Intersections of random intervals and disks

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Abstract

Consider a finite number of balls $D_1, \dots, D_n$ with fixed radius in $\mathbb{R}^d$, but with centers sampled iid from a fixed distribution F on $\mathbb{R}^d$. Let $I_n = \cap_n D_n$ denote their intersection: how does I_n behave? We show that, for intervals on the line, when F is uniform measure on $[-1, 1]$ and the intervals have radius 1 that $\frac{n}{2}\lambda(I_n)$ converges to a Gamma(2,1) random variable, where λ denotes Lebesgue measure. The case in $\mathbb{R}^2$ is more challenging. We show $\lim_{n\to\infty} n^2\lambda(I_n) = Z \in (0, \infty)$ almost surely for some random variable Z with $\mathbb{E}Z = \frac{\pi^3}{2}$. In fact, for large n, I_n is a polygon with a bounded number of sides.

1 Introduction

Processes involving intersections of random sets have been studied before (diminishing segment process). Other set-valued processes involving random walks have been studied earlier (e.g. Wiener sausage, interval erosion). Our model differs from the diminishing segment process in that the distribution of the centers of the intervals remains fixed: the sets we intersect are iid and independent of time. The analysis is somewhat simpler as a result, and the difficulty mainly lies in understanding the geometry of the intersection.

Our first goal will be to understand the average behavior of the intersection of iid random sets. Generally, we will have a fixed distribution F on $\mathbb{R}^d$, and we form 'random' sets D_i as balls of a fixed radius $r > 0$ about points C_i, where the C_i are drawn iid from F. Denote by (Ω, μ_F) the probability space where the C_i are defined: so typically, $\Omega = \mathbb{R}^d$, and μ_F is a probability measure with $\mu_F \ll \lambda$, where λ is Lebesgue measure. For each $\underline{\omega} = (\omega_1, \dots, \omega_n) \in \Omega^n$, think of $I_n = I_n(\underline{\omega})$ as a function of $\underline{\omega}$, i.e. as a random variable on Ω^n. A simple application of Fubini's theorem yields

$$
\begin{aligned}
\mathbb{E}\lambda(I_n) &= \int_{\Omega^n} \lambda(I_n(\underline{\omega}))\, d\underline{\omega} \\
&= \int_{\Omega^n} \int_{\mathbb{R}^d} 1[x \in I_n(\underline{\omega})]\, d\lambda(x)\, d\underline{\omega} \\
&= \int_{\mathbb{R}^d} \int_{\Omega^n} 1[x \in I_n(\underline{\omega})]\, d\underline{\omega}\, d\lambda(x) \\
&= \int_{\mathbb{R}^d} \mathbb{P}_{\mu_F^n}(x \in I_n)\, d\lambda(x) \\
&= \int_{\mathbb{R}^d} \mathbb{P}_{\mu_F}(x \in D_1)^n\, d\lambda(x). \qquad (1.1)
\end{aligned}
$$

Thus, computing the expected volume of I_n boils down to understanding the pointwise density $\mathbb{P}(x \in I_n)$, i.e. the probability that a fixed $x \in \mathbb{R}^d$ lies in one of the balls B_i.

Unfortunately, obtaining more detailed information about I_n turns out to be a complicated matter.

2 Results in $\mathbb{R}$

We first consider the case in $\mathbb{R}$. Let F denote any distribution, and for $n \geq 1$ let C_n be iid $\sim F$. Fix any $r > 0$, and let D_n denote the interval of radius r centered at C_n, set $I_0 = \mathbb{R}$, and for $n \geq 1$

$$I_n = \bigcap_{m \leq n} D_m.$$

Also, let $M_n = \max_{j \leq n} C_j$ and $m_n = \min_{j \leq n} C_j$ denote the running max and min of the C_j. Since I_n is contained in both the leftmost and rightmost intervals, these intervals determine the length of I_n, namely

$$\lambda(I_n) = 1[\lambda(I_n) > 0]\Big((m_n + r) - (M_n - r)\Big) = (2r - M_n + m_n)1[M_n - m_n < 2r]. \tag{2.1}$$

