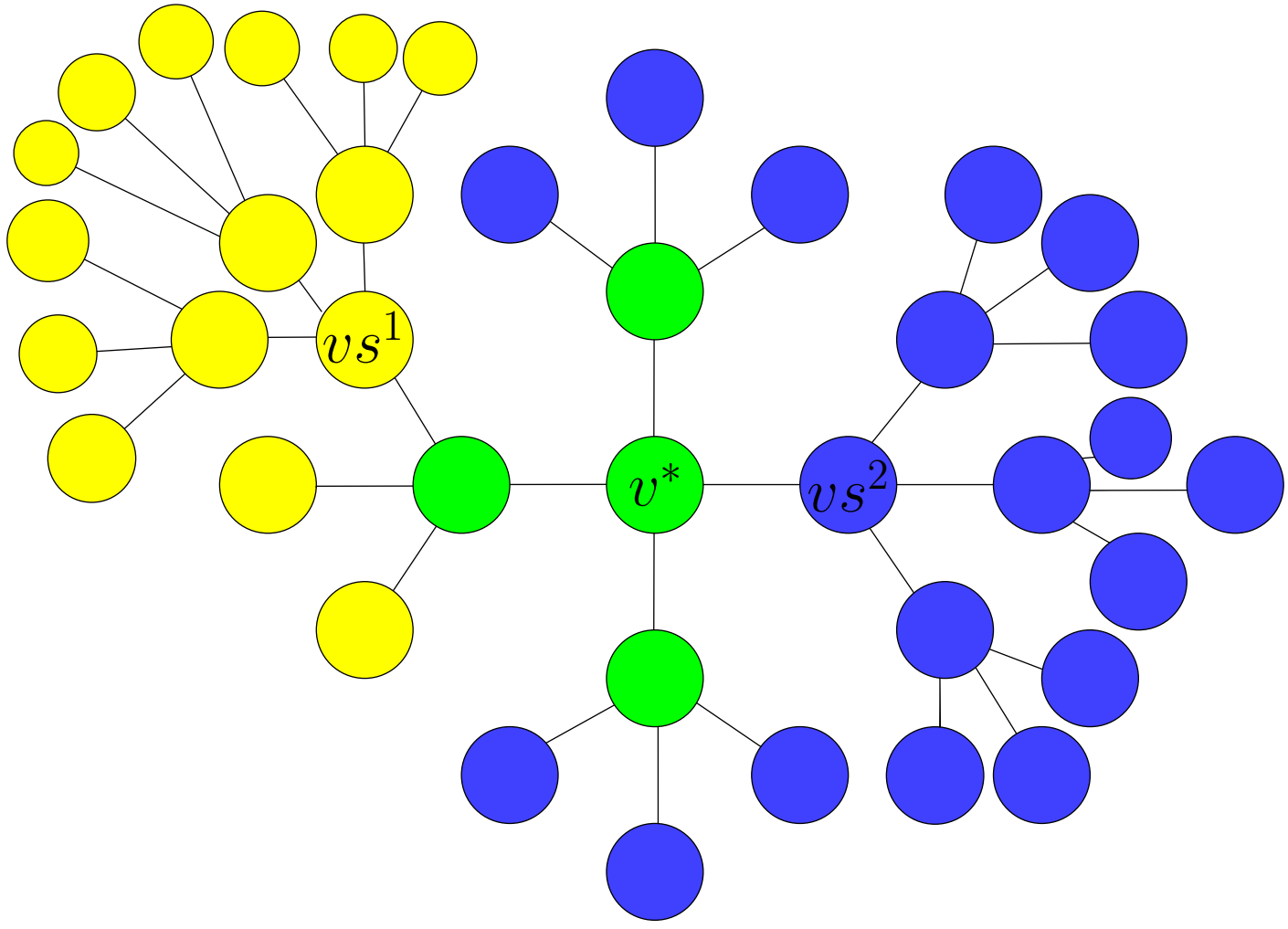
For every $\gamma \in (0, 1)$, there exists an adaptive diffusion with both polynomial obfuscation of order γ,

$$\mathbb{P}(\widehat{v}_{MLE} = v^*) = O(N_t^{-\gamma}),$$

and order optimal local spreading

$$\mathbb{E}[R_t] \geq (1 - \gamma)\frac{t}{2}.$$

Suppose the observer has access to $k > 1$ independent snapshots $\{G_t^i\}_{i=1}^k$ of the diffusion started from the same source v^*.



