MATH 303 Practice Problems Solutions - Week 1 *January 15, 2021*

Problem 1

1. As suggested, we proceed by induction. We denote by Q_n the following matrix, indexed by n which is our candidate n-step transition matrix:

$$Q_n := \frac{1}{2}\begin{pmatrix} 1+(2p-1)^n & 1-(2p-1)^n \\ 1-(2p-1)^n & 1+(2p-1)^n \end{pmatrix}.$$

The transition matrix $\widetilde{P}$ of the Markov chain writes

$$\widetilde{P} = \begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix}.$$

For $n = 1$, it is easy to check that Q_1 is equal to $\widetilde{P}$. We know from the lectures that the n-step transition matrix is given by $\widetilde{P}^n$.
Let us assume as our induction hypothesis that $Q_n = \widetilde{P}^n$, and prove that $\widetilde{P}^{n+1} = Q_{n+1}$:

$$\begin{aligned}
\widetilde{P}^{(n+1)} &= \widetilde{P}^{n+1} \quad \text{(using the Chapman-Kolmogorov eq.)} \\
&= \widetilde{P}^n\widetilde{P} \quad \text{(using the induction hypothesis)} \\
&= Q_n\widetilde{P} \\
&= \frac{1}{2}\begin{pmatrix} 1+(2p-1)^n & 1-(2p-1)^n \\ 1-(2p-1)^n & 1+(2p-1)^n \end{pmatrix}\begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} p(1+(2p-1)^n)+(1-p)(1-(2p-1)^n) & (1-p)(1+(2p-1)^n)+p(1-(2p-1)^n) \\ p(1-(2p-1)^n)+(1-p)(1+(2p-1)^n) & (1-p)(1-(2p-1)^n)+p(1+(2p-1)^n) \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} 1+(2p-1)^{n+1} & 1-(2p-1)^{n+1} \\ 1-(2p-1)^{n+1} & 1+(2p-1)^{n+1} \end{pmatrix} \\
&= Q_{n+1}.
\end{aligned}$$

Remark: For those familiar with matrix decomposition and symmetric real matrices, it is possible to show the result directly after diagonalizing $\widetilde{P}$.

2. Since $0 < p < 1$, we have that $-1 < 2p-1 < 1$, so $(2p-1)^n \to 0$ as n goes to infinity, and using question 1,

$$\lim_{n\to\infty} \widetilde{P}^n = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

For any given initial distribution $\mu_0 = (P(X_0 = 0), P(x_0 = 1))$, $(P(X_n = 0), P(x_n = 1)) = \mu_0 \widetilde{P}^n$,

$$\text{so} \lim_{n\to\infty} (P(X_n = 0), P(X_n = 1)) = \lim_{n\to\infty} \mu_0 P^n$$
$$= \frac{1}{2}\mu_0 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$
$$= \frac{1}{2}(\mu_0(0) + \mu_0(1), \mu_0(0) + \mu_0(1))$$
$$= \left(\frac{1}{2}, \frac{1}{2}\right) \quad (\ \mu_0(0) + \mu_0(1) = 1 \text{ since } \mu_0 \text{ is a probability distribution}),$$

and this is exactly the statement that $(X_n)_{n\geq 0}$ converges in law to the uniform distribution over $\{0, 1\}$ for any initial distribution.

Problem 2

We consider the random walk $(X_n)_{n\geq 0}$ on $\{-1, 0, 1\}$ such that at each step, the walker moves from 0 to -1 or 1 with equal probability, and from -1 and 1, the walker moves to 0 only.
1.

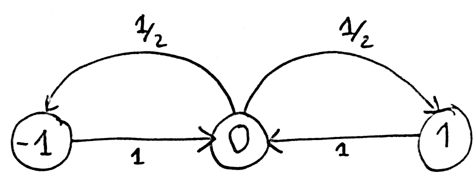


Figure 1: Transition diagram of the Markov chain in Problem 2.

The transition matrix $\widetilde{P}$ of $(X_n)_{n\geq 0}$ is

$$\widetilde{P} := \frac{1}{2} \begin{pmatrix} 0 & 2 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & 0 \end{pmatrix}.$$

(*Remark*: We use the natural convention that the first row corresponds to the transition probabilities from state -1 to states $-1, 0, 1$ in this order, the second row corresponds to the transition probabilities from -0 to states $-1, 0, 1$ in this order, and the third corresponds row to the transition probabilities from state 1 to states $-1, 0, 1$ in this order.)

