MATH 303 Practice Problems Solutions - Week 3 *Jamuary 29, 2021*

Problem 1

A communication class C is said to be *closed* if $P_{ij} = 0$ whenever $i \in C$ and $j \notin C$ (i.e. there is no escape from C).

1. Consider the stochastic matrices

$$M_1 = \begin{pmatrix} * & * & 0 & 0 \\ * & * & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix}, \qquad M_2 = \begin{pmatrix} * & * & 0 & * \\ * & * & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix}, \qquad M_3 = \begin{pmatrix} * & * & 0 & * \\ * & * & 0 & 0 \\ 0 & 0 & * & * \\ * & 0 & * & * \end{pmatrix},$$

where all entries marked with $*$ are greater than 0. For each of them, find the number of communicating classes, if they are closed (briefly justify).

2. Show that if a state j is accessible from i but i is not accessible from j, then i is transient (remember how transience of some states was proved in the Gambler's ruin problem). This also proves that if i is recurrent and j is accessible from i, then i is accessible from j.

3. Show that if i is recurrent then it belongs to a closed class. Conclude that if the state space is finite, then there is at least one closed communicating class.

4. Reciprocally, if the state space is finite, show that all the states of a closed communicating class are recurrent.

5 Is this still true in general (i.e. for all state space)? Justify.

Solution

1.

M_1: There are two closed communicating classes.
(1 only communicates with 2, and 3 only communicates with 4.)
M_2: There are two communication classes, but only $\{3, 4\}$ is closed.
($1 \leftrightarrow 2, 3 \leftrightarrow 4, 1 \to 4$ but $4 \not\leftarrow 1$ and otherwise there is no transition.)
M_3: The chain is irreducible, so there is only one closed class.
($1 \leftrightarrow 2, 3 \leftrightarrow 4$ and $1 \leftrightarrow 4$.)

2.

$\mathbb{P}(X_n = i \text{ for some } n > 0 | X_0 = i) \leq 1 - \mathbb{P}(X_n = j \text{ for some } n > 0 | X_0 = i) < 1$ since $i \to j$, i.e., $\mathbb{P}(X_n = j \text{ for some } n > 0 | X_0 = i) > 0$, and $j \not\leftarrow i$.

3.

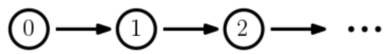
Assume the class of i, denoted by $C(i)$, is not closed, i.e., $\exists\, i_0 \in C(i)$ and $j \notin C(i)$ such that $P_{ij} > 0$. Then $i_0 \to j$, but since i_0 is recurrent, then $j \to i_0$, so $j \in C(i)$. Contradiction!
If the state space is finite, there is at least one recurrent state, belonging to a closed class.

4.

If a state of a finite closed communicating class was transient, then the Markov chain starting from that state can only visit each state a finite number of times, which contradicts the fact that it communicates with all the states of the class.

5.
When the state space is infinite, the previous statements do not hold. For example the asymmetric random walk contains one closed communicating class ($\mathbb{Z}$), which is transient.

Remark: We also saw in class that in finite space, there must be at least one recurrent class. This is not true when the state space is infinite; Aside from the previous example, you can also consider the Markov chain defined by the following transition diagram on $\mathbb{N}$



, which only has transient states.

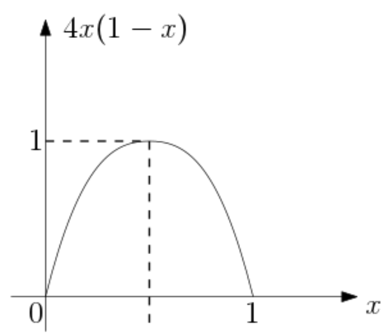
Problem 2

Consider the asymmetric random walks in $\mathbb{Z}$, where one moves one step to the right with probability p, and to the left with probability $1-p$, where $p \neq \frac{1}{2}$. Show that the walk is transient. (Hint: use a similar method as shown in class for a 1-D symmetric random walk.)

Solution

$$P_{00}^{2n} = p^n(1-p)^n \binom{2n}{n} \sim p^n(1-p)^n \left(\frac{2n}{e}\right)^{2n} \sqrt{2\pi 2n} \left(\frac{e}{n}\right)^{2n} \frac{1}{2\pi n} = \frac{1}{\sqrt{\pi n}}(4p(1-p))^n$$

using Stirling's formula. The graph of $x \mapsto 4x(1-x)$ for $0 < x < 1$ is depicted below, showing that $0 < 4p(1-p) \leq 1$.
For large $n \in \{1, 2, ...\}$, it holds that $0 < P_{00}^{2n} < \alpha^n$ where $|\alpha| < 1$. Hence, $\sum_{n=1}^{\infty} P_{00}^{2n} < \infty$.



Besides, $P_{00}^{2n+1} = 0$, so $\sum_{n=1}^{\infty} P_{00}^{n} < \infty$, and the random walk is transient.

Problem 3

Consider the Markov Chain in state space $\{0, 1, 2, 3, 4\}$. If the state is 0 then the next state is 4 w.p. 1. If the state is $i \neq 0$ then the next state is equally likely to be any of the states $\{0, \ldots, i-1\}$.
1. Determine the transition matrix for this Markov chain.
2. Calculate the stationary distribution of this Markov chain.