This implies, for example, that if F does not have compact support, then I_n will be empty for sufficiently large n almost surely. In fact:

Lemma 2.1. *$\lambda(I_n) > 0$ for all n almost surely $\iff$ F is supported in an open interval of length at most $2r$. Moreover, if the support of F has diameter exactly $2r$, then I_n converges to a point almost surely.*

Proof. Easy. □

If F has a density f, the distribution of $M_n - m_n$ can be written down explicitly in terms of F and f:

$$\mathbb{P}(M_n = x, m_n = y) = n(n-1)f(x)f(y)(F(x) - F(y))^{n-2}. \tag{2.2}$$

We also have a simple formula for $\mathbb{E}\lambda(I_n)$ from (1.1). Note that, for a fixed point $x \in \mathbb{R}$,

$$\mathbb{P}(x \in D_1) = F(x + r) - F(x - r).$$

Thus

$$\mathbb{E}\lambda(I_n) = \int_{\mathbb{R}} \Big(F(x + r) - F(x - r)\Big)^n \, dx. \tag{2.3}$$

2.1 Uniform distribution

We now turn to the particular case where F is the uniform distribution on $[-1, 1]$, and $r = 1$, so $F(x) = \frac{x+1}{2}$ for $x \in [-1, 1]$. Then (2.2) becomes

$$\mathbb{P}(M_n = x, m_n = y) = 2^{-n} n(n-1)(x - y)^{n-2}.$$

We can easily use this to compute moments of $\lambda(I_n)$: a simple computation shows

$$\mathbb{E}(M_n - m_n) = 2^{-n}n(n-1)\int_{-1}^{1}\int_{y}^{1}(x-y)^{n-1}dxdy$$
$$= 2\frac{n-1}{n+1}.$$

Thus we have

$$\mathbb{E}\lambda(I_n) = 2 - 2\frac{n-1}{n+1} = \frac{4}{n+1} \sim 4n^{-1}.$$

Note that we can also use (2.3) to get the expected length of I_n. We have

$$F(x+r) - F(x-r) = 1 - \frac{|x|}{2},$$

if $|x| \leq 2$, and thus

$$\mathbb{E}\lambda(I_n) = \int_{[-2,2]}\left(1 - \frac{|x|}{2}\right)^n dx$$
$$= \frac{4}{n+1}.$$

The variance is also easy to calculate by integrating against the density of $M_n - m_n$: this yields

$$\mathrm{Var}(\lambda(I_n)) = \mathrm{Var}[M_n - m_n] = \frac{8(n-1)}{(n+1)^2(n+2)} \sim 8n^{-2}.$$

These computations are suggestive of the limiting distribution. In fact,

Proposition 2.2. $\frac{n}{2}\lambda(I_n) \to_d Gamma(2,1)$.

Proof. Simply write

$$\mathbb{P}(\lambda(I_n) \leq z) = \mathbb{P}(M_n - m_n \geq 2 - z)$$
$$= 2^{-n}n(n-1)\int_{-1}^{1}\int_{y+z}^{1}(x-y)^{n-2}dxdy$$
$$= 1 - \left(1 - \frac{z}{2}\right)^{n-1}\left(1 + \frac{z}{2}(n-1)\right).$$

Thus

$$\lim_{n\to\infty}\mathbb{P}(\lambda(I_n) > 2z/n) = \lim_{n\to\infty}\left(1 - \frac{1}{n}\right)^{n-1}\left(1 + z\frac{n-1}{n}\right) = (1+z)e^{-z}.$$