2. The 2- and 3-step transition matrix are given by $\widetilde{P}^{(2)} = \widetilde{P}^2$ and $\widetilde{P}^{(3)} = \widetilde{P}^3$ respectively. Their expression is, after matrix multiplication,

$$\widetilde{P}^2 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix},$$

$$\widetilde{P}^3 = \widetilde{P}^2\widetilde{P} = \frac{1}{2}\begin{pmatrix} 0 & 2 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & 0 \end{pmatrix}.$$

We notice that $\widetilde{P}^3 = \widetilde{P}$. If n is odd, $n = 2k+1$ with $k \in \mathbb{N}$ and so

$$\widetilde{P}^{2k+1} = \widetilde{P}^{2k-2}\widetilde{P}^3 = \widetilde{P}^{2k-2}\widetilde{P} = \widetilde{P}^{2k-1} = \cdots = \widetilde{P}.$$

Similarly, we obtain for $n = 2k$ even,

$$\widetilde{P}^{2k} = \widetilde{P}^{2k-3}\widetilde{P}^3 = \widetilde{P}^{2k-3}\widetilde{P} = \widetilde{P}^{2(k-1)} = \cdots = \widetilde{P}^2.$$

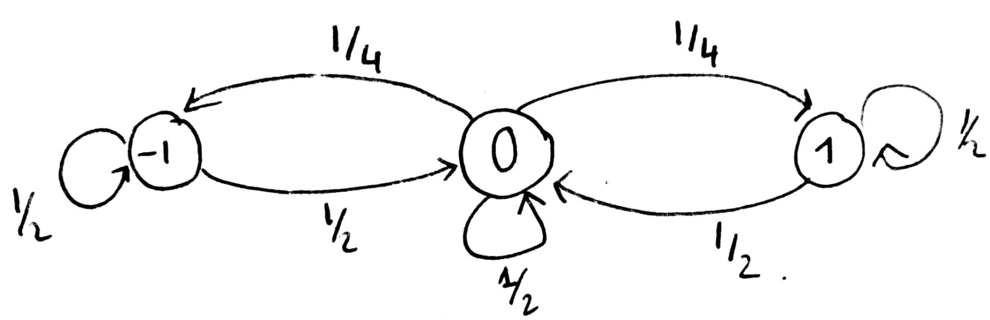
In conclusion,

$$\widetilde{P}^n = \begin{cases} \widetilde{P}^2 & \text{if } n \text{ is even,} \\ \widetilde{P} & \text{if } n \text{ is odd.} \end{cases}$$

3. The new transition matrix that we keep denoting $\widetilde{P}$ is

$$\widetilde{P} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 0 \\ \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & 1 & 1 \end{pmatrix},$$

and the transition diagram is



4. We proceed by induction. We see from Question 3 that the formula is correct for $n = 1$. We denote our candidate n-step transition matrix by Q_n, as

$$Q_n := \frac{1}{2}\begin{pmatrix} \frac{2^{n-1}+1}{2^n} & 1 & \frac{2^{n-1}-1}{2^n} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{2^{n-1}-1}{2^n} & 1 & \frac{2^{n-1}+1}{2^n} \end{pmatrix}.$$

Let us assume the induction hypothesis that $\widetilde{P}^{(n)} = Q_n$, and prove that this implies $\widetilde{P}^{(n+1)} = Q_{n+1}$. As previously in Problem 1:

$$
\begin{aligned}
\widetilde{P}^{(n+1)} &= \widetilde{P}^{n+1} \\
&= \widetilde{P}^{n}\widetilde{P} \\
&= Q_n\widetilde{P} \\
&= \frac{1}{4}\begin{pmatrix} \frac{2^{n-1}+1}{2^n} & 1 & \frac{2^{n-1}-1}{2^n} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{2^{n-1}-1}{2^n} & 1 & \frac{2^{n-1}+1}{2^n} \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & 1 & 1 \end{pmatrix} \\
&= \frac{1}{4}\begin{pmatrix} \frac{2^{n-1}+1}{2^n} + \frac{1}{2} & \frac{2^{n-1}+1}{2^n} + 1 + \frac{2^{n-1}-1}{2^n} & \frac{2^{n-1}-1}{2^n} + \frac{1}{2} \\ \frac{1}{2} + \frac{1}{2} & \frac{1}{2} + 1 + \frac{1}{2} & \frac{1}{2} + \frac{1}{2} \\ \frac{2^{n-1}-1}{2^n} + \frac{1}{2} & \frac{2^{n-1}+1}{2^n} + 1 + \frac{2^{n-1}-1}{2^n} & \frac{2^{n-1}+1}{2^n} + \frac{1}{2} \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} \frac{2^n+1}{2^{n+1}} & 1 & \frac{2^n-1}{2^{n+1}} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{2^n-1}{2^{n+1}} & 1 & \frac{2^n+1}{2^{n+1}} \end{pmatrix} \\
&= Q_{n+1}
\end{aligned}
$$

