MATH 303 Practice Problems Solutions - Week 11-12

April, 2021

Problem 1

1. We have $v_i = 1$ for all i, and $p_{ij} = \frac{1}{4}$ for all i and j from 1 to 5 such that $i \neq j$.

2. If we start from lily number 1, then the lilies 2, 3, 4 and 5 play similar roles in the problem, so the chances to be on each of them should be the same.

3. The forwards Chapman–Kolmogorov equation can be written:

$$p'_{11}(t) = \sum_{k=2}^{5} p_{1k}(t) \times \frac{1}{4} - p_{11}(t) \times 1.$$

Since the $p_{1k}(t)$ for $k = 2, 3, 4, 5$ are all equal and $\sum_{k=1}^{5} p_{1k}(t) = 1$, we have

$$p_{1k}(t) = \frac{1}{4}(1 - p_{11}(t))$$

for $k = 2, 3, 4, 5$, so the forwards Chapman–Kolmogorov equation becomes

$$p'_{11}(t) = 4 \times \frac{1}{4}(1 - p_{11}(t)) \times \frac{1}{4} - p_{11}(t) = \frac{1}{4} - \frac{5}{4} p_{11}(t).$$

4. We have a linear differential equation of the form $p'(t) = ap(t) + \theta$, with $a = -\frac{5}{4}$ and $\theta = \frac{1}{4}$. The solutions are of the form

$$p_{11}(t) = Ce^{-5t/4} + \frac{1}{5},$$

where C is a constant (see MATH 215/255, or look for how to solve first order differential equations with constant coefficients). To find C, we use $p_{11}(0) = 1$, so $C + \frac{1}{5} = 11$, so $C = \frac{4}{5}$ so

$$p_{11}(t) = \frac{4}{5} e^{-5t/4} + \frac{1}{5}.$$

Problem 2

We will denote by $\{i, j\}$ the state where there are i cells of type A and j of type B.
If we have i individuals of type A and j individuals of type B, the rate at which there is an A cell switching to B is $i\alpha$, and the rate at which there is a B cell that splits is $j\beta$, so we have

$$v_{\{i,j\}} = i\alpha + j\beta.$$

Moreover, when an event occurs, the probability that is is an A cell switching to B is $\frac{i\alpha}{i\alpha + j\beta}$, and the probability that it is a B cell splitting is $\frac{j\beta}{i\alpha + j\beta}$. In the first case, the new state of the population is $\{i - 1, j + 1\}$. In the second, the new state is $\{i + 2, j - 1\}$. Therefore, we have

$$p_{\{i,j\},\{i-1,j+1\}} = \frac{i\alpha}{i\alpha + j\beta}, \quad p_{\{i,j\},\{i+2,j-1\}} = \frac{j\beta}{i\alpha + j\beta}$$

and $p_{\{i,j\},\{k,\ell\}} = 0$ in all other cases.

Problem 3

1. New students can only become aware of the rumour one at a time, so the process has to start from 1 and then jump to 2, then 3 ad finally 4. In other words, we have $p_{12} = p_{23} = p_{34} = 1$. Therefore, the only thing we need to compute are the v_i.
When $A(t) = 1$, only one student (let us call her Alice) is aware of the rumour, so three possible students meetings can make $A(t)$ jump to 2 (the meetings where Alice meets one of the three other students). Each meeting occurs at rate 1, so such meetings occur at rate 3, so $v_1 = 3$.
When $A(t) = 2$, four possible meetings can make $A(t)$ jump to 3 (one of the two aware students meets one of the two unaware ones), so $v_2 = 4$.
When $A(t) = 3$, three possible meetings can make $A(t)$ jump to 4 (one of the three aware students meets the last unaware one), so $v_3 = 3$.

2. The time it takes to jump from 1 to 2 is $Exp(3)$, so it has expectation $\frac{1}{3}$. Similarly, the time it takes to jump from 2 to 3 has expectation $\frac{1}{4}$ and the time it takes to jump from 3 to 4 has expectation $\frac{1}{3}$. Since T is the sum of these three durations, it has expectation

$$\mathbb{E}[T] = \frac{1}{3} + \frac{1}{4} + \frac{1}{3} = \frac{11}{12}.$$

Problem 4

1. We denote by 0 the state where the machine is working, by 1 the state where it has a failure of type 1 and by 2 the state where it has a failure of type 2. The statement of the problem can be translated as:

$$v_0 = \lambda, \quad v_1 = \mu_1 \quad \text{and} \quad v_2 = \mu_2,$$

with

$$p_{0,1} = p, \quad p_{0,2} = 1 - p, \quad p_{1,0} = p_{2,0} = 1 \quad \text{and} \quad p_{1,2} = p_{2,1} = 0.$$

