MATH 303 Midterm Exam2A Solution

Problem 1

See Notebook report.

Problem 2

We consider a population of cells, starting from one cell at day 0, and such that from one day to the next, each cell can independently die, remain as it is, or divide into two cells. The probabilities of these events are given by running the notebook.

The notebook will give the following probabilities:

- Case (a): $[P(\text{death}), P(\text{remain}), P(\text{divide})] = [0.1, 0.6, 0.3]$
- Case (b): $[P(\text{death}), P(\text{remain}), P(\text{divide})] = [0.1, 0.5, 0.4]$
- Case (c): $[P(\text{death}), P(\text{remain}), P(\text{divide})] = [0.2, 0.2, 0.6]$
- Case (d): $[P(\text{death}), P(\text{remain}), P(\text{divide})] = [0.2, 0.5, 0.3]$

a. Find the mean and generating function of the reproduction law of the associated branching process (justify your answers).

Solution: Let x, y, z denote the Probabilities of death, remain and divide, respectively. The mean μ and generating function $G(s)$ of the reproduction law are then given by

$$\begin{aligned} \mu &= y + 2z \\ G(s) &= x + ys + zs^2. \end{aligned}$$

For the different cases, solutions are:

- Case (a): $\mu = 1.2$ and $G(s) = 0.1 + 0.6s + 0.3s^2$,
- Case (b): $\mu = 1.3$ and $G(s) = 0.1 + 0.5s + 0.4s^2$,
- Case (c): $\mu = 1.4$ and $G(s) = 0.2 + 0.2s + 0.6s^2$,
- Case (d): $\mu = 1.1$ and $G(s) = 0.2 + 0.5s + 0.3s^2$.

b. Find the probability of extinction (justify your answer).

Solution: From class, the probability of extinction P_e is the smallest fixed point of the generating function in [0,1]. Solving $G(s) = s$ (detailed calculations required in the handwritten solution) yields:

- Case (a): $P_e = \frac{1}{3}$,
- Case (b): $P_e = \frac{1}{4}$,
- Case (c): $P_e = \frac{1}{3}$,

- Case (d): $P_e = \frac{2}{3}$.

c. Assume that from one day to the next, each cell which didn't divide is removed from the population with a probability p. Find the value of p such that the probability of extinction of the population is equal to the value displayed in the notebook (justify your answer).

The notebook will give the following values:

- Case (a): $P_e = \frac{2}{3}$,
- Case (b): $P_e = \frac{1}{2}$,
- Case (c): $P_e = \frac{1}{2}$,
- Case (d): $P_e = \frac{4}{5}$.

Solution: This new process is also a branching process, for which the generating function of the reproduction law is $G(s) = x + py + (1-p)ys + zs^2$. Let's call X the value given by the notebook. Since X is the probability of extinction, it satisfies $x + py + (1-p)yX + zX^2 = X$. Solving this equation for p yields for the different cases:

- Case (a): $p = \frac{X(1-y-zX)-x}{y-yX} = \frac{1}{6}$,
- Case (b): $p = \frac{1}{5}$,
- Case (c): $p = \frac{1}{2}$,
- Case (d): $p = \frac{2}{25}$.

Problem 3

Let $N(t)$ be a Poisson process of rate λ. Recall that for random variables X and Y, the *covariance* of X and Y is given by

$$\mathrm{Cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y].$$

a. For $0 < s < t$, express $N(t)$ in terms of $N(s)$ and some independent random variable(s).

Solution: Note that $N(t) = N(s) + N(t) - N(s)$. We have $N(t) - N(s) \sim \mathrm{Poisson}(\lambda(t-s))$, and it is independent of $N(s)$ by independence of increments.

b. Use your computation from part a to show that for $0 < s < t$,

$$\mathbb{E}[N(s)N(t)] = \lambda s + \lambda^2 st.$$

Solution: Let $Z = N(t) - N(s)$. Plug in and compute:

$$
\begin{aligned}
\mathbb{E}[N(s)N(t)] &= \mathbb{E}[N(s)^2 + N(s) \cdot Z] \\
&= \lambda^2 s^2 + \lambda s + \mathbb{E}[N(s)]\mathbb{E}[Z] \\
&= \lambda^2 s^2 + \lambda s + \lambda s \cdot \lambda(t-s) \\
&= \lambda^2 s^2 + \lambda s + \lambda^2 st - \lambda^2 s^2 \\
&= \lambda s + \lambda^2 st.
\end{aligned}
$$

Here we used the fact that for a Poisson(μ) random variable Y, $\mathbb{E}[Y] = \mu$, $\mathbb{E}[Y^2] = \mu^2 + \mu$, and the independence from part a.

c. Conclude that

$$\mathrm{Cov}(N(s), N(t)) = \lambda s.$$

Make sure to justify all reasoning.

Solution: Just plug into the covariance formula:

$$\mathrm{Cov}(N(s), N(t)) = \lambda s + \lambda^2 st - (\lambda s)(\lambda t) = \lambda s.$$

Problem 4

Jacob likes to collect pocket watches. Suppose he finds gold watches and silver watches at times according to two independent Poisson processes, with rate 2 per month for silver watches and 1 per month for gold watches. Run the notebook to display how many watches has collected after exactly a year (12 months).

a. Find the probability displayed in the notebook. (you can leave an answer that is calculator ready)

The notebook will give the following for the number of watches collected after a year and the probability to find:

- Case (a): Jacob collected 10 watches / Probability that 4 out of the ten watches are gold
- Case (b): Jacob collected 20 watches / Probability that 10 out of the 20 watches are gold
- Case (c): Jacob collected 15 watches / Probability that 10 out of the 15 watches are gold

Solution: Given that N watches were collected, the number of gold watches is distributed as Binomial $(N, 1/3)$.(We proved this in class as a corollary to the superposition principle.) Thus the probability that exactly m are gold is $\binom{N}{m}(\frac{1}{3})^m(\frac{2}{3})^{N-m}$

b. What is the probability that Jacob collected no watches during March? (Assume for simplicity that all months are the same length.)

Solution: The number of watches collected by time t is a Poisson process $N(t)$ with rate 3 per month. Conditionally on $N(12) = N$, the times when the watches were collected are independent uniform random variables on $(0, 12)$. Thus, the probability that a given watch was collected during March is $1/12$; so the probability that no watches were collected during march is $(11/12)^N$.