MATH 303 Practice Problems - Week 7

March 5, 2021

Problem 1 (if you haven't done week 6 problem 4)

Let X be an exponential variable with unknown parameter λ. We know that $\mathbb{P}(X \leq 10) = \frac{5}{9}$.
1. Compute $\mathbb{P}(X \geq 15)$.
2. Compute $\mathbb{P}(X \geq 15 | X \geq 10)$ (use the memoryless property).
3. Compute $\mathbb{P}(X \geq 15 | 10 \leq X \leq 20)$.
4. Find the number $y > 0$ for which $\mathbb{P}(y \leq X \leq 2y)$ is the largest.

Problem 2 (Minimum of independent exponential random variables)

1. Let X_1, X_2 and X_3 be independent exponential r.v.'s with respective rates λ_1, λ_2 and λ_3 (not necessarily equal). Find the probability that $X_1 = \min(X_i,\ 1 \leq i \leq 3)$ (*hint*: Compute $P(X_1 < \min(X_2, X_3))$. Find the distribution of the minimum of n independent exponential r.v.'s.
2. 8 runners R_i $(i = 1, \ldots, 8)$ enter a race, with a time to complete respectively $\sim Exp(i)$ (and independent from the other runners). Suppose the winner of the race (with the shortest time) earns 10 dollars and other runners lose 1. What is the expected gain of runner 1?
3. If instead the winner of the race earns e^{-at}, where t is the time of the winner and $a > 0$ (losers make 0), what is the expected gain of runner 1? (*hint:* Condition on the winning time)

Problem 3 (From 2020 Final exam - Using Exponential distribution properties)

Five customers are being independently served with exponential random time with the same parameter λ.
1. Let T be the first time at which we observe that two customers have left. What are the mean and variance of T?
2. Given that it takes t units of time to serve the first 2 customers, what is the mean time to serve the first 4?

Problem 4 (From 2020 Final exam - Using Exponential distribution properties)

We consider a counter with two servers 1 and 2, for which the service times are exponential variables with respective parameters λ_1 and λ_2. Becca needs to be serviced by both servers (in the order she wishes). When she arrives, Merlin is currently being served by server 1, and Emmanuel is being served by server 2. Both Merlin and Emmanuel will leave as soon as they are done with their current server. We also assume that Becca goes first to the first server that is available.

1. Compute the probability that Becca goes first to server 1, and needs to wait for Emmanuel once she is done with server 1.
2. Compute the expected total time before Becca can leave the counter (*hint:* Think of all the possible events that happen before Becca leaves).

Problem 5 (Branching process with immigrants)

We prove here some results of last week's jupyter notebook problem 5.3, (see also Problem 6.1). We consider the process Z_n, where $Z_0 = 1$, and at each generation $n \geq 1$,

$$Z_{n+1} = I_n + \sum_{i=1}^{Z_n} X_{n,i},$$

where the $X_{n,i}$'s are i.i.d. with common distribution X, and the I_n's are i.i.d and independent from the $X_{n,i}$'s, with common distribution I (this can be seen as a branching process where an immigrant population arrives at each generation and independently reproduces at the next one).

1. Find the generating function of Z_n, as a function of the generating functions of I, X, and n.

2. Let $X \sim \mathit{Binomial}(1, p)$ and $I \sim \mathit{Poisson}(\mu)$, where $0 < p < 1$ and $\mu > 0$. Compute the generating function of Z_n as a function of μ, p and n.

3. Use the results from last week (Problem 1) to find the expectation and variance of Z_n.

4. Identify the distribution associated with the generating function of Z_n as $n \to +\infty$.

Problem 6 (Jupyter Notebook)

1. Branching process with immigrants: Use the solution to Problem 5.3 from last week to extract the empirical expectation and variance of the Branching process with immigrants, and compare them with the results of Problem 5.

2. A queuing model: People are entering a bank and wait in line before being served. We assume that the arrival of the next customer follows an exponential law of parameter β, and that there are two tellers, with service times following two exponential laws of parameters λ_1 and λ_2. When one customer is served, the next in line takes their place. We also assume that arrival and service events are independent. At a given time, can you tell the probabilities associated with the next event (a customer arrives, teller 1 or 2 finishes serving a client) and the time it happens? Use this to simulate the length of the line over time.