MATH 303 Practice Problems - Week 9

March 19, 2021

Problem 1

Let $N(t)$ be a Poisson process of rate λ. For fixed t, $u \in [0,t]$, and $n = 1, 2, \ldots$, find the conditional distribution of $N(u)$ given $N(t) = n$, i.e. find a formula for $\mathbb{P}(N(u) = k | N(t) = n)$ for $k = 0, 1, 2, \ldots, n$.

Solution:

Assume that $t > 0$. By Bayes' rule we have

$$\begin{aligned} P(N(u) = k | N(t) = n) &= \frac{P(N(u) = k, N(t) = n)}{P(N(t) = n)} \\ &= \frac{P(N(t) = n | N(u) = k) P(N(u) = k)}{P(N(t) = n)}. \end{aligned}$$

Then, by the memoryless property we have

$$P(N(t) = n | N(u) = k) = P(N(t-u) = n-k),$$

and hence we obtain

$$\begin{aligned} P(N(u) = k | N(t) = n) &= \frac{P(N(t-u) = n-k) P(N(u) = k)}{P(N(t) = n)} \\ &= \frac{e^{-\lambda(t-u)} \frac{(\lambda(t-u))^{n-k}}{(n-k)!} e^{-\lambda u} \frac{(\lambda u)^k}{k!}}{e^{-\lambda t} \frac{(\lambda t)^n}{n!}} \\ &= e^{-\lambda(t-u+u-t)} \frac{n!}{(n-k)!k!} \frac{(t-u)^{n-k} u^k}{t^n} \\ &= e^{-\lambda(0)} \binom{n}{k} \frac{(t-u)^{n-k} u^k}{t^n} \\ &= \binom{n}{k} \frac{(t-u)^{n-k} u^k}{t^n} \\ &= \binom{n}{k} \left(\frac{u}{t}\right)^k \left(\frac{t-u}{t}\right)^{n-k} \end{aligned}$$

which we may recognise as the pmf of the Binomial distribution with parameters n and $\frac{t}{u}$. Rather than solving directly as we did above, we may solve this question more quickly using superposition and the independent increments property of the Poisson process:

Note that we may write

$$P(N(u) = k | N(t) = n) = P(N(u) = k | (N(t) - N(u)) + N(u) = n).$$

Since $(N(t) - N(u))$ and $N(u)$ are independent by the independent increments property, by superposition we have

$$(N(t) - N(u)) + N(u) \sim \text{Poisson}(\lambda(t-u+u)) = \text{Poisson}(\lambda t).$$

Hence, as we saw for this general set-up in class, the conditional distribution of $N(u)$ given $(N(t) - N(u)) + N(u) = n$ is Binomial$(n, \frac{u}{t})$, and we obtain the same formula for the conditional probability as above.

See the class notes for another alternative way to solve/understand this question which involves the arrival times of the Poisson process.

Problem 2

Let $N(t)$ be a Poisson process of rate λ. Given that $N(t) = 3$, determine the conditional distributions of the first three arrival times S_1, S_2, S_3.
Solution:

In class we showed the result that the conditional joint distribution of (S_1, S_2, S_3) is given by the joint distribution of a point process of ordered uniform r.v. on $[0, t]$. In particular, we have that the pdf of the conditional joint distribution is given by

$$f(s_1, s_2, s_3) = \frac{3!}{t^3}$$

for $0 < s_1 < s_2 < s_3 < t$. We may now find the conditional distributions for each of the three arrival times by marginalizing over the other two. We obtain:

$$\begin{aligned} f_{S_1}(s_1) &= \int_{s_1}^{t} \int_{s_1}^{s_3} \frac{3!}{t^3} \, \mathrm{d}s_2 \mathrm{d}s_3 \\ &= \frac{3!}{t^3}(\frac{1}{2}t^2 - s_1 t + \frac{1}{2}s_1^2) \end{aligned}$$

