II. The Poisson process

The Poisson process describes a class of counting processes in real time, that count the number of occurences of some event:

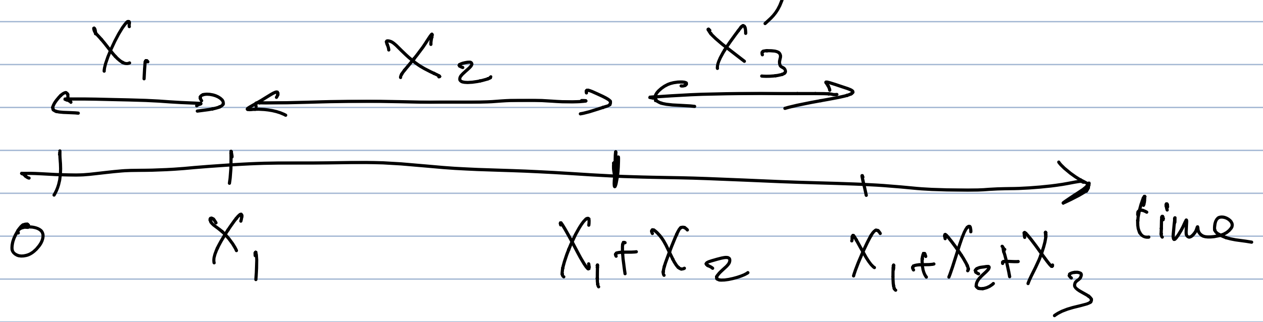
Ex: cf. Jupyter Notebook, # cars that pass a highway toll; # jobs completed by a busy computer server; # fish caught from a fishing boat ...

Main assumption: time between occurences is always $Exp(\lambda)$

$\Rightarrow$ The time for n events to occur is then (cf. last week)

$$\sum_{i=1}^{n} X_i \sim Gamma(n, \lambda), \quad \text{where } X_i \sim \text{iid } Exp(\lambda)$$

(X_i describes the waiting time for the i-th occurence, called interarrival time)



Remarks : • By time t, how many occurences of the event should we expect in average?

• What is the average interarrival time?

→ The average interarrival time is the Expectation of X_i → $\frac{1}{\lambda}$

→ It takes in average $\frac{1}{\lambda}$ units of time for an event to happen

⇒ there are λ events per unit of time in average (λ represents the average occurence rate or frequency)

(there is λ events in one minute (in average)
so 2λ events in 2 minutes
3λ ——— 3 ———
etc.)

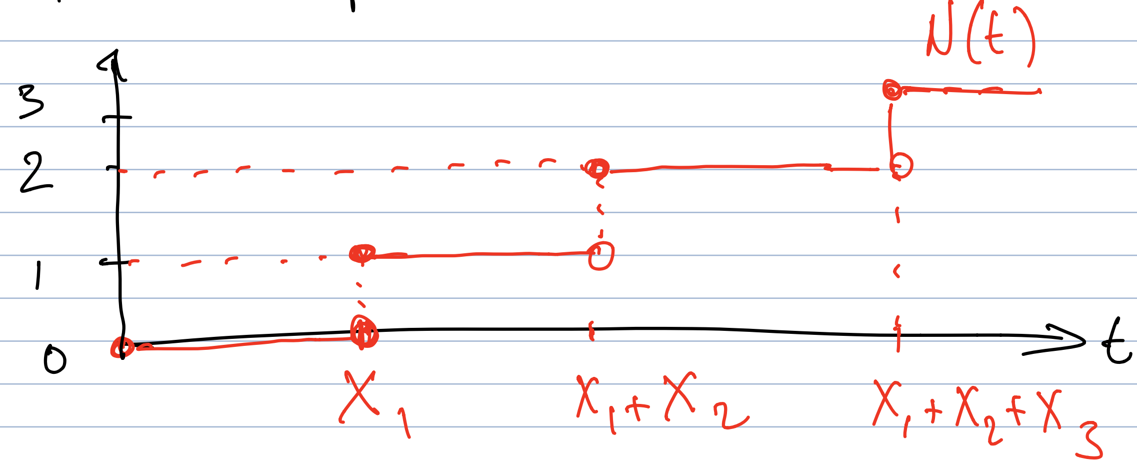
So by time t, we expect λt events to happen.

