MATH 303 Final Exam Solution

April 29, 2021

Problem 1

Part 1

- $P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0.4 & 0 & 0.6 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0.5 & 0.5 \end{bmatrix}$; (a) 2 ; (b) $[3, 4]$; (c) $[1, 2]$; (d) [ ]; (e) [ ]

- $P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0.6 & 0 & 0 & 0.4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$; (a) 2 ; (b) $[4]$; (c) [ ]; (d) $[1, 2, 3]$; (e) $[4]$

- $P = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0.6 & 0.4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$; (a) 3 ; (b) $[4]$; (c) $[2, 3]$; (d) [ ]; (e) $[4]$

- $P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.6 & 0 & 0.4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$; (a) 2 ; (b) $[4]$; (c) $[1, 2, 3]$; (d) [ ]; (e) $[4]$

Part 2

- $x_0 = 1$: (f) $r = 1/2$; (g) $Pi = [0, 0.25, 0.25, 0.5]$
- $x_0 = 2$: (f) $r = 1/3$; (g) $Pi = [0.4, 0, 0.2, 0.4]$
- $x_0 = 3$: (f) $r = 1/5$; (g) $Pi = [0.4, 0, 0.3, 0.3]$
- $x_0 = 1$: (f) $r = 1/4$; (g) $Pi = [0.5, 0, 0.5, 0]$

Problem 2

We study the weather and amount of snow every year in Seattle. A year can be either *rainy* (R) or *snowy* (S). We assume that whether a given year is snowy or rainy depends only on the previous year. Besides, if one year was S, the next year will be S or R with equal probability. Run the notebook for problem 2 with your student ID to find the transition probabilities between R and S, and the average number of inches of snow associated with each state. Justify all your answers.

Variants: The notebook displays the average amount of snow for snowy years $\mu(F)$; rainy years $\mu(G)$ and p_{RR} (other transition probabilities are also displayed below).

b=0 $\mu(F) = 6, \mu(G) = 3$

$$p_{RR} = 2/3 \qquad p_{RS} = 1/3 \qquad p_{SR} = 1/2 \qquad p_{SS} = 1/2.$$

b=1 $\mu(F) = 8, \mu(G) = 2$

$$p_{RR} = 1/3 \qquad p_{RS} = 2/3 \qquad p_{SR} = 1/2 \qquad p_{SS} = 1/2$$

b=2 $\mu(F) = 8, \mu(G) = 4$

$$p_{RR} = 1/4 \qquad p_{RS} = 3/4 \qquad p_{SR} = 1/2 \qquad p_{SS} = 1/2$$

b=3 $\mu(F) = 6, \mu(G) = 2$

$$p_{RR} = 3/4 \qquad p_{RS} = 1/4 \qquad p_{SR} = 1/2 \qquad p_{SS} = 1/2$$

a. Find the probability that it is a rainy year two years after a rainy year.

Solution: This is a Markov chain on two states, with transition matrix

$$P = \begin{bmatrix} p_{RR} & p_{RS} \\ p_{SR} & p_{SS} \end{bmatrix}.$$

Calculate P^2 and find the $(1,1)$ entry:

– b=0:

$$P = \begin{bmatrix} 2/3 & 1/3 \\ 1/2 & 1/2 \end{bmatrix} \qquad P^2 = \begin{bmatrix} 11/18 & 7/18 \\ 7/12 & 5/12 \end{bmatrix}$$

– b=1:

$$P = \begin{bmatrix} 1/3 & 2/3 \\ 1/2 & 1/2 \end{bmatrix} \qquad P^2 = \begin{bmatrix} 4/9 & 5/9 \\ 5/12 & 7/12 \end{bmatrix}$$

– b=2:

$$P = \begin{bmatrix} 1/4 & 3/4 \\ 1/2 & 1/2 \end{bmatrix} \qquad P^2 = \begin{bmatrix} 7/16 & 9/16 \\ 3/8 & 5/8 \end{bmatrix}$$

– b=3:

$$P = \begin{bmatrix} 3/4 & 1/4 \\ 1/2 & 1/2 \end{bmatrix} \qquad P^2 = \begin{bmatrix} 11/16 & 5/16 \\ 5/8 & 3/8 \end{bmatrix}$$

b. Suppose that the first year was rainy. Find the expected total number of inches of snow in the third year.

