17. If we divide R into mn subrectangles, $\iint_R k\,dA \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f\left(x_{ij}^*, y_{ij}^*\right)\Delta A$ for any choice of sample points $\left(x_{ij}^*, y_{ij}^*\right)$.

But $f\left(x_{ij}^*, y_{ij}^*\right) = k$ always and $\sum_{i=1}^{m}\sum_{j=1}^{n}\Delta A = \text{area of } R = (b-a)(d-c)$. Thus, no matter how we choose the sample points, $\sum_{i=1}^{m}\sum_{j=1}^{n} f\left(x_{ij}^*, y_{ij}^*\right)\Delta A = k\sum_{i=1}^{m}\sum_{j=1}^{n}\Delta A = k(b-a)(d-c)$ and so

$\iint_R k\,dA = \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f\left(x_{ij}^*, y_{ij}^*\right)\Delta A = \lim_{m,n\to\infty} k\sum_{i=1}^{m}\sum_{j=1}^{n}\Delta A = \lim_{m,n\to\infty} k(b-a)(d-c) = k(b-a)(d-c)$.

18. Because $\sin\pi x$ is an increasing function for $0 \le x \le \frac{1}{4}$, we have $\sin 0 \le \sin\pi x \le \sin\frac{\pi}{4} \Rightarrow 0 \le \sin\pi x \le \frac{\sqrt{2}}{2}$. Similarly, $\cos\pi y$ is a decreasing function for $\frac{1}{4} \le y \le \frac{1}{2}$, so $0 = \cos\frac{\pi}{2} \le \cos\pi y \le \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$. Thus on R, $0 \le \sin\pi x\,\cos\pi y \le \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2}}{2} = \frac{1}{2}$. Property (9) gives $\iint_R 0\,dA \le \iint_R \sin\pi x\cos\pi y\,dA \le \iint_R \frac{1}{2}\,dA$, so by Exercise 17 we have $0 \le \iint_R \sin\pi x\,\cos\pi y\,dA \le \frac{1}{2}\left(\frac{1}{4}-0\right)\left(\frac{1}{2}-\frac{1}{4}\right) = \frac{1}{32}$.

15.2 Iterated Integrals

1. $\int_0^5 12x^2y^3\,dx = \left[12\,\frac{x^3}{3}\,y^3\right]_{x=0}^{x=5} = 4x^3y^3\Big]_{x=0}^{x=5} = 4(5)^3y^3 - 4(0)^3y^3 = 500y^3$,

$\int_0^1 12x^2y^3\,dy = \left[12x^2\,\frac{y^4}{4}\right]_{y=0}^{y=1} = 3x^2y^4\Big]_{y=0}^{y=1} = 3x^2(1)^4 - 3x^2(0)^4 = 3x^2$

2. $\int_0^5 (y + xe^y)\,dx = \left[xy + \frac{x^2}{2}e^y\right]_{x=0}^{x=5} = \left(5y + \frac{25}{2}e^y\right) - (0+0) = 5y + \frac{25}{2}e^y$,

$\int_0^1 (y + xe^y)\,dy = \left[\frac{y^2}{2} + xe^y\right]_{y=0}^{y=1} = \left(\frac{1}{2} + xe^1\right) - \left(0 + xe^0\right) = \frac{1}{2} + ex - x$

3. $\int_1^4\int_0^2 (6x^2y - 2x)\,dy\,dx = \int_1^4 \left[3x^2y^2 - 2xy\right]_{y=0}^{y=2}\,dx = \int_1^4 (12x^2 - 4x)\,dx = \left[4x^3 - 2x^2\right]_1^4 = (256-32) - (4-2) = 222$

4. $\int_0^1\int_1^2 (4x^3 - 9x^2y^2)\,dy\,dx = \int_0^1 \left[4x^3y - 3x^2y^3\right]_{y=1}^{y=2}\,dx = \int_0^1 \left[(8x^3 - 24x^2) - (4x^3 - 3x^2)\right]dx$

$= \int_0^1 (4x^3 - 21x^2)\,dx = \left[x^4 - 7x^3\right]_0^1 = (1-7) - (0-0) = -6$

5. $\int_0^2\int_0^4 y^3e^{2x}\,dy\,dx = \int_0^2 e^{2x}\,dx\int_0^4 y^3\,dy$ [as in Example 5] $= \left[\frac{1}{2}e^{2x}\right]_0^2\left[\frac{1}{4}y^4\right]_0^4 = \frac{1}{2}(e^4-1)(64-0) = 32(e^4-1)$

6. $\int_{\pi/6}^{\pi/2}\int_{-1}^{5} \cos y\,dx\,dy = \int_{-1}^{5} dx\int_{\pi/6}^{\pi/2}\cos y\,dy$ [by Equation 5]

$= \left[x\right]_{-1}^{5}\left[\sin y\right]_{\pi/6}^{\pi/2} = [5-(-1)](\sin\frac{\pi}{2} - \sin\frac{\pi}{6}) = 6(1-\frac{1}{2}) = 3$

7. $\int_{-3}^{3}\int_0^{\pi/2} (y + y^2\cos x)\,dx\,dy = \int_{-3}^{3}\left[xy + y^2\sin x\right]_{x=0}^{x=\pi/2}\,dy$

$= \int_{-3}^{3}\left(\frac{\pi}{2}y + y^2\right)dy = \left[\frac{\pi}{4}y^2 + \frac{1}{3}y^3\right]_{-3}^{3}$

$= \left[\frac{9\pi}{4} + 9 - \left(\frac{9\pi}{4} - 9\right)\right] = 18$

8. $\int_1^3 \int_1^5 \frac{\ln y}{xy}\,dy\,dx = \int_1^3 \frac{1}{x}\,dx \int_1^5 \frac{\ln y}{y}\,dy$ [as in Example 5]

$= [\ln|x|]_1^3 \left[\frac{1}{2}(\ln y)^2\right]_1^5$ [substitute $u = \ln y \Rightarrow du = (1/y)\,dy$]

$= (\ln 3 - 0)\cdot\frac{1}{2}[(\ln 5)^2 - 0] = \frac{1}{2}(\ln 3)(\ln 5)^2$

9. $\int_1^4 \int_1^2 \left(\frac{x}{y} + \frac{y}{x}\right) dy\,dx = \int_1^4 \left[x\ln|y| + \frac{1}{x}\cdot\frac{1}{2}y^2\right]_{y=1}^{y=2} dx = \int_1^4 \left(x\ln 2 + \frac{3}{2x}\right) dx = \left[\frac{1}{2}x^2\ln 2 + \frac{3}{2}\ln|x|\right]_1^4$

$= 8\ln 2 + \frac{3}{2}\ln 4 - \frac{1}{2}\ln 2 = \frac{15}{2}\ln 2 + 3\ln 4^{1/2} = \frac{21}{2}\ln 2$

10. $\int_0^1\int_0^3 e^{x+3y}\,dx\,dy = \int_0^1\int_0^3 e^x e^{3y}\,dx\,dy = \int_0^3 e^x\,dx \int_0^1 e^{3y}\,dy = [e^x]_0^3\left[\frac{1}{3}e^{3y}\right]_0^1$

$= (e^3 - e^0)\cdot\frac{1}{3}(e^3 - e^0) = \frac{1}{3}(e^3-1)^2$ or $\frac{1}{3}(e^6 - 2e^3 + 1)$

11. $\int_0^1\int_0^1 v(u+v^2)^4\,du\,dv = \int_0^1 \left[\frac{1}{5}v(u+v^2)^5\right]_{u=0}^{u=1} dv = \frac{1}{5}\int_0^1 v\left[(1+v^2)^5 - (0+v^2)^5\right] dv$

$= \frac{1}{5}\int_0^1 \left[v(1+v^2)^5 - v^{11}\right] dv = \frac{1}{5}\left[\frac{1}{2}\cdot\frac{1}{6}(1+v^2)^6 - \frac{1}{12}v^{12}\right]_0^1$

[substitute $t = 1+v^2 \Rightarrow dt = 2v\,dv$ in the first term]

$= \frac{1}{60}\left[(2^6-1) - (1-0)\right] = \frac{1}{60}(63-1) = \frac{31}{30}$

