Apr 1 ①

II. Transition probabilities and Kolmogorov equations

Recall: We characterize a CTMC by

- P_{ij} : Proba. next jump (when it occurs) is to state j, given current state is i
- v_i: Exp. rate at which jump occurs.

$$\boxed{\text{Goal: Study } P_{ij}(t) = P(X(t)=j \mid X(0)=i)}$$

Recall: For discrete time M-C, we saw (chap. 1)

C-K eq: $P_{ij}^{n+m} = \sum_{k \in S} P_{ik}^{n} \cdot P_{kj}^{m}$

In the continuous-time setting, we have the analog

Prop (Chapman-Kolmogorov eq.)

$$\forall (s,t) \geq 0 \quad P_{ij}(s+t) = \sum_{k} P_{ik}(s) \cdot P_{kj}(t)$$

Proof: $\forall s, t \geq 0$

$$P_{ij}(s+t) = P(X(t)=j \mid X(0)=i)$$

$$= \sum_{k \in S} P(X(s+t)=j, X(s)=k \mid X(0)=i)$$

Markov property

$$= \sum_k P\left(X(s+t)=j \mid X(s)=k, \cancel{X(0)=i}\right) \quad (2)$$

$$\times \underbrace{P(X(s)=k \mid X(0)=i)}_{P_{ik}(s)}$$

$$= \sum_k \underbrace{P(X(s+t)=j \mid X(s)=k)}_{P_{kj}(t) \text{ (by stationarity)}} \times P_{ik}(s)$$

$$P(X(s+t)=j \mid X(s)=k) = P(X(t)=j \mid X(0)=k)$$

$$= \sum_k P_{kj}(t) \cdot P_{ik}(s) \qquad \boxtimes$$

Q : How to compute and solve $P_{ij}(t)$?

ex: $X(t)$ = Poisson process (λ) ($0 \to 1 \to 2 \ldots\ldots$)

Let's study $P_{i,i+1}(t+\Delta t) = P(X(t+\Delta t)=i+1 \mid X(0)=i)$

$$P_{i,i+1}(t+\Delta t) = P(X(t)=i, \text{a jump occurs in } [t, t+\Delta t] \mid X(0)=i)$$
$$+ P(X(t)=i+1, \text{ no jump } \_\_\_\_\_\_\_\_\_\_)$$

independent increments

$$= \underbrace{P(X(t)=i \mid X(0)=i)}_{P_{ii}(t)} P(\text{jump in } [t, t+\Delta t] \mid \cancel{X(0)=i})$$
$$+ \underbrace{P(X(t)=i+1 \mid X(0)=i)}_{P_{i\,i+1}(t)} P(\text{no jump in } [t, t+\Delta t] \mid \cancel{X(0)=i})$$

by independence

$$P(X(t+\Delta t) - X(t) = 1)$$

· Using the 2^{nd} def. of the P.P $\longrightarrow P(\text{jump in } [t, t+\Delta t]) = \lambda \Delta t + o(\Delta t)$

• Similarly, $P(\text{No jump in } [t, t+\Delta t]) = 1 - \lambda\Delta t + o(\Delta t)$ (3)

$$\left(\begin{aligned} &= 1 - P(1 \text{ jump in } [t, t+\Delta t]) - P(\geq 2 \text{ jumps in } [t, t+\Delta t]) \\ &= 1 - (\lambda\Delta t + o(\Delta t)) - o(\Delta t) = 1 - \lambda\Delta t + o(\Delta t) \end{aligned} \right)$$

2nd def. of the P.P.

$$\Rightarrow P_{i,i+1}(t+\Delta t) = P_{ii}(t)(\lambda\Delta t + o(\Delta t)) + P_{ii+1}(t)(1 - \lambda\Delta t + o(\Delta t))$$

$$\Leftrightarrow P_{i,i+1}(t+\Delta t) - P_{ii+1}(t) = \lambda(P_{ii}(t) - P_{ii+1}(t))\Delta t + o(\Delta t)$$

$$\Leftrightarrow \frac{P_{i,i+1}(t+\Delta t) - P_{ii+1}(t)}{\Delta t} = \lambda(P_{ii}(t) - P_{ii+1}(t)) + o(1)$$