Solution: Denoting $\mu(F)$ and $\mu(G)$ the average number of inches of snow on a snowy year, respectively on a rainy year, this amounts to computing the quantity :

$$\begin{bmatrix} 1 & 0 \end{bmatrix} P^2 \begin{bmatrix} \mu(G) \\ \mu(F) \end{bmatrix}.$$

- b=0: $\mu(F) = 6, \mu(G) = 3$ gives

$$\mathbb{E}(\text{Inches of snow in year 3}|\text{Year 1 is rainy}) = \frac{25}{6}.$$

- b=1: $\mu(F) = 8, \mu(G) = 2$ gives

$$\mathbb{E}(\text{Inches of snow in year 3}|\text{Year 1 is rainy}) = \frac{16}{3}.$$

- b=2: $\mu(F) = 8, \mu(G) = 4$ gives

$$\mathbb{E}(\text{Inches of snow in year 3}|\text{Year 1 is rainy}) = \frac{25}{4}.$$

- b=3: $\mu(F) = 6, \mu(G) = 2$ gives

$$\mathbb{E}(\text{Inches of snow in year 3}|\text{Year 1 is rainy}) = \frac{13}{4}.$$

c. What is the the long run average number of inches of snow in Seattle?

Solution: Find the stationary distribution of the chain i.e. find

$$\pi P = \pi \quad \text{and} \quad \pi(R) + \pi(S) = 1.$$

Then, the long run average number of inches of snow is

$$\pi \begin{bmatrix} \mu(G) \\ \mu(F) \end{bmatrix}$$

- b=0: We obtain

$$\pi = [3/5, 2/5].$$

And therefore, with $\mu(F) = 6, \mu(G) = 3$

$$\mathbb{E}_\pi(\text{Inches of snow}) = 3/5 \times 3 + 2/5 \times 6 = 21/5.$$

- b=1: We obtain

$$\pi = [3/7, 4/7].$$

And therefore, $\mu(F) = 8, \mu(G) = 2$

$$\mathbb{E}_\pi(\text{Inches of snow}) = 3/7 \times 2 + 4/7 \times 8 = 38/7.$$

- b=2: We obtain

  $$\pi = [2/5,\ 3/5].$$

  And therefore, $\mu(F) = 8, \mu(G) = 4$

  $$\mathbb{E}_\pi(\text{Inches of snow}) = 2/5 \times 4 + 3/5 \times 8 = 32/5.$$

- b=3: We obtain

  $$\pi = [2/3,\ 1/3].$$

  And therefore, $\mu(F) = 6, \mu(G) = 2$

  $$\mathbb{E}_\pi(\text{Inches of snow}) = 2/3 \times 2 + 1/3 \times 6 = 10/3.$$

Problem 3

The number of followers of Jacob's video game stream on Twitch follows a Poisson process with rate 5 per day. Assume that each follower is a subscriber with probability 1/5, all independently.

a. Run the notebook and find the probability asked for to Problem 3a. Justify your answer. The solution can be left as calculator ready.

Variants:

- b=0: probability that Jacob got at least one subscriber, and no followers who are not subscribers after a week
- b=1: probability that Jacob has exactly no subscriber and 2 followers who are not subscribers after 4 days.
- b=2: probability that Jacob has at most 1 follower who is not subscriber and exactly 2 subscribers after 4 days

Solution: By thinning, the total number of subscribers after n days is Poisson with mean $5 \times n/5 = n$; the total number of followers who are not subscribers after after n days is Poisson with mean $5 \times n \times 4/5 = 4n$. By thinning, these two variables are also independent. We obtain:

- b=0: $P = P(Poisson(7) \geq 1) \times P(Poisson(28) = 0) = (1 - e^{-7})e^{-28}$
- b=1: $P = P(Poisson(4) = 0) \times P(Poisson(16) = 2) = 128e^{-20}$
- b=2: $P = P(Poisson(16) \leq 1) \times P(Poisson(4) = 2) = 136e^{-20}$

b. Run the notebook and find the probability asked for to Problem 3b. Justify your answer. The solution can be left as calculator ready.