□

The appearance of a gamma variable in the limit has a simple explanation. Just as in the proof of proposition 2.2, one can directly show that

$$\frac{1}{2}n(1 - M_n) \to_d Exp(1),$$

and

$$\frac{1}{2}n(1+m_n) \to_d Exp(1).$$

Indeed, we have

$$\begin{aligned}
\mathbb{P}\Big(\frac{1}{2}n(1-M_n) \geq z\Big) &= \mathbb{P}(M_n \leq 1 - 2z/n) \\
&= \mathbb{P}(U_1 \leq 1 - 2z/n)^n \\
&= (1 - z/n)^n \\
&\to e^{-z}.
\end{aligned}$$

Moreover, this convergence in distribution occurs jointly, namely $\frac{n}{2}\Big(1-M_n, 1+m_n\Big) \to_d (Y_1, Y_2)$, where Y_1 and Y_2 are independent exponential mean 1 random variables. This is easily verified by carefully plugging into (2.2) and using the chain rule:

$$\begin{aligned}
\mathbb{P}\Big(\frac{n}{2}(1-M_n) = x, \frac{n}{2}(1+m_n) = y\Big) &= \mathbb{P}\Big(M_n = 1 - \frac{2x}{n}, m_n = \frac{2y}{n} - 1\Big) \\
&= 2^{-n}n(n-1)(2 - \frac{x}{n} - \frac{y}{n})^{n-2}\frac{4}{n^2} \\
&= \frac{n-1}{n^2}\Big(1 - \frac{x+y}{n}\Big)^{n-2} \\
&\to e^{-x-y}.
\end{aligned}$$

Proposition 2.2 follows immediately, since Gamma(2,1) is the distribution of a sum of two independent Exponential(1) random variables. In other words, the intersection process in this case can be described as follows: the right and left endpoints shrink to 0 at rate $O(n^{-1})$, and they do so roughly independently, a step being taken on the left or right endpoint roughly half the time.

Now consider a slightly larger uniform distribution: let F denote the uniform distribution on $[-a, a]$, for some $a > 1$. (The case $a < 1$ is essentially contained in the analysis of the case $a = 1$.) In this case, by (2.1), the intersection will be empty at some finite stage almost surely. Define

$$\tau = \inf\{n > 0 : I_n = \emptyset\},$$

with $\tau = \infty$ if $I_n \neq \emptyset$ for all n, for definite-ness. Note that, by (2.1), we have

$$\tau = \inf\{n > 0 : M_n - m_n > 2\}.$$

This allows us to compute directly: for $k \in \mathbb{N}$, we have

$$\begin{aligned}
\mathbb{P}(\tau \leq k) &= \mathbb{P}(M_k - m_k > 2) \\
&= \int_{2-a}^{a}\int_{-a}^{x-2} k(k-1)\frac{1}{4a^2}\Big(\frac{x+a}{2a} - \frac{y+a}{2a}\Big)^{k-2}\,dy\,dx \\
&= 1 - a^{-k}(1 + k(a-1)).
\end{aligned}$$

2.2 Normal distribution

Suppose instead that F is a normal $(0, \sigma^2)$ distribution, and suppose the length of the D_i is any fixed $r > 0$. (There is no loss of generality by centering F.) As usual, we start by using (2.3): recall

$$\mathbb{P}(x \in D_1) = \int_{x-r}^{x+r} \frac{1}{\sigma\sqrt{2\pi}} e^{-t^2/2\sigma}\, dt,$$

so that by Fubini's theorem,

$$\begin{aligned}
\mathbb{E}\lambda(I_n) &= \int_{\mathbb{R}} \int_{x-r}^{x+r} (\sigma\sqrt{2\pi})^{-n} e^{-nt^2/2\sigma^2}\, dt\, dx \\
&= \int_{\mathbb{R}} \int_{t-r}^{t+r} (\sigma\sqrt{2\pi})^{-n} e^{-nt^2/2\sigma^2}\, dx\, dt \\
&= 2r(\sigma\sqrt{2\pi})^{-n+1} n^{-1/2}.
\end{aligned}$$

The expectation decays extremely quickly! This is because at each step, there is a constant probability that the intersection will become empty.

