Recall: For $\{X(t), t \geq 0\}$ a general CTMC (not necessarily a birth-death process)

We characterized $X(t)$ with

v_i : rate of transitions from state i

P_{ij} : probability of transition to state $j \neq i$ (from state i) $\hookrightarrow \sum_{j \neq i} P_{ij}$

Let $q_{ij} := v_i P_{ij}$

→ If we know the q_{ij}, we also know v_i & P_{ij}:

$$v_i = v_i \underbrace{\sum_{j \neq i} P_{ij}}_{1} = \sum_{j \neq i} q_{ij} \; ; \; P_{ij} = \frac{q_{ij}}{v_i} = \frac{q_{ij}}{\sum_{k \neq i} q_{ik}}$$

(& this is always the case when the state space is finite)

→ We will assume for the rest of the chapter that the v_i's are "small enough" so the probability that an interval of time contains ∞-ly many transitions is 0 (otherwise, the process is explosive, and may not be defined for all $t \geq 0$)

Lemma: As $h \to 0$, $P_{ii}(h) = 1 - \underbrace{v_i}_{= \sum_{j \neq i} q_{ij}} h + o(h)$, i.e. $\underbrace{\lim_{h \to 0} \frac{P_{ii}(h) - 1}{h}}_{\frac{d}{dh} P_{ii}(h) |_{h=0}} = -v_i$

& for $i \neq j$, $P_{ij}(h) = q_{ij} h + o(h)$, i.e. $\underbrace{\lim_{h \to 0} \frac{P_{ij}(h) - 0}{h}}_{\frac{d}{dh} P_{ij}(h) |_{h=0}} = q_{ij}$

Proof: $P(\geq 2 \text{ events in } [0, h]) = o(h)$ (because transition time is Exp. so memoryless & there is no explosion)

$\Rightarrow$ For $j \neq i$, $P_{ij}(h) = P(1 \text{ jump to } j \text{ in } [0, h] \mid X_0 = i) + o(h)$

$= \underbrace{q_{ij} h}_{(1 \text{ event})} + \underbrace{o(h)}_{\geq 2 \text{ events}}$ (because this jump time is exponentially distributed with parameter q_{ij})

$\Rightarrow P_{ii}(h) = 1 - \sum_{j \neq i} P_{ij}(h) = 1 - \sum_{j \neq i} q_{ij} h + o(h)$

$$P_{ii}(h) = 1 - \underbrace{\Big(\sum_{j \neq i} q_{ij}\Big)}_{v_i} h + o(h) = 1 - v_i h + o(h) \quad \boxtimes$$

- This leads to the Kolmogorov Backward equations:

From the CK eq: $P_{ij}(t+h) - P_{ij}(t) = \sum_k P_{ik}(h) P_{kj}(t) - P_{ij}(t)$

$$= \sum_{k \neq i} \underbrace{P_{ik}(h)}_{q_{ik} h + o(h)} P_{kj}(t) - \underbrace{(1 - P_{ii}(h))}_{v_i h + o(h)} P_{ij}(t)$$

Assuming $\lim_{h \to 0} \frac{\sum_{k \neq i} P_{ik}(h)}{h} < \infty$ (the process is called conservative, & this is always true for finite state space)

(so $\sum_{k \neq i} q_{ik} P_{kj}(t) \underset{h \to 0}{\sim} \sum_{k \neq i} (q_{ik} + o(1)) P_{kj}(t)$)

$$P'_{ij}(t) = \lim_{h \to 0} \frac{P_{ij}(t+h) - P_{ij}(t)}{h} = \sum_{k \neq i} q_{ik} P_{kj}(t) - v_i P_{ij}(t)$$

So we obtain the Kolmogorov Backward eq.:

Thm:

$$\boxed{P'_{ij}(t) = \sum_{k \neq i} q_{ik} P_{kj}(t) - v_i P_{ij}(t)}$$

we look "backward" from $k \to j$

Example: Birth-death process $P_{ij} = 0$ if $j \neq i \pm 1$,

$$P_{i\,i+1} = \frac{\lambda_i}{\lambda_i + \mu_i}, \quad P_{i,i-1} = \frac{\mu_i}{\lambda_i + \mu_i}$$

$(\mu_0 = 0)$

$$v_i = \lambda_i + \mu_i$$

$$q_{ij} = v_i P_{ij} = \begin{cases} \lambda_i & j = i+1 \\ \mu_i & j = i-1 \\ 0 & j \neq i \pm 1 \end{cases}$$

so the backward eq is

$$P_{ij}'(t) = \lambda_i P_{i+1,j}(t) + \mu_i P_{i-1,j}(t) - (\lambda_i + \mu_i) P_{ij}(t) \quad i \neq 0$$

$$\big(i = 0 \quad P_{0j}'(t) = \lambda_0 P_{1,j}(t) - \lambda_0 P_{0j}(t)\big)$$