Variants:

- b=0: Probability that exactly three followers are subscribers, given that Jacob got eight total followers in two days.
- b=1: Probability that exactly three followers are not subscribers, given that Jacob got six total followers in three days.
- b=2: Probability that exactly four followers are subscribers, given that Jacob got eight total followers in four days.

Solution: Given that there were n followers, the number of subscribers is Binomial$(n, 1/5)$, and followers who are not subscribers is Binomial$(n, 4/5)$.

- b=0: $P = P(Binom(8, 1/5) = 3) = \binom{8}{3} \frac{1}{5^3} \left(\frac{4}{5}\right)^5 = \frac{56 \times 4^5}{5^8}$
- b=1: $P = P(Binom(6, 4/5) = 3) = \binom{6}{3} \frac{1}{5^3} \left(\frac{4}{5}\right)^3 = \frac{4^4}{5^5}$
- b=2: $P = P(Binom(8, 1/5) = 4) = \binom{8}{4} \frac{1}{5^4} \left(\frac{4}{5}\right)^4 = 14\frac{4^4}{5^7}$

c. Run the notebook and find the value asked for to Problem 3c. Justify your answer. The solution can be left as calculator ready.

Variants:

- b=0: Expected number of followers who are not subscribers, given that Jacob got eight total followers in two days.
- b=1: Expected number of subscribers, given that Jacob got six total followers in three days.
- b=2: Expected number of followers who are not subscribers, given that Jacob got eight total followers in four days.

Solution: From the binomial $B(n, p)$ found in question 2b, take the mean of the binomial with parameters n and $1 - p$ or n and p.

- b=0: $E = \frac{32}{5}$
- b=1: $E = \frac{6}{5}$
- b=2: $E = \frac{32}{5}$

Problem 4

Becca only watches three TV channels: news A, news B, and sports. While watching the news, she switches channels after an exponential amount of time with mean μ_{news} minute(s). While watching sports, she switches after an exponential amount of time with mean μ_{sports} minute(s). After watching a news channel, Becca switches to the other news channel with probability p and to sports with probability $1-p$. After watching sports, Becca switches to news A with probability p' and news B with probability $1-p'$. Run the notebook to find the values of $\mu_{\text{news}}, \mu_{\text{sports}}, p$ and p'.
Variants:

