Math 302

Final exam

Friday, June 25, 12pm

Instructions

- There are 7 questions on this exam.
- You have 120 minutes to complete the exam, then an additional 20 minutes to upload pictures/scans of your solutions to Canvas.
- Write your name on the top of each page of work that you submit.
- **You must show your work on all problems.** The correct answer with no supporting work may result in no credit. Put a box around your FINAL ANSWER for each problem and cross out any work that you don't want to be graded.
- Give exact answers.
- Any student found engaging in academic misconduct will receive a score of 0 on this exam.

GOOD LUCK!

1. (10 points) A slot machine randomly generates sequences of three symbols. There are four possibilities for each symbol: bar, cherry, seven, and star. For example, one possible sequence is (Bar, Bar, Seven). Assume that all sequences are equally likely.

   (a) (3pts) Find the probability that exactly two of the symbols are cherries.

   Solution: The number of cherries is Binomial$(3, 1/4)$, so it's $\mathbb{P}(\text{Bin}(3, 1/4) = 2) = \binom{3}{2} \cdot (1/4)^2 \cdot (3/4)$

   (b) (4pts) Find the probability that all three symbols are the same.

   Solution: There are exactly four ways this can happen (one for each symbol), and any given sequence has probability 4^{-3}, so it's $4 \cdot 4^{-3} = 1/16$.

   (c) (3pts) Suppose you win \$1 for each cherry, bar, or star, and \$10 if all three symbols are sevens. What is your expected winnings on a single play?

   Solution: we expect to win $3/4$ on each symbol from the 1 dollar condition, and $10/4^3$ dollars for the whole sequence from the sevens condition. So by linearity of expectation, expected winnings is $3 \cdot \frac{3}{4} + \frac{10}{64} = 154/64 = \frac{77}{32}$.

2. (10 points) The Yankees and the Red Sox are playing a five game series. The first three games are played at Yankee Stadium, and the last two are played at Fenway Park. Assume that the home team wins with probability 3/5, there are no ties, and the outcomes of the five games are independent. Let Y_n be the number of wins for the Yankees in the first n games, for $n = 1, 2, 3, 4, 5$.

   (a) (5 pts) Compute the conditional expectation $\mathbb{E}[Y_5|Y_1]$.

   Solution: $Y_5 - Y_1$ is independent of Y_1, with expectation $2 \cdot (3/5 + 2/5) = 2$. So by linearity, $\mathbb{E}[Y_5|Y_1] = 2 + Y_1$.

   (b) (2 pts) Compute the conditional probability $\mathbb{P}(Y_5 \geq 3|Y_2 = 0)$.

   Solution: to have $Y_5 \geq 3$ and $Y_2 = 0$, the Yanks have to lose games 1 and 2, and win games 3, 4 and 5. Note that $\mathbb{P}(\text{win games } 3, 4, 5) = \frac{3 \cdot 2 \cdot 2}{5^3} = 12/125$ by independence. Similarly, $\mathbb{P}(Y_2 = 0) = \frac{4}{25}$. Thus $\mathbb{P}(Y_5 \geq 3|Y_2 = 0) = 12/125$.

   (c) (3 pts) Compute the conditional probability $\mathbb{P}(Y_2 = 1|Y_3 = 2)$.

   Solution: Note $\mathbb{P}(Y_3 = 2) = \mathbb{P}(\text{Bin}(3, 3/5) = 2) = 3 \cdot (\frac{3}{5})^2 \frac{2}{5} = \frac{54}{125}$, and the event $Y_2 = 1$ and $Y_3 = 2$ occurs if and only if the Yanks win game 3, and exactly one of games 1 or 2. The probability of the latter event is $2 \cdot (\frac{3}{5})^2 \frac{2}{5} = \frac{36}{125}$. So $\mathbb{P}(Y_2 = 1|Y_3 = 2) = \frac{36/125}{54/125} = 36/54 = 2/3$.

3. (10 points) Let (X, Y) be a uniformly randomly chosen point from the set

$$S = [0,1]^2 \cup [-1,0]^2 = \{(x,y) : 0 \le x, y \le 1\} \cup \{(x,y) : -1, \le x, y \le 0\},$$

i.e. the joint density function is

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{2}, & (x,y) \in S \\ 0, & (x,y) \notin S \end{cases}$$

(a) (3 pts) Find the marginal distribution function of X. Give your formula as a PDF.