Let $\tau = \tau(r, \sigma)$ be the stopping time $\tau = \inf\{n > 0 : I_n = \emptyset\}$. Numerical computations suggest the following conjecture:

Conjecture 2.3. *The distribution of $\tau(\sigma, \sigma)$ does not depend on the value of σ.*

The analogous statement for the uniform distribution is trivial: the two processes where F is uniform on $[-ra, ra]$ with interval length r, and where F is uniform on $[-a, a]$ with interval length 1 are distributionally equivalent, since those models have a simple scaling property. The same should hold for the class of normal distributions.

TODO: Discrete distributions F? For example, we could show that if F is Poisson(α), then there are phase transitions for the process when the interval radius r is at a half-integer. How do they depend on α? Maybe this is boring, but should be easy.

Also: distributions with densities of the form $f(x) = c(\beta)(1 - |x|^\beta)$, for any $\beta \in \mathbb{R}$ and $r = 1$. Perhaps this class exhibits nice limiting behavior, just as in the uniform distribution case, i.e. $\beta = 0$.

3 Disks in $\mathbb{R}^2$

The challenge begins when we consider disks in $\mathbb{R}^2$. Suppose the D_i have centers uniformly sampled from area measure in the unit disk $\mathbb{D}$. A straightforward computation shows that for fixed $|x| \le 2$, (where $|x| = ||x||_2$),

$$\mathbb{P}(x \in D_1) = \frac{2}{\pi} \arccos(|x|/2) - \frac{|x|}{2\pi}\sqrt{4 - |x|^2},$$

So (2.3) leads to (in polar coordinates)

$$\mathbb{E}\lambda(I_n) = 2\pi \int_0^2 r \Big(\frac{2}{\pi} \arccos(r/2) - \frac{r}{2\pi}\sqrt{4 - r^2}\Big)^n dr.$$

Although this integral is difficult to evaluate directly, numerical evaluations (up to $n = 10^5$) yield $E\lambda(I_n) \approx 15.4568 n^{-2}$. Surprisingly, the formula for $\mathbb{P}\Big(x \in D_1 \Big| |x| = r\Big)$, i.e. the integrand of the above, is extremely close to a lienar function. Specifically, set $g(r) = 1 - r/2$, and $f(r) = \frac{2}{\pi}\arccos(r/2) - \frac{r}{2\pi}\sqrt{4-r^2}$; g is the linear function with $g(0) = f(0)$ and $g(2) = f(2)$. Mathematica verifies that

$$\max_{r\in[0,2]} |g(r) - f(r)| \leq .0623,$$

and in fact

$$f(r) \leq g(r) \text{ for } r \in [0,2].$$

Simply replacing f by g yields the integral

$$2\pi \int_0^2 r g(r)^n \, dr = \frac{8\pi}{(n+1)(n+2)} \sim 8\pi n^{-2}.$$

So it seems that the error term in making this approximation is not asymptotically negligible, but only changes the asymptotic answer by a constant factor. One can also show that the linear approximation to f at $r = 0$, given by $h(r) = 1 - \frac{2}{\pi}r$, is a lower bound for f on $[0,2]$, and

$$2\pi \int_0^2 r h(r)^n 1\{h(r) \geq 0\} \, dr = \frac{\pi^3}{2(n+1)(n+2)} \sim \frac{\pi^3}{2} n^{-2}.$$

In other words:

Proposition 3.1. *For all n sufficiently large, we have*

$$\pi^3/2 \leq n^2 \mathbb{E}\lambda(I_n) \leq 8\pi$$

An immediate consequence is:

Corollary 3.2. *If $n^2\lambda(I_n) \to_d X$ for some random variable X, then $\mathbb{E}X \in [\pi^3/2, 8\pi]$. In particular, X is non-zero and has finite first moment.*

We will show later that the lower bound is actually the correct limiting value. This isn't surprising, since roughly speaking, for n large only values of r near 0 will contribute to the integral, and the function h is the tangent approximation to f near 0.

