Math 302

Midterm exam

Friday, May 28, 10am

Instructions

- There are 4 questions on this exam.
- You have 60 minutes to complete the exam, then an additional 20 minutes to upload pictures/scans of your solutions to Canvas.
- Write your name on the top of each page of work that you submit.
- **You must show your work on all problems.** The correct answer with no supporting work may result in no credit. Put a box around your FINAL ANSWER for each problem and cross out any work that you don't want to be graded.
- Give exact answers.
- Any student found engaging in academic misconduct will receive a score of 0 on this exam.

GOOD LUCK!

1. (12 points; 4 each) A gumball machine has 100 gumballs. 30 are green, and the other 70 are red. You insert five quarters into the machine and get five gumballs. Assume the gumballs are drawn uniformly at random (without replacement).

   (a) What is the probability that the first three gumballs are red, and the last two are green?

   A: Order matters here. There are 70 choices for the first red, 69 for the second, and 68 for the third; then 30 for the fourth (first green), and 29 for the fifth (second green). So the desired probability is $\frac{70*69*68*30*29}{100*98*97*96*95}$.

   (b) What is the probability of getting exactly four red gumballs?

   A: This is equivalent to getting four red and one green. We can compute without worrying about order. So the probability is $\frac{\binom{70}{4}\binom{30}{1}}{\binom{100}{5}}$.

   (c) What is the probability of getting exactly two green gumballs, conditioned on the event that you get at least one green gumball?

   A: The probability of getting at least one green gumball is $1-\mathbb{P}(0 \text{ green gumballs}) = \frac{\binom{70}{5}}{\binom{100}{5}}$, while the probability of getting exactly two green is, by a calculation similar to the one in part (b), $\frac{\binom{70}{3}\binom{30}{2}}{\binom{100}{5}}$. The desired probability is $\frac{\mathbb{P}(2 \text{ green})}{\mathbb{P}(\geq 1 \text{ green})}$.

2. (12 points; 3 each) For statements (a) and (b), either prove the statement or give a counterexample. For statements (c) and (d), decide if it is true or false; no justification necessary.

   (a) For any pairwise disjoint events A, B, C in a probability space $(\Omega, \mathbb{P})$ with $A \cup B \cup C = \Omega$,

   $$\mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) = 1.$$

   A: True, by the additivity axiom and the fact that $\mathbb{P}(\Omega) = 1$. Since A, B, C are independent, $\mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) = \mathbb{P}(A \cup B \cup C) = \mathbb{P}(\Omega) = 1$.

   (b) If A and B are any two events in a probability space $(\Omega, \mathbb{P})$, then the conditional probability $\mathbb{P}(A \cup B | A \cap B)$ is 1.

   A: False, since if $A \cap B = \emptyset$, the conditional probability isn't well defined.

   (c) Let X be a Binomial(n, p) random variable for some positive integer n and $p \in (0, 1)$. For any $k \in \{0, 1, \ldots, n\}$,

   $$\mathbb{P}(X \geq k) = \sum_{j=k}^{n} \binom{n}{j} p^j (1-p)^{n-j}.$$

   True

   (d) Let Y be a Bernoulli(p) random variable for some $p \in (0, 1)$. The random variable Y^4 has the same CDF as Y.

   True, since Y takes values in $\{0, 1\}$, and both values satisfy $0^4 = 0, 1^4 = 1$, so $Y^4 = Y$.

3. (14 points) A party is attended by n guests. Each guest puts their coat in the coat room when they arrive, but when they go to leave the light is off, so they take random coats. Assume that all $n!$ possible ways of assigning one coat to each guest are equally likely. Let X_i be the indicator random variable of the event that guest i gets their own coat, and let $X = \sum_{i=1}^{n} X_i$ be the total number of guests that get their own coats.

   (a) (3 pts) Compute $\mathbb{E}[X_i]$.

   A: There are exactly $(n-1)!$ ways to assign guest i their own coat and the remaining $n-1$ guests any of the remaining $n-1$ coats. So $\mathbb{E}X_i = (n-1)!/n! = 1/n$.