Two independent observations (Racz, Richey '18)

Suppose the observer has access to two independent observations G_t^1 and G_t^2 started from a fixed source v^*. There exists a nice estimator $\widehat{v}$, not depending on the spreading algorithm, such that for any t,

$$\mathbb{P}(\widehat{v} = v^*) \geq \frac{d-1}{d} \cdot \frac{2}{t}.$$

Moreover, there exists a protocol such that for any t,

$$\mathbb{P}(\widehat{v}_{ML} = v^*) \leq \frac{d-1}{d} \cdot \frac{7}{t}.$$

Only weak obfuscation now!

It gets worse:

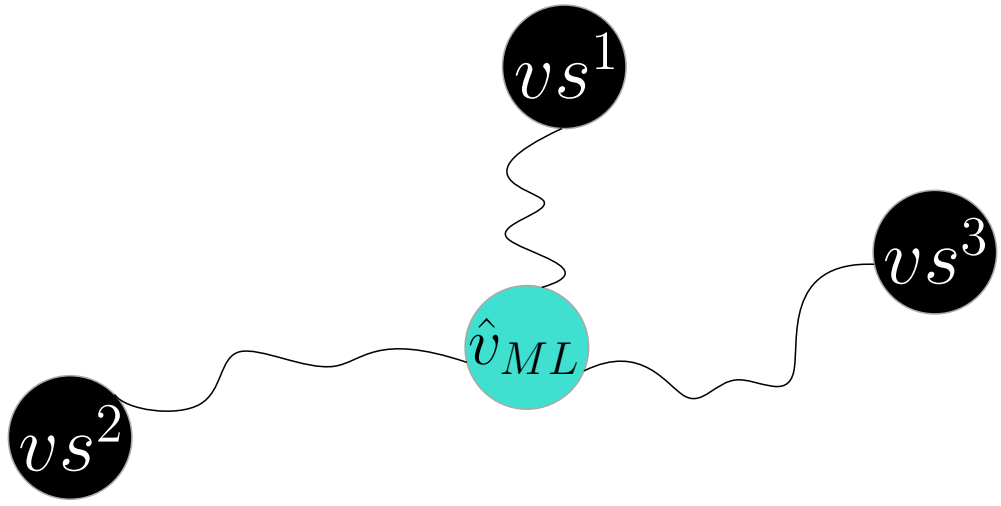
Three or more independent observations (Racz, Richey '18)

Suppose the observer has access to $k \geq 3$ independent observations G_t^i, $i \in [k]$. There exists a nice estimator $\widehat{w}$, not depending on the spreading algorithm, such that for any t,

$$\mathbb{P}(\widehat{w} = v^*) \geq 1 - d \exp\left(-\frac{(d-2)^2}{2d^2} k\right).$$

