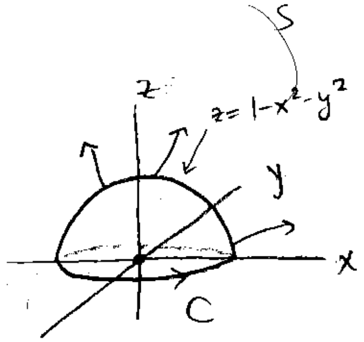
1. Verify Stokes' thm for S = surface of $z = 1 - x^2 - y^2$

for $x^2+y^2 \le 1$, $F = \langle 2xy,\ x,\ y+z \rangle$.

<u>Solution</u>: With the upward orientation, we can use the formula for a surface of the form $z = f(x,y)$ to get



$$\iint_S \mathrm{curl}(F)\cdot dS = \iint_S \langle 1,\ 0,\ 1-2x \rangle \cdot dS$$

$$= \iint_{x^2+y^2\le 1} \left[-(-2x)(1) - (-2y)(0) + (1-2x)\right] dA$$

$$= \iint_{x^2+y^2\le 1} 1\, dA = \boxed{\pi.}$$

The boundary curve C is the circle of radius 1 oriented cc-wise, so

$$\int_C F\cdot dr = \int_0^{2\pi} F(x(t),y(t),z(t)) \cdot \langle x'(t),\ y'(t),\ z'(t) \rangle\, dt$$

$x = \cos t$

$y = \sin t$

$z = 0$

$$= \int_0^{2\pi} \langle 2\cos t \sin t,\ \cos t,\ \sin t \rangle \cdot \langle -\sin t,\ \cos t,\ 0 \rangle\, dt$$

$$= \int_0^{2\pi} \Big(\underbrace{-2\cos t \sin^2 t}_{\substack{\text{odd function} \\ \text{about } t=\pi}} + \cos^2 t \Big)\, dt = \int_0^{2\pi} \cos^2 t\, dt = \boxed{\pi.}$$

$[\cos(t-\pi)\sin^2(t-\pi) = \cos(-t-\pi)\sin^2(-t-\pi)]$

\# 2 $\quad F = \langle yz, 0, x \rangle$, S = the plane $x/2 + y/3 + z = 1$ for $x, y, z \geq 0$.

Verify Stokes' theorem: $\displaystyle\int_{\partial S} F \cdot dr = \iint_S \operatorname{curl}(F) \cdot dS$.

Solution: First, find curl(F).

$$\operatorname{curl}(F) = \det \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ yz & 0 & x \end{vmatrix} = 0\hat{\imath} - (1-y)\hat{\jmath} + (0-z)\hat{k} = (y-1)\hat{\imath} - z\hat{k}.$$

S is parameterized by $r(x,y) = x\hat{\imath} + y\hat{\jmath} + (1 - \frac{x}{2} - \frac{y}{3})\hat{k}$ for $x, y \geq 0$, and

$z \geq 0$, or $1 - \frac{x}{2} - \frac{y}{3} \geq 0 \implies y \leq 3 - \frac{3}{2}x$; note $3 - \frac{3}{2}x = 0$ when $x=2$,

so $0 \leq x \leq 2$.

So

$$\iint_S \nabla \times F \cdot dS = \int_0^2 \int_0^{3-\frac{3}{2}x} \left[-\frac{\partial z}{\partial x} \cdot 0 - \frac{\partial z}{\partial y} \cdot (y-1) - (1 - \frac{x}{2} - \frac{y}{3}) \right] dy\, dx$$

$$= \int_0^2 \int_0^{3-\frac{3}{2}x} \left(\frac{y}{3} - \frac{1}{3} - 1 + \frac{x}{2} + \frac{y}{3} \right) dy\,dx = -1.$$

The boundary of S consists of 3 curves: when $\underset{C_1}{x=0}$, $\underset{C_2}{y=0}$, or $\underset{C_3}{z=0}$.

These are the lines 1) $z = 1 - y/3$, $x=0$, 2) $z = 1 - x/2$, $y=0$; 3) $y = 3 - \frac{3}{2}x$, $z=0$.

