MATH 303 Midterm Exam A Solution

Problem 1

Consider the transition matrix P obtained from running the notebook with your student ID.

The notebook will give the following possible outputs

- Case (a): $P = \begin{pmatrix} 0.0 & 0.0 & 0.0 & 1.0 & 0.0 \\ 0.0 & 0.5 & 0.0 & 0.0 & 0.5 \\ 0.2 & 0.0 & 0.6 & 0.0 & 0.2 \\ 1.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.4 & 0.0 & 0.0 & 0.6 \end{pmatrix}, i = 5$
- Case (b): $P = \begin{pmatrix} 0.8 & 0.0 & 0.2 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 1.0 & 0.0 \\ 0.4 & 0.0 & 0.6 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.1 & 0.1 & 0.8 \end{pmatrix}, i = 3$
- Case (c): $P = \begin{pmatrix} 0.2 & 0.6 & 0.2 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 1.0 \\ 0.0 & 0.0 & 0.2 & 0.8 & 0.0 \\ 0.0 & 0.0 & 0.4 & 0.6 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 & 0.0 \end{pmatrix}, i = 4$
- Case (d): $P = \begin{pmatrix} 0.4 & 0.6 & 0.0 & 0.0 & 0.0 \\ 0.5 & 0.5 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 1.0 \\ 0.3 & 0.0 & 0.0 & 0.3 & 0.4 \\ 0.0 & 0.0 & 1.0 & 0.0 & 0.0 \end{pmatrix}, i = 2$

a. Draw the transition diagram associated with P with the states corresponding to their row index in the matrix (i.e. first row correspond to state 1, second to state 2 etc.).

Solution: See Figure 1

b. Determine all the communication states (no need to justify).

Solution: Overall, There are 3 classes.

- Case (a): $\{\{1,4\},\{2,5\},\{3\}\}$
- Case (b): $\{\{1,3\},\{2,4\},\{5\}\}$
- Case (c): $\{\{3,4\},\{2,5\},\{1\}\}$
- Case (d): $\{\{1,2\},\{3,5\},\{4\}\}$

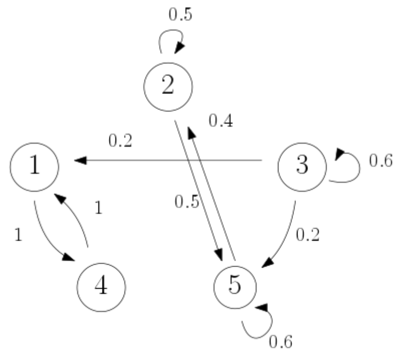


Figure 1: Transition diagram for case (a). Other cases are similar (permute the states and change transion probabilites accordingly)

c. Determine which states are recurrent and which are transient (briefly justify).

Solution: Overall, There are 2 recurrent classes and one transient class (the one with only one state).

- Case (a): $\{\{1,4\}r, \{2,5\}r, \{3\}t\}$. Justification: Starting from 1 the only possible trajectory is $1,4,1,4$, etc. so 1 is recurrent and 4 is too since they communicate. The class $\{2,5\}$ is closed and finite thus recurrent. There is no transition to state 3 apart from $p_{3,3}$ which means that if one starts from 3 and leaves the state it is not possible to come back. Since leaving 3 happens with positive probability the state 3 is transient.
- Case (b): $\{\{1,3\}r, \{2,4\}r, \{5\}t\}$ (similar justification as in (a))
- Case (c): $\{\{3,4\}r, \{2,5\}r, \{1\}t\}$ (similar justification as in (a))
- Case (d): $\{\{1,2\}r, \{3,5\}r, \{4\}t\}$ (similar justification as in (a))

d. Determine the period of each state (briefly justify).

Solution: Overall, There is a class with period 2 and two aperiodic classes (including the one with only one state).