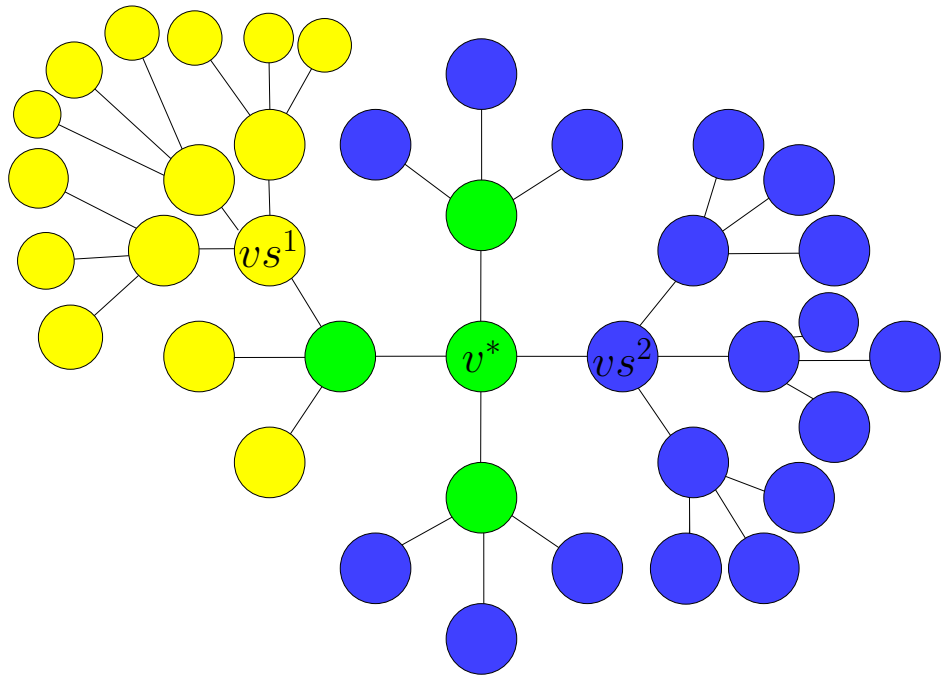
Not even weak obfuscation!

Proof: Pick any three virtual sources and draw the paths between them.



When the three virtual sources lie in different sub-trees away from the root, there will be a unique intersection point $\widehat{w}$.

Simple estimator: guess a green vertex!



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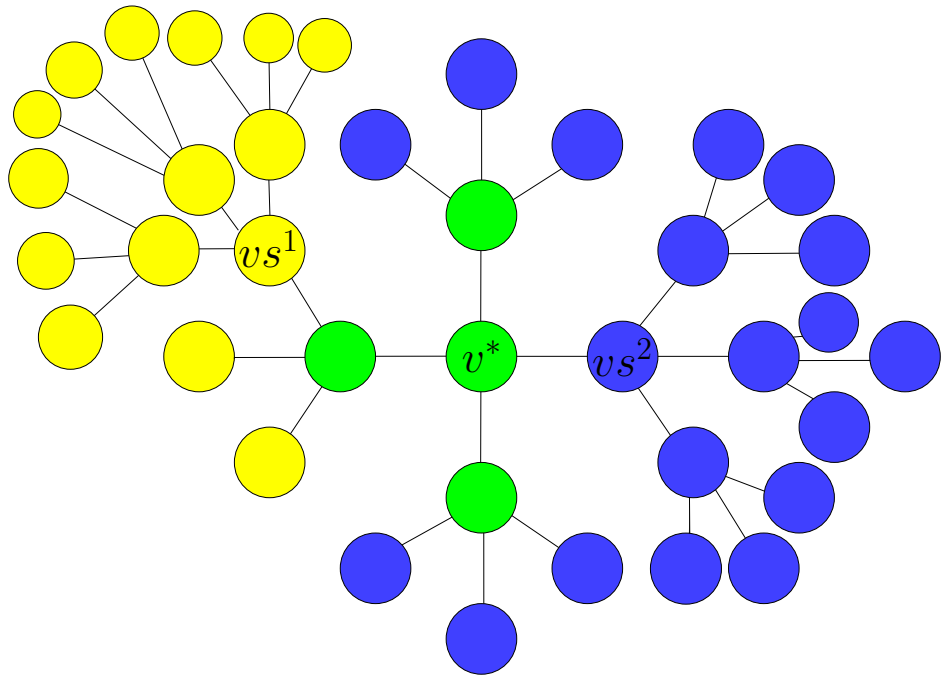

Obfuscation and local spreading are **positively linked** in this case:

$$\mathbb{P}(\hat{v}_{MLE} = v^*) \geq \mathbb{E}\left[\left|\bigcap_{i=1}^{k} G_t^i\right|^{-1}\right]$$

Question

Does there exist a spreading algorithm that achieves order-optimal spreading and polynomial obfuscation under ≥ 2 observations?

Should look at algorithms that have order-optimal local spreading.

Also, need more randomness: adaptive diffusion is indexed by a single vertex. Too simple!