$$\begin{aligned} f_{S_2}(s_2) &= \int_{s_2}^{t} \int_{0}^{s_2} \frac{3!}{t^3} \, \mathrm{d}s_1 \mathrm{d}s_3 \\ &= \frac{3!}{t^3} s_2 (t - s_2) \end{aligned}$$

$$\begin{aligned} f_{S_3}(s_3) &= \int_{0}^{s_3} \int_{0}^{s_2} \frac{3!}{t^3} \, \mathrm{d}s_1 \mathrm{d}s_2 \\ &= \frac{3!}{t^3} \frac{1}{2} s_3^2 \\ &= \frac{3 s_3^2}{t^3} \end{aligned}$$

Problem 3

Customers arrive at a theme park according to a Poisson process $N(t)$ of rate λ. Each customer pays \$1 on arrival. At time t, the *discounted value* of the total sum collected so far is

$$D_t = \sum_{i=1}^{N(t)} e^{-\beta S_i},$$

where S_i is the ith arrival time, and $\beta > 0$ is the discount rate. Compute $\mathbb{E}D_t$.

Solution:

By the law of total probability,

$$\begin{aligned}
\mathbb{E}[D_t] &= \mathbb{E}\left[\sum_{i=1}^{N(t)} e^{-\beta S_i}\right], \\
&= \sum_{n=0}^{\infty} \mathbb{E}\left[\sum_{i=1}^{N(t)} e^{-\beta S_i} | N(t) = n\right] P(N(t) = n)
\end{aligned}$$

Let $U_1, U_2, \ldots, U_n$ be i.i.d Uniform($(0,t]$) random variables. Then since the conditional distribution of the first n arrival times is the same as the distribution of $(U_1, U_2, \ldots, U_n)$, we have

$$\begin{aligned}
\mathbb{E}\left[\sum_{i=1}^{N(t)} e^{-\beta S_i} | N(t) = n\right] &= \mathbb{E}\left[\sum_{i=1}^{n} e^{-\beta U_i}\right] \\
&= \sum_{i=1}^{n} \mathbb{E}[e^{-\beta U_i}] \\
&= \sum_{i=1}^{n} \int_0^t e^{-\beta u} \frac{1}{t} \mathrm{d}t \\
&= \frac{n}{\beta t}(1 - e^{-\beta t}).
\end{aligned}$$

Hence we obtain that the expectation of $D(t)$ is given by

$$\begin{aligned}
\mathbb{E}[D_t] &= \sum_{n=0}^{\infty} \frac{n}{\beta t}(1 - e^{-\beta t}) e^{-\lambda t} \frac{(\lambda t)^n}{n!} \\
&= (1 - e^{-\beta t}) e^{-\lambda t} \frac{\lambda}{\beta} \sum_{n=1}^{\infty} \frac{(\lambda t)^{n-1}}{(n-1)!} \\
&= (1 - e^{-\beta t}) e^{-\lambda t} \frac{\lambda}{\beta} e^{\lambda t} \\
&= \frac{\lambda}{\beta}(1 - e^{-\beta t})
\end{aligned}$$

Problem 4

Alpha particles are emitted by a radioactive source according to a Poisson process of rate λ. Each alpha particle independently survives for a random amount of time and then is annihilated. The lifetimes $Y_1, Y_2, \ldots$ of the particles have common distribution function $G(y) = \mathbb{P}(Y_k \leq y)$. Let $M(t)$ denote the number of alpha particles in existence at time t.

a. Determine the distribution of $M(t)$.

