Jan 21

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Def (aperiodicity). We say that a state i is aperiodic if its period is 1.

. A M-C is aperiodic if all its states are aperiodic

3) Recurrence and transience

Def: (Recurrence and transience)

Let $f_i = P(X_n = i \text{ for some } n \mid X_0 = i)$

We say that i is $\begin{cases} \text{transient} & \text{if } f_i < 1 \\ \text{recurrent} & \text{if } f_i = 1 \end{cases}$

Interpretation: This defines the ability to re-visit a state.

Rmk:
- A state is either recurrent or transient, so we can split the state into recurrent or transient states
- If $p_{ii} = 1$, i is recurrent and more specifically, i is called an absorbing state

Before studying in more details these properties, let us first see how all the concepts in this section apply in the context of classical problems.

Ex: the Gambler ruin's problem

Problem: Smith has $\$n$, and plays a game with probability p of winning $\$1$ at each round, and $1-p$ of losing $\$1$ $(0<p<1)$.

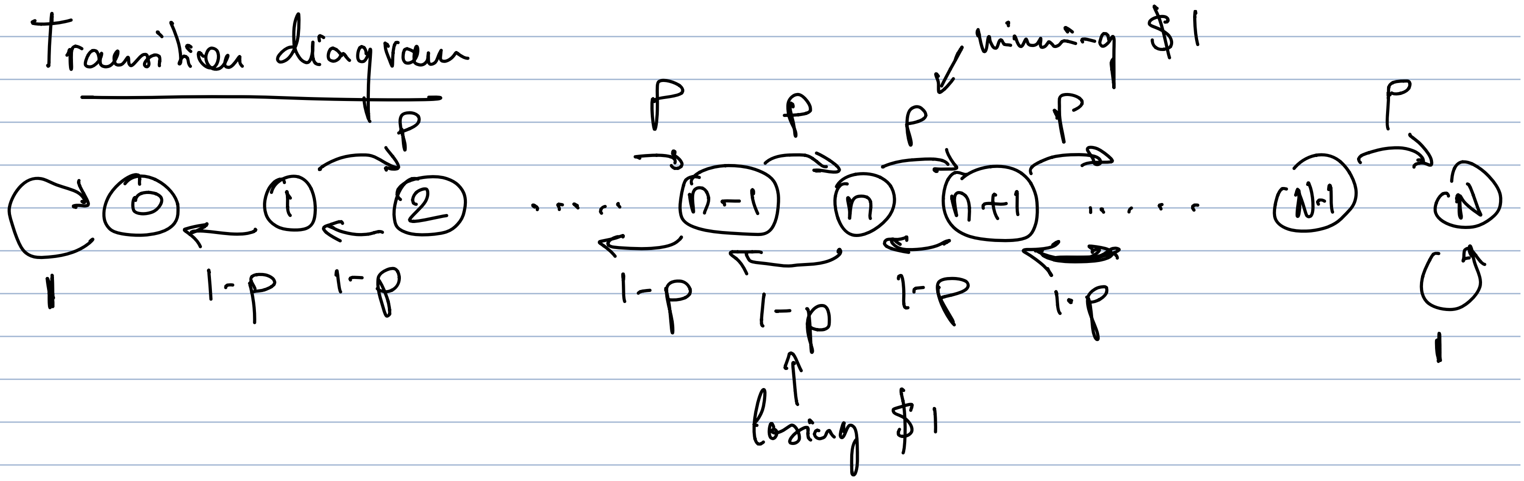
Smith plays until he gets broke, or reach a goal of $\$N$ $(N \geq n)$.

Question: what is the probability that Smith gets broke? (or equivalently that Smith reaches the goal?)

- let's denote X_i Smith's wealth in the i^{th} round

$(X_i)_{i\geq 0}$ follows a Markov Chain.

Transition diagram



Transition Matrix:

$$
\begin{array}{c|c}
 & \begin{matrix} 0 & 1 & \cdots & \cdots & \cdots & N \end{matrix} \\
\begin{matrix} 0 \\ 1 \\ \vdots \\ \vdots \\ N-1 \\ N \end{matrix} &
\begin{pmatrix}
1 & 0 & \cdots & \cdots & \cdots & 0 \\
1-p & 0 & p & 0 & \cdots & 0 \\
 & 1-p & 0 & p & & (0) \\
 & (0) & \ddots & \ddots & \ddots & \\
 & & & 1-p & 0 & p \\
 & & & & 0 & 1
\end{pmatrix}
\end{array}
$$

