Math 302, PSET 1 Solutions

(1) Ex 1.26
Solution:
(a) $\frac{\binom{10}{2}\binom{5}{2}}{\binom{15}{4}}$
(b) $\frac{\binom{13}{2}}{\binom{15}{4}}$
(c) $\frac{\binom{13}{3}}{\binom{15}{4}}$

(2) Ex 1.28
Solution:
(a) $\frac{m(m-1)+n(n-1)}{(m+n)(m+n-1)}$. Verify that this can be expressed as a fraction of combinations!
(b) $\frac{m^2+n^2}{(m+n)^2}$
(c) This is equivalent to

$$\frac{m(m-1)+n(n-1)}{(m+n)(m+n-1)} = \frac{(m^2+n^2)-(m+n)}{(m+n)^2-(m+n)} < \frac{m^2+n^2}{(m+n)^2},$$

which holds for all $m, n \in \mathbb{N}$. Note that when you replace a ball, you are more likely to choose another ball of the same color (versus if you did not replace the ball)

(3) Ex 1.30
Solution: There are 8 rooks and 64 squares, so there are $\binom{64}{8}$ ways to position the rooks. A valid positioning of the rooks requires that each row and column has exactly one rook. We can enumerate the possibilities by choosing a rook anywhere in the first row, then a rook anywhere in the second row *except* the one column occpupied by the first rook, and so on. Thus there are $8 \cdot 7 \cdot 6 \cdots 1 = 8!$ ways to position the rooks so they are mutually non-attacking, giving an answer of $\frac{8!}{\binom{64}{8}}$

(4) Ex 1.42
Solution: Note that $\mathbb{P}(A \cup B) \leq 1$ since $A \cup B$ is an event in the probability space. By the inclusion-exclusion principle,

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cup B) \geq .8 + .5 - 1 = .3$$

(5) Ex 1.50
Solution:
(a) Let $E = \{5 \text{ tails in a row eventually occurs}\}$. Then

$$E^c = \{5 \text{ tails in a row never occurs}\},$$

and it suffices to show that $P(E^c) = 0$. For each $n \in \mathbb{N}$, define the event

$F_n = \{\text{flips } 5n+1, 5n+2, 5n+3, 5n+4, 5n+5 \text{ are not all tails}\}$.

Note that $\mathbb{P}(F_n) = 1 - (1/2)^5 = 31/32$, and $E^c \subset F_n$ for each n. Since the events F_n are independent, for any n,

$P(E^c) \leq \mathbb{P}(F_1 \cap F_2 \cap \cdots \cap F_n) = (31/32)^n \to 0$ as $n \to \infty$.

Thus $\mathbb{P}(E^c) = 0$.

(b) Let X_n denote the outcome of the nth flip. Apply the same argument as in part (a), but with

$$F_n = \{X_{nr+1} = a_1, X_{nr+2} = a_2, \cdots, X_{nr+r} = a_r\},$$

noting that $\mathbb{P}(F_n) = 1 - 2^{-r}$, and for fixed r, $(1-2^{-r})^n \to 0$ as $n \to \infty$.

(6) Ex 1.52

Solution: Let A_i be the event that couple i sits together. Note that for $i = 1, 2, 3$, $|A_i| = 6 \cdot 2 \cdot 4! = 288$, since we can first choose the wife (6 spots), then choose the husband (in one of the two adjacent spots), then seat the remaining 4 guests. Similarly, for distinct $i, j \in \{1, 2, 3\}$,

$$|A_i \cap A_j| = 6 \cdot 2 \cdot (2 \cdot 1 \cdot 2! + 2 \cdot 2 \cdot 2!) = 144.$$

Seating the first couple is the same (12 choices). The second couple is a bit tricky now, because if we seat the wife in the second couple first, and she sits next to someone from the first couple, we only have one choice for her husband; otherwise, he has two possible seats. Finally,

$$|A_1 \cap A_2 \cap A_3| = 6 \cdot 2 \cdot (4 \cdot 2!) = 96$$

The desired event is $A_1 \cup A_2 \cup A_3$, which by inclusion-exclusion, has size

$$|A_1 \cup A_2 \cup A_3| = 3 * 288 - 3 * 144 + 96 = 528.$$

Thus $\mathbb{P}(A_1 \cup A_2 \cup A_3) = 528/6! = 11/15$.

