MATH 303 Practice Problems - Week 1

Jamuary 15, 2021

Problem 1

1. We consider the 2-state Markov chain $(X_n)_{n\geq 0}$ on $\{0,1\}$ seen in class, such that $P_{00} = P_{11} = p$, where $0 < p < 1$. By induction, prove that the n-step transition matrix of $(X_n)_{n\geq 0}$ is

$$\frac{1}{2}\begin{pmatrix} 1+(2p-1)^n & 1-(2p-1)^n \\ 1-(2p-1)^n & 1+(2p-1)^n \end{pmatrix}$$

2. Show that for any initial distribution, $P(X_n)$ converges to the uniform distribution on $\{0,1\}$ when $n \to +\infty$.

Problem 2

We consider the random walk $(X_n)_{n\geq 0}$ on $\{-1,0,1\}$ such that at each step, the walker moves from 0 to -1 or 1 with equal probability, and from -1 and 1, the walker moves to 0 only.
1. Draw the transition diagram and write the transition matrix of $(X_n)_{n\geq 0}$.
2. Compute the 2- and 3-step transition matrix. Deduce the n-step transition matrix.
3. We modify $(X_n)_{n\geq 0}$ as follows: At each step the walker first decides to move or stay by flipping a fair coin first and follows the same rules as above if the decision is to move. Draw the transition diagram and write the transition matrix of this new Markov chain $(Y_n)_{n\geq 0}$.
4. By induction, prove that the n-step transition matrix of $(Y_n)_{n\geq 0}$ is

$$\frac{1}{2}\begin{pmatrix} \frac{2^{n-1}+1}{2^n} & 1 & \frac{2^{n-1}-1}{2^n} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{2^{n-1}-1}{2^n} & 1 & \frac{2^{n-1}+1}{2^n} \end{pmatrix}$$

5. We assume that $(Y_n^2)_{n\geq 0}$ is a Markov chain (on $\{0,1\}$). Find the transition matrix of $(Y_n^2)_{n\geq 0}$ and study its asymptotic behavior (remark: you can use Problem 1)

Problem 3

We consider an urn with two blue balls and two red balls.
1. What is the probability of picking 2 balls of the same color?
2. We repeat the process of picking 2 balls (and place them back) 4 times. What is the probability of having picked 2 balls of the same color 2 times in a row?

Problem 4 (Jupyter Notebook)

Use the Jupyter Notebooks seen in class (link available on Canvas) to solve the following questions:
1a. Modify the script for the symmetric random walk to make it asymmetric, i.e. the probability to move from i to $i+1$ is $p \neq 0.5$
1b. For $p = 0.25$, get the empirical average position of the walker as a function of the time step, upon running 100 simulations for 100 time steps. What can you say about the average position over time?
2. Modify the script of the jupyter notebook to simulate the Example 4.3 given in Ross textbook:

"On a given day Gary is either cheerful (C), so-so (S), or glum (G). If he is cheerful today, then he will be C, S, or G tomorrow with respective probabilities 0.5, 0.4, 0.1. If he is feeling so-so today, then he will be C, S, or G tomorrow with respective probabilities 0.3, 0.4, 0.3. If he is feeling glum today, then he will be C, S, or G tomorrow with respective probabilities 0.2, 0.3, 0.5."

3. We modify the walk Y_n from Problem 2 so that at position 1 the walker directly moves back to 0. Draw the transition diagram of the new chain $(Z_n)_{n\geq 0}$. By simulating the chain, compare the empirical probabilities $P(Z_3^2 = 0 \mid Z_2^2 = 1)$ and $P(Z_3^2 = 0 \mid Z_2^2 = 1 \ , \ Z_1^2 = 0)$. Does it suggest that $(Z_n^2)_{n\geq 0}$ is a Markov chain? (You can also try to prove it analytically, i.e., compare $P(Z_2^2 = 0 \mid Z_1^2 = 1)$ with $P(Z_2^2 = 0 \mid Z_1^2 = 1 \ , \ Z_0^2 = 0)$. We have shifted the index in the original question since we are explicitly setting $Z_0 = 0$ in the simulation.)