2. This is an irreducible CTMC with finite state space, so it has limiting probabilities (π_i) satisfying the equations:

$$\lambda\pi_0 = \mu_1\pi_1 + \mu_2\pi_2,$$
$$\mu_1\pi_1 = p\lambda\pi_0,$$
$$\mu_2\pi_2 = (1-p)\lambda\pi_0,$$
$$\pi_0 + \pi_1 + \pi_2 = 1.$$

In particular, the second and third equations give $\pi_1 = \frac{p\lambda}{\mu_1}\pi_0$ and $\pi_2 = \frac{(1-p)\lambda}{\mu_2}\pi_0$, so the fourth equations gives

$$\left(1 + \frac{p\lambda}{\mu_1} + \frac{(1-p)\lambda}{\mu_2}\right)\pi_0 = 1,$$

so

$$\pi_0 = \frac{1}{1 + \frac{p\lambda}{\mu_1} + \frac{(1-p)\lambda}{\mu_2}}, \quad \pi_1 = \frac{p\lambda/\mu_1}{1 + \frac{p\lambda}{\mu_1} + \frac{(1-p)\lambda}{\mu_2}} \quad \text{and} \quad \pi_2 = \frac{(1-p)\lambda/\mu_2}{1 + \frac{p\lambda}{\mu_1} + \frac{(1-p)\lambda}{\mu_2}}.$$

Note: It is also possible to find π by using reversibility.

Problem 5

1. We first describe the problem as a CTMC $X(t)$, where $X(t)$ is the number of broken machines at time t. The state space is $\{0,1,2,3,4\}$. There are $4-X(t)$ working machines, so the rate at which one of them gets broken is $q_{i,i+1}=\frac{1}{20}(4-X(t))$, so

$$q_{0,1}=\frac{1}{5},\quad q_{1,2}=\frac{3}{20},\quad q_{2,3}=\frac{1}{10}\quad\text{and}\quad q_{3,4}=\frac{1}{20}.$$

On the other hand, the rate at which one of the broken machines gets repaired is $\frac{1}{5}X(t)$ if $X(t)\leq 2$ (since then there is a repairman on each broken machine), and $\frac{1}{5}\times 2$ if $X(t)\geq 2$ (since then both repairmen are busy). In other words, we have:

$$q_{1,0}=\frac{1}{5}\quad\text{and}\quad q_{2,1}=q_{3,2}=q_{4,3}=\frac{2}{5}.$$

Therefore, we have

$$\begin{aligned}
v_0&=q_{0,1}=\frac{1}{5},\\
v_1&=q_{1,0}+q_{1,2}=\frac{7}{20},\\
v_2&=q_{2,1}+q_{2,3}=\frac{1}{2},\\
v_3&=q_{3,2}+q_{3,4}=\frac{9}{20},\\
v_4&=q_{4,3}=\frac{2}{5}.
\end{aligned}$$

We could also compute the probabilities $p_{ij}=\frac{q_{ij}}{v_i}$, but this will not be necessary here.
To study the long run behviour of $X(t)$, we now compute the limiting probabilities π_i of $X(t)$ (they exist because the chain is irreducible and has finite state space). We get the equations:

$$\frac{1}{5}\pi_0=\frac{1}{5}\pi_1,\quad\text{so}\quad\pi_1=\pi_0,$$

$$\begin{aligned}
\frac{7}{20}\pi_1&=\frac{1}{5}\pi_0+\frac{2}{5}\pi_2,\quad\text{so}\quad\pi_2=\frac{7}{8}\pi_1-\frac{1}{2}\pi_0=\frac{3}{8}\pi_0,\\
\frac{1}{2}\pi_2&=\frac{3}{20}\pi_1+\frac{2}{5}\pi_3,\quad\text{so}\quad\pi_3=\frac{5}{4}\pi_2-\frac{3}{8}\pi_1=\frac{3}{32}\pi_0,\\
\frac{9}{20}\pi_3&=\frac{1}{10}\pi_2+\frac{2}{5}\pi_4,\quad\text{so}\quad\pi_4=\frac{9}{8}\pi_3-\frac{1}{4}\pi_2=\frac{3}{256}\pi_0.
\end{aligned}$$

