Recall:
- We introduced the generating function associated with a r.v. X on $\mathbb{N}$

$$G_X(s) = \mathbb{E}(s^X)$$

- $G_X(0) = P(X=0)$

- We're interested in the probability of extinction of the Branching process

- We showed that for the B.P. $(Z_n)_{n\geq 0}$

$$G_n(s) = G_1^n(s) = G_\xi^n(s),$$

generating fnctn of Z_n — n-th iteration of Z_1

where ξ is the reproduction law

(We now simply write $G_\xi := G$)

$\rightarrow$ Extinction $= \bigcup_{n=0}^{+\infty} \{Z_n = 0\}$

Let's consider $\bigcup_{n=0}^{N} \{Z_n = 0\}$

$P(\{Z_n=0\}) = P(\text{extinction occurs before time } n)$

and $\{Z_n=0\} \subset \{Z_{n+1}=0\}$

so $\bigcup_{n=0}^{N} \{Z_n = 0\} = \{Z_N = 0\}$

and $P(Z_N = 0) = G_N(0)$ (see Recall)

so $\eta = P\left(\bigcup_{n=0}^{+\infty} \{Z_n = 0\}\right) = \lim_{N\to+\infty} P\left(\bigcup_{n=0}^{N} \{Z_n = 0\}\right)$

$= \lim_{N\to+\infty} P(\{Z_N = 0\}) = \lim_{N\to+\infty} G_N(0)$

so $\boxed{\eta = \lim_{N\to+\infty} G^N(0)}$ (because $G_N(s) = G^n(s)$)

→ To study η, we thus simply now need to study what happens when you iterate G, starting from 0.

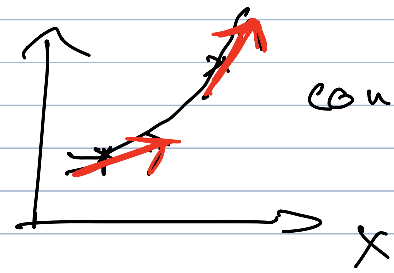
→ Let's see some further properties of G first.

- For a given reproduction law ξ with generating function G, we have the following
  - $G(1) = E(1^\xi) = 1$
  - $G'(1) = E(\xi)$ (exercise, see homework problem)
- G is non-decreasing and convex on $[0,1]$

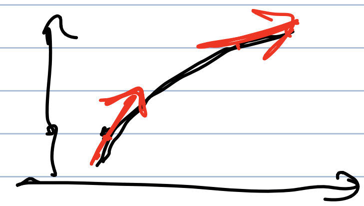
$(G' \geq 0)$ $\qquad$ $(G'' \geq 0)$

$\Updownarrow$

G' is non decreasing

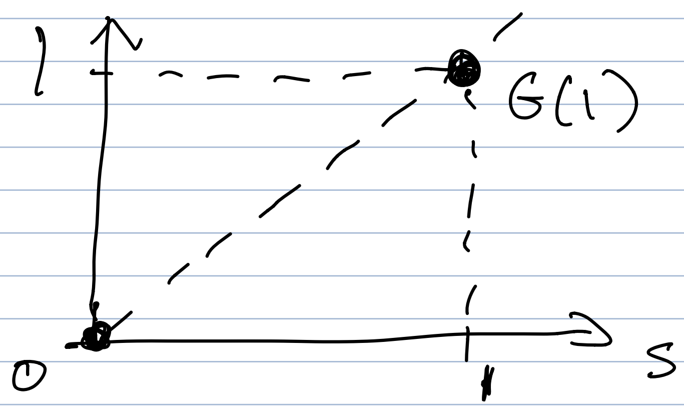


convex



non convex

- Let's visualize $G_n(0) = G^n(0)$ using the graph of G (generating function of the reproduction law)

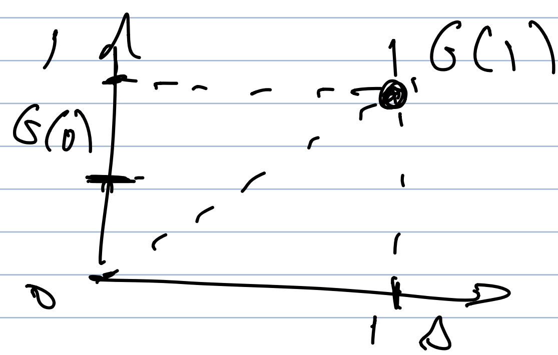


- we know that $G(1) = 1$
- we also know that $G(0) = P(\xi = 0) := p_0 \in [0,1]$