C_1 is parameterized by $r(y) = \langle 0, y, 1 - y/3 \rangle$ for $0 \leq y \leq 3$

C_2 " " " $s(x) = \langle x, 0, 1 - x/2 \rangle$ for $0 \leq x \leq 2$

C_3 " " " $q(x) = \langle x, 3 - \frac{3}{2}x, 0 \rangle$ for $0 \leq x \leq 2$.

So $\displaystyle\int_{C_1} F \cdot dr = \int_0^3 \langle y(1 - y/3), 0, 0 \rangle \cdot \langle 0, 1, -1/3 \rangle \cdot dy = 0.$

$$\int_{C_2} F \cdot dr = \int_0^2 \langle 0,0,x\rangle \cdot \langle 1,0,-\tfrac{1}{2}\rangle \, dx = \int_0^2 -\tfrac{1}{2}x \, dx = -1.$$

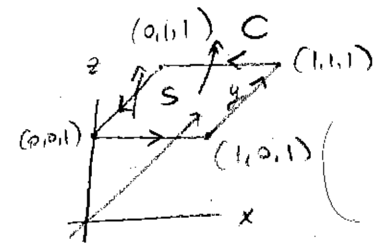
And $$\int_{C_3} F \cdot dr = \int_0^2 \langle 0,0,x\rangle \cdot \langle 1, -\tfrac{3}{2}, 0\rangle \, dx = 0.$$

Thus $$-1 = \iint_S \nabla \times F \cdot dS = \int_{C_1} F \cdot dr + \int_{C_2} F \cdot dr + \int_{C_3} F \cdot dr = 0 + (-1) + 0 = -1$$

So Stokes' theorem holds for this surface and vector field.

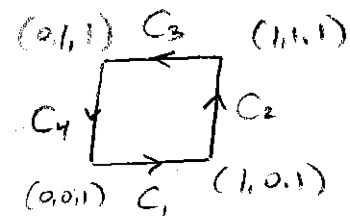
#3 $F = e^{y-z}\hat{\imath}$, S = square w/vertices $(1,0,1)$, $(1,1,1)$, $(0,1,1)$ and $(0,0,1)$. Verify Stokes'!

<u>Solution</u>: Note $\operatorname{curl}(F) = 0\hat{\imath} - e^{y-z}\hat{\jmath} - e^{y-z}\hat{k}$.



The normal vector to S is $\hat{k}$, so

$$\iint_S \operatorname{curl}(F)\cdot dS = \int_0^1\int_0^1 -e^{y-1}\,dx\,dy = \boxed{\frac{1}{e} - 1.}$$



For the boundary curve C,

note $\int_{C_2} F\cdot dr = \int_{C_4} F\cdot dr = 0$, since

F only has non-zero x-component, and x is constant on C_2 and C_4 (so $x'(t) = 0$ there, if we were to parameterize).

$$\int_{C_1} F\cdot dr = \int_0^1 e^{0-1}\hat{\imath}\cdot\hat{\imath}\,dt = \frac{1}{e}, \quad \text{and}$$

$$\int_{C_3} F\cdot dr = \int_0^1 e^{1-1}\hat{\imath}\cdot(-\hat{\imath})\,dt = -1.$$

Thus $\int_C F\cdot dr = 0 + 0 + \frac{1}{e} - 1 = \boxed{\frac{1}{e} - 1}.$

5 Use Stokes' thm to evaluate

$$\iint_S \operatorname{curl}(x\hat{j} + xz\hat{k}) \cdot dS,$$ where S is the spherical cap $x^2+y^2+z^2=1$, $z \geq \frac{1}{2}$.

<u>Solution</u>: By Stokes theorem,

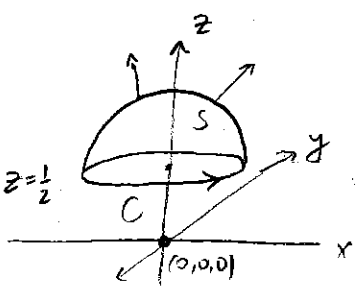
$$\iint_S \operatorname{curl}(x\hat{j} + xz\hat{k}) \cdot dS = \int_{\partial S} (x\hat{j} + xz\hat{k}) \cdot dr.$$

$C = \partial S$ is the circle $x^2+y^2+(\frac{1}{2})^2 = 1$, or $x^2+y^2 = (\frac{\sqrt{3}}{2})^2$.