Finally, we have

$$1=\pi_0+\pi_1+\pi_2+\pi_3+\pi_4=\left(1+1+\frac{3}{8}+\frac{3}{32}+\frac{3}{256}\right)\pi_0=\frac{635}{256},$$

so

$$\pi_0=\frac{256}{635},\quad\pi_1=\frac{256}{635},\quad\pi_2=\frac{96}{635},\quad\pi_3=\frac{24}{635}\quad\text{and}\quad\pi_4=\frac{3}{635}.$$

Finally, both repairmen are busy when $X(t)$ is 2, 3 or 4, so the proportion of time where both repairmen are busy is

$$\pi_2+_p i_3+\pi_4=\frac{123}{635}.$$

Note: It was also possible (and quicker) to notice that $X(t)$ is a birth and death chain, and to use the formula from the course about birth-and-death chains. Of course, we obtain the same result.

2. The average number of broken machines is

$$\pi_1 + 2\pi_2 + 3\pi_3 + 4\pi_4 = \frac{256 + 192 + 72 + 12}{635} = \frac{532}{635}.$$

Problem 6

1. The possible states are the following:

- there is no customer at all (we call this state 0),
- there is exactly one customer, being serviced by server 1 (we call this state 1_a),
- there is exactly one customer, being serviced by server 2 (we call this state 1_b),
- there are $i \geq 2$ customers, (we call each of this these states i for $i \geq 2$).

Our state space is then $\{0, 1_a, 1_b, 2, 3, 4, \dots\}$.

2. The transition rates q_{ij} are as follows:

$$q_{0,1a} = q_{0,1b} = \frac{1}{2}\lambda,$$

$$q_{1a,0} = \mu_1 \quad \text{and} \quad q_{1a,2} = \lambda$$

$$q_{1b,0} = \mu_2 \quad \text{and} \quad q_{1b,2} = \lambda$$

$$q_{2,1a} = \mu_2, \quad q_{2,1b} = \mu_1 \quad \text{and} \quad q_{2,3} = \lambda,$$

$$\text{for all } i \geq 3: \quad q_{i,i-1} = \mu_1 + \mu_2 \quad \text{and} \quad q_{i,i+1} = \lambda.$$

From here, we can deduce the v_i and the p_{ij}:

$$v_0 = \lambda, \quad v_{1a} = \lambda + \mu_1, \quad v_{1b} = \lambda + \mu_2 \quad \text{and} \quad v_i = \lambda + \mu_1 + \mu_2 \quad \text{for } i \geq 2.$$

3. The chain is irreducible, so if it has a stationary distribution π, then it has limiting probabilities given by π. In order to have lighter computations, we try to find π using reversibility. We want:

$$\frac{\lambda}{2}\pi_0 = \mu_1 \pi_{1a},$$

$$\frac{\lambda}{2}\pi_0 = \mu_2 \pi_{1b},$$

$$\lambda \pi_{1a} = \mu_2 \pi_2,$$

$$\lambda \pi_{1b} = \mu_1 \pi_2,$$

$$\lambda \pi_i = (\mu_1 + \mu_2)\pi_{i+1} \quad \text{for all } i \geq 2.$$

From the first three equations, we get $\pi_{1a} = \frac{\lambda}{2\mu_1}\pi_0$, $\pi_{1b} = \frac{\lambda}{2\mu_2}\pi_0$ and $\pi_2 = \frac{\lambda}{\mu_2}\pi_{1a} = \frac{\lambda^2}{2\mu_1\mu_2}$. Note that then the fourth equation is automatically satisfied. Finally, using the last equation, we get, for $i \geq 2$:

$$\pi_i = \left(\frac{\lambda}{\mu_1 + \mu_2}\right)^{i-2} \pi_2 = \frac{\lambda^i}{2\mu_1\mu_2(\mu_1 + \mu_2)^{i-2}}\pi_0.$$

Using $\sum_i \pi_i = 1$, we finally obtain

$$\pi_0 = \frac{1}{1 + \frac{\lambda}{2\mu_1} + \frac{\lambda}{2\mu_2} + \frac{\lambda^2}{2\mu_1\mu_2} \times \frac{1}{1-\frac{\lambda}{\mu_1+\mu_2}}},$$

and the other limiting probabilities follow.

Note: It was not clear a priori if the chain was going to be reversible or note. Here is a quick computation that could let us think that it is. When we draw the transition rates diagram, there is basically only one cycle, namely $0 \to 1a \to 2 \to 1b \to 0$. If we compute the product of the transition rates long this cycle, we get $\frac{\mu_1\mu_2\lambda^2}{2}$, and in the reverse direction we get the same result.

