1. (10 pts) Let $f(x, y)$ be a function on $\mathbb{R}^2$. Assume that the minimum value of $D_u f(1, -3)$ is attained when $u = -\frac{\sqrt{2}}{2}\hat{\imath} + \frac{\sqrt{2}}{2}\hat{\jmath}$. Also, assume $\frac{\partial f}{\partial x}(1, -3) = +4$.

(a) (5 pts) Find $\nabla f(1, -3)$.

$-|\nabla f|$ $\qquad \nabla f = a\hat{\imath} + b\hat{\jmath}$

$\nabla f(1,-3) \cdot u = D_u f(1,-3) =$ $\qquad a = 4$

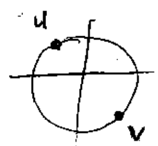
$-\frac{\sqrt{2}}{2} \cdot 4 + \frac{\sqrt{2}}{2} \cdot b = -\left|\sqrt{16+b^2}\right|$ $\qquad \frac{\partial f}{\partial x}(1,-3) = f_x(1,-3) = a$

$$\boxed{\nabla f(1,-3) = 4\hat{\imath} - 4\hat{\jmath}}$$

$\frac{-b\sqrt{2}}{2} \cdot 2\sqrt{2} \cdot 2$ $\qquad 2\sqrt{2} + \frac{(-b)\sqrt{2}}{2} = \sqrt{16+b^2}$

$8 - 4b + \frac{b^2}{2} = 16 + b^2$ $\qquad \frac{b^2}{2} + 4b + 8 = 0$ $\qquad b^2 + 8b + 16 = 0$ $\qquad (b+4)^2 = 0$ $\qquad b = -4$

(b) (2 pts) What unit vector v maximizes $D_v f(1, -3)$?



$$\boxed{v = \frac{\sqrt{2}}{2}\hat{\imath} - \frac{\sqrt{2}}{2}\hat{\jmath}}$$

(c) (3 pts) Given your answer from part (a), is it possible that $f(x, y) = 10 - x^2 - 3xy + x$? Explain.

$\nabla f(x,y) = (-2x - 3y + 1)\hat{\imath} + (-3x)\hat{\jmath}$

$\nabla f(1,-3) = (-2(1) - 3(-3) + 1)\hat{\imath} + (-3(1))\hat{\jmath} = 8\hat{\imath} - 3\hat{\jmath} \neq 4\hat{\imath} - 4\hat{\jmath}$

$\nabla f(x,y) = f_x(x,y)\hat{\imath} + f_y(x,y)\hat{\jmath}$

$\boxed{\text{No}}$ because plugging in the point (1,-3) gives an answer different from part A.

2. (15 pts) Let D be the region in the plane under the parabola $y = 4 - x^2$ and above the line $y = 3x$.

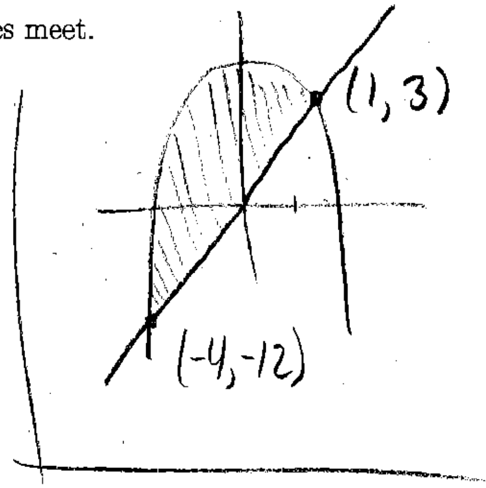
(a) (3 pts) Draw D, and find the points where the two bounding curves meet.

$4 - x^2 = 3x$

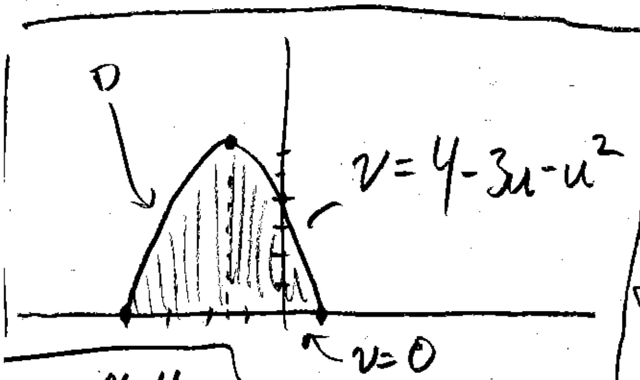
$0 = x^2 + 3x - 4$

$\frac{-3 \pm \sqrt{9 - 4(-4)}}{2} = \frac{-3 \pm \sqrt{25}}{2} = -4, 1$

$\boxed{(-4,-12) \; ; \; (1,3)}$



(b) (7 pts) Consider the change of coordinates $u = x, v = y - 3x$. Draw the image of D in the u-v plane, and find the Jacobian of the transformation.