5. We assume that Y_n^2 is a Markov chain. Since Y_n takes value in $\{-1, 0, 1\}$, Y_n^2 must take values in $\{0, 1\}$ only. The transition probabilities are

$$
\begin{aligned}
P(Y_{n+1}^2 = 0 \mid Y_n^2 = 0) &= P(Y_{n+1} = 0 \mid Y_n = 0) \\
&= \frac{1}{2}. \\
P(Y_{n+1}^2 = 0 \mid Y_n^2 = 1) &= P(Y_{n+1} = 0 \mid \{Y_n = 1\} \cup \{Y_n = -1\}) \\
&= \frac{P(\{Y_{n+1} = 0\} \cap (\{Y_n = 1\} \cup \{Y_n = -1\}))}{P(\{Y_n = 1\} \cup \{Y_n = -1\})} \\
&= \frac{P(\{Y_{n+1} = 0\} \cap \{Y_n = 1\}) + P(\{Y_{n+1} = 0\} \cap \{Y_n = -1\}))}{P(\{Y_n = 1\} \cup \{Y_n = -1\})} \\
&= \frac{\frac{1}{2}P(Y_n = 1) + \frac{1}{2}P(Y_n = -1)}{P(Y_n = 1) + P(Y_n = -1)} \\
&= \frac{1}{2}
\end{aligned}
$$

These two transition probabilities are enough since we assume that Y_n^2 is a Markov chain, and we complete the stochastic transition matrix for the Markov chain $(Y_n^2)_{n\geq 0}$, as

$$
\widetilde{P} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.
$$

We recognize the set-up of Problem 1 in the special case $p = \frac{1}{2}$ and we can therefore apply its results to deduce that $(Y_n^2)_{n\geq 0}$ converges in law to the uniform distribution over $\{0, 1\}$ for any given initial distribution.

(*Remark:* You can try to prove that $(Y_n^2)_{n\geq 0}$ is a Markov chain)

Problem 3

1. This event is satisfied if and only if, either you *first* pick a blue ball (with probability $\frac{1}{2}$) and *then* pick another blue ball (with probability $\frac{1}{3}$) or you *first* pick a red ball (with probability $\frac{1}{2}$) and *then* pick another red ball (with probability $\frac{1}{3}$). Therefore the probability of picking two balls of the same color is equal to $2 \times \frac{1}{2} \times \frac{1}{3} = \frac{1}{3}$.

2. Let X_n be the Markov chain with state-space $S = \{0, 1, 2\}$ which reflects how close we are to have two balls of the same color drawn two times in a row. We adopt the convention that, letting $s \in S$,

- if $s = 0$ we need two more draws;
- if $s = 1$ we need one more draw;
- if $s = 2$ we do not need any more draw.

The Markov chain's initial value satisfies $P(X_0 = 0) = 1$. In other terms the initial distribution is $(1\ 0\ 0)$. This is indeed a Markov chain since the successive drawings are independant. Using the first question, the transition matrix $\widetilde{P}$ of this Markov chain is given by:

$$\widetilde{P} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & 0 \\ \frac{2}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 1 \end{pmatrix}.$$

We are looking for the last coordinate ($s = 2$) of the vector $(100)\widetilde{P}^4$. So it only remain to compute $\widetilde{P}^4$.

$$\widetilde{P}^2 = \begin{pmatrix} \frac{2}{3} & \frac{2}{9} & \frac{1}{9} \\ \frac{4}{9} & \frac{2}{9} & \frac{1}{3} \\ 0 & 0 & 1 \end{pmatrix}$$

$$\widetilde{P}^4 = \begin{pmatrix} \frac{44}{81} & \frac{16}{81} & \frac{22}{81} \\ \frac{32}{81} & \frac{12}{81} & \frac{37}{81} \\ 0 & 0 & 1 \end{pmatrix}$$

And thus the probability that we are looking for is equal to $\frac{7}{27}$.

Problem 4 (Jupyter Notebook)

See the notebook.