MATH 303 Practice Problems - Week 8

March 12, 2021

Problem 1

Let $N(t)$ be a rate λ Poisson process, and let $S_n = \min\{t : N(t) \geq n\}$ be the time of the nth event. Compute the following:

a. $\mathbb{E}[N(3)]$

b. $\text{Var}[N(4)]$

c. $\mathbb{E}[S_5]$

d. $\mathbb{P}(S_3 > 4)$

e. $\mathbb{P}(S_2 > 4 | N(1) = 3)$

Problem 2

Arrivals of the number 14 bus form a Poisson process with rate 4 per hour, while arrivals of the number 99 bus form an (independent) Poisson process with rate 10 per hour.

a. What is the probability that at least 8 buses total arrive in one hour?

b. Suppose you arrive at the bus stop, and will only take a number 99 bus. What is the probability that exactly two number 14 busses go by while you wait?

c. On an icy day, only half of the bus drivers show up to work, cutting the rates of each bus line in half. What is the probability that no bus arrives between $8:30$ and $8:45$ AM?

Problem 3

It's Halloween, and fireworks are going off across the city. Per the covid-19 guidelines, you stand at your window watching. Assume fireworks are set off at times following a Poisson process with rate 1 per minute. If there is a span of 4 minutes with no fireworks, you will get bored and stop.

a. On average, how long do you watch for before getting bored? [Hint: condition on the time at which the first customer arrives.]

b. Let T be the time at which you stop watching. Argue that for any positive integer k,

$$\left(1 - \frac{1}{e^2}\right)^{2k} \leq \mathbb{P}(T > 4k) \leq \left(1 - \frac{1}{e^4}\right)^k$$

[Hint: find a necessary condition and a (different) sufficient condition for $T > 4k$, and use the 'independent increments' property of the Poisson process.]

Problem 4

Jacob and Khanh are on a skiing trip at Whistler mountain. Tired from shredding the halfpipe, they take a breather by the gondola, and watch skiers and snowboarders go by. Assume that people arrive at the times of a Poisson process with rate ten per minute, and that $\frac{2}{3}$ of riders are skiers, and $\frac{1}{3}$ are snowboarders.

a. Khanh's goggles fall off, and while he reattaches them, Jacob watches four riders go by. Jacob tells Khanh that four more people went by, but not whether they were skiing or boarding. What is the probability that k of the four were skiers for $k = 0, 1, 2, 3, 4$?

b. Jacob bets Khanh that at least two skiers will arrive before one snowboarder arrives. Should Khanh take the bet? What proportion of riders would have to be skiers for it to be a fair bet?

c. Khanh notices that in a one minute period, exactly four snowboarders went by. What is the probability that exactly six skiers went by in the same period?

Problem 5

Complete the following exercises from lecture.

a. If $X_1, X_2 \sim \text{Exp}(\lambda)$, then the probability density function of $X_1 + X_2$ is $f(x) = \lambda^2 x \exp(-\lambda x)$, and

$$\mathbb{P}(X_1 < X_2) = \frac{\lambda_1}{\lambda_1 + \lambda_2}.$$

b. Complete the proof of the fact that any memory-less random variable supported on the positive reals is distributed as $\text{Exp}(\lambda)$ for some λ. (Hint: re-write the memory-less property as a functional equation involving the cumulative distribution function.)

c. Let $N(t)$ be a Poisson process with rate λ. Show using integration by parts on the formula

$$\mathbb{P}(N(t) \geq k) = \frac{1}{(k-1)!} \int_0^t \lambda^k u^{k-1} e^{-\lambda u} \, \mathrm{d}u$$

that

$$\mathbb{P}(N(t) \geq k) - \mathbb{P}(N(t) \geq k+1) = e^{-\lambda t} \frac{(\lambda t)^k}{k!}.$$

d. Prove the distributional convergence $\text{Binomial}(n, \mu/n) \to \text{Poisson}(\mu)$ as $n \to \infty$ by showing that for any $k = 0, 1, \ldots$,

$$\lim_{n \to \infty} \binom{n}{k} \left(\frac{\mu}{n}\right)^k \left(1 - \frac{\mu}{n}\right)^{n-k} = e^{-\mu} \frac{\mu^k}{k!}.$$

Problem 6 (Jupyter Notebook)

1. Use the week 8 notebook to compare the empirical distribution of a Poisson process $N(t)$ of intensity (rate) λ (for example, use $t = 100$, $\lambda = 0.2$) with the theoretical distribution found in class.

2. In this problem we will simulate non-homogeneous Poisson processes using the week 8 Jupyter notebook. Recall that for a non-negative function $\lambda : \mathbb{R}^+ \to \mathbb{R}^+$, the non-homogeneous Poisson process with intensity function λ is the unique counting process satisfying

   - $N(0) = 0$
   - $\{N(t) : t \geq 0\}$ has independent increments
   - $\mathbb{P}(N(t + dt) - N(t) \geq 2) = o(dt)$
   - $\mathbb{P}(N(t + dt) - N(t) = 1) = \lambda(t)dt + o(dt)$

   See Ross, section 5.4, for more details. Use the Jupyter notebook to simulate the non-homogeneous Poisson process with the following functions:

   a. $\lambda(t) = ct$ for a constant $c > 0$. Generate a histogram to find the distribution of $N(t)$ for a fixed t – what do you expect the distribution to be?

   b. $\lambda(t) = e^{-ct}$ for a constant $c > 0$. Argue that $N(t)$ is bounded as $t \to \infty$. Generate a histogram for $\lim_{t\to\infty} N(t)$ by sampling at a large time t – what do you expect the distribution to be?

   c. $\lambda(t) = c \cdot 1\{\lfloor t \rfloor \text{ is even}\}$ for a constant $c > 0$. Generate a histogram for $N(t)$: is it Poisson? How is this process related to the usual Poisson process with rate c?