12. $\int_0^1\int_0^1 xy\sqrt{x^2+y^2}\,dy\,dx = \int_0^1 x\left[\frac{1}{3}(x^2+y^2)^{3/2}\right]_{y=0}^{y=1} dx = \frac{1}{3}\int_0^1 x[(x^2+1)^{3/2} - x^3]\,dx = \frac{1}{3}\int_0^1 [x(x^2+1)^{3/2} - x^4]dx$

$= \frac{1}{3}\left[\frac{1}{5}(x^2+1)^{5/2} - \frac{1}{5}x^5\right]_0^1 = \frac{1}{15}\left[2^{5/2} - 1 - 1 + 0\right] = \frac{2}{15}(2\sqrt{2}-1)$

13. $\int_0^2\int_0^\pi r\sin^2\theta\,d\theta\,dr = \int_0^2 r\,dr \int_0^\pi \sin^2\theta\,d\theta$ [as in Example 5] $= \int_0^2 r\,dr\int_0^\pi \frac{1}{2}(1-\cos 2\theta)\,d\theta$

$= \left[\frac{1}{2}r^2\right]_0^2\cdot\frac{1}{2}\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^\pi = (2-0)\cdot\frac{1}{2}\left[\left(\pi - \frac{1}{2}\sin 2\pi\right) - \left(0 - \frac{1}{2}\sin 0\right)\right]$

$= 2\cdot\frac{1}{2}[(\pi - 0) - (0-0)] = \pi$

14. $\int_0^1\int_0^1 \sqrt{s+t}\,ds\,dt = \int_0^1\left[\frac{2}{3}(s+t)^{3/2}\right]_{s=0}^{s=1} dt = \frac{2}{3}\int_0^1[(1+t)^{3/2} - t^{3/2}]\,dt = \frac{2}{3}\left[\frac{2}{5}(1+t)^{5/2} - \frac{2}{5}t^{5/2}\right]_0^1$

$= \frac{4}{15}[(2^{5/2}-1) - (1-0)] = \frac{4}{15}\left(2^{5/2}-2\right)$ or $\frac{8}{15}(2\sqrt{2}-1)$

15. $\iint_R \sin(x-y)\,dA = \int_0^{\pi/2}\int_0^{\pi/2}\sin(x-y)\,dy\,dx = \int_0^{\pi/2}[\cos(x-y)]_{y=0}^{y=\pi/2}\,dx = \int_0^{\pi/2}\left[\cos(x-\frac{\pi}{2}) - \cos x\right] dx$

$= \left[\sin(x - \frac{\pi}{2}) - \sin x\right]_0^{\pi/2} = \sin 0 - \sin\frac{\pi}{2} - \left[\sin(-\frac{\pi}{2}) - \sin 0\right]$

$= 0 - 1 - (-1-0) = 0$

16. $\iint_R (y + xy^{-2})\,dA = \int_1^2\int_0^2 (y+xy^{-2})\,dx\,dy = \int_1^2\left[xy + \frac{1}{2}x^2y^{-2}\right]_{x=0}^{x=2} dy = \int_1^2 (2y + 2y^{-2})\,dy$

$= \left[y^2 - 2y^{-1}\right]_1^2 = (4-1) - (1-2) = 4$

17. $\iint_R \frac{xy^2}{x^2+1}\,dA = \int_0^1\int_{-3}^3 \frac{xy^2}{x^2+1}\,dy\,dx = \int_0^1 \frac{x}{x^2+1}\,dx \int_{-3}^3 y^2\,dy = \left[\frac{1}{2}\ln(x^2+1)\right]_0^1 \left[\frac{1}{3}y^3\right]_{-3}^3$

$= \frac{1}{2}(\ln 2 - \ln 1)\cdot\frac{1}{3}(27+27) = 9\ln 2$

18. $\iint_R \frac{1+x^2}{1+y^2}\,dA = \int_0^1 \int_0^1 \frac{1+x^2}{1+y^2}\,dy\,dx = \int_0^1 (1+x^2)\,dx \int_0^1 \frac{1}{1+y^2}\,dy = \left[x + \frac{1}{3}x^3\right]_0^1 \left[\tan^{-1} y\right]_0^1$
$= \left(1 + \frac{1}{3} - 0\right)\left(\frac{\pi}{4} - 0\right) = \frac{\pi}{3}$

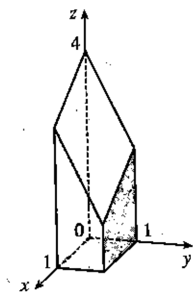
19. $\int_0^{\pi/6} \int_0^{\pi/3} x \sin(x+y)\,dy\,dx$
$= \int_0^{\pi/6} \left[-x\cos(x+y)\right]_{y=0}^{y=\pi/3} dx = \int_0^{\pi/6} \left[x\cos x - x\cos\left(x + \frac{\pi}{3}\right)\right] dx$
$= x\left[\sin x - \sin\left(x + \frac{\pi}{3}\right)\right]_0^{\pi/6} - \int_0^{\pi/6} \left[\sin x - \sin\left(x + \frac{\pi}{3}\right)\right] dx$ [by integrating by parts separately for each term]
$= \frac{\pi}{6}\left[\frac{1}{2} - 1\right] - \left[-\cos x + \cos\left(x + \frac{\pi}{3}\right)\right]_0^{\pi/6} = -\frac{\pi}{12} - \left[-\frac{\sqrt{3}}{2} + 0 - \left(-1 + \frac{1}{2}\right)\right] = \frac{\sqrt{3}-1}{2} - \frac{\pi}{12}$

20. $\iint_R \frac{x}{1+xy}\,dA = \int_0^1 \int_0^1 \frac{x}{1+xy}\,dy\,dx = \int_0^1 \left[\ln(1+xy)\right]_{y=0}^{y=1} dx = \int_0^1 \left[\ln(1+x) - \ln 1\right] dx$
$= \int_0^1 \ln(1+x)\,dx = \left[(1+x)\ln(1+x) - x\right]_0^1$ [by integrating by parts]
$= (2\ln 2 - 1) - (\ln 1 - 0) = 2\ln 2 - 1$

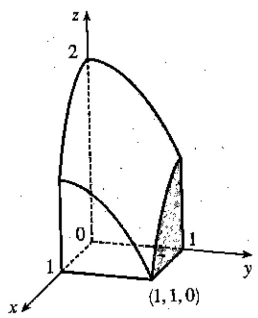
21. $\iint_R y e^{-xy}\,dA = \int_0^3 \int_0^2 y e^{-xy}\,dx\,dy = \int_0^3 \left[-e^{-xy}\right]_{x=0}^{x=2} dy = \int_0^3 (-e^{-2y} + 1)\,dy = \left[\frac{1}{2}e^{-2y} + y\right]_0^3$
$= \frac{1}{2}e^{-6} + 3 - \left(\frac{1}{2} + 0\right) = \frac{1}{2}e^{-6} + \frac{5}{2}$

22. $\iint_R \frac{1}{1+x+y}\,dA = \int_1^3 \int_1^2 \frac{1}{1+x+y}\,dy\,dx = \int_1^3 \left[\ln(1+x+y)\right]_{y=1}^{y=2} dx = \int_1^3 \left[\ln(x+3) - \ln(x+2)\right] dx$
$= \left[\left((x+3)\ln(x+3) - (x+3)\right) - \left((x+2)\ln(x+2) - (x+2)\right)\right]_1^3$
[by integrating by parts separately for each term]
$= (6\ln 6 - 6 - 5\ln 5 + 5) - (4\ln 4 - 4 - 3\ln 3 + 3) = 6\ln 6 - 5\ln 5 - 4\ln 4 + 3\ln 3$

23. $z = f(x,y) = 4 - x - 2y \geq 0$ for $0 \leq x \leq 1$ and $0 \leq y \leq 1$. So the solid is the region in the first octant which lies below the plane $z = 4 - x - 2y$ and above $[0,1] \times [0,1]$.



24. $z = 2 - x^2 - y^2 \geq 0$ for $0 \leq x \leq 1$ and $0 \leq y \leq 1$. So the solid is the region in the first octant which lies below the circular paraboloid $z = 2 - x^2 - y^2$ and above $[0,1] \times [0,1]$.