4. The chain is now the same, except that $q_{0,1a} = \lambda$ and $q_{0,1b} = 0$. We now have $q_{0,1b} = 0$ but $q_{1b,0} = \mu_2$, so the chain is not reversible anymore. Hence, our only choice is to write the equations for the stationary distribution:

$$\lambda\pi_0 = \mu_1\pi_{1a} + \mu_2\pi_{1b},$$

$$(\lambda + \mu_1)\pi_{1a} = \lambda\pi_0 + \mu_2\pi_2,$$

$$(\lambda + \mu_2)\pi_{1b} = \mu_1\pi_2.$$

Our goal is to compare π_{1a} and π_{1b}. We first get rid of π_0 by summing up the first two equations. We obtain:

$$\lambda\pi_{1a} = \mu_2\pi_{1b} + \mu_2\pi_2.$$

Using the third equation, we can replace π_2 by $\frac{\lambda+\mu_2}{\mu_1}\pi_{1b}$, so the last equation becomes

$$\lambda\pi_{1a} = \mu_2\pi_{1b} + \frac{\mu_2(\lambda+\mu_2)}{\mu_1}\pi_{1b} = \frac{\mu_2(\lambda+\mu_1+\mu_2)}{\mu_1}\pi_2,$$

so $\pi_{1a} = \frac{\mu_2(\lambda+\mu_1+\mu_2)}{\mu_1\lambda}\pi_{1a}$. In particular, server 1 is busier than server 2 if and only if

$$\mu_2(\lambda + \mu_1 + \mu_2) > \mu_1\lambda,$$

which is equivalent to $\lambda < \frac{\mu_2(\mu_1+\mu_2)}{\mu_1-\mu_2}$. If $\mu_1 < 2\mu_2$, then we have $\mu_2 > \mu_1 - \mu_2$, so

$$\lambda < \mu_1 + \mu_2 < \frac{\mu_2(\mu_1+\mu_2)}{\mu_1 - \mu_2},$$

i.e. 1 is always busier. If $\mu_1 > 2\mu_2$, then $\mu_2 < \mu_1 - \mu_2$, so $\frac{\mu_2(\mu_1+\mu_2)}{\mu_1-\mu_2} < \mu_1 + \mu_2$, so we can find some λ in between for which server 2 is busier than server 1.

Problem 7

1. The transition rates q_{ij} are the following:

$$q_{01} = q_{12} = q_{23} = 4 \quad \text{(reparation)},$$

$$q_{10} = 1 \quad \text{(the last machine breaks)},$$

$$q_{21} = q_{20} = \frac{1}{2} \times 2 = 1 \quad \text{(one of the last two fails, the other may break as well)}$$

$$q_{32} = q_{30} = \frac{1}{2} \times 3 = \frac{3}{2} \quad \text{(one of the three, the others may break as well)},$$

and the other transitions are zero. From here, we can deduce the v_i and p_{ij}:

$$v_0 = 4, \quad v_1 = 5, \quad v_2 = 6, \quad v_3 = 3.$$

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1/5 & 0 & 4/5 & 0 \\ 1/6 & 1/6 & 0 & 2/3 \\ 1/2 & 0 & 1/2 & 0 \end{pmatrix}$$

2. It is possible to jump from state 3 to state 0 but not from state 0 to state 3, so the chain is not reversible.

3. The chain is irreducible and has finite state space, so it has limiting probabilities given by the unique stationary distribution π.
Since the chain is not a birth-death chain and is not reversible, our only choice is to write down the stationarity equations:

$$\begin{aligned} 4\pi_0 &= \pi_1 + \pi_2 + \frac{3}{2}\pi_3, \\ 5\pi_1 &= 4\pi_0 + \pi_2, \\ 6\pi_2 &= 4\pi_1 + \frac{3}{2}\pi_3, \\ 3\pi_3 &= 4\pi_2, \\ \pi_0 + \pi_1 &+ \pi_2 + \pi_3 = 1. \end{aligned}$$

The fourth equation gives $\pi_3 = \frac{4}{3}\pi_2$. Substituting in the third one, we get $6\pi_2 = 4\pi_1 + 2\pi_2$, so $\pi_1 = \pi_2$. Substituting in the second equation, we get $5\pi_2 = 4\pi_0 + \pi_2$, so $\pi_0 = \pi_1 = \pi_2$. Finally, using the last equation, we find

$$\left(1 + 1 + 1 + \frac{4}{3}\right)\pi_2 = 1,$$

so $\pi_0 = \pi_1 = \pi_2 = \frac{3}{13}$ and $\pi_3 = \frac{4}{13}$.