$\rightarrow$ If $p_0 = 0$, the population cannot decrease so $\eta = 0$

$\rightarrow$ If $p_0 = 1$, $\eta = 1$

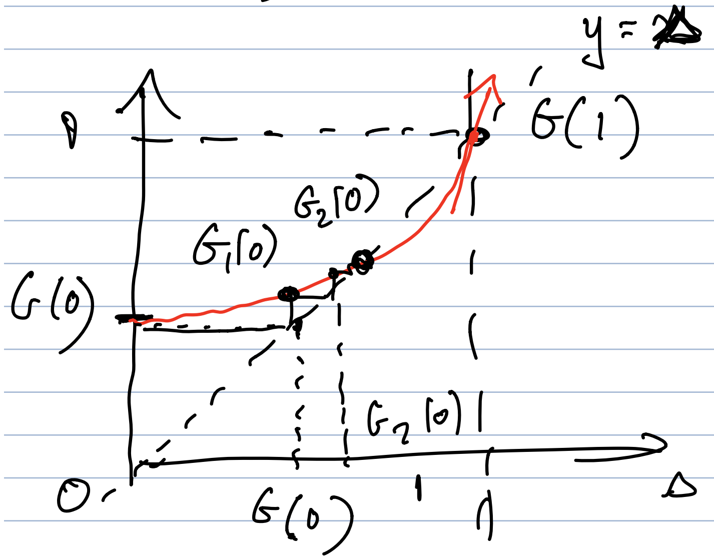
$\rightarrow$ Assume $0 < p_0 < 1 \Rightarrow$



- we know that $\begin{cases} G \text{ is non-decreasing} \\ G \text{ is convex} \\ G'(1) = \mathbb{E}(\xi) \end{cases}$

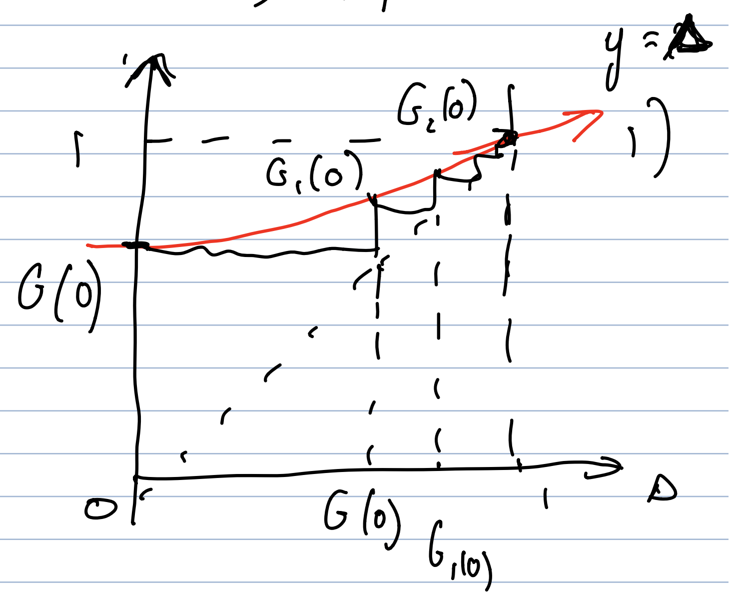
We distinguish two cases: $G'(1) \leq 1$ and $G'(1) > 1$

$G'(1) > 1$:



$\to G_n(0) \xrightarrow[n \to +\infty]{}$ smallest fixed point of $G \in]0,1[$

$G'(1) \leq 1$



$\to G_n(0) \xrightarrow[n \to +\infty]{} 1$

Thm: For a (non-trivial - $p_0 \neq 0, 1$) branching process with reproduction law ξ, the probability of extinction η is the smallest non-negative root of $s = G_\xi(s)$, and

$$\eta = 1 \text{ if } \mathbb{E}(\xi) \leq 1$$
$$\eta < 1 \text{ if } \mathbb{E}(\xi) > 1$$

Rmk: This theorem matches our intuition of the process: If individuals are not reproducing enough to renew the population ($\mathbb{E}(\xi) \leq 1$), then it will go extinct at some point

Week 6 Notebook: $\xi \sim \text{Uniform}(\{0,1,2,3\})$

$$G_\xi(s) = \mathbb{E}(s^\xi) = 0.25(s^0 + s^1 + s^2 + s^3)$$

→ By replacing s by 0.41 (see simulations or graph) we can see that $0.25(1 + 0.41 + 0.41^2 + 0.41^3) \simeq 0.25(1 + 0.41 + 0.17 + 0.06) = 0.41$

So $\boxed{\eta \simeq 0.41}$