Math 302, PSET 5

(1) An evil mathematician has trapped you in a dungeon behind 5 doors. Every door is locked with a keypad, in which you must enter a number between 1 and 6000. You enter one random number (with replacement) into the keypad every second. The lock will open if you enter one of 15 special numbers the evil mathematician has selected for each lock, and once you open one lock, you move to the next door and keypad.

(a) Use an exponential random variable to approximate the probability that it'll take you longer than 25 minutes to open the first door.

(b) Use a Poisson random variable to approximate the probability that after one hour, you have escaped the dungeon.

(2) Let X be a Poisson random variable with unknown parameter λ.

(a) Which $n = n(\lambda) \geq 0$ is the most likely value of X, i.e. maximizes $\mathbb{P}(X = n)$?

(b) Suppose the experiment described by X has returned the value $n \geq 0$. Which parameter $\lambda = \lambda(n)$ maximizes $\mathbb{P}(X = n)$?

(3) For each of the sequences of random variables given, decide if it converges in distribution/probability/almost surely, and if so, identify the limiting distribution or random variable.

(a) $W_n \sim$ Uniform$(0, n)$ for $n \geq 1$.

(b) $X_n \sim$ independent Normal$(0, n^{-2})$ for $n \geq 1$.

(c) Let Y_1 be a Uniform$(0, 1)$ random variable, and for $n \geq 2$, define random variables Y_n on the same probability space by $Y_n = 1 - Y_{n-1}$.

(d) $Z_n \sim$ independent Bernoulli(n^{-1}) for $n \geq 1$.

(4) Let $X_1, X_2, \ldots$ be iid random variables with $X_1 \geq a$ almost surely for some $a > 0$ and $\mathbb{E}X_1 < \infty$.

(a) Use a method similar to the one used in class to show that

$$P_n = \left(\prod_{i=1}^{n} X_i\right)^{1/n}$$

converges almost surely to a constant random variable.

(b) Assuming $X_n \sim$ Geo$(1/2)$, find an expression for the limiting constant of the form

$$\prod_{n=1}^{\infty} a_n,$$

and compute its approximate value.

(5) Suppose X_n are non-negative iid random variables with $\mathbb{E}[X_1] > 0$. Show that $S_n = X_1 + X_2 + X_3 + \cdots + X_n$ converges in probability to ∞. (In other words, show that $\lim_{n\to\infty} \mathbb{P}(S_n > s) = 1$ for every $s > 0$.)

(6) Let X_n be iid Poisson(1) random variables, and let $Z_n = \frac{X_1+X_2+\cdots+X_n-n}{\sqrt{n}}$.

(a) Show that Z_n converges in distribution to a Normal$(0, 1)$ random variable by using the central limit theorem.

(b) Use your result from part (a) to show that

$$\lim_{n\to\infty} e^{-n} \sum_{k=0}^{n} \frac{n^k}{k!} = 1/2.$$

[Hint: identify the LHS as a Poisson probability, then use (a).]

(7) A surveyor is measuring the height of a cliff known to be about 1000 feet. She assumes her instrument is properly calibrated and that her measurement errors are independent, with mean $\mu = 0$ and variance $\sigma^2 = 10$. She plans to take n measurements and form the average. Estimate how large n should be to be 95% sure that the average falls within 1 foot of the true value
 (a) using Chebychev's inequality.
 (b) using the normal approximation.

Now estimate, in the same two ways, what value σ^2 should have if she wants to only make $n = 10$ measurements.

(8) A fair die is rolled 24 times. Use the CLT to estimate the probability that
 (a) The sum is greater than 89.
 (b) The sum is equal to 84.

Additional exercises: Anderson 7.41, 9.20, 9.21, 9.25, 9.27, 9.29, 9.31