One way to get probabilistic bounds on $\lambda(I_n)$ is to consider the probability that a fixed circle centered at the origin is contained in I_n. Fix a circle C_s of radius $s > 0$ centered at $0 \in \mathbb{D}$. Then

$$\mathbb{P}(C_s \subset I_n) = \mathbb{P}(C_s \subset D_1)^n = (1-s)^{2n}. \tag{3.1}$$

Thus, if we choose $s = n^{-2-\epsilon}$ for any $\epsilon > 0$, then simple exponential bounds give

$$\mathbb{P}(C_s \subset I_n) = (1 - n^{-2-\epsilon})^{2n} \geq \exp(-n^{-1-\epsilon}) \geq 1 - n^{-1-\epsilon}.$$

Thus, by the first Borel Cantelli lemma, the event $[C_{n^{-2-\epsilon}} \not\subset I_n]$ occurs for finitely many n almost surely, or in other words, $C_{n^{-2-\epsilon}} \subset I_n$ for all n sufficiently large almost surely. This gives the crude lower bound

$$\lambda(I_n) \geq \pi n^{-4-2\epsilon}$$

for all n sufficiently large. The estimate (3.1) also gives a quick lower bound on the expectation:

$$
\begin{aligned}
\mathbb{E}\lambda(I_n) &\geq \mathbb{E}[\lambda(I_n)1[C_{n^{-1}} \subset I_n]] \\
&\geq \pi n^{-2}\mathbb{P}(C_{n^{-1}} \subset D)^n \\
&= \pi n^{-2}(1-n^{-1})^{2n} \\
&\geq (1-o(1))\pi e^{-2}n^{-2}.
\end{aligned}
$$

We also note the bound

$$
\mathbb{P}(\lambda(I_n) > a) > \mathbb{P}(C_s \subset I_n),
$$

where $s = \left(\frac{a}{\pi}\right)^{1/2}$. Scaling appropriately leads to a distributional inequality in the limit. Let $s_n = a^{1/2}/n$. Then we have

$$
\mathbb{P}\left(\frac{n^2}{\pi}\lambda(I_n) > a\right) > \mathbb{P}(C_{s_n} \subset I_n) = \left(1 - \frac{a^{1/2}}{n}\right)^{2n}.
$$

We have

$$
\lim_{n\to\infty}\left(1 - \frac{1}{n}a^{1/2}\right)^{2n} = e^{-2\sqrt{a}},
$$

so if $\frac{n^2}{\pi}\lambda(I_n) \to_d X$ where X is a r.v. with cdf $F(x)$, then

$$
F(x) \leq 1 - e^{-2\sqrt{x}}.
$$

3.1 The radius of I_n

A natural statistic of I one might study is the 'radius' of I in a given direction, namely for $\theta \in [0, 2\pi]$,

$$
R_n(\theta) = \sup\{t > 0 : te^{i\theta} \in I_n\}.
$$

Note that $\{R_n(\theta)\}_{\theta\in[0,2\pi]}$ are identically distributed because of symmetry, but of course these variables are not independent. Let $R_n = R_n(0)$ for convenience: R_n exhibits the same behavior as in the one dimensional case:

Proposition 3.3. *$\frac{2}{\pi}nR_n \to_d Exp(1)$.*

Proof. The process $\{R_n\}_{n\geq 0}$ with $R_0 = 1$ is a non-increasing markov chain. To get the transition probabilities, note that for each $z \in [0,1]$, $R_{n+1} = z$ either if $R_n = z$ and the center of D_n lies inside the circle $(x-z)^2 + y^2 \leq 1$, or if $R_n > z$ and the center of D_n is on the boundary of that circle. This leads to

$$
\mathbb{P}(R_{n+1} = x | R_n = y) = \begin{cases} \frac{1}{\pi}\sqrt{4-x^2}, x < y \\ \frac{1}{2\pi}y\sqrt{4-y^2} + \frac{2}{\pi}\arcsin(y/2), x = y \end{cases}
$$