(7) Ex 1.56

Solution:

(a) Let E be the event that every box contains at least one ball, and let E_i be the event that the ith box contains a ball. Then

$$E = \bigcap_{i=1}^{n} E_i$$

By De Morgan's Law,

$$\mathbb{P}(E) = 1 - \mathbb{P}\left(\bigcup_{i=1}^{n} E_i{}^c\right).$$

Note that for each distinct subset of indices $i_1, i_2, \ldots, i_j \in \{1, 2, \ldots, n\}$,

$$\mathbb{P}(E_{i_1}^c \cap E_{i_2}^c \cap \ldots \cap E_{i_j}^c) = \left(\frac{n-j}{n}\right)^k,$$

since on this event, the k balls can go into any of the $n-j$ bins *not* labeled $i_1, \ldots, i_j$. Applying the inclusion-exclusion formula yields

$$\begin{aligned}
\mathbb{P}(E) &= 1 - \frac{1}{n^k} \sum_{j=1}^{n} (-1)^{j-1} \binom{n}{j} (n-j)^k \\
&= 1 - \frac{(-1)^n}{n^k} \sum_{j=0}^{n-1} (-1)^{j-1} j^k \binom{n}{j} \\
&= \frac{(-1)^{n-1}}{n^k} \sum_{j=1}^{n} (-1)^{j-1} j^k \binom{n}{j}.
\end{aligned}$$

Here we (in order): used the fact that $\binom{n}{j} = \binom{n}{n-j}$, replaced j by $n-j$ (reversing the order of the sum), added and subtracted the $j = n$ term, and did some algebra.

(b) Note that if $k < n$ then $\mathbb{P}(E) = 0$ since there aren't enough balls to fill each box, while if $k = n$, then exactly one ball goes in each box, so in that case $\mathbb{P}(E) = \frac{n!}{n^n}$. Plugging this into the previous equation, we obtain the nice formulas

$$\sum_{j=1}^{n} (-1)^{j-1} j^n \binom{n}{j} = \begin{cases} (-1)^{n-1} n!, & k = n \\ 0, & k < n \end{cases}$$

(8) Ex 1.57

Solution: This domino tiling setup is counted by Fibonacci numbers. Let a_n be the number of such tilings on the $2 \times n$ grid G_n. To obtain a tiling of G_n by dominos, the rightmost tile must be vertical, or the rightmost two tiles are both horizontal. The number of tilings in the first case is exactly a_{n-1}, and the number in the second case is exactly a_{n-2} (as long as $n \geq 3$). Thus $a_n = a_{n-1} + a_{n-2}$ with $a_1 = 1, a_2 = 2$, so a_n are the Fibonacci numbers.

(a) The total number of configurations is $a_9 = 55$, so the probability of obtaining the 'all vertical' tiling is $1/55$.

(b) If the middle tile is vertical, then we are left with two disjoint tilings of 2×4 grids on the left and right. The number of domino tilings of G_4 is $a_4 = 5$, so the total number of tilings with the center tile vertical is $5 \times 5 = 25$. Thus the probability of the middle tile being vertical is $5 \times 5/55 = 25/55$.

Aside: Note that this is slightly larger than $1/2$. Intuitively, if n is large, any given tile should be vertical or horizontal with probability roughly $1/2$.

Challenge problem: Suppose we tile the grid G_n, i.e. the $2 \times n$ grid. Show that for any $k \in \mathbb{N}$, the probability that the kth tile is vertical converges, and compute the limit.