25. The solid lies under the plane $4x + 6y - 2z + 15 = 0$ or $z = 2x + 3y + \frac{15}{2}$ so

$$V = \iint_R (2x + 3y + \tfrac{15}{2})\,dA = \int_{-1}^{1}\int_{-1}^{2}(2x + 3y + \tfrac{15}{2})\,dx\,dy = \int_{-1}^{1}\left[x^2 + 3xy + \tfrac{15}{2}x\right]_{x=-1}^{x=2}dy$$
$$= \int_{-1}^{1}\left[(19 + 6y) - (-\tfrac{13}{2} - 3y)\right]dy = \int_{-1}^{1}(\tfrac{51}{2} + 9y)\,dy = \left[\tfrac{51}{2}y + \tfrac{9}{2}y^2\right]_{-1}^{1} = 30 - (-21) = 51$$

26. $$V = \iint_R (3y^2 - x^2 + 2)\,dA = \int_{-1}^{1}\int_{1}^{2}(3y^2 - x^2 + 2)\,dy\,dx = \int_{-1}^{1}\left[y^3 - x^2y + 2y\right]_{y=1}^{y=2}dx$$
$$= \int_{-1}^{1}\left[(12 - 2x^2) - (3 - x^2)\right]dx = \int_{-1}^{1}(9 - x^2 dx) = \left[9x - \tfrac{1}{3}x^3\right]_{-1}^{1} = \tfrac{26}{3} + \tfrac{26}{3} = \tfrac{52}{3}$$

27. $$V = \int_{-2}^{2}\int_{-1}^{1}\left(1 - \tfrac{1}{4}x^2 - \tfrac{1}{9}y^2\right)dx\,dy = 4\int_0^2\int_0^1\left(1 - \tfrac{1}{4}x^2 - \tfrac{1}{9}y^2\right)dx\,dy$$
$$= 4\int_0^2\left[x - \tfrac{1}{12}x^3 - \tfrac{1}{9}y^2x\right]_{x=0}^{x=1}dy = 4\int_0^2\left(\tfrac{11}{12} - \tfrac{1}{9}y^2\right)dy = 4\left[\tfrac{11}{12}y - \tfrac{1}{27}y^3\right]_0^2 = 4\cdot\tfrac{83}{54} = \tfrac{166}{27}$$

28. $$V = \int_{-1}^{1}\int_0^{\pi}(1 + e^x\sin y)\,dy\,dx = \int_{-1}^{1}\left[y - e^x\cos y\right]_{y=0}^{y=\pi}dx = \int_{-1}^{1}(\pi + e^x - 0 + e^x)\,dx$$
$$= \int_{-1}^{1}(\pi + 2e^x)\,dx = \left[\pi x + 2e^x\right]_{-1}^{1} = 2\pi + 2e - \tfrac{2}{e}$$

29. Here we need the volume of the solid lying under the surface $z = x\sec^2 y$ and above the rectangle $R = [0, 2] \times [0, \pi/4]$ in the xy-plane.

$$V = \int_0^2\int_0^{\pi/4} x\sec^2 y\,dy\,dx = \int_0^2 x\,dx\int_0^{\pi/4}\sec^2 y\,dy = \left[\tfrac{1}{2}x^2\right]_0^2\left[\tan y\right]_0^{\pi/4}$$
$$= (2 - 0)(\tan\tfrac{\pi}{4} - \tan 0) = 2(1 - 0) = 2$$

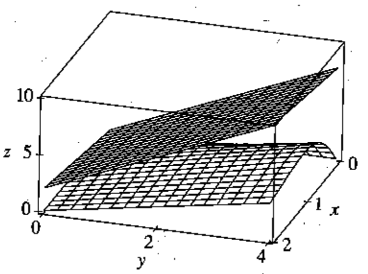
30. The cylinder intersects the xy-plane along the line $x = 4$, so in the first octant, the solid lies below the surface $z = 16 - x^2$ and above the rectangle $R = [0, 4] \times [0, 5]$ in the xy-plane.

$$V = \int_0^5\int_0^4(16 - x^2)\,dx\,dy = \int_0^4(16 - x^2)\,dx\int_0^5 dy = \left[16x - \tfrac{1}{3}x^3\right]_0^4\left[y\right]_0^5 = (64 - \tfrac{64}{3} - 0)(5 - 0) = \tfrac{640}{3}$$

31. The solid lies below the surface $z = 2 + x^2 + (y - 2)^2$ and above the plane $z = 1$ for $-1 \le x \le 1$, $0 \le y \le 4$. The volume of the solid is the difference in volumes between the solid that lies under $z = 2 + x^2 + (y - 2)^2$ over the rectangle $R = [-1, 1] \times [0, 4]$ and the solid that lies under $z = 1$ over R.

$$V = \int_0^4\int_{-1}^{1}[2 + x^2 + (y - 2)^2]\,dx\,dy - \int_0^4\int_{-1}^{1}(1)\,dx\,dy = \int_0^4\left[2x + \tfrac{1}{3}x^3 + x(y - 2)^2\right]_{x=-1}^{x=1}dy - \int_{-1}^{1}dx\int_0^4 dy$$
$$= \int_0^4\left[(2 + \tfrac{1}{3} + (y - 2)^2) - (-2 - \tfrac{1}{3} - (y - 2)^2)\right]dy - [x]_{-1}^{1}\,[y]_0^4$$
$$= \int_0^4\left[\tfrac{14}{3} + 2(y - 2)^2\right]dy - [1 - (-1)][4 - 0] = \left[\tfrac{14}{3}y + \tfrac{2}{3}(y - 2)^3\right]_0^4 - (2)(4)$$
$$= \left[(\tfrac{56}{3} + \tfrac{16}{3}) - (0 - \tfrac{16}{3})\right] - 8 = \tfrac{88}{3} - 8 = \tfrac{64}{3}$$

32.



The solid lies below the plane $z = x + 2y$ and above the surface $z = \dfrac{2xy}{x^2 + 1}$ for $0 \le x \le 2$, $0 \le y \le 4$. The volume of the solid is the difference in volumes between the solid that lies under $z = x + 2y$ over the rectangle $R = [0, 2] \times [0, 4]$ and the solid that lies under $z = \dfrac{2xy}{x^2 + 1}$ over R.

[continued]

6. $\int_0^1 \int_0^{e^v} \sqrt{1+e^v}\,dw\,dv = \int_0^1 \left[w\sqrt{1+e^v}\right]_{w=0}^{w=e^v} dv = \int_0^1 e^v\sqrt{1+e^v}\,dv = \frac{2}{3}(1+e^v)^{3/2}\Big]_0^1$

$$= \tfrac{2}{3}(1+e)^{3/2} - \tfrac{2}{3}(1+1)^{3/2} = \tfrac{2}{3}(1+e)^{3/2} - \tfrac{4}{3}\sqrt{2}$$

7. $\iint_D y^2\,dA = \int_{-1}^1 \int_{-y-2}^{y} y^2\,dx\,dy = \int_{-1}^1 \left[xy^2\right]_{x=-y-2}^{x=y} dy = \int_{-1}^1 y^2\,[y-(-y-2)]\,dy$

$$= \int_{-1}^1 (2y^3+2y^2)dy = \left[\tfrac{1}{2}y^4 + \tfrac{2}{3}y^3\right]_{-1}^1 = \tfrac{1}{2}+\tfrac{2}{3}-\tfrac{1}{2}+\tfrac{2}{3} = \tfrac{4}{3}$$

8. $\displaystyle\iint_D \frac{y}{x^5+1}\,dA = \int_0^1\int_0^{x^2} \frac{y}{x^5+1}\,dy\,dx = \int_0^1 \frac{1}{x^5+1}\left[\frac{y^2}{2}\right]_{y=0}^{y=x^2} dx = \frac{1}{2}\int_0^1 \frac{x^4}{x^5+1}\,dx = \tfrac{1}{2}\left[\tfrac{1}{5}\ln\left|x^5+1\right|\right]_0^1$

$$= \tfrac{1}{10}(\ln 2 - \ln 1) = \tfrac{1}{10}\ln 2$$

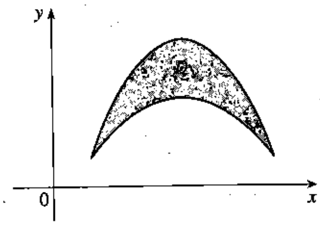
9. $\iint_D x\,dA = \int_0^\pi \int_0^{\sin x} x\,dy\,dx = \int_0^\pi [xy]_{y=0}^{y=\sin x}\,dx = \int_0^\pi x\sin x\,dx$ $\left[\begin{array}{c}\text{integrate by parts}\\ \text{with } u=x,\ dv=\sin x\,dx\end{array}\right]$

$$= \left[-x\cos x + \sin x\right]_0^\pi = -\pi\cos\pi + \sin\pi + 0 - \sin 0 = \pi$$