3. Suppose the Markov chain has been running for a long time. What fraction of time has it spent in state 0?

Solution
1.
The transition matrix reads

$$\mathrm{P} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 \end{pmatrix}.$$

2.
$\pi = \pi \mathrm{P}$ becomes

$$\begin{aligned} \pi_0 &= \pi_1 + \frac{1}{2}\pi_2 + \frac{1}{3}\pi_3 + \frac{1}{4}\pi_4 \\ \pi_1 &= \frac{1}{2}\pi_2 + \frac{1}{3}\pi_3 + \frac{1}{4}\pi_4 \\ \pi_2 &= \frac{1}{3}\pi_3 + \frac{1}{4}\pi_4 \\ \pi_3 &= \frac{1}{4}\pi_4 \\ \pi_4 &= \pi_0, \end{aligned}$$

which, with $\sum_i \pi_i = 1$, has solution $\pi = (\frac{12}{37}, \frac{6}{37}, \frac{4}{37}, \frac{3}{37}, \frac{12}{37})$.

3.
The fraction is $\pi_0 = \frac{12}{37}$.

Problem 4

We consider a Markov chain $(X_n)_{n\geq 0}$ on states 1, 2, 3 and 4, with transition matrix

$$\begin{pmatrix} 1/5 & 2/5 & 1/5 & 1/5 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ * & * & * & * \\ * & * & * & * \end{pmatrix}$$

Assuming $X_0 = 1$,
1. what is the probability to enter state 3 before state 4?
2. what is the mean number of transitions until either state 3 or 4 is entered? (hint: for both questions, do a one-step analysis, i.e. condition on the outcome of X_1)
3. Solve **2.** using the method introduced last week to find the mean time in transient states (see week 2 notebook).

Solution

1.

Let $p(i) = \mathbb{P}(X_n \text{ enters 3 before } 4 | X_0 = i)$ where $i \in \{1, 2\}$. Then

$$\left\{ \begin{array}{l} p(1) = p(1)P_{11} + p(2)P_{12} + P_{13} \\ p(2) = p(1)P_{21} + p(2)P_{22} + P_{23} \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} \frac{4}{5}p(1) = \frac{2}{5}p(2) + \frac{1}{5} \\ \frac{3}{4}p(2) = \frac{1}{4}p(1) + \frac{1}{4} \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} p(1) = \frac{p(2)}{2} + \frac{1}{4} \\ p(1) = \frac{p(1)}{3} + \frac{1}{3} \end{array} \right\}$$

$$\Leftrightarrow \left\{ \begin{array}{l} p(1) = \frac{1}{2} \\ p(2) = \frac{1}{2} \end{array} \right\}.$$

Thus, the probability to enter state 3 before state 4 is $\frac{1}{2}$.

2.

Let $N(i)$ = mean number of transitions before entering 3 or 4, given $X_0 = i$ where $i \in \{1, 2\}$. Then

$$\begin{array}{ll} & \left\{ \begin{array}{l} N(1) = (1 + N(1))p_{11} + (1 + N(2))p_{12} + 1 \cdot (p_{13} + p_{14}) \\ N(2) = (1 + N(1))p_{21} + (1 + N(2))p_{22} + 1 \cdot (p_{23} + p_{24}) \end{array} \right\} \\ \Leftrightarrow & \left\{ \begin{array}{l} N(1) = (1 + N(1))\frac{1}{5} + (1 + N(2))\frac{2}{5} + \frac{2}{5} \\ N(2) = (1 + N(1))\frac{1}{4} + (1 + N(2))\frac{1}{4} + \frac{1}{2} \end{array} \right\} \\ \Leftrightarrow & \left\{ \begin{array}{l} 4N(1) = 2N(2) + 5 \\ 3N(2) = N(1) + 4 \end{array} \right\} \\ \Leftrightarrow & \left\{ \begin{array}{l} N(1) = \frac{23}{10} \\ N(2) = \frac{21}{10} \end{array} \right\} \end{array}$$

Therefore, the mean number of steps before entering 3 or 4 starting from 1 is $\frac{23}{10}$.

3.

We consider $\mathrm{P_T} = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} = \begin{pmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{1}{4} & \frac{1}{4} \end{pmatrix}$, so $(\mathrm{I} - \mathrm{P_T})^{-1} = \begin{pmatrix} \frac{4}{5} & -\frac{2}{5} \\ -\frac{1}{4} & \frac{3}{4} \end{pmatrix}^{-1}$.

Calculating $(\mathrm{I} - \mathrm{P_T})^{-1}$ (for example, using $M^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{\det(M)} \operatorname{adj}(A) = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$), we find

$$(\mathrm{I} - \mathrm{P_T})^{-1} = \begin{pmatrix} 1.5 & 0.8 \\ 0.5 & 1.6 \end{pmatrix}.$$

Thus, the number of steps before reaching 3 or 4 starting from 1 is

$$(\mathrm{I} - \mathrm{P_T})^{-1}_{11} + (\mathrm{I} - \mathrm{P_T})^{-1}_{12} = 1.5 + 0.8 = 2.3.$$