So C can be parameterized by $s(t) = \frac{\sqrt{3}}{2}\cos t\,\hat{i} + \frac{\sqrt{3}}{2}\sin t\,\hat{j} + \frac{1}{2}$, $0 \leq t \leq 2\pi$

Thus $$\int_C (x\hat{j} + xz\hat{k}) \cdot dr = \int_0^{2\pi} \langle 0, \tfrac{\sqrt{3}}{2}\cos t, \tfrac{\sqrt{3}}{4}\cos t \rangle \cdot \langle -\tfrac{\sqrt{3}}{2}\sin t, \tfrac{\sqrt{3}}{2}\cos t, 0 \rangle \, dt$$

$$= \int_0^{2\pi} \left(\tfrac{3}{4}\cos^2 t\right) dt = \frac{3\pi}{4}.$$



Thus $$\boxed{\iint_S \nabla \times (x\hat{j} + xz\hat{k}) \cdot dS = \frac{3\pi}{4}.}$$

#6 Verify the divergence theorem for the vector field $F = \langle y, x, z \rangle$, and the region E = ball of radius 2 centered at the origin.

Solution: Note that div $F = 1$, so

$$\iiint_E \operatorname{div}(F)\, dV = \iiint_E dV = \frac{4\pi}{3}(2)^3 = \frac{32\pi}{3}.$$

We can parameterize the sphere of radius 2 by

$r(\phi,\theta) = 2\cos\theta \sin\phi\, \hat{\imath} + 2\sin\theta\sin\phi\, \hat{\jmath} + 2\cos\phi\, \hat{k}$, so that

$r_\phi \times r_\theta = 4\sin\phi \langle \cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi \rangle$. Thus

$$\iint_{\partial E} F \cdot dS = \int_0^{\pi}\int_0^{2\pi} F(r(\phi,\theta)) \cdot r_\phi \times r_\theta \, d\theta\, d\phi$$

$$= \int_0^{\pi}\int_0^{2\pi} \langle 2\sin\theta\sin\phi, 2\cos\theta\sin\phi, 2\cos\phi \rangle \cdot 4\sin\phi \langle \cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi \rangle \, d\theta\, d\phi$$

$$= 8\int_0^{\pi}\int_0^{2\pi} (\underbrace{2\cos\theta\sin\theta}_{[\,=\sin(2\theta);\ \int_0^{2\pi}\sin 2\theta\, d\theta = 0.\,]} \sin^2\phi + \sin\phi\cos^2\phi)\, d\theta\, d\phi$$

$$= 16\pi \int_0^{\pi} \sin\phi\cos^2\phi\, d\phi = 16\pi \int_{-1}^{1} u^2\, du = \frac{32\pi}{3}.$$

$u = \cos\phi \quad du = -\sin\phi\, d\phi.$

#8 Suppose $E \subset \mathbb{R}^3$ is a region, and

$$\iint_{\partial E} \langle x+2xy+z,\ e^x-3z^2-y^2,\ 4z \rangle \cdot dS = 85.$$

Find the volume of E.

<u>Solution</u>: Use the divergence theorem!

$$\iint_{\partial E} \langle x+2xy+z;\ e^x-3z^2-y^2,\ 4z \rangle\, dS$$

$$= \iiint_E \operatorname{div}\left((x+2xy+z)\hat{\imath} + (e^x-3z^2-y^2)\hat{\jmath} + 4z\hat{k}\right) \cdot dV \quad \left(\text{by the div. thm}\right)$$

$$= \iiint_E \left((1+2y+0)+(0-0-2y)+(4)\right) dV$$

$$= \iiint_E 5\, dV.$$

Therefore, $5\iiint_E dV = 5 \cdot \text{volume}(E) = 85,$

so the volume of E is 17.