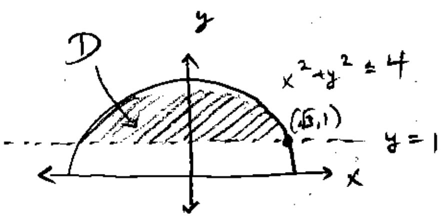
1. (10 points) Consider the region $D = \{(x, y) \in \mathbb{R}^2 : y \geq 1, x^2 + y^2 \leq 4\}$. Draw a picture of D, and evaluate

$$\iint_D \frac{y^2}{x^2+y^2}\, dA.$$

D is part of a semi-circle.



The circle & line intersect at

$x^2 + (1)^2 = 4 \Rightarrow x = \pm\sqrt{3}$

In polar, $(\sqrt{3}, 1)$ is $\theta = \pi/6$, $r = 2$.

Thus $D = \left\{ (r,\theta) : \begin{array}{c} \pi/6 \leq \theta \leq 5\pi/6 \\ \frac{1}{\sin\theta} \leq r \leq 2 \end{array} \right\}$, since

$y = 1 \rightarrow r\sin\theta = 1$, or $r = \frac{1}{\sin\theta}$ in polar;

So $\displaystyle\iint_D \frac{y^2}{x^2+y^2}\, dA = \int_{\pi/6}^{5\pi/6}\int_{1/\sin\theta}^{2} \frac{r^2\sin^2\theta}{r^2}\, r\,dr\,d\theta = \int_{\pi/6}^{5\pi/6}\int_{1/\sin\theta}^{2} r\sin^2\theta\, dr\,d\theta = \boxed{\frac{\pi}{3} + \frac{\sqrt{3}}{2}}$

2. (14 pts) Suppose there is a cylindrical can of beer occupying the region

$$\{(x, y, z) \in \mathbb{R}^3 : y^2 + z^2 \leq 1, 0 \leq x \leq 4\}.$$

A nervous freshman is swirling the can, concentrating the beer near the edge. Assume the beer has density function $f(x, y, z) = x^2y^2$. Find the total volume of beer in the can by integrating f over the region occupied by the can.

Use cylindrical coordinates : $x = x$, $y = r\cos\theta$, $z = r\sin\theta$.

So the beer-can-region is

$$E = \left\{ (x, r, \theta) : \begin{array}{l} 0 \leq x \leq 4 \\ 0 \leq \theta \leq 2\pi \\ 0 \leq r \leq 1 \end{array} \right\}.$$

The total volume is $\iiint_E x^2y^2 \, dV = \int_0^4 \int_0^{2\pi} \int_0^1 x^2(r^2\cos^2\theta) \cdot r \, dr \, d\theta \, dx$

$$= \boxed{\frac{16\pi}{3}}$$

3. (14 points) Find the volume of the "ice cream cone" region

$$E = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1, z \geq 0, z^2 \geq x^2 + y^2\}$$

using spherical coordinates. (No credit will be awarded for using any other coordinate system!)

In spherical, ① $x^2+y^2+z^2 \leq 1 \longrightarrow \rho \leq 1$

② $z^2 \geqslant x^2+y^2 \longrightarrow \rho^2 \cos^2\phi \geqslant \rho^2 \sin^2\phi\,(\sin^2\theta + \cos^2\theta)$

$\rho^2 \cos^2\phi \geqslant \rho^2 \sin^2\phi$

$1 \geqslant \tan\phi$

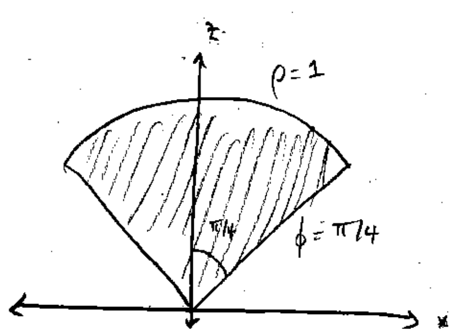
$\pi/4 \geqslant \phi$

③ $z \geqslant 0 \rightarrow \rho\cos\phi \geqslant 0$

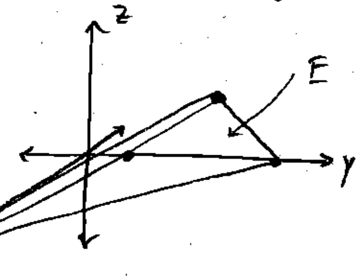
$\phi \leq \pi/2$ (since we always have $\phi \in (0, \pi)$)

So $\text{Volume} = \iiint_E 1\, dV = \int_0^{\pi/4} \int_0^{2\pi} \int_0^1 \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$

$= \boxed{\frac{2\pi}{3}\left(1 - \frac{\sqrt{2}}{2}\right)}$



4. (12 pts) Consider the region E bounded by the planes $z = 0, x = 0, x + 2y + 3z = 6$ and $y = z + 1$. Paramaterize **but do not evaluate** the integral

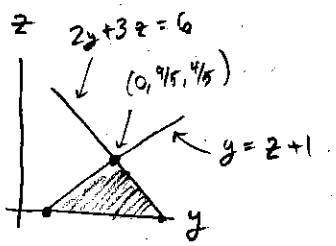


$$\iiint_E xyz\, dV$$

in the two orders given below.

(a) (6 points)

$$\int_{0}^{4/5} \int_{z+1}^{3-\frac{3}{2}z} \int_{0}^{6-2y-3z} xyz \; dx\, dy\, dz$$



projection of E in the y-z plane

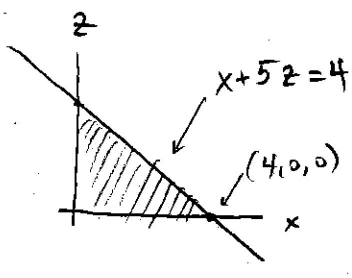
Intersection of $2y+3z=6$ and $y=z+1 \longrightarrow 5z+2=6$

$z = 4/5$

$2y+3z=6 \longrightarrow y = \frac{6-3z}{2} = 3 - \frac{3}{2}z.$

(b) (6 points)

$$\int_{0}^{4} \int_{0}^{\frac{4}{5}-\frac{x}{5}} \int_{z+1}^{3-\frac{3}{2}z-\frac{x}{2}} xyz \; dy\, dz\, dx$$



proj. of E into the x-z plane.

Find the intersection of $x+2y+3z=6$ and $y=z+1$:

$x + 2z + 2 + 3z = 6$

$x + 5z = 4$, or $z = \frac{4}{5} - \frac{x}{5}.$

$x+2y+3z=6 \longrightarrow y = 3 - \frac{3}{2}z - \frac{x}{2}.$

(Note: this is not the same as setting $y=0$!)