Communicating classes: $\{0\}$, $\{N\}$, $\{1, \dots, N-1\}$

Periodicity: $d(0) = 1$ (0 is aperiodic)

$d(N) = 1$ (N " ")

$d(1) = \dots d(N-1) = 2$

(if a trajectory $X_0 \to X_1 \to \dots \to X_\ell$ s.t. $X_0 = X_\ell$ had k winning events, then it must also have k losing events so ℓ is even)

Recurrence and transience:

- 0 is recurrent (since it's an absorbing state)
- N " " (")
- Let's consider $0 < i < N$
  and the path $\mathcal{P} = i \to i+1 \to \dots \to N \to N \to \dots$
  The probability of observing this path is

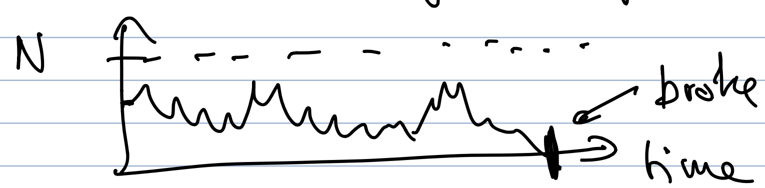
$$P(\mathcal{P}) = p^{N-i} > 0$$

so $f_i \leq 1 - P(\mathcal{P}) < 1$

and i is transient

Q: What is $Pr(\underbrace{\text{Smith goes broke}}_{\text{"R"}} \mid X_0 = n)$?

Rmk: . We can also interpret this problem as 1-D random walk with absorbing boundaries 0 and N, s.t. we want to find
$Pr(\text{hitting } 0 \text{ before } N)$



see (Jupyter Notebook)

A: Let $p(n) = P(R \mid X_0 = n)$

- $p(0) = 1$, $p(N) = 0$ (boundary conditions)
- For $1 \leq n \leq N-1$

Key idea: To find an eq. satisfied by $p(n)$, we condition on the 1st event

$$p(n) = P(R \mid 1^{st} \text{ round is a win}) \cdot P(1^{st} \text{ round is a win})$$
$$+ P(R \mid 1^{st} \text{ round is a loss}) \cdot P(1^{st} \text{ round is a loss})$$

$$p(n) = P(R \mid X_1 = n+1) \cdot p + P(R \mid X_1 = n-1) \cdot (1-p)$$

$$\Rightarrow p(n) = p(n+1) \cdot p + p(n-1) \cdot (1-p)$$

This is a 2^{nd} order linear recurrence equation

To solve it, we solve the characteristic equation (in X)

$$\underset{\substack{\updownarrow \\ p(n)}}{X} = \underset{\substack{\updownarrow \\ p(n+1)}}{X^2} \cdot p + \underset{\substack{\updownarrow \\ p(n-1)}}{1} \cdot (1-p)$$

$$\Leftrightarrow pX^2 - X + (1-p) = 0 \qquad (aX^2 + bX + c = 0)$$

$$\Delta = (-1)^2 - 4(1-p) \cdot p = 4p^2 - 4p + 1$$

$$= (2p-1)^2 \geqslant 0$$

1st case : $\Delta > 0$, i.e $\boxed{p \neq \frac{1}{2}}$

2 roots $\begin{cases} X_1 = \dfrac{1 + 2p - 1}{2p} & \left(\dfrac{-b + \sqrt{\Delta}}{2a}\right) \\ X_2 = \dfrac{1 - 2p + 1}{2p} & \left(\dfrac{-b - \sqrt{\Delta}}{2a}\right) \end{cases}$

$$\Leftrightarrow \begin{cases} X_1 = 1 \\ X_2 = \dfrac{1-p}{p} = a \text{ (for short)} \end{cases}$$

For all values α and β, $v(n) = \alpha X_1^n + \beta X_2^n$ solves
$v(n) = (1-p) \cdot v(n+1) + p \cdot v(n-1)$.

To find $p(n)$ we need to find α, β, which are determined by the boundary conditions

$$\begin{cases} p(0) = 1 \\ p(N) = 0 \end{cases} \Rightarrow \begin{cases} \alpha = \dfrac{a^N}{1-a^N} \\ \beta = \dfrac{1}{1-a^N} \end{cases}$$

(exercise)

$$\text{so } p(n) = \frac{a^N - a^n}{a^N - 1} = \boxed{1 - \frac{a^n - 1}{a^N - 1}}$$

(exercise)