MATH 303 Midterm 2 Exam A

March 25, 2021

INSTRUCTIONS: PLEASE READ ALL CAREFULLY BEFORE STARTING

After inputing your student ID, we recommend to write down the values set by the script. If you show results obtained with values different from those set by the Jupyter notebook script, your question **will not be graded.**

1. **Problem 1**: Complete the Problem 1 set in `midterm2A_students.ipynb` (there are five parts). Submit the completed notebook as `midterm2A_complete.ipynb` (you don't have to finish all the questions to submit).
   Remark: If you are running the notebook on syzygy, we recommend that you make a copy of the original notebook to edit it, save after each new answer, and take a screenshot of your work as soon as you are done. If your connection to the server broke, simply log back to the server.

2. **Problems 2-4**: For each problem, you will need to separately assemble a pdf file of handwritten solutions. We recommend to not use more than 2 pages per problem. **Write your name** on top of each page and do not forget to indicate which subquestion (a, b...) you are solving.

Grading

We recommend spending an equivalent amount of time on problems 1, 2, 3 and 4.

Problem 1

Complete the Problem 1 set in `midterm2A_students.ipynb` (there are five parts).

Remark: If you are running the notebook on syzygy, we recommend that you make a copy of the original notebook to edit it, save after each new answer, and take a screenshot of your work as soon as you are done. If your connection to the server broke, simply log back to the server and re-open your saved notebook.

Warning: Having "Success" displayed when running the cells does not necessarily mean that the solution is correct (this will be evaluated after you submit the notebook).

Problem 2

We consider a population of cells, starting from one cell at day 0, and such that from one day to the next, each cell can independently die, remain as it is, or divide into two cells. The probabilities of these events are given by running the notebook.

a. Find the mean and generating function of the reproduction law of the associated branching process (justify your answers).

b. Find the probability of extinction (justify your answer).

c. Assume that from one day to the next, each cell which remains and hasn't divided is removed from the population with a probability p. Find the value of p such that the probability of extinction of the population is 0.5 (justify your answer).

Problem 3

Let $N(t)$ be a Poisson process of rate λ. Recall that for random variables X and Y, the *covariance* of X and Y is given by

$$\mathrm{Cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y].$$

a. For $0 < s < t$, express $N(t)$ in terms of $N(s)$ and some independent random variable(s).

b. Use your computation from part a to show that for $0 < s < t$,

$$\mathbb{E}[N(s)N(t)] = \lambda s + \lambda^2 st.$$

(*Remark:* for a Poisson(μ) random variable Y, $\mathbb{E}[Y^2] = \mu^2 + \mu$.)

c. Conclude that

$$\mathrm{Cov}(N(s), N(t)) = \lambda s.$$

Make sure to justify all reasoning.

Problem 4

Jacob likes to collect pocket watches. Suppose he finds gold watches and silver watches at times according to two independent Poisson processes, with rate 2 per month for silver watches and 1 per month for gold watches. Run the notebook to display how many watches Jacob has collected after exactly a year (12 months).

a. Find the probability displayed in the notebook. (You can leave an answer that is calculator ready.)

b. What is the probability that Jacob collected no watches during March? (Assume for simplicity that all months are the same length.)