- b=0: $\{\{1,4\}d = 2, \{2,5\}d = 1, \{3\}d = 1\}$. Justification: The chain started from 1 is deterministic and alternates between 1 and 4 in particular $p_{1,1}^{(2k)} = 1$ for all $k \geq 1$ and $p_{1,1}^{(2k+1)} = 0$ for all $k \geq 0$ so the gcd of all the even numbers is 2. Thus the period of 1 is equal to 2 and the same holds for 4. For the class $\{2,5\}$, from the transition diagram we see that $p_{2,2}^{(1)} > 0$ and thus 2 has period 1. Since the state 5 is in the same class and since the period is a class property we see that the period of 5 is 1. Finally since $p_{3,3}^{(1)} > 0$ we have that the period of 3 is equal to 1.
- b=1: $\{\{1,3\}\, d = 1\,, \{2,4\}\, d = 2\,, \{5\}\, d = 1\}$ (similar justification as in (a))
- b=2: $\{\{3,4\}\, d = 1\,, \{2,5\}\, d = 2\,, \{1\}\, d = 1\}$ (similar justification as in (a))
- b=3: $\{\{1,2\}\, d = 1\,, \{3,5\}\, d = 2\,, \{4\}\, d = 1\}$ (similar justification as in (a))

e. Starting from i, what is the mean number of steps to re-visit i? (justify your calculation with key steps; answers directly written won't be accepted).

Solution: By doing a one-step analysis, and with $\mathbb{E}(N(i)|X_0 = j)$ being the expected number of steps to visit i given that the chain starts at j,

- Case (a)

$$\mathbb{E}(N(5)|X_0 = 5) = (1 + \mathbb{E}(N(5)|X_0 = 2))p_{52} + p_{55}$$
$$\mathbb{E}(N(5)|X_0 = 2) = (1 + \mathbb{E}(N(5)|X_0 = 2))p_{22} + p_{25}.$$

$$\begin{aligned}\mathbb{E}(N(5)|X_0 = 5) &= (1 + \frac{1}{p_{25}})p_{52} + p_{55} \\ &= (1 + \frac{1}{0.5})0.4 + 0.6 \\ &= 1.8\end{aligned}$$

- Case (b)

$$\mathbb{E}(N(3)|X_0 = 3) = (1 + \mathbb{E}(N(3)|X_0 = 1))p_{31} + p_{33}$$
$$\mathbb{E}(N(3)|X_0 = 1) = (1 + \mathbb{E}(N(3)|X_0 = 1))p_{11} + p_{13}.$$

$$\begin{aligned}\mathbb{E}(N(3)|X_0 = 1) &= (1 + \frac{1}{p_{13}})p_{31} + p_{33} \\ &= (1 + \frac{1}{0.2})0.4 + 0.6 \\ &= 3\end{aligned}$$

- Case (c)

$$\mathbb{E}(N(4)|X_0 = 4) = (1 + \mathbb{E}(N(4)|X_0 = 3))p_{43} + p_{44}$$
$$\mathbb{E}(N(4)|X_0 = 3) = (1 + \mathbb{E}(N(4)|X_0 = 3))p_{33} + p_{34}.$$

$$\begin{aligned}\mathbb{E}(N(4)|X_0 = 4) &= (1 + \frac{1}{p_{34}})p_{43} + p_{44} \\ &= (1 + \frac{1}{0.8})0.4 + 0.6 \\ &= 1.5\end{aligned}$$

- Case (d)

$$\mathbb{E}(N(2)|X_0 = 2) = (1 + \mathbb{E}(N(2)|X_0 = 2))p_{21} + p_{22}$$
$$\mathbb{E}(N(2)|X_0 = 1) = (1 + \mathbb{E}(N(2)|X_0 = 1))p_{11} + p_{12}.$$

$$
\begin{aligned}
\mathbb{E}(N(2)|X_0 = 2) &= (1 + \frac{1}{p_{12}})p_{21} + p_{22} \\
&= (1 + \frac{1}{0.6})0.5 + 0.5 \\
&= 11/6 \simeq 1.83
\end{aligned}
$$

Problem 2

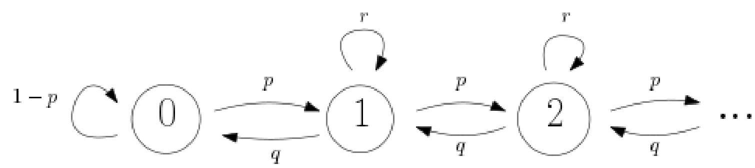
Consider the Markov Chain $(X_n)_{n\geq 0}$ defined on $\mathbb{N} = \{0, 1, \ldots\}$ with transition probabilities obtained from running the notebook.