Solution: $f_X(x) = \int_{-1}^{1} f_{X,Y}(x,y)dy = \frac{1}{2}1\{x \in [-1,1]\}$. So $X \sim \text{Unif}(-1,1)$.

(b) (4 pts) Find the conditional distribution function $f_{Y|X}(y|x)$. Are X and Y independent?

Solution: the conditional distribution is the quotient of the joint distribution by the marginal of X. So $f_{Y|X}$ is 0 outside of S, and takes value 1 on S. By symmetry, the marginal of Y is also uniform on $[-1,1]$, so the product $f_X f_Y$ is non-zero outside of S (e.g. at the point $(-1/2, 1/2)$). Thus $f_X f_Y \neq f_{X,Y}$, so X and Y are not independent.

(c) (3 pts) Compute $\mathbb{E}[X^2 Y^2]$.

Solution: by the Law of the unconscious statistician, and using the fact that the integral over the positive part of S is the same as over the negative part,

$$\mathbb{E}[X^2Y^2] = \frac{1}{2} \cdot 2 \int_0^1 \int_0^1 x^2 y^2 \, dx \, dy = \frac{1}{9}$$

4. (10 points) Let $X_1, X_2, \ldots$ be an iid sequence with common distribution Geometric(1/2), and for each $n \geq 1$, let Y_n be the random variable

$$Y_n = \begin{cases} 1, & \text{if } X_n = 1 \\ 2, & \text{if } X_n = 2 \\ 3, & \text{if } X_n > 2 \end{cases}.$$

Also, for each (integer) $n \geq 1$, let $S_n = \sum_{i=1}^n Y_i$.

(a) (2 pts) Compute $\mathbb{E}[Y_n]$ and $\mathbb{E}[S_n]$ for each $n \geq 1$.
Solution: $\mathbb{E}[Y_n] = 1 \cdot 1/2 + 2 \cdot 1/4 + 3 \cdot 1/4 = 7/4$. By linearity, $\mathbb{E}[S_n] = \frac{7n}{4}$.

(b) (2 pts) Compute $\mathrm{Var}(Y_n)$ and $\mathrm{Var}(S_n)$ for each $n \geq 1$.
Solution: $\mathbb{E}[Y^2] = 1 \cdot 1/2 + 4 \cdot 1/4 + 9 \cdot 1/4 = 15/4$, so $\mathrm{Var}(Y) = 15/4 - (7/4)^2 = 11/16$. Since variances add for independent r.v.'s, $\mathrm{Var}(S_n) = \frac{11n}{16}$.

(c) (2 pts) Compute the conditional probability $\mathbb{P}[X_1 = 3 | Y_1 = 3]$.
Solution: Note that $\mathbb{P}(Y_1 = 3) = \mathbb{P}(X_1 \geq 3) = \frac{1}{4}$. Thus $\mathbb{P}(X_1 = 3 | Y_1 = 3) = \frac{\mathbb{P}(X_1 = 3)}{\mathbb{P}(X_1 \geq 3)} = 1/2$.

(d) (4 pts) Does $\frac{1}{\sqrt{n}} S_n$ converge in distribution as $n \to \infty$? Justify. (If it does converge, identify the limit distribution.)
Solution: by the WLLN, $\frac{1}{n} S_n$ converges in probability to $\frac{7}{4}$. Using the definition of convergence in probability,

$$\mathbb{P}(|\frac{1}{\sqrt{n}} S_n - \frac{7}{4}\sqrt{n}| \leq \epsilon\sqrt{n}) = \mathbb{P}(|\frac{1}{n} S_n - \frac{7}{4}| \leq \epsilon) \to 1 \text{ as } n \to \infty$$

for any $\epsilon > 0$. Taking $\epsilon = 3/4$, and re-arranging the absolute value, we get that

$$\mathbb{P}(\frac{1}{\sqrt{n}} S_n > \sqrt{n}) \to 1 \text{ as } n \to \infty,$$

or in particular, for any fixed number a,

$$\mathbb{P}(\frac{1}{\sqrt{n}} S_n > a) \to 1 \text{ as } n \to \infty.$$

This implies that the CDF of S_n converges to the 0 function, i.e. that $\frac{1}{\sqrt{n}} S_n$ does not converge in distribution.