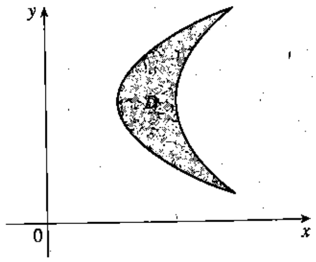
10. $\iint_D x^3\,dA = \int_1^e \int_0^{\ln x} x^3\,dy\,dx = \int_1^e \left[x^3 y\right]_{y=0}^{y=\ln x} dx = \int_1^e x^3\ln x\,dx$ $\left[\begin{array}{c}\text{integrate by parts}\\ \text{with } u=\ln x,\ dv=x^3\,dx\end{array}\right]$

$$= \left[\tfrac{1}{4}x^4\ln x - \tfrac{1}{16}x^4\right]_1^e = \tfrac{1}{4}e^4 - \tfrac{1}{16}e^4 - 0 + \tfrac{1}{16} = \tfrac{3}{16}e^4 + \tfrac{1}{16}$$

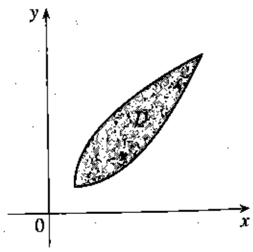
11. (a) At the right we sketch an example of a region D that can be described as lying between the graphs of two continuous functions of x (a type I region) but not as lying between graphs of two continuous functions of y (a type II region). The regions shown in Figures 6 and 8 in the text are additional examples.



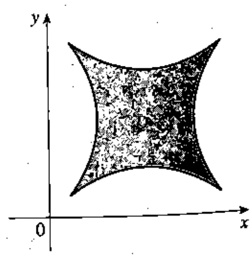
(b) Now we sketch an example of a region D that can be described as lying between the graphs of two continuous functions of y but not as lying between graphs of two continuous functions of x. The first region shown in Figure 7 is another example.



12. (a) At the right we sketch an example of a region D that can be described as lying between the graphs of two continuous functions of x (a type I region) and also as lying between graphs of two continuous functions of y (a type II region). For additional examples see Figures 9, 10, 12, and 14–16 in the text.



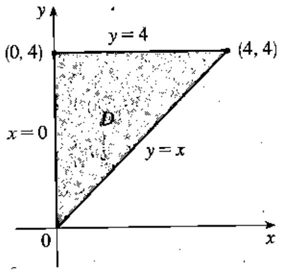
(b) Now we sketch an example of a region D that can't be described as lying between the graphs of two continuous functions of x or between graphs of two continuous functions of y. The region shown in Figure 18 is another example.



more simply described as a type II region, giving one iterated integral rather than a sum of two, so we evaluate the latter integral:

$$\iint_D y\,dA = \int_{-1}^{2}\int_{y^2}^{y+2} y\,dx\,dy = \int_{-1}^{2}\left[xy\right]_{x=y^2}^{x=y+2} dy = \int_{-1}^{2}(y+2-y^2)y\,dy = \int_{-1}^{2}(y^2+2y-y^3)\,dy$$
$$= \left[\tfrac{1}{3}y^3 + y^2 - \tfrac{1}{4}y^4\right]_{-1}^{2} = \left(\tfrac{8}{3}+4-4\right) - \left(-\tfrac{1}{3}+1-\tfrac{1}{4}\right) = \tfrac{9}{4}$$

16.

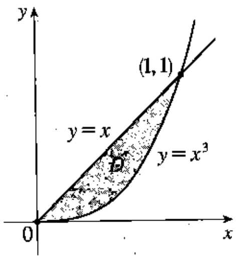


As a type I region, $D = \{(x,y) \mid 0 \le x \le 4,\, x \le y \le 4\}$ and $\iint_D y^2 e^{xy}\,dA = \int_0^4\int_x^4 y^2 e^{xy}\,dy\,dx$. As a type II region, $D = \{(x,y) \mid 0 \le y \le 4,\, 0 \le x \le y\}$ and $\iint_D y^2 e^{xy}\,dA = \int_0^4\int_0^y y^2 e^{xy}\,dx\,dy$. Evaluating $\int y^2 e^{xy}\,dy$ requires integration by parts whereas $\int y^2 e^{xy}\,dx$ does not, so the iterated integral corresponding to D as a type II region appears easier to evaluate.

$$\iint_D y^2 e^{xy}\,dA = \int_0^4\int_0^y y^2 e^{xy}\,dx\,dy = \int_0^4\left[ye^{xy}\right]_{x=0}^{x=y} dy = \int_0^4\left(ye^{y^2} - y\right)dy$$
$$= \left[\tfrac{1}{2}e^{y^2} - \tfrac{1}{2}y^2\right]_0^4 = \left(\tfrac{1}{2}e^{16} - 8\right) - \left(\tfrac{1}{2} - 0\right) = \tfrac{1}{2}e^{16} - \tfrac{17}{2}$$

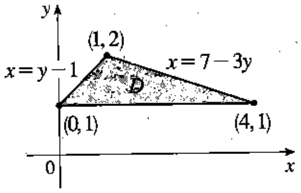
17. $\int_0^1\int_0^{x^2} x\cos y\,dy\,dx = \int_0^1\left[x\sin y\right]_{y=0}^{y=x^2} dx = \int_0^1 x\sin x^2\,dx = -\tfrac{1}{2}\cos x^2\Big]_0^1 = \tfrac{1}{2}(1-\cos 1)$

18.



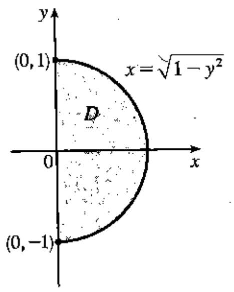
$$\iint_D (x^2+2y)\,dA = \int_0^1\int_{x^3}^{x}(x^2+2y)\,dy\,dx = \int_0^1\left[x^2y + y^2\right]_{y=x^3}^{y=x} dx$$
$$= \int_0^1(x^3+x^2-x^5-x^6)\,dx = \left[\tfrac{1}{4}x^4 + \tfrac{1}{3}x^3 - \tfrac{1}{6}x^6 - \tfrac{1}{7}x^7\right]_0^1$$
$$= \tfrac{1}{4} + \tfrac{1}{3} - \tfrac{1}{6} - \tfrac{1}{7} = \tfrac{23}{84}$$

19.



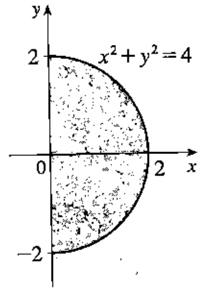
$$\iint_D y^2\,dA = \int_1^2\int_{y-1}^{7-3y} y^2\,dx\,dy = \int_1^2\left[xy^2\right]_{x=y-1}^{x=7-3y} dy$$
$$= \int_1^2\left[(7-3y)-(y-1)\right]y^2\,dy = \int_1^2(8y^2-4y^3)\,dy$$
$$= \left[\tfrac{8}{3}y^3 - y^4\right]_1^2 = \tfrac{64}{3} - 16 - \tfrac{8}{3} + 1 = \tfrac{11}{3}$$

20.



$$\iint_D xy^2\,dA = \int_{-1}^{1}\int_0^{\sqrt{1-y^2}} xy^2\,dx\,dy$$
$$= \int_{-1}^{1} y^2\left[\tfrac{1}{2}x^2\right]_{x=0}^{x=\sqrt{1-y^2}} dy = \tfrac{1}{2}\int_{-1}^{1} y^2(1-y^2)\,dy$$
$$= \tfrac{1}{2}\int_{-1}^{1}(y^2-y^4)\,dy = \tfrac{1}{2}\left[\tfrac{1}{3}y^3 - \tfrac{1}{5}y^5\right]_{-1}^{1}$$
$$= \tfrac{1}{2}\left(\tfrac{1}{3} - \tfrac{1}{5} + \tfrac{1}{3} - \tfrac{1}{5}\right) = \tfrac{2}{15}$$

46.



