Recall: We study the Gambler's ruin problem

$$p(n) = P(R \mid X_0 = n)$$

We showed (by conditioning on the first event) that $p(n) = p(n+1)p + p(n-1)(1-p)$ $0 \leq n \leq N$

with boundary conditions $p(0) = 1$ and $p(1) = N$

. To solve this recurrence equation, the general method is to first solve the associated characteristic equation (see week 2 online page)

$$X = X^2 p + (1-p) \iff pX^2 - X + (1-p) = 0$$

→ the roots allow to find a general solution of $p(n) = p(n+1)p + p(n-1)(1-p)$ that depends on 2 constants.

→ To determine the constant, we use the boundary conditions.

$\boxed{\text{Case } p = \frac{1}{2}}$: By solving $pX^2 - X + (1-p) = 0$ with $p = \frac{1}{2}$

we find $\Delta = 0 \Rightarrow$ one root $X = \frac{1}{2p} = 1$

In this case the general solution is

$$p(n) = \alpha X^n + \beta . n X^n,$$ where α and β are constants

$$\Rightarrow p(n) = \alpha + \beta \cdot n \text{ , and}$$

$$\begin{cases} p(0) = 1 \\ p(N) = 0 \end{cases} \iff \begin{cases} \alpha \cdot 1^0 + \beta \cdot 0 \, 1^0 = 1 \\ \alpha \cdot 1^N + \beta \cdot N \cdot 1^N = 0 \end{cases}$$

$$\iff \begin{cases} \alpha = 1 \\ \beta = -\frac{1}{N} \end{cases}$$

$$\Rightarrow \boxed{p(n) = 1 - \frac{n}{N}}$$

Conclusion: $P(\text{ruin} \mid X_0 = n) = \begin{cases} 1 - \frac{a^n - 1}{a^N - 1} & \text{if } p \neq \frac{1}{2} \text{ and where } a = \frac{1-p}{p} \\ 1 - \frac{n}{N} & \text{if } p = \frac{1}{2} \end{cases}$

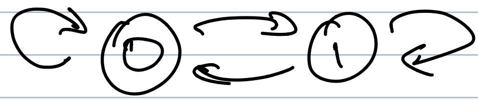
- We saw that the states in $\{1, \dots, N-1\}$ were transient (i.e the probability of revisiting the state is < 1). Can we find this probability f_i ?

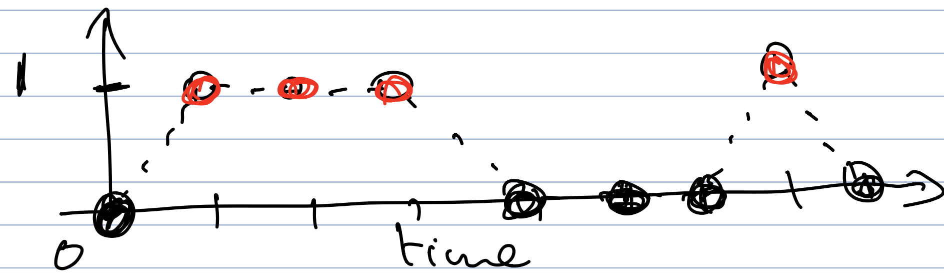
Answer: YES! $\to$ see Jupyter Notebook and more properties on recurrence and transience.

Properties of transience and recurrence

- Let $N_i = \# \{ n \geq 0 \mid X_n = i \}$

ex: If X_n is a M-C with transition diagram:





- N_0 = How many times X_n was eq. to 0
- N_1 = " " " " " " " 1

Rmk: • N_i is a random variable on $\mathbb{N} \cup \{+\infty\}$

• $N_i = \sum_{n=0}^{+\infty} \mathbb{1}_{(X_n = i)}$,

where $\mathbb{1}_{(X_n=i)}$ (indicator function) $= \begin{cases} 1 & \text{if } X_n = i \\ 0 & \text{else} \end{cases}$

• $E\left(\mathbb{1}_{(X_n=i)}\right) = Pr(X_n = i)$

$$\left(= 1.\,Pr\left(\mathbb{1}_{(X_n=i)} = 1\right) + \cancel{0.\,Pr\left(\mathbb{1}_{(X_n=i)} = 0\right)} \right)$$

Prop: (i) If i is recurrent, then $P(N_i=\infty \mid X_0=i)=1$

(ii) If i is transient and $X_0=i$,
then $N_i \sim \text{Geom}(1-f_i)$
(i.e. $P(N_i=m)=(1-f_i).f_i^{m-1}$, $m\geq 1$)
and in particular $E(N_i)=\dfrac{1}{1-f_i}$

Interpretation: A recurrent state is re-visited ∞.ly many times, while for a transient state, the chain doesn't revisit the state after a certain time