- b=0,
  Data:
  $$\mu_{\text{news}} = 1 \quad \mu_{\text{sports}} = 1/2 \quad p = 1/2 \quad p' = 1/3$$
  Corresponding transition rates and probabilities:
  $$p_{AB} = 1/2 \quad p_{BA} = 1/2 \quad p_{AS} = 1/2 \quad p_{SA} = 1/3 \quad p_{SB} = 2/3 \quad p_{BS} = 1/2,$$
  $$\nu_A = 1 \qquad \nu_B = 1 \qquad \nu_S = 2,$$
  $$q_{AB} = 1/2 \quad q_{BA} = 1/2 \quad q_{AS} = 1/2 \quad q_{SA} = 2/3 \quad q_{SB} = 4/3 \quad q_{BS} = 1/2.$$
- b=1,
  Data:
  $$\mu_{\text{news}} = 1/2 \quad \mu_{\text{sports}} = 1/3 \quad p = 1/3 \quad p' = 1/2$$
  Corresponding transition rates and probabilities:
  $$p_{AB} = 1/3 \quad p_{BA} = 1/3 \quad p_{AS} = 2/3 \quad p_{SA} = 1/2 \quad p_{SB} = 1/2 \quad p_{BS} = 2/3,$$
  $$\nu_A = 2 \qquad \nu_B = 2 \qquad \nu_S = 3,$$
  $$q_{AB} = 2/3 \quad q_{BA} = 2/3 \quad q_{AS} = 4/3 \quad q_{SA} = 3/2 \quad q_{SB} = 3/2 \quad q_{BS} = 4/3.$$
- b=2,
  Data:
  $$\mu_{\text{news}} = 1/2 \quad \mu_{\text{sports}} = 1/3 \quad p = 1/4 \quad p' = 2/3$$
  Corresponding transition rates and probabilities:
  $$p_{AB} = 1/4 \quad p_{BA} = 1/4 \quad p_{AS} = 3/4 \quad p_{SA} = 2/3 \quad p_{SB} = 1/3 \quad p_{BS} = 3/4,$$
  $$\nu_A = 2 \qquad \nu_B = 2 \qquad \nu_S = 3,$$
  $$q_{AB} = 1/2 \quad q_{BA} = 1/2 \quad q_{AS} = 3/2 \quad q_{SA} = 2 \quad q_{SB} = 1 \quad q_{BS} = 3/2.$$
- b=3,
  Data:
  $$\mu_{\text{news}} = 1/4 \quad \mu_{\text{sports}} = 1/2 \quad p = 1/3 \quad p' = 1/4$$
  Corresponding transition rates and probabilities:
  $$p_{AB} = 1/3 \quad p_{BA} = 1/3 \quad p_{AS} = 2/3 \quad p_{SA} = 1/4 \quad p_{SB} = 3/4 \quad p_{BS} = 2/3,$$
  $$\nu_A = 4 \qquad \nu_B = 4 \qquad \nu_S = 2,$$
  $$q_{AB} = 4/3 \quad q_{BA} = 4/3 \quad q_{AS} = 8/3 \quad q_{SA} = 1/2 \quad q_{SB} = 3/2 \quad q_{BS} = 8/3.$$

a. Model Becca's news watching habits by a continuous time Markov chain, with state 1 = news A, state 2 = news B, state 3 = sports. Draw the transition diagram, and add the rates q_{ij}, as defined in the course, to each arrow of the diagram.

Solution: See above for q_{ij}.

b. Write down the two forward Kolmogorov equations with initial condition being the sports channel.

Solution: For this chain, with initial state S, the two forward equations are

$$P'_{SA}(t) = q_{BA}P_{SB}(t) - v_A P_{SA}(t)$$

and

$$P'_{SB}(t) = q_{AB}P_{SA}(t) - v_B P_{SB}(t).$$

- b=0,

$$P'_{SA}(t) = 1/2P_{SB}(t) - P_{SA}(t)$$

  and

$$P'_{SB}(t) = 1/2P_{SA}(t) - P_{SB}(t).$$

- b=1,

$$P'_{SA}(t) = 2/3P_{SB}(t) - 2P_{SA}(t)$$

  and

$$P'_{SB}(t) = 2/3P_{SA}(t) - 2P_{SB}(t).$$

- b=2,

$$P'_{SA}(t) = 1/2P_{SB}(t) - 2P_{SA}(t)$$

  and

$$P'_{SB}(t) = 1/2P_{SA}(t) - 2P_{SB}(t).$$

- b=3,

$$P'_{SA}(t) = 4/3P_{SB}(t) - 4P_{SA}(t)$$

  and

$$P'_{SB}(t) = 4/3P_{SA}(t) - 4P_{SB}(t).$$

c. Is the chain reversible? Justify your answer.

Solution: The detailed balance equations take the form

$\pi_A = \pi_B$, and $\pi_S = \frac{q_{AS}}{q_{SA}}\pi_A = \frac{q_{BS}}{q_{SB}}\pi_B$.

Thus there is a solution if and only if $\frac{q_{AS}}{q_{SA}} = \frac{q_{BS}}{q_{SB}}$.

- b=0,

  No solution. Not reversible.

- b=1,

  Solution exists: $\pi = \frac{9}{26}(1, 1, 8/9)$.

- b=2,

  No solution. Not reversible.

- b=3,

  No solution. Not reversible.