Math 308 E - Spring 2018

Midterm exam 2

Friday, May 18th, 2018

Name: ______________________________

| | | |
|---|---|---|
| Problem 1 | 15 | |
| Problem 2 | 10 | |
| Problem 3 | 15 | |
| Problem 4 | 10 | |
| Total | 50 | |

- There are 4 questions on this exam. Make sure you have all four.
- Always explain your reasoning clearly and concisely.
- Any student found engaging in academic misconduct will receive a score of 0 on this exam.
- You have 50 minutes to complete the exam.

GOOD LUCK!

1. (15 points) Consider the matrices

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & -2 & 0 \\ 2 & 3 & 4 \\ 0 & 7 & 4 \end{bmatrix}$$

(a) (9 points) For both $A$ and $B$, find the inverse or explain why it doesn't exist.

(b) (3 points) Does there exist a $3 \times 3$ matrix $C$ such that $\det(AC) = 8$? Give an example, or explain why this can't happen.

(c) (3 points) Does there exist a $3 \times 3$ matrix $D$ such that $\det(BD) = -1$? Give an example, or explain why this can't happen.

2. (10 points) Consider the following set of vectors in $\mathbb{R}^4$:

$$\mathcal{B} = \left\{ v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, v_4 = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

(a) (5 points) Prove that $\mathcal{B}$ is a basis for $\mathbb{R}^4$. (Hint: it helps to show that $e_1, e_2, e_3, e_4 \in \text{span}(\mathcal{B})$.)

(b) (5 points) Suppose $v \in \mathbb{R}^4$ is a vector satisfying $[v]_{\mathcal{B}} = \begin{bmatrix} 4 \\ 2 \\ 0 \\ -1 \end{bmatrix}$. Find $v$.

3. (15 points) Let $A$ be any $2 \times 6$ matrix, and let $B$ be any $6 \times 2$ matrix.

(a) (5 points) Is the set of vectors $v \in \mathbb{R}^2$ satisfying $(AB)^{51}v = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ a subspace of $\mathbb{R}^2$? Explain.

(b) (5 points) Assume $AB$ is invertible (for part (b) only). Is the set of vectors $v \in \mathbb{R}^2$ satisfying $((AB)^{-1})^3 v = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$ a subspace of $\mathbb{R}^2$? Explain.

(c) (5 points) Suppose that the nullity of $A$ is 4, and the rank of $B$ is 2. What are the dimensions of the row spaces of $A$ and $B$?

(d) (Extra credit, 3 points) Suppose that, in addition to the assumptions in part (c),

$$Null(A) \cap Range(B) = \{0\}.$$

($\cap$ is notation for the intersection of sets.) What is the rank of $AB$?

4. (10 points) Let $\pi : \mathbb{R}^3 \to \mathbb{R}^3$ be the projection map

$$\pi\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix},$$

and let $R : \mathbb{R}^3 \to \mathbb{R}^3$ be the rotation map

$$R\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \frac{\sqrt{2}}{2}\begin{bmatrix} x - y \\ x + y \\ z \end{bmatrix}.$$

(a) (2 points) Find the matrix representing the linear transformation $T = R \circ \pi$. What is the rank of $T$?

(b) (4 points) Find a basis for the kernel of $\pi$.

(c) (4 points) Show that the set of all vectors $v \in \mathbb{R}^3$ such that $Rv = v$ is a subspace of $\mathbb{R}^3$, and find a basis for it.