MATH 303 Midterm Exam A

February 11, 2021

INSTRUCTIONS: PLEASE READ ALL CAREFULLY BEFORE STARTING

After inputing your student ID, we recommend to write down the values set by the script. If you show results obtained with values different from those set by the Jupyter notebook script, your question **will not be graded.**

1. **Problems 1-3**: For each problem, you will need to separately assemble a pdf file of handwritten solutions. We recommend to not use more than 2 pages per problem. **Write your name** on top of each page and do not forget to indicate which subquestion (a, b...) you are solving.
2. **Problem 4**: Complete the Problem 4 set in `midtermA_students.ipynb` (there are two questions). Submit the completed notebook as `midtermA_complete.ipynb` (you don't have to finish all the questions to submit).
   Warning: Having "Success" displayed when running the cells does not necessarily mean that the solution is correct (this will be evaluated after you submit the notebook).

Grading

Problems 1-3 will count for $\sim 85\%$ of the grade. We recommend spending an equivalent amount of time on problems 1, 2 and 3.

Problem 1

Consider the transition matrix P obtained from running the notebook with your student ID.

a. Draw the transition diagram associated with P with the states corresponding to their row index in the matrix (i.e. first row corresponds to state 1, second to state 2 etc.).
b. Determine all the communication states (no need to justify).
c. Determine which states are recurrent and which are transient (briefly justify).
d. Determine the period of each state (briefly justify).
e. Starting from i, what is the mean number of steps to re-visit i? (justify your calculation with key steps; answers directly written won't be accepted).

Problem 2

Consider the Markov Chain $(X_n)_{n\geq 0}$ defined on $\mathbb{N} = \{0, 1, \ldots\}$ with transition probabilities obtained from running the notebook with your student ID.

a. Draw the transition diagram.
b. Suppose that the chain admits a stationary distribution π. Find a relation between π_0 and π_1 (justify your answer).
c. For $i > 1$, find a relation between π_{i-1}, π_i and π_{i+1} (justify your answer).
d. Show by induction that $\frac{\pi_{i+1}}{\pi_i}$ is a constant (to be determined), and deduce π_i as a function of i and π_0. Is the chain positive-recurrent? Justify your answer.

Problem 3

You have a bag containing four marbles. Marbles come in two colors: red and blue. At each step, you put your hand in the bag, remove a marble (selecting one uniformly at random from those in the bag), and replace it with a marble of the *opposite* color. Let X_n be the number of blue marbles in the bag after n steps.

a. Draw the transition diagram for X.

b. Consider the state i obtained from running the notebook. Find $\mathbb{E}(X_2|X_0 = i)$ (justify your calculation with key steps; answers directly written won't be accepted).

c. Show that distribution $\pi = \frac{1}{16}(1, 4, 6, 4, 1)$ is stationary for X, and that the process is reversible.

Consider a different Markov chain defined by slightly modifying the marble process: namely, when all the marbles in the bag are the same color, you take *two* out and replace them with marbles of the opposite color. Let Y_n be the number of blue marbles after n steps in the modified process.

d. We assume that $\sigma = \frac{1}{18}(1, 4, 8, 4, 1)$ is stationary for Y. Use a result from class to argue that Y_n converges in distribution to σ as $n \to \infty$. Is Y reversible? Justify.

Problem 4

Complete the Problem 4 set in `midtermA_students.ipynb` (there are two questions).
Submit the completed notebook as `midtermA_complete.ipynb` (you don't have to finish all the questions to submit).

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