Def : The (homogeneous) Poisson process is defined by $N(t)$ = # occurences by time t

$$= \operatorname{Max}\left\{ n \mid \sum_{i=1}^{n} X_i \leq t \right\}$$

where $X_i \sim$ iid $\operatorname{Exp}(\lambda)$

Graphical interpretation



(By the remark, $\mathbb{E}(N(t)) = \underbrace{\lambda}_{\substack{\#\text{ occurrences} \\ \text{per unit of time.} \\ (\text{frequency})}} t$) ← time that has passed

We call $N(t)$ a Poisson process because:

<u>Thm</u>: $N(t) \sim \text{Poisson}(\lambda t)$, i.e.

for any $t > 0$ and $k = 0, 1, 2 \dots$

$$P(N(t) = k) = e^{-\lambda t} \frac{(\lambda t)^k}{k!}$$

<u>Proof</u>: Note that $N(t) \geq k \iff \sum_{i=1}^{k} X_i \leq t$

So using $\sum_{i=1}^{k} X_i \sim \text{Gamma}(k, \lambda)$,

$$P(N(t) \geq k) = P(\text{Gamma}(k, \lambda) \leq t)$$

$$= \int_0^t \lambda e^{-\lambda s} \frac{(\lambda s)^{k-1}}{(k-1)!} ds$$

$$= \frac{1}{(k-1)!} \int_0^{\lambda t} u^{k-1} e^{-u} \, du \quad (u = \lambda s)$$

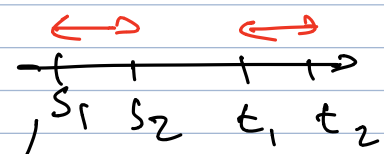
By integration by parts (exercise) one can show that

$$P(N(t)=k) = P(N(t) \geq k+1) - P(N(t) \geq k)$$

$$= e^{-\lambda t} \frac{(\lambda t)^k}{k!} \qquad \boxtimes$$

<u>Rmk</u>: $\mathbb{E}(N(t)) = Var(N(t)) = \lambda t$

Properties of $N(t)$



1) (independent increments)

for $s_1 < s_2 < t_1 < t_2$

$N(t_2) - N(t_1)$ and $N(s_2) - N(s_1)$ are independent.

(total number of occurences in disjoint intervals are independent)

2) (stationarity)

$N(t+s) - N(t)$ doesn't depend on t; it has the same distribution as $N(s) - N(0) = N(s)$.

<u>Rmk</u>: the proof relies on the memoryless property of the Exponential r.v.

<u>Exercises</u>: (let's $N(t)$ be a Poisson Process of intensity λ)

1) $P(N(2)=3 \mid N(1)=2) = ?$

2) $P(N(4)-N(3)=2 \mid N(1)=6) = ?$

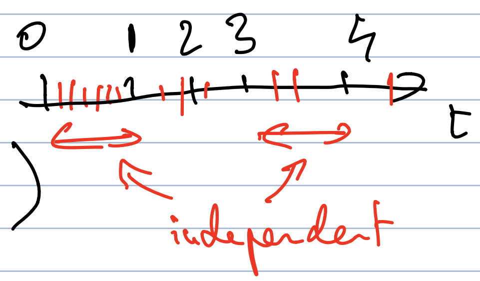
<u>Answer</u>: 1) $P(N(2)=3 \mid N(1)=2) = P(N(1)=1)$

(memoryless property)

$\left(P(N(t)=k) = e^{-\lambda t}\dfrac{(\lambda t)^k}{k!}\right)$

$= P(\text{Poisson}(\lambda)=1)$

$= \boxed{\lambda e^{-\lambda}}$



2) $P(N(4)-N(3)=2 \mid N(1)=6)$

$= P(N(4)-N(3))$

by independent increment property

$= P(N(1)=2)$

by the stationarity property

$= \boxed{\dfrac{\lambda^2 e^{-\lambda}}{2}}$