Feb 4 (1)

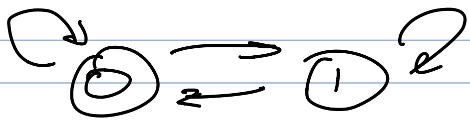
Recall : thm : $Y_n = X_{N-n}$; $(X_n)_{0 \le n \le N}$ has a stationary distribution π.

Then, (i) Y_n is a MC ✓ (see previous lecture)

(ii) π is a stationary distrib. of Y_n

(iii) The transition probabilities of Y_n are

$$Q_{ij} = P_{ji} \cdot \frac{\pi_j}{\pi_i}$$

ex : 

$$P = \begin{pmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{pmatrix}$$

What is $P(X_0 = 1 \mid X_1 = 0)$?

$$= \frac{P(X_0 = 1, X_1 = 0)}{P(X_1 = 0)} = \frac{P(X_1 = 0 \mid X_0 = 1) \cdot P(X_0 = 1)}{P(X_1 = 0)}$$

$$= \frac{p_{10} \cdot P(X_0 = 1)}{P(X_1 = 0)}$$

Proof : (iii) Transition probabilities

$$\underbrace{Q_{ij}}_{P(Y_1 = j \mid Y_0 = i)} = \frac{P(X_{N-n+1} = i \mid X_{N-n} = j) \cdot \overbrace{P(X_{N-n} = j)}^{= \pi_j}}{\underbrace{P(X_{N-n+1} = i)}_{= \pi_i}}$$

$$= \frac{P_{ji} \cdot \pi_j}{\pi_i}$$

because $X_0 \sim \pi$
$\Rightarrow X_1 \sim \pi$ etc.

(ii) $\underline{\Pi}$ is stationary:

$$(\Pi Q)_j = \sum_i \Pi_i Q_{ij} = \sum_i \cancel{\Pi_i} \cdot \frac{P_{ji} \cdot \Pi_j}{\cancel{\Pi_i}}$$

$$= \sum_i P_{ji} \cdot \Pi_j = \Pi_j \underbrace{\sum_i P_{ji}}_{=1} = \Pi_j \quad \boxtimes$$

Def: A MC is time reversible if $Q_{ij} = P_{ij}$

Rmk: The existence of the stationary distribution guarantees that the reversed process is homogeneous (see proof of (iii) in the thm)

Rmk:
- $Q_{ii} = P_{ii}$
- For a time reversible MC — i.e. $Q_{ij} = P_{ij}$ $\left(Q_{ij} = \frac{P_{ji} \cdot \Pi_j}{\Pi_i}\right)$

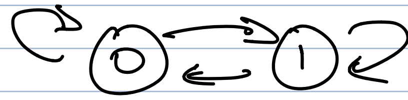
$$\boxed{\forall i \neq j, \quad \Pi_i P_{ij} = \Pi_j P_{ji}}$$

→ This forms a set of equations that are called detailed balance equations.

(Interpretation: Left and right hand sides represent the "fraction of jumps at stationarity"

$i \to j : \Pi_i P_{ij}$

$j \to i : \Pi_j P_{ji}$)

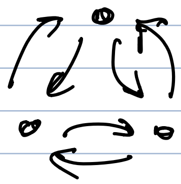


Example: 1) 2-state MC

$$\underline{P} = \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix} \quad (0 < p, q < 1)$$

$$\underline{\Pi} = \left(\frac{q}{p+q}, \frac{p}{p+q} \right)$$: detailed balance is satisfied: $\Pi_0 P_{01} = \Pi_1 P_{10}$

2) R-W on a triangle with $r = 0$ (HW 2)



$$\underline{P} = \begin{bmatrix} 0 & p & 1-p \\ 1-p & 0 & p \\ p & 1-p & 0 \end{bmatrix}$$

Is a chain irreducible? Y

What is the period? the chain is aperiodic

The chain is positive recurrent

$\Rightarrow$ the chain is ergodic

so we know that there is a unique stationary distribution.

$\underline{P}$ is doubly stochastic $\Rightarrow \underline{\Pi} = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$

the MC is reversible $\Leftrightarrow \Pi_i P_{ij} = \Pi_j P_{ji} \quad \forall i, j$

$\Leftrightarrow P_{ij} = P_{ji} \quad \forall i, j$

$\iff p = 1-p \iff \boxed{p = \frac{1}{2}}$ (4)

Prop: Let X_n an irreducible ergodic MC.
If we can find $x_i \geq 0$ st. $\begin{cases} x_i P_{ij} = x_j P_{ji} \\ \sum x_i = 1 \end{cases}$

then $x_i = \pi_i$ and the MC is reversible

Ex: Becca has 3 umbrellas (in total) at home and at her office.

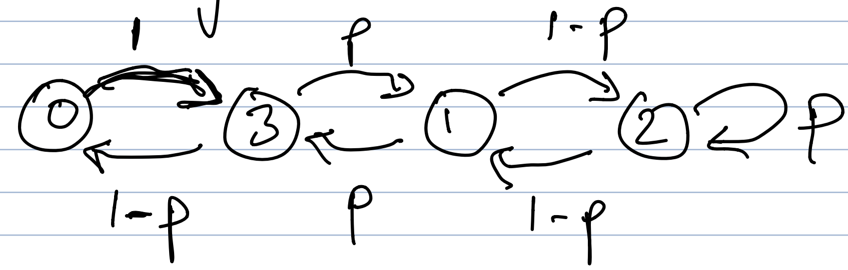
She:
- takes an umbrella if it is raining and if there one at her current location
- doesn't take one if it is not raining
- at each trip, it rains with probability p

Q: What fraction of time (in the long run) does Becca get wet?

A: Let X_n = # of umbrellas at the current location at time n

- State space = $\{0, 1, 2, 3\}$

- Transition diagram



- The chain is irreducible and ergodic

So the answer is $\pi_0 \cdot p$ where $\underline{\pi}$ is the stationary distribution

Proba of having no umbrella

proba. of raining

Assuming that the chain is time reversible, we can try to solve

$$\begin{cases} \pi_i P_{ij} = \pi_j P_{ji} \\ \sum \pi_i = 1 \end{cases}$$

(this is more simple than $\underline{\pi}\,\underline{P} = \underline{\pi}$)

(do it as an exercise)

$$\Rightarrow \pi_0 = \frac{1-p}{4-p} \Rightarrow \boxed{\text{The answer is } p \cdot \frac{1-p}{4-p}}$$