5. (12 points) Let $U_1, U_2, \ldots$ be an iid sequence with common distribution Uniform$(0,4)$, and let

$$Q_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} [U_i - 2] .$$

(a) (2 pts) Compute the moment generating function of U_1.
Solution: $M_U(t) = \mathbb{E}[e^{tU}] = \frac{1}{4}\int_0^4 e^{tu}\,du = \frac{e^{4t}-1}{4t}$.

(b) (1 pt) Find a number c such that $e^{tQ_n} = e^{\frac{t}{\sqrt{n}}(U_1+U_2+\cdots+U_n)}e^{-ct\sqrt{n}}$. Justify.
Solution: Note $Q_n = \frac{t}{\sqrt{n}}\sum_{i=1}^n U_i - \frac{t}{\sqrt{n}}(2n) = \frac{t}{\sqrt{n}}\sum_{i=1}^n U_i - 2t\sqrt{n}$. So $c = 2$.

(c) (5 pts) Use your formula from part (b) to compute the moment generating function of Q_n.
Solution: By independence,

$$\mathbb{E}[e^{tQ_n}] = M_U(t/\sqrt{n})^n e^{-2t\sqrt{n}} = e^{-2t\sqrt{n}}(\frac{4t}{\sqrt{n}})^{-n}(e^{4tn^{-1/2}} - 1)^n$$

(d) (4 pts) Does Q_n converge in distribution as $n \to \infty$? Justify. (If it does converge, identify the limit distribution.)
Solution: Yes. By the CLT, Q_n converges to Normal$(0, \sigma^2)$, where $\sigma^2 = \mathrm{Var}(U) = 16\mathrm{Var}(\mathrm{Unif}(0,1)) = 16/12 = 4/3$, since U has the same distribution as 4 times a Uniform$(0,1)$ r.v.

6. (11 points) Let $W_1, W_2, \ldots$ be an iid sequence with common distribution Binomial$(4, \frac{1}{3})$, and let

$$T_n = \sum_{i=1}^{n} W_i.$$

(a) (3 pts) Does T_n have Binomial(m, p) distribution for some values of m, p? Justify. (If you answer 'yes', find m and p.)

Solution: Yes, T is a sum of binomials, each of which is a sum of iid Bernoullis. So T is Binomial$(4n, 1/3)$, so $m = 4n, p = 1/3$.

(b) (4 pts) Show that $\frac{1}{n^2} T_n$ converges in probability to 0.

Solution: Simple application of Chebychev.

(c) (4 pts) Show that $\frac{1}{n} T_n$ converges almost surely to a constant random variable, and identify the constant.

Solution: SLLN $\implies$ $\frac{1}{n} T_n \to \mathbb{E}[W] = 4/3$.

7. (7 points) Let X, Y, Z be any random variables on the same probability space satisfying the following:

$$\mathbb{E}X = 1, \mathrm{Cov}(X,Z) = 3, \mathbb{E}[XY] = 4, \mathrm{Var}(Z) = 2, \mathrm{Cov}(X,Y) = -1.$$

(a) (2 pts) Compute $\mathbb{E}(2X)$ and $\mathrm{Var}(2Z)$.
Solution: $\mathbb{E}[2X] = 2\mathbb{E}[X] = 2$, $\mathrm{Var}(2Z) = 4\ \mathrm{Var}(Z) = 8$.

(b) (3 pts) Compute $\mathrm{Cov}(2X, 2Y - Z)$.
Solution: By bilinearity, $\mathrm{Cov}(2X, 2Y - Z) = 4\mathrm{Cov}(X,Y) - 2\mathrm{Cov}(X,Z) = -4 - 6 = -10$.

(c) (2 pts) Compute $\mathrm{Var}(Y) - \mathbb{E}[Y^2]$.
Solution: This is another way to write $-\mathbb{E}[Y]^2$. Also, $-1 = \mathrm{Cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = 4 - \mathbb{E}[Y]$, so $\mathbb{E}[Y] = 5$, i.e. $\mathrm{Var}(Y) - \mathbb{E}[Y^2] = -25$.