   (b) (3 pts) Compute $\mathbb{E}[X]$.

   A: By linearity of expectation, $\mathbb{E}[X] = n \cdot \mathbb{E}[X_1] = 1$.

   (c) (3 pts) Compute $\mathbb{E}[X_i X_j]$ for $i \neq j$.

   A: There are exactly $(n-2)!$ ways to assign guests i and j their own coats and the remaining $n-2$ guests any of the remaining $n-2$ coats. So $\mathbb{E}[X_i X_j] = \frac{(n-2)!}{n!} = 1/(n^2 - n)$.

   (d) (3 pts) Compute $\mathbb{E}[X^2]$.

   A: Note that

$$X^2 = \sum_{i=1}^{n} X_i^2 + \sum_{i \neq j} X_i X_j.$$

   By linearity of expectation, and the calculations in parts (a) and (c), and using the fact that $X_i^2 = X_i$, we get $\mathbb{E}[X] = n \cdot \frac{1}{n} + n(n-1) \cdot \frac{1}{n^2 - n} = 2$.

   (e) (2 pts) Compute $\mathrm{Var}(X)$.

   A: $\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = 2 - 1^2 = 1$.

4. (12 points; 4 each) Let Y be the random variable with PDF

$$f_Y(y) = \frac{2}{\pi}\frac{1}{1+y^2}, \text{ for } y > 0.$$

(a) Show that $\mathbb{E}[Y^2] = \mathbb{E}[Y] = \infty$.

A: $\mathbb{E}[Y^2] = \int_0^\infty \frac{2}{\pi}\frac{y^2}{1+y^2}\,dy \geq \frac{2}{\pi}\int_1^\infty \frac{1}{2}\,dy = \infty$, since $\frac{y^2}{1+y^2}$ is positive and increasing on $(0,\infty)$, and takes value $1/2$ at $y=1$. Similarly, $\mathbb{E}[Y] = \int_0^\infty \frac{2}{\pi}\frac{y}{1+y^2}\,dy \geq \frac{2}{\pi}\int_1^\infty \frac{1}{y}\,dy = \infty$ by the 'p-test.

(b) Define the random variable $Z = (Y-1)(Y-\sqrt{3})$. Compute $\mathbb{P}(Z \geq 0)$.

A: Note that $Z \geq 0$ if and only if $Y \in (0,1]$ or $Y \in [\sqrt{3},\infty)$. Thus

$$\begin{aligned}
\mathbb{P}(Z \geq 0) &= \mathbb{P}(Y \in (0,1]) + \mathbb{P}(Y \in [\sqrt{3},\infty)) \\
&= \frac{2}{\pi}\left(\int_0^1 \frac{dy}{1+y^2} + \int_{\sqrt{3}}^\infty \frac{dy}{1+y^2}\right) \\
&= \frac{2}{\pi}\left(\arctan(1) - \arctan(0) + \arctan(\infty) - \arctan(\sqrt{3})\right) \\
&= \frac{2}{\pi}\left(\frac{\pi}{4} - 0 + \frac{\pi}{2} - \frac{\pi}{3}\right) \\
&= \frac{5}{24}.
\end{aligned}$$

(c) Show that the PDF of the random variable Y^{-1} is also f_Y.

A: The CDF of Y^{-1} is $\mathbb{P}(Y^{-1} \leq y) = \mathbb{P}(Y \geq y^{-1}) = 1 - \mathbb{P}(Y \leq y^{-1})$. Thus, by the chain rule, the PDF $g(y)$ of Y^{-1} is given by

$$g(y) = \frac{d}{dy}\mathbb{P}(Y \leq y^{-1}) = -f_Y(y^{-1}) \cdot \frac{-1}{y^2} = \frac{2}{\pi}\frac{1}{1+y^{-2}} \cdot \frac{1}{y^2} = \frac{2}{\pi}\frac{1}{y^2+1}.$$