Math 302
Midterm exam
Friday, May 28, 5pm

Instructions

- There are 4 questions on this exam.
- You have 60 minutes to complete the exam, then an additional 20 minutes to upload pictures/scans of your solutions to Canvas.
- Write your name on the top of each page of work that you submit.
- **You must show your work on all problems.** The correct answer with no supporting work may result in no credit. Put a box around your FINAL ANSWER for each problem and cross out any work that you don't want to be graded.
- Give exact answers.
- Any student found engaging in academic misconduct will receive a score of 0 on this exam.

GOOD LUCK!

1. (12 points; 3 each) Deal a hand of 4 cards from a standard 52 card deck – that is, sample 4 cards from the deck without replacement.

   (a) What is the probability of getting two pairs, but not four of a kind? (e.g. 4H, 4S, JC, JS)

   A: Count the number of hands that have two pairs but not four of a kind. We have to choose the two values – there are $\binom{13}{2}$ possibilities – then the suits for each pair – there are $\binom{4}{2}$ suit combinations for each pair. So the desired probability is $\frac{\binom{13}{2}\binom{4}{2}^2}{\binom{52}{4}}$.

   (b) What is the probability that no two of the cards have the same suit?

   A: The only way this can happen is if we have one card from each suit. So for each of the four suits we pick a value. Thus the desired probability is $\frac{13^4}{\binom{52}{4}}$.

   (c) What is the probability that two of the cards are the ace and king of spades?

   A: The number of such hands is $\binom{50}{2}$, since we just have to pick the remaining two cards. So the desired probability is $\frac{\binom{50}{2}}{\binom{52}{4}}$.

   (d) What is the probability of getting four twos, conditioned on the event that the hand has at least two twos?

   A: The probability of having exactly j twos is $p_j = \frac{\binom{4}{j}\binom{48}{4-j}}{\binom{52}{4}}$. The desired probability is $\frac{p_4}{p_2+p_3+p_4}$.

2. (12 points; 4 each) Let (Ω, P) be a probability space, and let A, B be events in Ω. Assume that $\mathbb{P}(A \cap B) = 1/2$ and $\mathbb{P}(A) = 2/3$. Find $\mathbb{P}(B)$ in each of the following cases:

   (a) A, B are independent

   A: If A, B are independent, then $1/2 = \mathbb{P}(A \cap B) = \mathbb{P}(A) \cap \mathbb{P}(B) = 2/3\mathbb{P}(B)$, so $\mathbb{P}(B) = 3/4$ in this case.

   (b) $\mathbb{P}(A \cup B) = 1$

   A: By the inclusion-exclusion formula, $\mathbb{P}(B) = \mathbb{P}(A \cup B) - \mathbb{P}(A) + \mathbb{P}(A \cap B) = 1 - 2/3 + 1/2 = 5/6$.

   (c) $\mathbb{P}(A|B) = 3/4$.

   A: By the conditional probability formula, $3/4 = \mathbb{P}(A|B) = \mathbb{P}(A \cap B)/\mathbb{P}(B) = \frac{1}{2\mathbb{P}(B)}$, so $\mathbb{P}(B) = 2/3$.

3. (14 points) There are $2n$ seats at a circular dinner table, labeled clockwise from 1 to $2n$. Suppose that of the $2n$ guests, exactly n are vegetarians, and they choose random seats in such a way that all $\binom{2n}{n}$ arrangements of vegetarians are equally likely. For $i \in \{1, 2, \ldots, 2n\}$, let X_i be the indicator random variable of the event that there is a vegetarian in both seats i and $i+1$.

   (a) (3pts) Give the probability mass function of X_i for any $i \in \{1, 2, \ldots, 2n\}$.
   A: X_i takes values 0 or 1. For $X_i = 1$, seats i and $i+1$ must be occupied. This has probability $\frac{\binom{2n-2}{n-2}}{\binom{2n}{n}} = \frac{n(n-1}{2n(2n-1)} = \frac{n-1}{4n-2}$. Thus $\mathbb{P}(X_i = 0) = 1 - \mathbb{P}(X_i = 1) = \frac{3n-1}{4n-2}$.

   (b) (3pts) Compute $\mathbb{E}X_i$.
   A: From the calculation in part (a), $\mathbb{E}[X_i] = \frac{n-1}{4n-2}$.

   (c) (3pts) Compute $\mathbb{E}[X_1 X_2]$.
   A: $X_1 X_2$ is the indicator of the event that seats $1, 2$, and 3 are all occupied. This has probability $\frac{\binom{2n-3}{n-3}}{\binom{2n}{n}}$.

   (d) (3pts) Let X denote the total number of pairs of vegetarians sitting next to each other (each pair counts once). Give a formula for X using the X_i's.
   A: $X = \sum_{i=1}^{2n} X_i$.

   (e) (2pts) Compute $\mathbb{E}[X]$.
   A: By linearity, $\mathbb{E}[X] = 2n\mathbb{E}[X_i] = \frac{n^2-n}{2n-1}$.

4. (12 points; 4 each) Consider the random variable Y with PDF given by

$$f_Y(y) = \frac{e}{2} y e^{-y}, y \geq 1.$$

(a) Let $Z = e^Y$. Find the PDF of Z.

A: Note that the function e^y is increasing. Thus, $\mathbb{P}(Z \leq z) = \mathbb{P}(Y \leq \log z)$. By the chain rule, we obtain

$$f_Z(z) = \frac{d}{dz}\mathbb{P}(Y \leq \log z) = f_Y(\log z) \cdot \frac{d}{dz}(\log z) = \frac{e}{2}(\log z)e^{-\log z} \cdot \frac{1}{z} = \frac{e}{2}\frac{\log z}{z^2}.$$

(b) Use your PDF from part (a) to give an integral expression for $\mathbb{E}[Z]$, and show that the integral diverges (i.e. that it is $+\infty$).

A: Using the PDF from (a) and the law of the unconscious statistician, $\mathbb{E}[Z] = \int_e^\infty \frac{e}{2}\frac{\log z}{z}\, dz \geq \frac{e}{2}\int_e^\infty \frac{1}{z}\, dz = \infty$, where we have used the fact that the integrand is non-negative, and $\log z \geq 1$ for $z \geq e$. That the final integral is infinite follows from the 'p-test.'

(c) Let $W = \log Y$. Compute $\mathbb{P}(1 \leq W \leq 2)$. You can leave your answer as an integral.

A: Since the function e^y is increasing in y,

$$\mathbb{P}(1 \leq W \leq 2) = \mathbb{P}(e \leq Y \leq e^2) = \int_e^{e^2} f_Y(y)\, dy.$$