When $\Delta t \to 0$, $o(1) \to 0$

$$\frac{P_{i,i+1}(t+\Delta t) - P_{ii+1}(t)}{\Delta t} \longrightarrow P_{ii+1}'(t) \quad \left(\frac{d}{dt} P_{ii+1}(t)\right)$$

(Reminder: $o(1) = f(\Delta t) \cdot 1$, where $f(\Delta t) \xrightarrow[\Delta t \to 0]{} 0$

$$\lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t} = f'(t)$$
)

So $\boxed{P_{ii+1}'(t) = \lambda(P_{ii}(t) - P_{ii+1}(t))}$

and similarly (exercise) $\boxed{P'_{ii}(t) = \lambda\left(P_{i-1,i}(t) - P_{ii}(t)\right)}$ (4)

Cc: Transition probabilities satisfy a system of linear differential equations. This is also the case in general for any given CTMC.

First, let's denote (⚠ important notation)

$$\boxed{q_{ij} := v_i \cdot p_{ij}}$$

Rmk (important): If we know the q_{ij}'s, this is sufficient to define the CTMC, as we can deduce v_i and p_{ij}:

- $v_i = v_i \underbrace{\sum_{j \neq i} p_{ij}}_{=1} = \sum_{j \neq i} q_{ij}$

- $p_{ij} = \dfrac{q_{ij}}{v_i} = \dfrac{q_{ij}}{\sum_{k \neq i} q_{ik}}$

Then, we have (similarly as above, consider the process at $t + \Delta t$, using the Chapman Kolmogorov eq., and $\Delta t \to 0$)

Thm: (Kolmogorov backwards eq.)

$$P_{ij}'(t) = \sum_{k \neq i} q_{ik} \cdot P_{kj}(t) - v_i P_{ij}(t) \qquad \forall t, i, j$$

we look "backwards" from $k \to j$

Thm: (Kolmogorov forward eq.)

$$P_{ij}'(t) = \sum_{k \neq j} P_{ik}(t)\, q_{kj} - v_j P_{ij}(t) \qquad \forall t, i, j$$

we look "forward" from $i \to k$

Rmk: These thms hold for any CTMC defined over a finite state space. In infinite space one has to be careful that the v_i's are "small enough" so the probability that an interval of time contains infinitely many transitions is 0 (otherwise the process is "explosive" in time, and may not be defined for all $t \geq 0$)

$\Rightarrow$ In this chapter, we will assume this never happens.

ex: • 2-state CTMC

$$0 \underset{\mu}{\overset{\lambda}{\rightleftarrows}} 1$$

$v_0 = \lambda$, $v_1 = \mu$ $\quad P_{01} = 1$

$P_{10} = 1$

$$\begin{cases} q_{01} = \lambda & (v_0 p_{01} = \lambda) \\ q_{10} = \mu & (v_1 p_{10} = \mu) \end{cases}$$

K-Backwards eq.
$$\begin{cases} P_{00}'(t) = \underset{q_{01}}{\lambda} P_{10}(t) - \underset{(v_0)}{\lambda} P_{00}(t) \\ P_{10}'(t) = \underset{q_{10}}{\mu} P_{00}(t) - \underset{v_1}{\mu} P_{10}(t) \end{cases}$$

Rmk: If you're familiar with elementary ODE's (Math 215/255), this system is simple enough to be solved (exercise).

$$\rightarrow P_{00}(t) = \frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)t} + \frac{\mu}{\lambda+\mu}.$$

- Birth-Death processes (to be seen on Tuesday)

Q: What is the backward Kolmogorov. eq.?

$p_{i\,i+1} = ?$ $\quad p_{i,i-1} = ?$

$v_i ?$ $\quad q_{i\,i+1} = ?$ $\quad q_{i,i-1} = ?$