The notebook gives $\forall i > 0$ $p_{i,i+1} = p$; $p_{i,i-1} = q$; $p_{i,i} = r$ and $p_{0,1} = p$; $p_{0,0} = 1 - p$. Values of p, q, r are the following:

- Case (a): $(p, q, r) = (0.25, 0.5, 0.25)$
- Case (b): $(p, q, r) = (0.5, 0.25, 0.25)$
- Case (c): $(p, q, r) = (0.2, 0.4, 0.4)$
- Case (d): $(p, q, r) = (0.6, 0.2, 0.2)$

a. Draw the transition diagram.

Solution:



b. Suppose that the chain admits a stationary distribution π. Find a relation between π_0 and π_1 (justify your answer).

Solution: (replace p, q by their numerical values) Since $\pi = \pi P$, where P is the transition matrix, we have $\pi_0 = P_{00}\pi_0 + P_{10}\pi_1 = (1-p)\pi_0 + q\pi_1$

c. For $i > 1$, find a relation between by π_{i-1}, π_i and π_{i+1} (justify your answer).

Solution: (replace p, q, r by their numerical values) Similarly, $\pi_i = P_{i-1,i}\pi_{i-1} + P_{i,i}\pi_i + P_{i+1,i}\pi_{i+1} = p\pi_{i-1} + r\pi_i + q\pi_{i+1}$

d. Show by induction that $\dfrac{\pi_{i+1}}{\pi_i}$ is a constant (to be determined), and deduce π_i in function of i and π_0. Is the chain positive-recurrent? Justify your answer.

Solution: (replace p, q by their numerical values) The recurrence initializes at $i = 0$ with $\dfrac{\pi_1}{\pi_0} = \dfrac{p}{q}$. Assuming $\dfrac{\pi_i}{\pi_{i-1}} = \dfrac{p}{q}$, rearranging the relation in **c** yields $\pi_i = q\pi_i + r\pi_i + q\pi_{i+1} \Leftrightarrow \pi_i(1 - q - r) = q\pi_{i+1}$. Since $p + q + r = 1$, we obtain that $\dfrac{\pi_{i+1}}{\pi_i} = \dfrac{p}{q}$ and the recurrence is proved. We hence obtain that $\pi_i = \left(\dfrac{p}{q}\right)\pi_{i-1} = \left(\dfrac{p}{q}\right)^2\pi_{i-2} = \ldots = \left(\dfrac{p}{q}\right)^i\pi_0$.
From class, we know that the chain is positive recurrent if and only if π also satisfies $\sum_i \pi_i = 1$. This is achieved if $\sum_i \left(\dfrac{p}{q}\right)^i < \infty$. If $\dfrac{p}{q} < 1$ (determined by the values of p and q set by the student ID), the series converges and the chain is positive recurrent. Else, it is not.

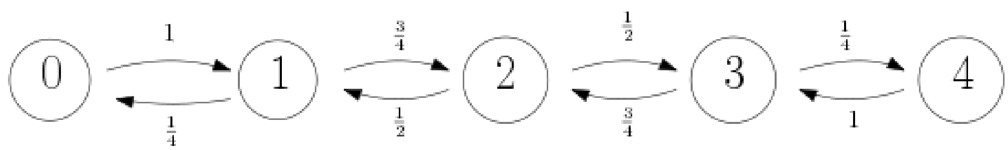
Problem 3

You have a bag containing four marbles. Marbles come in two colors: red and blue. At each step, you put your hand in the bag, remove a marble (selecting one uniformly at random from those in the bag), and replace it with a marble of the *opposite* color. Let X_n be the number of blue marbles in the bag after n steps.

The notebook gives four possible values of i to be used in question b.

a. Draw the transition diagram for X.

Solution:



b. Consider the state i obtained from running the notebook. Find $\mathbb{E}(X_2|X_0 = i)$ (justify your calculation with key steps; answers directly written won't be accepted).

