Recall: (Superposition) March 16 ①

Let $N(t)$ and $M(t)$ be independent PP's of intensity λ_1, λ_2, then $N+M$ is a PP of intensity $\lambda_1 + \lambda_2$

Ex: Cars traveling on a highway

$N(t) = \#$ cars going east up to $t \to$ P.P. (λ)

$M(t) =$ ____________ west ______ $\to$ P.P. (μ)

Q: What is the distribution of $N(t)$, given that $N(t) + M(t) = C$?

A: Intuitively Binomial $\left(C, \dfrac{\lambda}{\lambda+\mu}\right)$

More precisely:

$$P(N(t) = k \mid N(t) + M(t) = C)$$

$$= \frac{P(N(t) = k, M(t) = C - N(t))}{P(N(t) + M(t) = C)}$$

$$= \frac{P(N(t) = k, M(t) = C-k)}{P(N(t) + M(t) = C)}$$

$$= \frac{P(N(t) = k)\, P(M(t) = C-k)}{P(N(t) + M(t) = C)}$$

②

$$= \frac{e^{-\lambda t}(\lambda t)^k/k! \cdot e^{-\mu t}(\mu t)^{c-k}/(c-k)!}{e^{-(\lambda+\mu)t}\,\dfrac{((\lambda+\mu)t)^c}{c!}}$$

$N(t) \sim \text{Poisson}(\lambda t)$

$M(t) \sim \text{Poisson}(\mu t)$

$(N+M)(t) \sim \text{Poisson}((\lambda+\mu)t)$

superposition property

$$= \cancel{\frac{e^{-\lambda t}\,e^{-\mu t}}{e^{-(\lambda+\mu)t}}} \cdot \cancel{\frac{t^k\,t^{c-k}}{t^c}} \cdot \frac{\lambda^k \mu^{c-k}}{(\lambda+\mu)^c}\,\underbrace{\frac{c!}{k!\,(c-k)!}}_{\binom{c}{k}}$$

$$= \binom{c}{k}\left(\frac{\lambda}{\lambda+\mu}\right)^k \cdot \left(\frac{\mu}{\lambda+\mu}\right)^{c-k}$$

$$= \binom{c}{k}\left(\frac{\lambda}{\lambda+\mu}\right)^k \left(1-\frac{\lambda}{\lambda+\mu}\right)^{c-k}$$

$$\boxed{= P\left(\text{Bin}\left(c, \frac{\lambda}{\lambda+\mu}\right) = k\right)}$$

$\boxtimes$

Rmk: More generally if $N_1, N_2, \dots, N_\ell$ are ℓ independent P.P.'s with rates $\lambda_1, \dots, \lambda_\ell$ then $N_1 + \dots + N_\ell$ is also a P.P. of intensity is $\lambda_1 + \dots + \lambda_\ell$

The following property is a kind of reciprocal of the superposition property

Prop (Thinning): Suppose that each arrival of a P.P. $N(t)$ (with rate λ) is classified or labelled of either type 1 or type 2 independently with probability p and $1-p$, respectively

- Let $N_1(t) = \#$ of type 1 events by time t
- Let $N_2(t) =$ ______ 2 ______

Then N_1 and N_2 are Poisson processes with rates λp and $\lambda(1-p)$, respectively

Proof: To see this, we check that N_1 and N_2 satisfy the second definition (4 axioms)

1) $N_1(0) = 0$

2) Independent increment property.
$N_1(t) - N_1(s)$ is a function of $N(t) - N(s)$ for $t > s$. So the independence of increments for N $\Rightarrow$ same for N_1

3) $P(N_1(t+\Delta t) - N_1(t) = 1) \overset{?}{=} p\lambda \Delta t + o(\Delta t)$ (1)

$= P(N_1(\Delta t) = 1)$ (by stationarity)

$= P(N_1(\Delta t) = 1 \mid N(\Delta t) = 1) \cdot P(N(\Delta t) = 1)$

we condition on $N(\Delta t)$

$+ P(N_1(\Delta t) = 1 \mid N(\Delta t) > 1) \cdot P(N(\Delta t) > 1)$

$= p \cdot (\lambda \Delta t + o(\Delta t)) + P(N_1(\Delta t) = 1 \mid N(\Delta t) > 1) \cdot o(\Delta t)$

we used the second definition, knowing that $N(t)$ is P.P. (λ)

(rmk: $e^{-\lambda \cancel{\Delta} t}(\lambda \Delta t) = \lambda \Delta t + o(\Delta t)$

because $e^x = 1 + o(x)$)

we use the second def. (4)

$= p\lambda \Delta t + \left(p + P(N_1(\Delta t) = 1 \mid N(\Delta t) > 1)\right) \cdot o(\Delta t)$

$= p\lambda \Delta t + o(\Delta t)$ ✓

4) $P(N_1(t+\Delta t) - N_1(t) > 1) \leq P(N(\Delta t) > 1) = o(\Delta t)$

$$\Rightarrow P(N_1(t+\Delta t) - N_1(t) > 1) = o(\Delta t)$$ ⑤

Example: A vendor sells fruits, and the sales form a Poisson process of rate $100/\text{day}$. Suppose each fruit sold is rotten with a probability $\frac{1}{20}$ independently.

Then $N_1(t) = \#$ rotten fruits sold is a Poisson process (thinning property) with rate $100 \cdot \frac{1}{20} = 5/\text{day}$

and Prob(no rotten fruit is sold during 1 day)

$$= P(N_1(1) = 0) \qquad (N_1(t) \sim \text{Poisson}(5t))$$

$$= P(\text{Poisson}(5 \cdot 1) = 0)$$

$$= \boxed{e^{-5}}$$

We can also check this by direct computation.

P(No rotten fruit sold in a day)

$$= \sum_{k \geq 0} P(k \text{ fruits are sold}) \cdot P(\text{No fruit among these } k \text{ is rotten})$$

Condition on the total # of fruits sold $= k$

$$= \sum_{k \geq 0} P(k \text{ fruits are sold}) \cdot \left(\tfrac{19}{20}\right)^k$$

$$= \sum_{k \geq 0} P(\text{Poisson}(100) = k) \cdot \left(\tfrac{19}{20}\right)^k$$

$$= \sum_{k \geq 0} e^{-100} \frac{(100)^k}{k!} \cdot \left(\tfrac{19}{20}\right)^k$$

$$= \sum_{k \geq 0} \frac{e^{-100}}{k!} \left(\frac{100 \cdot 19}{20}\right)^k$$

$$= \sum_{k \geq 0} \frac{e^{-100}}{k!} (5 \cdot 19)^k$$

$$= e^{-100} \cdot \sum_{k \geq 0} \frac{(5 \cdot 19)^k}{k!}$$

$$= e^{-100} \cdot e^{5 \cdot 19} = e^{-100 + 5 \cdot 19} = \boxed{e^{-5}}$$

More generally (exercise)

$$P(\text{exactly } j \text{ fruits are rotten}) = e^{-5} \cdot \frac{5^j}{j!}$$

$$\left(\rightarrow \text{this shows } N_1(t) \sim \text{Poisson}(\underbrace{\lambda}_{100} \underbrace{p}_{\frac{1}{20}} t)\right)$$