$x = u$

$y = v + 3x = v + 3u$

(u, v)

$-4 = x \quad y = -12$

$+1 = x \quad 3 = y$

$u = -4$

$v = -12 + 12 = 0$

$u = 1$

$v = 3 - 3 = 0$

$(-4, 0)_{uv} \Leftrightarrow (-4, -$

$(1, 0)_{uv} \Leftrightarrow (1, 3)$

$\det \begin{vmatrix} 1 & 0 \\ +3 & 1 \end{vmatrix} = 1$

$y = 3x \Rightarrow v + 3u = 3u$

$v = 0$

$y = 4 - x^2 \Rightarrow v + 3u = 4 - u^2$

$v = 4 - 3u - u^2$

(c) (5 pts) Parameterize the double integral $\iint_D (x + y)\, dA$ in terms of u's and v's. **You do not need to evaluate it.**

$$\int_{-4}^{1} \int_{0}^{4-3u-u^2} (4u + v) \cdot (1) \, dv \, du$$

$\frac{-b}{2a} = -\frac{3}{2}$

$4 + \frac{9}{2} - \frac{9}{4} = 4 + \frac{9}{4} = 6\frac{1}{4}$

3. (10 pts) Consider the vector field $F = 3x^2y\hat{\imath} + x^3\hat{\jmath}$.

(a) (5 pts) Is F conservative? If so, find a potential function; if not, explain how you know it isn't conservative.

$F = \langle 3x^2y, x^3 \rangle$ Domain = $\mathbb{R}^2$ so open & simply connected.

$P = 3x^2y \quad Q = x^3$

$\frac{\partial P}{\partial y} = 3x^2 = \frac{\partial Q}{\partial x} = 3x^2$

PF? $f_x = 3x^2y$

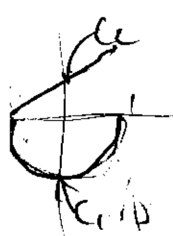
$F = x^3y$

$F_y = x^3$

$\rightarrow F_{\text{potential}} = x^3y + K$

so $F = \nabla F_{\text{potential}} \rightarrow F$ is conservative

(b) (5 pts) Let C be the curve consisting of the part of the circle $x^2 + y^2 = 1$ below the x-axis, from $(1,0)$ to $(-1,0)$, followed by the line segment from $(-1,0)$ to $(1,1)$. Evaluate $\int_C F \cdot dr$.



$\int_C 3x^2y\hat{\imath} + x^3\hat{\jmath}\, dr$

$\int_C F\cdot dr = \int_{C_1} F\cdot dr + \int_{C_2} F\cdot dr$ by fundamental theory

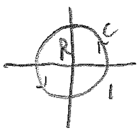
$\int_{C_1} F\cdot dr = f(-1,0) - f(1,0) = (-1)^3 0 - (1^3 \cdot 0)$

$\int_{C_2} F\cdot dr = f(1,1) - f(-1,0) \quad (1^3)1 - (-1)^3\cdot 0$

$0 + 1 = 1$

4. (15 pts) Let R be the circle of radius 1 centered at $(0,0)$, and let $C = \partial R$ be the boundary of R, oriented counter-clockwise.

(a) (10 pts) Use Green's theorem to evaluate



$$\int_C x^3\,dy.$$

$$\int_C 0\,dx + x^3\,dy$$

$$\iint_D 3x^2 - 0\;dA$$

$$\int_0^{2\pi}\int_0^1 3r^2\cos^2(\theta)\;r\,dr\,d\theta$$

$$3\int_0^{2\pi}\int_0^1 r^3\cos^2(\theta)\,dr\,d\theta$$

$$3\int_0^{2\pi}\left[\frac{r^4}{4}\cos^2(\theta)\right]_0^1 d\theta$$

$$3\int_0^{2\pi}\frac{1}{4}\cos^2(\theta)\,d\theta$$

$$\frac{3}{4}\int_0^{2\pi}\frac{1}{2}\left(1+\cos(2\theta)\right)d\theta$$

$$\frac{3}{4}\left[\frac{1}{2}\left(\theta + \frac{1}{2}\sin(2\theta)\right)\right]_0^{2\pi}$$

$$\frac{3}{4}\left[\frac{1}{2}\left(2\pi + 0\right)\right]$$

$$\frac{3}{4}\left[\frac{2\pi}{2}\right]$$

$$\boxed{\frac{3\pi}{4}}$$

(b) (5 pts) Verify your answer from part (a) by evaluating the line integral directly.

$x = \cos(t)$ $\quad y = \sin(t)$ $\quad 0 \le t \le 2\pi$

$dy = \cos(t)\,dt$

$$\int_C x^3\,dy = \int_0^{2\pi}\cos^3(t)\cdot\cos(t)\,dt$$

$$\int_0^{2\pi}\cos^4(t)\,dt$$

$$\int_0^{2\pi}\left(\tfrac{1}{2}\right)\left(1+\cos(2t)\right)\left(\tfrac{1}{2}\right)\left(1+\cos(2t)\right)dt$$

$$\frac{1}{4}\int_0^{2\pi} 1 + 2\cos(2t) + \cos^2(2t)\,dt$$

$$\frac{1}{4}\int_0^{2\pi} 1 + 2\cos(2t) + \frac{1}{2}\left(1+\cos(4t)\right)dt$$

$$\frac{1}{4}\left[t + \sin(2t) + \frac{1}{2}\left(t + \frac{1}{4}\sin(4t)\right)\right]_0^{2\pi}$$

$$\frac{1}{4}\left[2\pi + 0 + \frac{1}{2}\left(2\pi + 0\right) - 0\right]$$

$$\frac{1}{4}\left[2\pi + \pi\right]$$

$$\boxed{\frac{3\pi}{4}}$$