Solution:
For example, for $i = 2$ we notice that in two steps and starting from i, there are only three possible states that can be reached: 0, 2 and 4, so

$$\mathbb{E}[X_2|X_0 = i] = 2 \times P^2_{2,2} + 0 \times P^2_{2,0} + 4 \times P^2_{2,4},$$

with 2-step transition probabilities (no need to evaluate $P^2_{2,0}$)

$$\begin{aligned} P^2_{22} &= P_{2,3}P_{3,2} + P_{2,1}P_{1,2} = \frac{3}{4} \\ P^2_{i,i+2} &= P_{2,3}P_{3,4} = \frac{1}{8} \end{aligned}$$

Doing something similar for different possible values of i (check what states can be reached) yields

$$\mathbb{E}[X_2|X_0 = i] = \begin{cases} 7/4, & i = 1 \\ 2, & i = 2 \\ 9/4, & i = 3 \\ 5/2, & i = 4 \end{cases}$$

Remark: One can answer the question by calculating the 2-step transition matrix P^2, but it is faster to directly use the transition diagram.

c. Show that distribution $\pi = \frac{1}{16}(1, 4, 6, 4, 1)$ is stationary for X, and that the process is reversible.

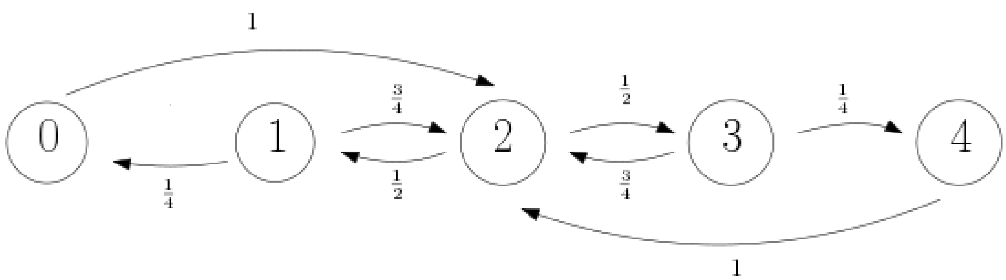
Solution: We just need to check that detailed balance is satisfied, i.e. $\sum_i \pi_i = 1$ (trivial since the question says that this is a distribution), and

$$\forall i \in \{0, 1, 2, 3\} \quad P_{i,i+1}\pi_i = P_{i+1,i}\pi_{i+1}$$

Remark: There was no need to show the general relation given by a stationary distribution.

Consider a different Markov chain defined by slightly modifying the marble process: namely, when all the marbles in the bag are the same color, you take *two* out and replace them with marbles of the opposite color. Let Y_n be the number of blue marbles after n steps in the modified process.
d. We assume that $\sigma = \frac{1}{18}(1, 4, 8, 4, 1)$ is stationary for Y. Use a result from class to argue that Y_n converges in distribution to σ as $n \to \infty$. Is Y reversible? Justify.

Solution: Now the transition diagram is



Y_n is aperiodic (since $P^3_{22} > 0$ and $P^2_{22} > 0$), and since the chain is irreducible and finite, it is positive recurrent. Thus it is ergodic, so by a result from class, $Y_n \to \sigma$. It is not reversible, since, for example,

$$\sigma_1 Q_{12} = \frac{1}{6} \neq \frac{2}{9} = \sigma_2 Q_{21}.$$

Problem 4

Complete the Problem 4 set in `midtermA.ipynb` (there are two questions).
Submit the completed notebook as `midtermA_complete.ipynb`

The notebook gives 3 values for p, q, r

Solution to question a: Enter the following matrix

$$P = \begin{pmatrix} r & (1-r)(1-p) & (1-r)p \\ 1-q & q(1-p) & qp \\ 1-q & q(1-p) & qp \end{pmatrix}$$

Marking: Points are possibly awarded if one enters a 3×3 matrix, the matrix is stochastic, and values are correct.

Solution to question b: Using the empirical distribution obtained by running the code $\mu = (\mu_1, \mu_2, \mu_3)$, enter the following value for the probability of having biked in the previous trip (being in state 'bs'), given that they take a bus (being in state 'Br' or 'Bs'):

$$proba = P_{13}\frac{\mu_3}{\mu_1} + P_{23}\frac{\mu_2}{\mu_1} = (1-r)p\frac{\mu_3}{\mu_1} + qp\frac{\mu_2}{\mu_1}$$

Marking: Points are possibly awarded if one enters a probability (proba $\in [0, 1]$), and the value is correct.