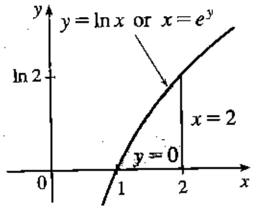
Because the region of integration is

$$D=\left\{(x,y)\mid 0\le x\le\sqrt{4-y^2},-2\le y\le 2\right\}$$
$$=\left\{(x,y)\mid -\sqrt{4-x^2}\le y\le\sqrt{4-x^2},0\le x\le 2\right\}$$

we have

$$\int_{-2}^{2}\int_{0}^{\sqrt{4-y^2}} f(x,y)\,dx\,dy=\iint_D f(x,y)\,dA=\int_0^2\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} f(x,y)\,dy\,dx.$$

47.



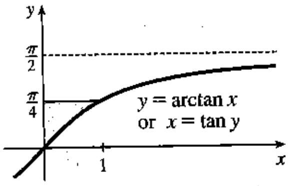
Because the region of integration is

$$D=\{(x,y)\mid 0\le y\le\ln x,1\le x\le 2\}=\{(x,y)\mid e^y\le x\le 2,0\le y\le\ln 2\}$$

we have

$$\int_1^2\int_0^{\ln x} f(x,y)\,dy\,dx=\iint_D f(x,y)\,dA=\int_0^{\ln 2}\int_{e^y}^2 f(x,y)\,dx\,dy$$

48.



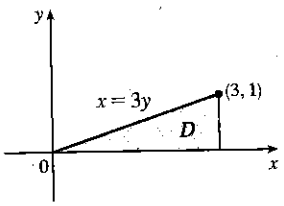
Because the region of integration is

$$D=\left\{(x,y)\mid \arctan x\le y\le\tfrac{\pi}{4},0\le x\le 1\right\}$$
$$=\left\{(x,y)\mid 0\le x\le\tan y,0\le y\le\tfrac{\pi}{4}\right\}$$

we have

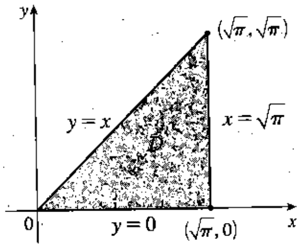
$$\int_0^1\int_{\arctan x}^{\pi/4} f(x,y)\,dy\,dx=\iint_D f(x,y)\,dA=\int_0^{\pi/4}\int_0^{\tan y} f(x,y)\,dx\,dy$$

49.



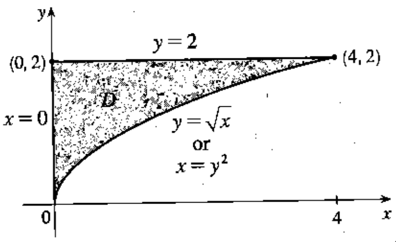
$$\int_0^1\int_{3y}^3 e^{x^2}\,dx\,dy=\int_0^3\int_0^{x/3} e^{x^2}\,dy\,dx=\int_0^3\left[e^{x^2}y\right]_{y=0}^{y=x/3}dx$$
$$=\int_0^3\left(\frac{x}{3}\right)e^{x^2}\,dx=\tfrac{1}{6}\,e^{x^2}\Big]_0^3=\frac{e^9-1}{6}$$

50.



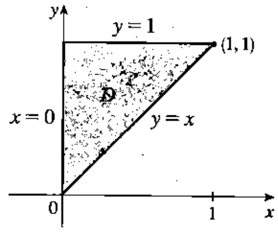
$$\int_0^{\sqrt{\pi}}\int_y^{\sqrt{\pi}}\cos(x^2)\,dx\,dy=\int_0^{\sqrt{\pi}}\int_0^x\cos(x^2)\,dy\,dx$$
$$=\int_0^{\sqrt{\pi}}\cos(x^2)\left[y\right]_{y=0}^{y=x}dx=\int_0^{\sqrt{\pi}}x\cos(x^2)\,dx$$
$$=\tfrac{1}{2}\sin(x^2)\Big]_0^{\sqrt{\pi}}=\tfrac{1}{2}(\sin\pi-\sin 0)=0$$

51.



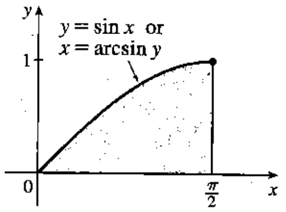
$$\int_0^4\int_{\sqrt{x}}^2\frac{1}{y^3+1}\,dy\,dx=\int_0^2\int_0^{y^2}\frac{1}{y^3+1}\,dx\,dy$$
$$=\int_0^2\frac{1}{y^3+1}\left[x\right]_{x=0}^{x=y^2}dy=\int_0^2\frac{y^2}{y^3+1}\,dy$$
$$=\tfrac{1}{3}\ln\left|y^3+1\right|\Big]_0^2=\tfrac{1}{3}(\ln 9-\ln 1)=\tfrac{1}{3}\ln 9$$

52.



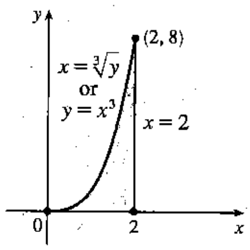
$$\int_0^1 \int_x^1 e^{x/y}\,dy\,dx = \int_0^1 \int_0^y e^{x/y}\,dx\,dy = \int_0^1 \left[ye^{x/y}\right]_{x=0}^{x=y} dy$$
$$= \int_0^1 (e-1)y\,dy = \tfrac{1}{2}(e-1)y^2\Big]_0^1$$
$$= \tfrac{1}{2}(e-1)$$

53.



$$\int_0^1 \int_{\arcsin y}^{\pi/2} \cos x\,\sqrt{1+\cos^2 x}\,dx\,dy$$
$$= \int_0^{\pi/2} \int_0^{\sin x} \cos x\,\sqrt{1+\cos^2 x}\,dy\,dx$$
$$= \int_0^{\pi/2} \cos x\,\sqrt{1+\cos^2 x}\,\big[\,y\,\big]_{y=0}^{y=\sin x}\,dx$$
$$= \int_0^{\pi/2} \cos x\,\sqrt{1+\cos^2 x}\,\sin x\,dx \qquad \left[\begin{array}{c}\text{Let } u = \cos x,\ du = -\sin x\,dx,\\ dx = du/(-\sin x)\end{array}\right]$$
$$= \int_1^0 -u\,\sqrt{1+u^2}\,du = -\tfrac{1}{3}\left(1+u^2\right)^{3/2}\Big]_1^0$$
$$= \tfrac{1}{3}\left(\sqrt{8}-1\right) = \tfrac{1}{3}\left(2\sqrt{2}-1\right)$$

54.



$$\int_0^8 \int_{\sqrt[3]{y}}^2 e^{x^4}\,dx\,dy = \int_0^2 \int_0^{x^3} e^{x^4}\,dy\,dx$$
$$= \int_0^2 e^{x^4}\Big[\,y\,\Big]_{y=0}^{y=x^3} dx = \int_0^2 x^3 e^{x^4}\,dx$$
$$= \tfrac{1}{4}e^{x^4}\Big]_0^2 = \tfrac{1}{4}(e^{16}-1)$$

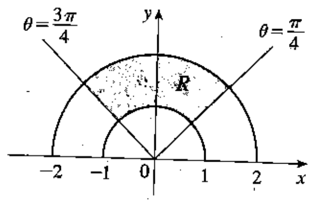
55. $D = \{(x,y) \mid 0 \le x \le 1,\ -x+1 \le y \le 1\} \cup \{(x,y) \mid -1 \le x \le 0, x+1 \le y \le 1\}$
$\cup \{(x,y) \mid 0 \le x \le 1,\ -1 \le y \le x-1\} \cup \{(x,y) \mid -1 \le x \le 0,\ -1 \le y \le -x-1\}$, all type I.

$$\iint_D x^2\,dA = \int_0^1 \int_{1-x}^1 x^2\,dy\,dx + \int_{-1}^0 \int_{x+1}^1 x^2\,dy\,dx + \int_0^1 \int_{-1}^{x-1} x^2\,dy\,dx + \int_{-1}^0 \int_{-1}^{-x-1} x^2\,dy\,dx$$
$$= 4\int_0^1 \int_{1-x}^1 x^2\,dy\,dx \qquad \text{[by symmetry of the regions and because } f(x,y) = x^2 \ge 0\text{]}$$
$$= 4\int_0^1 x^3\,dx = 4\left[\tfrac{1}{4}x^4\right]_0^1 = 1$$

56. $D = \left\{(x,y) \mid -1 \le y \le 0,\ -1 \le x \le y - y^3\right\} \cup \left\{(x,y) \mid 0 \le y \le 1, \sqrt{y} - 1 \le x \le y - y^3\right\}$, both type II.

