1. Give a parametric description of the plane containing $(-1,0,0)$ with normal vector $\langle 3,1,-1 \rangle$.

Solution: The equation of this plane is

$$3(x+1) + y - z = 0, \qquad \text{or}$$

$$z = y + 3x + 3.$$

So we can parameterize by

$$r(x,y) = x\hat{\imath} + y\hat{\jmath} + (y+3x+3)\hat{k},$$

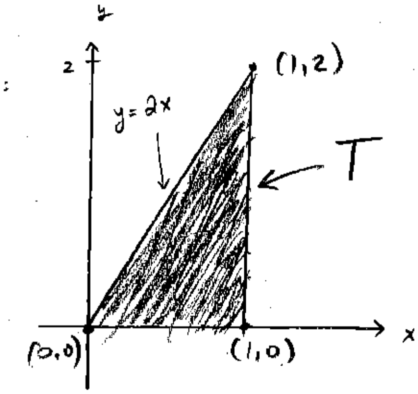
for any $(x,y) \in \mathbb{R}^2$.

Another valid parameterization is

$$s(x,z) = x\hat{\imath} + (z-3x-3)\hat{\jmath} + z\hat{k}.$$

3. Find the area of the part of the surface $z = x^2 + 2y$ lying above the triangle with vertices $(0,0)$, $(1,0)$, $(1,2)$.

<u>Solution</u>: Parameterize the surface by



$$r(x,y) = x\hat{\imath} + y\hat{\jmath} + (x^2+2y)\hat{k}, \quad (x,y) \in T.$$

Then $r_x = \hat{\imath} + 0\hat{\jmath} + 2x\hat{k}$,

$r_y = 0\hat{\imath} + \hat{\jmath} + 2\hat{k}$, so

$$r_x \times r_y = \det\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 0 & 2x \\ 0 & 1 & 2 \end{vmatrix} = -2x\hat{\imath} - 2\hat{\jmath} + \hat{k}, \text{ and}$$

$$|r_x \times r_y| = \sqrt{(-2x)^2 + (-2)^2 + (1)^2} = \sqrt{4x^2+5}.$$

Thus, the surface area is

$$SA = \iint_T 1 \cdot |r_x \times r_y| \, dA = \int_0^1 \int_0^{2x} \sqrt{4x^2+5} \, dy\,dx = \int_0^1 2x\sqrt{4x^2+5} \, dx$$

Let $u = 4x^2+5$, $du = 8x\,dx$, so $SA = \frac{1}{4}\int_5^9 \sqrt{u}\, du = \boxed{\frac{1}{6}\left[9\sqrt{9} - 5\sqrt{5}\right]}$

\# 5. Compute $\iint_S f\,dS$, where $f(x,y,z) = y$, and

S is the surface $x^2+y^2+z^2 = 4$, $0 \le y \le 1$.

Solution: We can parameterize by "rotated" spherical coordinates:

$$r(\theta,\phi) = 2\cos\theta\sin\phi\,\hat{\imath} + 2\cos\phi\,\hat{\jmath} + 2\sin\theta\sin\phi\,\hat{k},$$

for $0 \le \theta \le 2\pi$, $\pi/3 \le \phi \le \pi/2$. [When $\phi = \frac{\pi}{2}$, $y = 0$,

and when $\phi = \pi/3$, $y = 1$.]

Then
$$r_\theta = -2\sin\theta\sin\phi\,\hat{\imath} + 0\,\hat{\jmath} + 2\cos\theta\sin\phi\,\hat{k},$$
$$r_\phi = 2\cos\theta\cos\phi\,\hat{\imath} - \sin\phi\,\hat{\jmath} + 2\sin\theta\cos\phi\,\hat{k},$$

so
$$|r_\theta \times r_\phi| = \left|\det\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -2\sin\theta\sin\phi & 0 & 2\cos\theta\sin\phi \\ 2\cos\theta\cos\phi & -\sin\phi & 2\sin\theta\cos\phi \end{vmatrix}\right| = 4\sin\phi.$$

Thus
$$\iint_S f\,dS = \int_0^{2\pi}\int_{\pi/3}^{\pi/2} (2\cos\phi)\cdot(4\sin\phi)\,d\phi\,d\theta$$
$$= 16\pi\int_{\pi/3}^{\pi/2} \frac{1}{2}\sin(2\phi)\,d\phi = \boxed{6\pi}$$

#7. Evaluate $\iint_S F \cdot dS$, where $F = \langle 0, y, -z \rangle$,

and S consists of the paraboloid $y = x^2 + z^2$, $0 \le y \le 1$,

together with the disk $x^2 + z^2 \le 1$, $y = 1$, oriented outward.

Solution: Let S_1 denote the paraboloid part,

and S_2 the disk part.

Then S_1 is parameterized by $r(x,z) = x\hat{\imath} + (x^2+z^2)\hat{\jmath} + z\hat{k}$,

for $x^2 + z^2 \le 1$. Then $r_x \times r_z = 2x\hat{\imath} - \hat{\jmath} + 2z\hat{k}$,

and $$\iint_{S_1} F \cdot dS = \iint_{S_1} \langle 0, y, -z \rangle \cdot \langle 2x, -1, +2z \rangle \, dS = \iint_{\{x^2+z^2 \le 1\}} \left(-(x^2+z^2) - 2z^2 \right) dA$$

$$= \iint_{\{x^2+z^2 \le 1\}} \left(-x^2 - 3z^2 \right) dA = \int_0^{2\pi}\int_0^1 \left(-r^2\cos^2\theta - 3r^2\sin^2\theta \right) r\,dr\,d\theta$$

$$= -\int_0^1 r^3\,dr \int_0^{2\pi} \left(\cos^2\theta + 3\sin^2\theta \right) d\theta = \boxed{-\pi.}$$

Note that $r_x \times r_z$ points in the negative y-direction, so we have given S_1 the "outward" orientation.

For S_2, we can parameterize by

$$s(r,\theta) = r\sin\theta\,\hat{\imath} + \hat{\jmath} + r\cos\theta\,\hat{k}, \quad \text{for } 0 \le r \le 1,\ 0 \le \theta \le 2\pi.$$

Then $s_r = \sin\theta\,\hat{\imath} + 0\hat{\jmath} + \cos\theta\,\hat{k}$, and

$s_\theta = +r\cos\theta\,\hat{\imath} + 0\hat{\jmath} - r\sin\theta\,\hat{k}$, so

$$s_r \times s_\theta = \det\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \sin\theta & 0 & \cos\theta \\ r\cos\theta & 0 & -r\sin\theta \end{vmatrix} = (+r\cos^2\theta + r\sin^2\theta)\,\hat{\jmath} = +r\,\hat{\jmath}.$$

This gives the outward pointing orientation on S_2 (i.e. toward the positive y-axis). Thus

$$\iint_{S_2} F \cdot dS = \int_0^{2\pi}\int_0^1 \langle 0, 1, -r\cos\theta \rangle \cdot \langle 0, r, 0 \rangle \, dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 r\,dr\,d\theta = \boxed{\pi}$$

So $\displaystyle\iint_S F \cdot dS = \iint_{S_1} F \cdot dS + \iint_{S_2} F \cdot dS = \pi - \pi = \boxed{0}.$

[Note: Alternatively, by the divergence theorem,

$$\iint_S F \cdot dS = \iiint_E \operatorname{div}(F) \cdot dV = \iiint_E (0 + 1 - 1)\,dV = 0,$$

where E is the 3-D region "inside" S.]