April 13 (1)

Today → Conclusion of Chap. 3.
→ HW Pbm 4 → Model a stochastic process as a CTMC
↳ find the parameters
→ find the stationary distribution

→ Notebook → simulation of a CTMC

⚠ Read Announcements posted over the next week
↳ Important instructions to prevent technical issues for the final exam

Tomorrow → 2 office hours sessions
9 - 11
6 pm (usual)
→ links in Announcement

2) Time reversibility

Like for discrete time, we can also consider a time reversal of an irreducible CTMC with a stationary distribution π. If the chain is time reversible, then π satisfies the

detailed balance equations

Def: X is time reversible if it admits a stationary distribution π, and

$$\pi_i q_{ij} = \pi_j q_{ji} \quad \forall i, j$$

Rmk: To find limiting probabilities, one can try to find a distribution that satisfies detailed balance.

Prop: Any birth-death process is time reversible

(cf. stationary distribution seen last week).

HW problem 4

→ Machine with 3 states.

→ Working

→ failure of type 1

→ failure of type 2.

→ If failure happens it has probability p to be of type 1, and $(1-p)$ to be of type 2.

→ Time for a failure to happen is $Exp(\lambda)$

→ Time for failure 1 to repair is $Exp(\mu_1)$

→ ——— 2 ——— $Exp(\mu_2)$

(4 parameters p, λ, μ_1, μ_2.)

Q1 : Describe the CTMC associated with this process

Q2 : Proportion of time when machine is working, is down with type 1 failure / type 2 failure

↳ find the stationary distribution.

Q1 : $2 \rightleftarrows 0 \rightleftarrows 1$

"failure of type 2" "working" "failure of type 1"

- $v_0 = \lambda$; $v_1 = \mu_1$; $v_2 = \mu_2$
- $p_{01} = p$; $p_{02} = 1-p$; $p_{20} = 1$; $p_{10} = 1$

Based on the transition diagram: $p_{12} = p_{21} = 0$

$$\rightarrow Q = \begin{pmatrix} q_{00} & q_{01} & q_{02} \\ q_{10} & q_{11} & q_{12} \\ q_{20} & q_{21} & q_{22} \end{pmatrix}$$

Recall $\left(q_{ij} = \begin{cases} v_i p_{ij} & \text{if } i \neq j \\ -v_i & \text{if } i = j \end{cases} \right)$

$$Q = \begin{pmatrix} -\lambda & \lambda p & \lambda(1-p) \\ \mu_1 & -\mu_1 & 0 \\ \mu_2 & 0 & -\mu_2 \end{pmatrix}$$

Q 2: We solve $\pi Q = 0$

$$\begin{pmatrix}\pi_0 & \pi_1 & \pi_2\end{pmatrix}\begin{pmatrix}-\lambda & p\lambda & (1-p)\lambda \\ \mu_1 & -\mu_1 & 0 \\ \mu_2 & 0 & -\mu_2\end{pmatrix} = (0,0,0)$$

$$\Rightarrow \begin{cases} 0 = \pi_0(-\lambda) + \pi_1\mu_1 + \pi_2\mu_2 \\ 0 = \pi_0 p\lambda - \pi_1\mu_1 \\ 0 = \pi_0(1-p)\lambda - \pi_2\mu_2 \end{cases} \qquad (*)$$

In addition $\pi_0 + \pi_1 + \pi_2 = 1$

From $(*)$ (take eq. 2,3) $\Rightarrow \begin{cases} \pi_1 = \pi_0 \dfrac{p\lambda}{\mu_1} \\ \pi_2 = \pi_0 \dfrac{(1-p)\lambda}{\mu_2} \end{cases}$

so $\pi_0 + \pi_1 + \pi_2 = 1 \Rightarrow \pi_0\left(1 + \dfrac{p\lambda}{\mu_1} + \dfrac{(1-p)\lambda}{\mu_2}\right) = 1$

and $\begin{cases} \pi_0 = \dfrac{1}{1 + \dfrac{p\lambda}{\mu_1} + \dfrac{(1-p)\lambda}{\mu_2}} \\ \pi_1 = \dfrac{p\lambda/\mu_1}{1 + p\lambda/\mu_1 + (1-p)\lambda/\mu_2} \end{cases}$

$$\pi_2 = \frac{(1-p)\lambda/\mu_2}{1 + p\lambda/\mu_1 + (1-p)\lambda/\mu_2}$$

Answer: The proportion of time of a machine working is π_0, failure of type is π_1, failure of type 2 is π_2.

Rmk: Using reversibility, we find for the detailed balance equation:

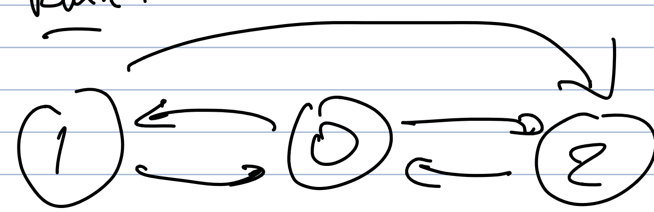
$$\begin{cases} \pi_0 q_{01} = \pi_1 q_{10} \\ \pi_0 q_{02} = \pi_2 q_{20} \end{cases} \iff \begin{cases} \pi_0 p\lambda = \pi_1 \mu_1 \\ \pi_0 (1-p)\lambda = \pi_2 \mu_2 \end{cases}$$

Same as eq. 2,3 in (*)

→ Notebook

week 12.

Rmk:



Not time reversible (direct flux of proba of proba from $1 \to 2$ but not $2 \to 1$)