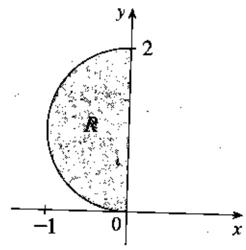
$$\iint_D y\,dA = \int_{-1}^0 \int_{-1}^{y-y^3} y\,dx\,dy + \int_0^1 \int_{\sqrt{y}-1}^{y-y^3} y\,dx\,dy = \int_{-1}^0 [xy]_{x=-1}^{x=y-y^3} dy + \int_0^1 [xy]_{x=\sqrt{y}-1}^{x=y-y^3} dy$$
$$= \int_{-1}^0 (y^2 - y^4 + y)\,dy + \int_0^1 (y^2 - y^4 - y^{3/2} + y)\,dy$$
$$= \left[\tfrac{1}{3}y^3 - \tfrac{1}{5}y^5 + \tfrac{1}{2}y^2\right]_{-1}^0 + \left[\tfrac{1}{3}y^3 - \tfrac{1}{5}y^5 - \tfrac{2}{5}y^{5/2} + \tfrac{1}{2}y^2\right]_0^1$$
$$= \left(0 - \tfrac{11}{30}\right) + \left(\tfrac{7}{30} - 0\right) = -\tfrac{2}{15}$$

5. The integral $\int_{\pi/4}^{3\pi/4}\int_1^2 r\,dr\,d\theta$ represents the area of the region $R=\{(r,\theta)\mid 1\le r\le 2, \pi/4\le\theta\le 3\pi/4\}$, the top quarter portion of a ring (annulus).



$$\int_{\pi/4}^{3\pi/4}\int_1^2 r\,dr\,d\theta=\left(\int_{\pi/4}^{3\pi/4}d\theta\right)\left(\int_1^2 r\,dr\right)$$
$$=\left[\theta\right]_{\pi/4}^{3\pi/4}\left[\tfrac{1}{2}r^2\right]_1^2=\left(\tfrac{3\pi}{4}-\tfrac{\pi}{4}\right)\cdot\tfrac{1}{2}(4-1)=\tfrac{\pi}{2}\cdot\tfrac{3}{2}=\tfrac{3\pi}{4}$$

6. The integral $\int_{\pi/2}^{\pi}\int_0^{2\sin\theta} r\,dr\,d\theta$ represents the area of the region $R=\{(r,\theta)\mid 1\le r\le 2\sin\theta, \pi/2\le\theta\le\pi\}$. Since $r=2\sin\theta \;\Rightarrow\; r^2=2r\sin\theta \;\Leftrightarrow\; x^2+y^2=2y \;\Leftrightarrow$ $x^2+(y-1)^2=1$, R is the portion in the second quadrant of a disk of radius 1 with center $(0,1)$.



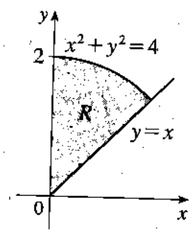
$$\int_{\pi/2}^{\pi}\int_0^{2\sin\theta} r\,dr\,d\theta=\int_{\pi/2}^{\pi}\left[\tfrac{1}{2}r^2\right]_{r=0}^{r=2\sin\theta}d\theta=\int_{\pi/2}^{\pi}2\sin^2\theta\,d\theta$$
$$=\int_{\pi/2}^{\pi}2\cdot\tfrac{1}{2}(1-\cos 2\theta)\,d\theta=\left[\theta-\tfrac{1}{2}\sin 2\theta\right]_{\pi/2}^{\pi}$$
$$=\pi-0-\tfrac{\pi}{2}+0=\tfrac{\pi}{2}$$

7. The half disk D can be described in polar coordinates as $D=\{(r,\theta)\mid 0\le r\le 5, 0\le\theta\le\pi\}$. Then

$$\iint_D x^2y\,dA=\int_0^{\pi}\int_0^5(r\cos\theta)^2(r\sin\theta)\,r\,dr\,d\theta=\left(\int_0^{\pi}\cos^2\theta\sin\theta\,d\theta\right)\left(\int_0^5 r^4\,dr\right)$$
$$=\left[-\tfrac{1}{3}\cos^3\theta\right]_0^{\pi}\left[\tfrac{1}{5}r^5\right]_0^5=-\tfrac{1}{3}(-1-1)\cdot 625=\tfrac{1250}{3}$$

8. The region R is $\frac{1}{8}$ of a disk, as shown in the figure, and can be described by $R=\{(r,\theta)\mid 0\le r\le 2, \pi/4\le\theta\le\pi/2\}$. Thus

$$\iint_R(2x-y)\,dA=\int_{\pi/4}^{\pi/2}\int_0^2(2r\cos\theta-r\sin\theta)\,r\,dr\,d\theta$$
$$=\left(\int_{\pi/4}^{\pi/2}(2\cos\theta-\sin\theta)\,d\theta\right)\left(\int_0^2 r^2\,dr\right)$$
$$=\left[2\sin\theta+\cos\theta\right]_{\pi/4}^{\pi/2}\left[\tfrac{1}{3}r^3\right]_0^2$$
$$=\left(2+0-\sqrt{2}-\tfrac{\sqrt{2}}{2}\right)\left(\tfrac{8}{3}\right)=\tfrac{16}{3}-4\sqrt{2}$$



9. $$\iint_R\sin(x^2+y^2)\,dA=\int_0^{\pi/2}\int_1^3\sin(r^2)\,r\,dr\,d\theta=\left(\int_0^{\pi/2}d\theta\right)\left(\int_1^3 r\sin(r^2)\,dr\right)$$
$$=\left[\theta\right]_0^{\pi/2}\left[-\tfrac{1}{2}\cos(r^2)\right]_1^3$$
$$=\left(\tfrac{\pi}{2}\right)\left[-\tfrac{1}{2}(\cos 9-\cos 1)\right]=\tfrac{\pi}{4}(\cos 1-\cos 9)$$

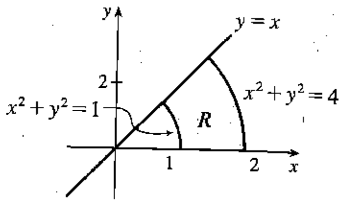
10. $$\iint_R\frac{y^2}{x^2+y^2}\,dA=\int_0^{2\pi}\int_a^b\frac{(r\sin\theta)^2}{r^2}\,r\,dr\,d\theta=\left(\int_0^{2\pi}\sin^2\theta\,d\theta\right)\left(\int_a^b r\,dr\right)$$
$$=\int_0^{2\pi}\tfrac{1}{2}(1-\cos 2\theta)\,d\theta\int_a^b r\,dr=\tfrac{1}{2}\left[\theta-\tfrac{1}{2}\sin 2\theta\right]_0^{2\pi}\left[\tfrac{1}{2}r^2\right]_a^b$$
$$=\tfrac{1}{2}(2\pi-0-0)\left[\tfrac{1}{2}(b^2-a^2)\right]=\tfrac{\pi}{2}(b^2-a^2)$$

$$= \left[\,\theta\,\right]_{-\pi/2}^{\pi/2}\left[-\tfrac{1}{2}e^{-r^2}\right]_0^2 = \pi\left(-\tfrac{1}{2}\right)(e^{-4}-e^0) = \tfrac{\pi}{2}(1-e^{-4})$$

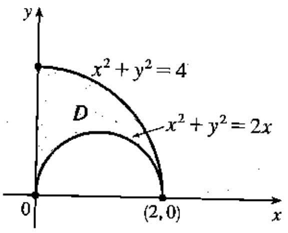
12. $\iint_D \cos\sqrt{x^2+y^2}\,dA = \int_0^{2\pi}\int_0^2 \cos\sqrt{r^2}\,r\,dr\,d\theta = \int_0^{2\pi} d\theta\,\int_0^2 r\cos r\,dr$. For the second integral, integrate by parts with $u = r$, $dv = \cos r\,dr$. Then $\iint_D \cos\sqrt{x^2+y^2}\,dA = \left[\,\theta\,\right]_0^{2\pi}\left[r\sin r + \cos r\right]_0^2 = 2\pi(2\sin 2 + \cos 2 - 1)$.

