Math 303: Introduction to stochastic processes

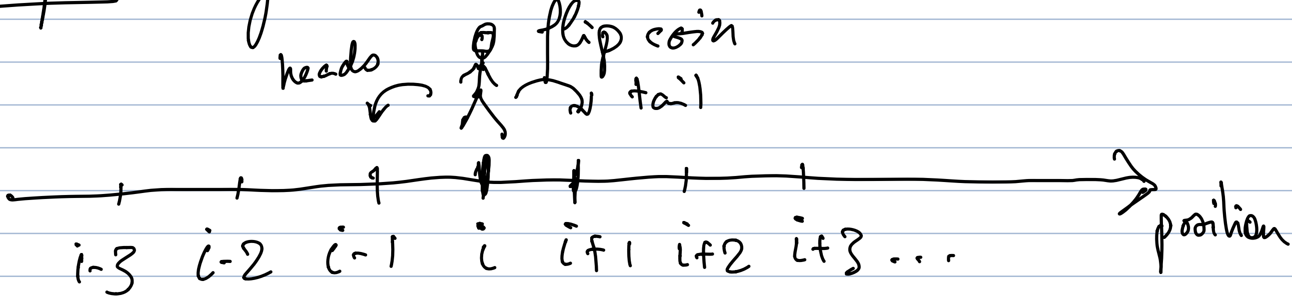
Week 1 content: Introduction to discrete time M-C, state space, Markov property, transition matrix, transition diagram, n-step transition probability, Chapman-Kolmogorov equation

Chap 1: Discrete-time Markov chains

I. Introduction

In this course, we are interested in the evolution of a random variable over time

Example: symmetric random walk in 1-D

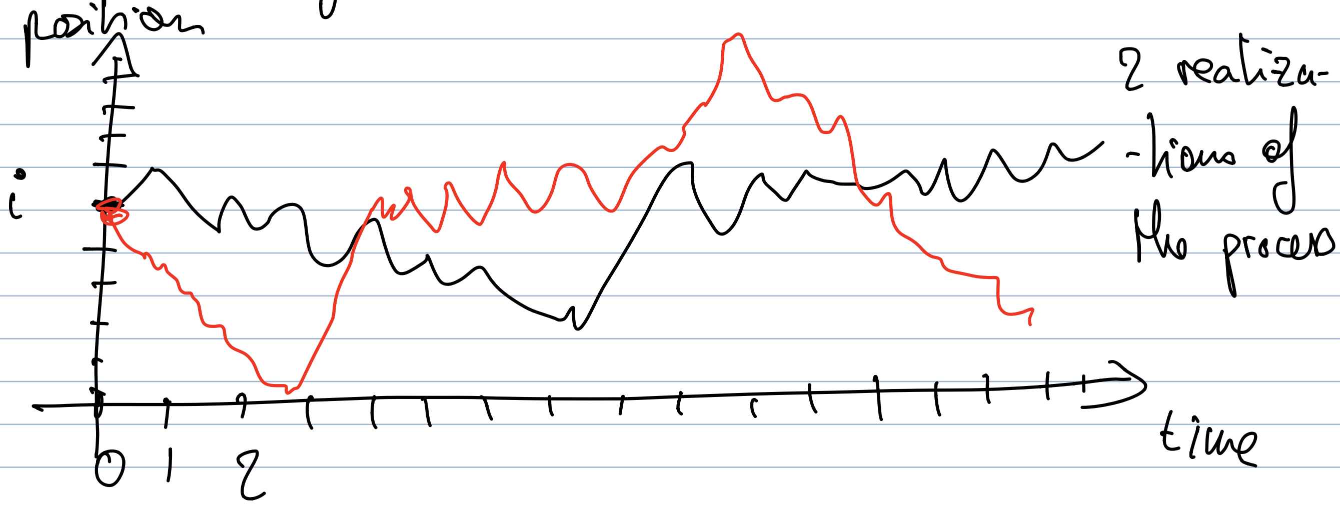


- Let's assume that the walker is at position i

at time $t = 0$

- The walker flips a coin and moves to $i+1$ if it's tail and $i-1$ if it's heads
- At time $= 1$, the walker is either at $i+1$ w.p. 0.5 or $i-1$ w.p. 0.5, and repeats the same process etc.

Upon simulating this process, we can obtain different realizations of it



Def: We call the state space the set of values taken by the random variable

ex: 1D-random walk: State space $S = \mathbb{Z}$
$= \{\dots, -2, -1, 0, 1, 2, \dots\}$

In this chapter, we will assume that time is DISCRETE (indexed by an integer $n \in \mathbb{N}$), and that the state space space is FINITE or COUNTABLE ($\simeq \mathbb{N}$).

Def: Let $(X_n)_{n \geq 0}$ be a sequence of random variables (r.v.'s) in state space S.
$(X_n)_{n \geq 0}$ is a Markov chain if it satisfies the MARKOV PROPERTY:

$\forall n \in \mathbb{N}$, $\forall (x_0, x_1, \dots, x_{n+1}) \in S^{n+2}$,

$$P(X_{n+1} = x_{n+1} \mid X_0 = x_0, \dots\dots X_n = x_n)$$
$$= P(X_{n+1} = x_{n+1} \mid X_n = x_n)$$

Interpretation: "What happens at time $n+1$ only depends on the state at time n"

Example: Symmetric 1-D random RW

$$P(X_{n+1} = x_{n+1} \mid X_0 = x_0, \dots\dots, X_n = x_n)$$

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$$= \begin{cases} 0.5 & \text{if } x_{n+1} = x_n + 1 \\ 0.5 & \text{if } x_{n+1} = x_n - 1 \\ 0 & \text{else} \end{cases}$$