Solution:

Let us denote the number of particles created by the emission process by time t as $N(t) \sim Poisson(\lambda t)$ where the distribution is determined by the Poisson process assumption. Let $0 \leq S_1, S_2, \dots, S_n \leq t$ be the emission times (i.e. arrival times) of the first n. Then a particle k exists at time t if and only if

$$S_k + Y_k \geq t.$$

With this observation we see that we may express $M(t)$ as:

$$M(t) = \sum_{k=0}^{N(t)} \mathbf{1}_{S_k+Y_k \geq t}.$$

Additionally, by the law of total probability we have, for any $m \in \mathbb{N}$,

$$P(M(t) = m) = \sum_{n=0}^{\infty} P(M(t) = m|N(t) = n)P(N(t) = n).$$

Let $U_1, U_2, \dots, U_n$ be i.i.d Uniform($(0,t]$) random variables. Then since the conditional distribution of the first n arrival times is the same as the distribution of the ordered uniform random variables $(U_{(1)}, U_{(2)}, \dots, U_{(n)})$, where $U_{(i)}$ denotes the ith smallest value out of $U_1, U_2, \dots, U_n$, we obtain the following result for the conditional probability of $M(t)$ given $N(t) = n$

$$\begin{aligned}
P(M(t) = m|N(t) = n) &= P(\sum_{k=0}^{N(t)} \mathbf{1}_{S_k+Y_k \geq t} = m|N(t) = n) \\
&= P(\sum_{k=0}^{n} \mathbf{1}_{U_k+Y_k \geq t} = m) \\
&= \binom{n}{m} P(U_k + Y_k \geq t)^m (1 - P(U_k + Y_k \geq t))^{n-m}
\end{aligned}$$

where we recognize the binomial distribution with parameter $p = P(U_k + Y_k \geq t)$. This probability p is given by

$$\begin{aligned}
p &= \int_0^t P(Y_k \geq t - U_k|U_k = u)P(U_k = u)\mathrm{d}u \\
&= \frac{1}{t}\int_0^t P(Y_k \geq t - u)\mathrm{d}u \\
&= \frac{1}{t}\int_0^t (1 - P(Y_k \leq t - u))\mathrm{d}u \\
&= \frac{1}{t}\int_0^t (1 - G(t - u))\mathrm{d}u \\
&= \frac{1}{t}\int_0^t (1 - G(y))\mathrm{d}y
\end{aligned}$$

Recognizing the pdf for the binomial, we have found $M(t)|N(t) = n \sim \text{Binomial}(n, \frac{\mu}{t})$. Returning to the distribution of $M(t)$ we now have

$$\begin{aligned}
P(M(t) = m) &= \sum_{n=0}^{\infty} P(M(t) = m|N(t) = n)P(N(t) = n) \\
&= \sum_{n=0}^{\infty} \binom{n}{m} p^m (1-p)^{n-m} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \\
&= e^{-\lambda t} p^m \sum_{n=0}^{\infty} \frac{n!}{(n-m)!m!} (1-p)^{n-m} \frac{(\lambda t)^n}{n!} \\
&= e^{-\lambda t} p^m \frac{(\lambda t)^m}{m!} \sum_{n=0}^{\infty} \frac{1}{(n-m)!} (\lambda t)^{n-m} (1-p)^{n-m} \\
&= e^{-\lambda t} \frac{(p\lambda t)^m}{m!} e^{\lambda t(1-p)} \\
&= e^{-p\lambda t} \frac{(p\lambda t)^m}{m!},
\end{aligned}$$

i.e. the number of alpha particles in existence at time t follows a Poisson distribution:

$$M(t) \sim \text{Poisson}(p\lambda t).$$

b. Show that as $t \to \infty$, the distribution you found in part a converges to Poisson$(\lambda\mu)$, where $\mu = \mathbb{E}Y$ is the mean lifetime of an alpha particle.

Solution:

Recall that we found in (a) that $M(t) \sim \text{Poisson}(p\lambda t)$ where

$$p = \frac{1}{t} \int_0^t (1 - G(y))\mathrm{d}y.$$

Then as $t \to \infty$ we have that the parameter of the distribution converges to $\lambda\mu$ since

$$\lim_{t\to\infty} \lambda p t = \lambda \int_0^{\infty} (1 - G(y))\mathrm{d}y = \lambda\mu$$

where we recognise the integral as the mean lifetime of an alpha particle $\mu = \mathbb{E}Y$. Hence the distribution converges to Poisson$(\lambda\mu)$.