13. R is the region shown in the figure, and can be described by $R = \{(r,\theta) \mid 0 \le \theta \le \pi/4, 1 \le r \le 2\}$. Thus $\iint_R \arctan(y/x)\,dA = \int_0^{\pi/4}\int_1^2 \arctan(\tan\theta)\,r\,dr\,d\theta$ since $y/x = \tan\theta$. Also, $\arctan(\tan\theta) = \theta$ for $0 \le \theta \le \pi/4$, so the integral becomes

$$\int_0^{\pi/4}\int_1^2 \theta\,r\,dr\,d\theta = \int_0^{\pi/4}\theta\,d\theta\,\int_1^2 r\,dr = \left[\tfrac{1}{2}\theta^2\right]_0^{\pi/4}\left[\tfrac{1}{2}r^2\right]_1^2 = \tfrac{\pi^2}{32}\cdot\tfrac{3}{2} = \tfrac{3}{64}\pi^2.$$



14.



$$\iint_D x\,dA = \iint_{\substack{x^2+y^2\le 4\\ x\ge 0,\,y\ge 0}} x\,dA - \iint_{\substack{(x-1)^2+y^2\le 1\\ y\ge 0}} x\,dA$$

$$= \int_0^{\pi/2}\int_0^2 r^2\cos\theta\,dr\,d\theta - \int_0^{\pi/2}\int_0^{2\cos\theta} r^2\cos\theta\,dr\,d\theta$$

$$= \int_0^{\pi/2}\tfrac{1}{3}(8\cos\theta)\,d\theta - \int_0^{\pi/2}\tfrac{1}{3}(8\cos^4\theta)\,d\theta$$

$$= \tfrac{8}{3} - \tfrac{8}{12}\left[\cos^3\theta\sin\theta + \tfrac{3}{2}(\theta + \sin\theta\cos\theta)\right]_0^{\pi/2}$$

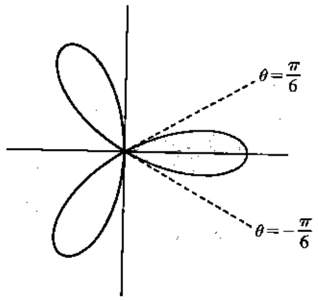
$$= \tfrac{8}{3} - \tfrac{2}{3}\left[0 + \tfrac{3}{2}\left(\tfrac{\pi}{2}\right)\right] = \tfrac{16-3\pi}{6}$$

15. One loop is given by the region $D = \{(r,\theta) \mid -\pi/6 \le \theta \le \pi/6, 0 \le r \le \cos 3\theta\}$, so the area is

$$\iint_D dA = \int_{-\pi/6}^{\pi/6}\int_0^{\cos 3\theta} r\,dr\,d\theta = \int_{-\pi/6}^{\pi/6}\left[\frac{1}{2}r^2\right]_{r=0}^{r=\cos 3\theta} d\theta$$

$$= \int_{-\pi/6}^{\pi/6}\frac{1}{2}\cos^2 3\theta\,d\theta = 2\int_0^{\pi/6}\frac{1}{2}\left(\frac{1+\cos 6\theta}{2}\right)d\theta$$

$$= \frac{1}{2}\left[\theta + \frac{1}{6}\sin 6\theta\right]_0^{\pi/6} = \frac{\pi}{12}$$



16. By symmetry, the area of the region is 4 times the area of the region D in the first quadrant enclosed by the cardiod $r = 1 - \cos\theta$ (see the figure). Here $D = \{(r,\theta) \mid 0 \le r \le 1 - \cos\theta, 0 \le \theta \le \pi/2\}$, so the total area is

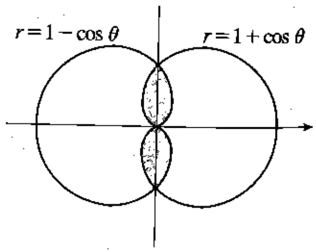
$$4A(D) = 4\iint_D dA = 4\int_0^{\pi/2}\int_0^{1-\cos\theta} r\,dr\,d\theta = 4\int_0^{\pi/2}\left[\tfrac{1}{2}r^2\right]_{r=0}^{r=1-\cos\theta} d\theta$$

$$= 2\int_0^{\pi/2}(1-\cos\theta)^2 d\theta = 2\int_0^{\pi/2}(1 - 2\cos\theta + \cos^2\theta)\,d\theta$$

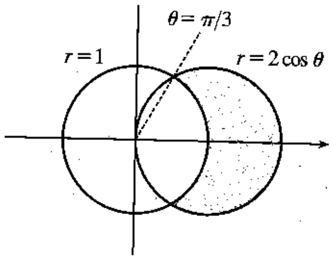
$$= 2\int_0^{\pi/2}\left[1 - 2\cos\theta + \tfrac{1}{2}(1+\cos 2\theta)\right]d\theta$$

$$= 2\left[\theta - 2\sin\theta + \tfrac{1}{2}\theta + \tfrac{1}{4}\sin 2\theta\right]_0^{\pi/2}$$

$$= 2\left(\tfrac{\pi}{2} - 2 + \tfrac{\pi}{4}\right) = \tfrac{3\pi}{2} - 4$$

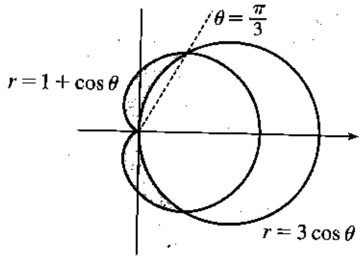


17. In polar coordinates the circle $(x-1)^2+y^2=1 \;\Leftrightarrow\; x^2+y^2=2x$ is $r^2=2r\cos\theta \;\Rightarrow\; r=2\cos\theta$, and the circle $x^2+y^2=1$ is $r=1$. The curves intersect in the first quadrant when $2\cos\theta=1 \;\Rightarrow\; \cos\theta=\frac{1}{2} \;\Rightarrow\; \theta=\pi/3$, so the portion of the region in the first quadrant is given by $D=\{(r,\theta)\mid 1\le r\le 2\cos\theta, 0\le\theta\le\pi/2\}$. By symmetry, the total area is twice the area of D:



$$2A(D)=2\iint_D dA=2\int_0^{\pi/3}\int_1^{2\cos\theta} r\,dr\,d\theta=2\int_0^{\pi/3}\left[\tfrac{1}{2}r^2\right]_{r=1}^{r=2\cos\theta}d\theta$$
$$=\int_0^{\pi/3}\left(4\cos^2\theta-1\right)d\theta=\int_0^{\pi/3}\left[4\cdot\tfrac{1}{2}(1+\cos 2\theta)-1\right]d\theta$$
$$=\int_0^{\pi/3}(1+2\cos 2\theta)\,d\theta=\left[\theta+\sin 2\theta\right]_0^{\pi/3}=\frac{\pi}{3}+\frac{\sqrt{3}}{2}$$