Proof: (i) $f_i=1 \Rightarrow$ the process returns to i w.p. 1
so $P(N_i\geq 1 \mid X_0=i)=1$, but after returning, the process essentially starts afresh, so it returns again
so $P(N_i\geq 2 \mid X_0=i)=1$ etc.

so $\forall m \; P(N_i\geq m \mid X_0=i)=1$

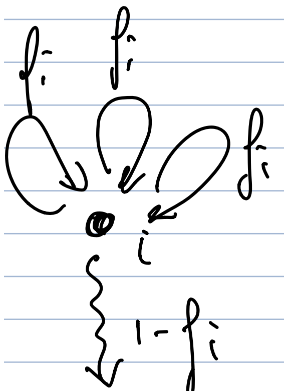
$\Rightarrow \forall k\in\mathbb{N} \; P(N_i=k \mid X_0=i)=0$

so $P(N_i=\infty \mid X_0=i)=1$ ✓

(ii) Let's assume $X_0 = i$, so $P(N_i = 1) = 1 - f_i$
(by def. of f_i)

- For $N_i = 2$
  $\Rightarrow$ the chain returns once and never after that, so $P(N_i = 2) = f_i \cdot (1 - f_i)$

- For $N_i = 3$
  $\Rightarrow$ the chain returns twice and never after, so $P(N_i = 3) = f_i^2 (1 - f_i)$



(Picture for $N_i = 4$)

(...) $P(N_i = n) = f_i^{n-1} (1 - f_i)$ ✓

$\boxtimes$

Rmk: • if i is recurrent $\mathbb{E}(N_i \mid X_0 = i) = +\infty$

• if the state space is finite, then some state must be recurrent

(Proof: if all states are transient, there is no state visited after a certain time)
Contradiction!

We now characterize recurrence and transience using n-step transition probabilities.

Prop: i is $\begin{cases} \text{recurrent if } \sum_{n=0}^{+\infty} P_{ii}^n = \infty \\ \text{transient if } \sum_{n=0}^{+\infty} P_{ii}^n < +\infty \end{cases}$

Proof: $\mathbb{E}(N_i \mid X_0 = i) = \sum_{n=0}^{+\infty} \mathbb{E}(\mathbb{1}_{(X_n = i)} \mid X_0 = i)$

$\sum_{n=0}^{+\infty} \mathbb{1}(X_n = i)$ by def.

$$= \sum_{n=0}^{+\infty} P(X_n = i \mid X_0 = i)$$

$$= \sum_{n=0}^{+\infty} P_{ii}^n \quad \boxtimes$$

Corollary: if i is recurrent and $i \leftrightarrow j$, then j is recurrent

Important consequence: Recurrence and transience are class properties (all states in a communicating class are either all recurrent or all transient).

Proof: (exercise) $\to$ (use the Chapman-Kolmogorov equation as in previous proofs)

$\to$ if $\sum P_{ii}^n > \infty$ and $i \leftrightarrow j$, show that $\sum P_{jj}^n = \infty$

As recurrence and transience are class properties, we can also directly conclude about these when we observe a certain type of class.

Def: A communication C is **closed** if $P_{ij} = 0$ whenever $i \in C$ and $j \notin C$
(i.e. there is no escape from C)

Rmk: if $i \to j$, but $j \not\to i$, then i is transient.
(exercise, see also practice problems for week 3)

Prop: (i) A communication class which is not closed is always transient

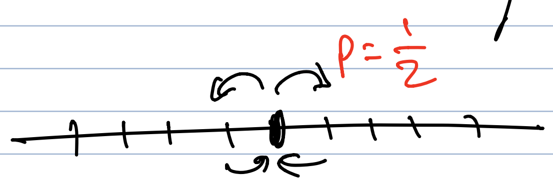
(ii) A **closed finite** communication class is always recurrent

Rmk: For an infinite communication class, it is harder to conclude, and we need to rely on the previous proposition, as we will see with the following example.

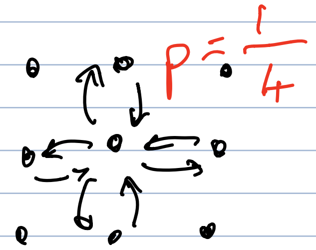
Example: Random walk in $\mathbb{Z}^d$

· We consider the uniform R-W in $\mathbb{Z}^d$

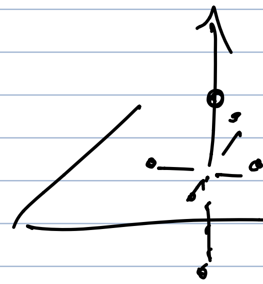
$d = 1$



$d = 2$



$d = 3$



$p = \frac{1}{6}$

In general, $p = \frac{1}{2d}$

Q: Is the process recurrent or transient?

$$\boxed{\text{Result: The walk is } \begin{cases} \text{recurrent if } d = 1, 2 \\ \text{transient if } d = 3, 4, 5, \ldots \end{cases}}$$