Technically, the transition measure has an atom at $R_{n+1} = R_n$, and a continuous density between 0 and R_n. Since the probability of transitioning to a value $R_n < x$ does not depend on the value of R_n, we can easily compute

$$
\begin{aligned}
\mathbb{P}(R_n > x) &= (1 - \int_0^x \frac{1}{\pi}\sqrt{4-z^2}dz)^n \\
&= (1 - \frac{1}{2\pi}x\sqrt{4-x^2} + \frac{2}{\pi}\arcsin(x/2))^n \\
&= (1 - \frac{2}{\pi}x + \frac{x^3}{12\pi} - \cdots)^n.
\end{aligned}
$$

Thus

$$
\mathbb{P}\left(\frac{2}{\pi}nR_n > x\right) = \mathbb{P}\left(R_n > \frac{\pi x}{2n}\right) = \left(1 - \frac{x}{n} + O(x^3n^{-3})\right)^n \to e^{-x}
$$

as $n \to \infty$, as desired. □

This immediately gives the limiting expected area:

Proposition 3.4. $\lim_{n\to\infty} n^2 \mathbb{E}\lambda(I_n) = \pi^3/2$.

Proof. Writing the area of I_n as a Riemann sum in polar coordinates, and using the fact that the distribution of $R_n(\theta)$ does not depend on θ,

$$
\begin{aligned}
\mathbb{E}\lambda(I_n) &= \mathbb{E}\Big[\lim_{m\to\infty}\sum_{j=1}^m \frac{1}{2}\frac{2\pi}{m}R_n\Big(\frac{2\pi j}{m}\Big)^2\Big] \\
&= \lim_{m\to\infty}\sum_{j=1}^m \frac{\pi}{m}\mathbb{E}\Big[R_n^2\Big] \\
&= \pi\mathbb{E}R_n^2.
\end{aligned}
$$

To justify the change of expecation and limit, note that the Riemann sum is increasing in m since I_n is convex for each n, and hence the monotone convergence theorem applies. Now by the proof of proposition 3.3,

$$
\begin{aligned}
\mathbb{E}R_n^2 &= \int_0^\infty \mathbb{P}(R_n^2 > y)\,dy \\
&= \frac{\pi^2}{4n^2}\int_0^\infty \mathbb{P}\left(R_n > \frac{\pi\sqrt{z}}{2n}\right)\,dz \\
&= \frac{\pi^2}{4n^2}\int_0^\infty \exp(-\sqrt{z})\exp(\cdots)^{n^{-2}}\,dz.
\end{aligned}
$$

A straightforward application of the dominated convergence theorem allows passing a limit in n through this integral, so we obtain

$$
\lim_{n\to\infty} n^2\mathbb{E}R_n^2 = \frac{\pi^2}{4}\int_0^\infty \exp(-\sqrt{z})\,dz = \frac{\pi^2}{2}. \tag{3.2}
$$

Combining this with the first calculation gives the result.

□

We can do better than computing the distribution of R_n. The next step would be to compute covariances between the $R_n(\theta)$'s: for example, one can show that

$$\frac{\pi n}{2}(R_n, R_n(\theta)) \to_d (Z_1, Z_2),$$

where $Z_1, Z_2 \sim Exp(1)$ are correlated somehow, say $Corr(Z_1, Z_2) = C(\theta)$ for some continuous $C : [0, \pi] \to [0, 1]$ with $C(0) = 1, C(\pi) = 0$. It is relatively easy to establish that $C(\pi) = 0$, i.e. the radii in polar opposite directions are uncorrelated:

Proposition 3.5. *$\frac{\pi n}{2}(R_n, R_n(\pi)) \to_d (Z_1, Z_2)$, where Z_1 and Z_2 are independent $\sim Exp(1)$.*