4. The transition rates are now the same as before, with the exception that $q_{12} = q_{23} = 0$. A consequence is that the chain is not irreducible anymore: after some time, it will stay only on the two states 0 and 1, so we can study the restriction of our chain to the state space $\{0, 1\}$. The transition rates are then $q_{01} = 4$ and $q_{10} = 1$, so the limiting probabilities π satisfy $4\pi_0 = \pi_1$, so $\pi_0 = \frac{1}{5}$ aand $\pi_1 = \frac{4}{5}$. In particular $\frac{1}{5} < \frac{3}{13}$, so this new strategy is better: it becomes rarer to have no working machine at all.

Problem 8

1. We will denote by A a working machine, B a half-broken one and C a fully broken one. We have 6 states, that we can write AA, AB, AC, BB, BC and CC. Note that the two machines are identical, so if one is broken we don't need to remember which one. This is why we don't need two different states AB and BA.

2. We give the q_{ij} matrix for this chain (the speeds v_i an probabilities p_{ij} can easily be deduced from here):

$$q_{AA,AB} = 2, \quad q_{AB,BB} = q_{AC,BC} = 1 \quad \text{(a machine passes from } A \text{ to } B\text{)},$$

$$q_{BB,BC} = 4, \quad q_{AB,AC} = q_{BC,CC} = 2 \quad \text{(a machine passes from } B \text{ to } C\text{)},$$

$$q_{AB,AA} = q_{BB,AB} = q_{BC,AC} = 2 \quad \text{(the repairman fixes a machine from } B \text{ to } A\text{)}$$

$$q_{AC,AB} = q_{CC,BC} = 2 \quad \text{(the repairman fixes a machine from } C \text{ to } B\text{)}.$$

Note that there is no transition from BC to BB, since in state BC the repairman fixes the B machine.

3. Note that it is not reversible since $q_{BB,BC} > 0$ whereas $q_{BC,BB} = 0$, so our only choice is to write down the stationarity equations:

$$2\pi_{AA} = 2\pi_{AB},$$

$$5\pi_{AB} = 2\pi_{AA} + 2\pi_{BB} + 2\pi_{AC},$$

$$3\pi_{AC} = 2\pi_{AB} + 2\pi_{BC},$$

$$6\pi_{BB} = \pi_{AB},$$

$$4\pi_{BC} = \pi_{AC} + 4\pi_{BB} + 2\pi_{CC},$$

$$2\pi_{CC} = 2\pi_{BC}.$$

Using the first, fourth and last equations, we get respectively $\pi_{AA} = \pi_{AB}$, $\pi_{BB} = \frac{1}{6}\pi_{AB}$ and $\pi_{CC} = \pi_{BC}$. Substituting these into the second equation, we obtain

$$5\pi_{AB} = 2\pi_{AB} + \frac{1}{3}\pi_{AB} + 2\pi_{AC}, \text{ so } 2\pi_{AC} = \frac{8}{3}\pi_{AC}, \text{ so } \pi_{AC} = \frac{4}{3}\pi_{AB}.$$

Finally, substituting in the third equation above, we get

$$4\pi_{AB} = 2\pi_{AB} + 2\pi_{BC}, \text{ so } \pi_{BC} = \pi_{AB},$$

so $\pi_{AA} = \pi_{AB} = \pi_{BC} = \pi_{CC}$. Finally, we can write

$$1 = \pi_{AA} + \pi_{AB} + \pi_{AC} + \pi_{BB} + \pi_{BC} + \pi_{CC} = \left(1 + 1 + \frac{4}{3} + \frac{1}{6} + 1 + 1\right)\pi_{AB} = \frac{11}{2}\pi_{AB},$$

so our limiting probabilities are:

$$\pi_{AA} = \pi_{AB} = \pi_{BC} = \pi_{CC} = 2/11, \quad \pi_{BB} = 1/33, \quad \text{and} \quad \pi_{AC} = 8/33.$$

4. We just compute the mean number of A in the limiting probabilities above:

$$2\pi_{AA} + \pi_{AB} + \pi_{AC} = \frac{4}{11} + \frac{2}{11} + \frac{8}{33} = \frac{26}{33}.$$