18. The region lies between the two polar curves in quadrants I and IV, but in quadrants II and III the region is enclosed by the cardiod. In the first quadrant, $1+\cos\theta=3\cos\theta$ when $\cos\theta=\frac{1}{2} \;\Rightarrow\; \theta=\frac{\pi}{3}$, so the area of the region inside the cardiod and outside the circle is



$$A_1=\int_{\pi/3}^{\pi/2}\int_{3\cos\theta}^{1+\cos\theta} r\,dr\,d\theta=\int_{\pi/3}^{\pi/2}\left[\tfrac{1}{2}r^2\right]_{r=3\cos\theta}^{r=1+\cos\theta}d\theta$$
$$=\tfrac{1}{2}\int_{\pi/3}^{\pi/2}(1+2\cos\theta-8\cos^2\theta)d\theta=\tfrac{1}{2}\left[\theta+2\sin\theta-8\left(\tfrac{1}{2}\theta+\tfrac{1}{4}\sin 2\theta\right)\right]_{\pi/3}^{\pi/2}$$
$$=\left[-\tfrac{3}{2}\theta+\sin\theta-\sin 2\theta\right]_{\pi/3}^{\pi/2}=\left(-\tfrac{3\pi}{4}+1-0\right)-\left(-\tfrac{\pi}{2}+\tfrac{\sqrt{3}}{2}-\tfrac{\sqrt{3}}{2}\right)=1-\tfrac{\pi}{4}.$$

The area of the region in the second quadrant is

$$A_2=\int_{\pi/2}^{\pi}\int_0^{1+\cos\theta} r\,dr\,d\theta=\int_{\pi/2}^{\pi}\left[\tfrac{1}{2}r^2\right]_{r=0}^{r=1+\cos\theta}d\theta=\tfrac{1}{2}\int_{\pi/2}^{\pi}(1+2\cos\theta+\cos^2\theta)d\theta$$
$$=\tfrac{1}{2}\left[\theta+2\sin\theta+\tfrac{1}{2}\theta+\tfrac{1}{4}\sin 2\theta\right]_{\pi/2}^{\pi}=\tfrac{1}{2}\left(\tfrac{3\pi}{4}-2\right)=\tfrac{3\pi}{8}-1.$$

By symmetry, the total area is $A=2(A_1+A_2)=2\left(1-\frac{\pi}{4}+\frac{3\pi}{8}-1\right)=\frac{\pi}{4}$.

19. $V=\iint_{x^2+y^2\le 4}\sqrt{x^2+y^2}\,dA=\int_0^{2\pi}\int_0^2\sqrt{r^2}\,r\,dr\,d\theta=\int_0^{2\pi}d\theta\int_0^2 r^2\,dr=\left[\theta\right]_0^{2\pi}\left[\frac{1}{3}r^3\right]_0^2=2\pi\left(\frac{8}{3}\right)=\frac{16}{3}\pi$

20. The paraboloid $z=18-2x^2-2y^2$ intersects the xy-plane in the circle $x^2+y^2=9$, so

$$V=\iint\limits_{x^2+y^2\le 9}\left(18-2x^2-2y^2\right)dA=\iint\limits_{x^2+y^2\le 9}\left[18-2\left(x^2+y^2\right)\right]dA=\int_0^{2\pi}\int_0^3\left(18-2r^2\right)r\,dr\,d\theta$$
$$=\int_0^{2\pi}d\theta\int_0^3\left(18r-2r^3\right)dr=\left[\theta\right]_0^{2\pi}\left[9r^2-\tfrac{1}{2}r^4\right]_0^3=(2\pi)\left(81-\tfrac{81}{2}\right)=81\pi$$

21. The hyperboloid of two sheets $-x^2 - y^2 + z^2 = 1$ intersects the plane $z = 2$ when $-x^2 - y^2 + 4 = 1$ or $x^2 + y^2 = 3$. So the solid region lies above the surface $z = \sqrt{1 + x^2 + y^2}$ and below the plane $z = 2$ for $x^2 + y^2 \le 3$, and its volume is

$$V = \iint_{x^2+y^2 \le 3} \left(2 - \sqrt{1 + x^2 + y^2}\right) dA = \int_0^{2\pi} \int_0^{\sqrt{3}} \left(2 - \sqrt{1 + r^2}\right) r\, dr\, d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^{\sqrt{3}} \left(2r - r\sqrt{1 + r^2}\right) dr = \left[\theta\right]_0^{2\pi} \left[r^2 - \tfrac{1}{3}(1 + r^2)^{3/2}\right]_0^{\sqrt{3}}$$

$$= 2\pi \left(3 - \tfrac{8}{3} - 0 + \tfrac{1}{3}\right) = \tfrac{4}{3}\pi$$

22. The sphere $x^2 + y^2 + z^2 = 16$ intersects the xy-plane in the circle $x^2 + y^2 = 16$, so

$$V = 2 \iint_{4 \le x^2+y^2 \le 16} \sqrt{16 - x^2 - y^2}\, dA \quad \text{[by symmetry]} \quad = 2 \int_0^{2\pi} \int_2^4 \sqrt{16 - r^2}\, r\, dr\, d\theta = 2 \int_0^{2\pi} d\theta \int_2^4 r(16 - r^2)^{1/2} dr$$

$$= 2\left[\theta\right]_0^{2\pi} \left[-\tfrac{1}{3}(16 - r^2)^{3/2}\right]_2^4 = -\tfrac{2}{3}(2\pi)(0 - 12^{3/2}) = \tfrac{4\pi}{3}\left(12\sqrt{12}\right) = 32\sqrt{3}\,\pi$$

23. By symmetry,

$$V = 2 \iint_{x^2+y^2 \le a^2} \sqrt{a^2 - x^2 - y^2}\, dA = 2 \int_0^{2\pi} \int_0^a \sqrt{a^2 - r^2}\, r\, dr\, d\theta = 2 \int_0^{2\pi} d\theta \int_0^a r\sqrt{a^2 - r^2}\, dr$$

$$= 2\left[\theta\right]_0^{2\pi} \left[-\tfrac{1}{3}(a^2 - r^2)^{3/2}\right]_0^a = 2(2\pi)\left(0 + \tfrac{1}{3}a^3\right) = \tfrac{4\pi}{3}a^3$$

24. The paraboloid $z = 1 + 2x^2 + 2y^2$ intersects the plane $z = 7$ when $7 = 1 + 2x^2 + 2y^2$ or $x^2 + y^2 = 3$ and we are restricted to the first octant, so

$$V = \iint_{\substack{x^2+y^2 \le 3,\\ x \ge 0,\, y \ge 0}} \left[7 - \left(1 + 2x^2 + 2y^2\right)\right] dA = \int_0^{\pi/2} \int_0^{\sqrt{3}} \left[7 - \left(1 + 2r^2\right)\right] r\, dr\, d\theta$$

$$= \int_0^{\pi/2} d\theta \int_0^{\sqrt{3}} (6r - 2r^3)\, dr = \left[\theta\right]_0^{\pi/2} \left[3r^2 - \tfrac{1}{2}r^4\right]_0^{\sqrt{3}} = \tfrac{\pi}{2} \cdot \tfrac{9}{2} = \tfrac{9}{4}\pi$$

25. The cone $z = \sqrt{x^2 + y^2}$ intersects the sphere $x^2 + y^2 + z^2 = 1$ when $x^2 + y^2 + \left(\sqrt{x^2 + y^2}\right)^2 = 1$ or $x^2 + y^2 = \frac{1}{2}$. So

$$V = \iint_{x^2+y^2 \le 1/2} \left(\sqrt{1 - x^2 - y^2} - \sqrt{x^2 + y^2}\right) dA = \int_0^{2\pi} \int_0^{1/\sqrt{2}} \left(\sqrt{1 - r^2} - r\right) r\, dr\, d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^{1/\sqrt{2}} \left(r\sqrt{1 - r^2} - r^2\right) dr = \left[\theta\right]_0^{2\pi} \left[-\tfrac{1}{3}(1 - r^2)^{3/2} - \tfrac{1}{3}r^3\right]_0^{1/\sqrt{2}} = 2\pi\left(-\tfrac{1}{3}\right)\left(\tfrac{1}{\sqrt{2}} - 1\right) = \tfrac{\pi}{3}\left(2 - \sqrt{2}\right)$$

26. The two paraboloids intersect when $3x^2 + 3y^2 = 4 - x^2 - y^2$ or $x^2 + y^2 = 1$. So

$$V = \iint_{x^2+y^2 \le 1} \left[(4 - x^2 - y^2) - 3(x^2 + y^2)\right] dA = \int_0^{2\pi} \int_0^1 4(1 - r^2)\, r\, dr\, d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^1 (4r - 4r^3)\, dr = \left[\theta\right]_0^{2\pi} \left[2r^2 - r^4\right]_0^1 = 2\pi$$