Proof. We need to understand the joint probability $\mathbb{P}(R_n > x, R_n(\pi) > y)$. For $z \in [0, 1]$, let A_z and B_z denote the sets outside the circles $(x-z)^2+y^2 \leq 1$ and $(x+z)^2+y^2 \leq 1$, and inside the unit disk, respectively. As in the proof of proposition 3.2, $\lambda(A_z) = \frac{1}{2\pi}z\sqrt{4-z^2} + \frac{2}{\pi}\arcsin(z/2) = \frac{2}{\pi}z + O(z^3)$ as $z \to 0$, and we have

$$\mathbb{P}(R_n > x, R_n(\pi) > y) = (1 - \lambda(A_x \cup B_y))^n = (1 - \lambda(A_x) - \lambda(B_y) + \lambda(A_x \cap B_y))^n.$$

It is easy to upper bound the area of the intersection by finding a circumscribing rectangle: this yields $\lambda(A_x \cap B_y) \leq 4\sqrt{2}\max(x, y)^3$. Thus,

$$\mathbb{P}(\frac{\pi}{2}nR_n > x, \frac{\pi}{2}nR_n(\pi) > y) = (1 - \frac{x}{n} - \frac{y}{n} + O(n^{-3}))^n \to e^{-x}e^{-y},$$

as $n \to \infty$, as desired.

$\square$

The correlation can be computed explicitly, though the formulas are a bit tedious. We have:

Proposition 3.6. *For $x, y \in \mathbb{R}^2$ and $\theta \in (0, \pi)$, define the function $\hat{\phi} = \hat{\phi}(x, y)$ via*

$$\cos(\hat{\phi}) = \frac{x^2 + y^2\cos^2\theta - 2xy\cos\theta}{x^2 + y^2 - 2xy\cos\theta}.$$

Then

$$\lim_{n\to\infty} \mathbb{P}(\frac{n}{\pi}R_n > x, \frac{n}{\pi}R_n(\theta) > y) = \exp(-(x + y) - x\sin\hat{\phi} - y\sin(\theta - \hat{\phi})).$$

3.2 Upper bounds and the Continuous Coupon Collector Problem

In the same spirit, we might hope to get an upper approximation for $\lambda(I_n)$ by considering a fixed circle C_s and asking for $\mathbb{P}(I_n \subset C_s)$. To get a handle on the event $I_n \subset C_s$, note that

$$[I_n \subset C_s] \subset [\partial C_s \subset \cup_{m\leq n}(C_s \setminus D_m)].$$

This says that once each point of the boundary of C_s does not lie in some D_m, the intersection I_n lies inside C_s.

It is natural to ask about the process of 'covering' the boundary of C_s with the complements of the circles, which we dub the 'continuous coupon collector process.' We describe a simpler version of this kind of process on the unit circle $\mathbb{S}^1$. Fix an angle $\theta \in [0, 2\pi]$, and sample iid points $w_1, w_2, \ldots$ from $\mathbb{S}^1$. Place arcs of length θ centered at the w_i: let A_i denote the arc centered at w_i. The process ends when $\mathbb{S}^1 = \bigcup_i A_i$. Let $N(\theta)$ denote the first time this occurs: what is the distribution of N? One easy fact is the following: for fixed $z \in \mathbb{S}^1$, we have $\mathbb{P}\Big(z \in \bigcup_{i=1}^n A_i\Big) = 1 - (1-\theta/2\pi)^n$. Idea: chop up $\mathbb{S}^1$ into $\lceil 2/\theta \rceil$ intervals of length $\theta/2$. Then once each interval contains at least one center w_i, the whole circle is covered. This is now a discrete coupon collector problem: each of the $2/\theta$ length $\theta/2$ intervals has equal probability of having any given center. Thus

$$\mathbb{E}N \leq 2\theta^{-1}\log(2\theta^{-1}).$$

To relate this back to the intersection process, some simple geometry shows that

$$\mathbb{P}\Big(|\partial C_s \setminus D_1| > \frac{2\pi}{3}s\Big) > s - s^2/4 \geq s/2.$$

To see this, observe that we have probability s of D_1 not completely containing C_s; further, there is probability $1-(1-s/2)^2 = s - s^2/4$ that the amount of arclength of C_s that D_1 does not contain is more than $2\pi s/3$, which comes from approximating the circle D_1 by a line. [Picture?] We can now couple the intersection process with the 'delayed' continuous coupon collector problem as follows. Let $\sigma_0 = 0$ and set

$$\sigma_j = \min\{s > \sigma_j : |\partial C_s \setminus D_j| > \frac{2\pi}{3}s\}$$

to be the times where we are guaranteed to get at least $1/3$ of the arclength of C_s 'left out' by the newly added disk. At each time $\sigma_j, j \geq 1$, we take a bite out of the circle of radius s of size $1/3$ of the total circle. Note that the waiting times satisfy $\mathbb{E}(\sigma_{j+1} - \sigma_j) \leq 2/s$. Write $T_s = \min\{t > 0 : I_n \subset C_s\}$ for the time it takes for the intersection to be contained in the circle of radius s. Then in expectation, by Wald's lemma,

$$\mathbb{E}T_s \leq \frac{2}{s}\mathbb{E}N(2\pi/3) \leq \Big(\frac{8\pi}{3}\log\frac{4\pi}{3}\Big)s^{-1}.$$

3.3 How many sets are 'relevant' to the intersection?

Another feature of this process in 2 or more dimensions, not present in the one dimensional setting, is the number of sets that are 'relevant' for the intersection. That is, given that we have already sampled n disks, we can compute the size of the smallest subset $A \subset [n]$ satisfying

$$\bigcap_{i\in A} D_i = \bigcap_{i\in[n]} D_i.$$

In words, some of the disks D_i will no longer 'contribute' to the intersection after some time: indeed, for each fixed i, there is a finite time where D_i can be removed from the intersection, and the set I_n will remain the same. It is natural to ask: what is the expected number of 'relevant' sets up to time n? Is it bounded in expectation, and if not, at what rate does it grow? For $j \geq 1$, define

$$\tau_j = \inf\{t \geq 0 : \bigcap_{\substack{1\leq i\leq j+t \\ i\neq j}} D_i \subset D_j\}$$

to be the amount of time it takes for D_j to 'become irrelevant.' We have:

Proposition 3.7. *For any $j \geq 1$ and $k \geq 1$,*

$$\mathbb{P}(\tau_j = k) = \mathbb{P}(\tau_1 = j + k - 1).$$

Thus understanding the τ_j's boils down to understanding τ_1. One consequence is

$$\mathbb{E}\tau_j = \mathbb{E}(\tau_1 - j + 1)1[\tau_j \geq j - 1],$$

which implies that for each $j > 1$, $\mathbb{E}\tau_1 < \infty \iff \mathbb{E}\tau_j < \infty$.

The distribution of τ is closely related to the geometry of I_n. Let S_n denote the number of 'sides' of I_n, i.e. the number of circles relevant to I_n. The S_n's have a distributional limit, a consequence of the following fact:

Proposition 3.8. *$\{S_n : n \in \mathbb{N}\}$ is tight, i.e. $\mathbb{P}(S_n > s)$ is uniformly bounded for all s sufficiently large.*

In fact, we conjecture that $\mathbb{E}S_n$ is uniformly bounded, so if $S_n \to_d S$, then $\mathbb{E}S < \infty$. This is very surprising: for example, it would imply that nI_n does not converge almost surely to a fixed set, since that set would have to be a circle by symmetry, but a circle does not have a finite number of sides. As n increases, nI_n oscillates between being different types of polygons, but keeping the number of sides bounded in expectation. Simulations suggest $\mathbb{E}S \approx 6$.