Chap. 3: Continuous Time Markov Chains
(Ross Chap 6)

We now consider a class of stochastic processes that contains the Poisson Process, but is also an analog of the discrete time M-C, in continuous time

I. Introduction

As a continuous-time analog of the Chap 1, the following Markovian property characterizes the process

Recall: In discrete time, the Markov property states that

$$P(X_{n+1}=j \mid (X_n, X_{n-1}, \dots) = (i_n, i_{n-1}, \dots))$$
$$= P(X_{n+1}=j \mid X_n = i_n)$$

Def: Let $\{X(t), t \geq 0\}$ be a collection of r.v.'s $X(t)$, each taking values in $\{0, 1, 2, \dots\}$ (discrete state space). $\{X(t), t \geq 0\}$ is a continuous-time Markov chain if

$$P(X(s+t)=j \mid X(s)=i, X(u)=x(u) \text{ for } 0<u<s)$$

$$= P(X(s+t)=j \mid X(s)=i)$$

$\forall s,t \geq 0$, $\forall$ states i,j $\forall x(u)$ $(0 \leq u < s)$

Rmk: • We will always assume here stationarity of the process, i.e.,

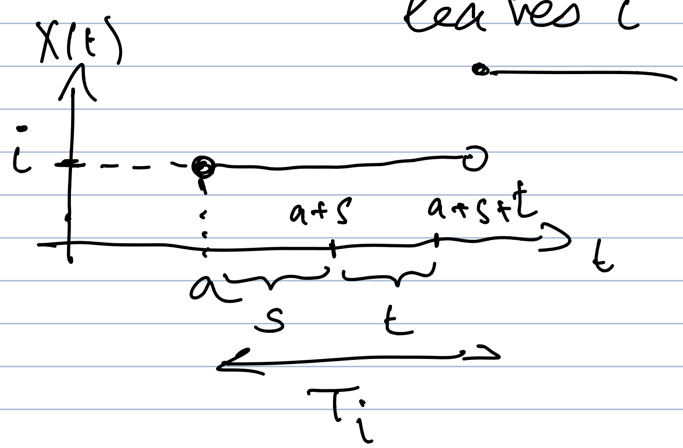
$P(X(s+t)=j \mid X(s)=i)$ is independent of s,

so the process is a homogeneous CTMC

- This has the following key consequence on interarrival times:

Suppose that the MC is in a state i at time a, and let

T_i = additional time until the MC leaves i



Then $P(T_i > s+t \mid T_i > s)$

$$= P(X(r)=i \ \forall r \in [a+s, a+s+t] \mid X(u)=i \ \forall u \in [a, a+s])$$

$$= P\big(X(r) = i \ \forall r \in [a+s, a+s+t] \ \big|\ X(a+s) = i\big)$$

Markov property

$$= P\big(X(a+s+v) = i \ \forall v \in [0,t] \ \big|\ X(a+s) = i\big)$$

$$= P\big(X(v) = i \ \forall v \in [0,t] \ \big|\ X(0) = i\big)$$

stationarity

$$= P(T_i > t)$$

$$\Rightarrow \boxed{P(T_i > s+t \mid T_i > s) = P(T_i > t)}$$

$\Rightarrow$ T_i has no-memory property (chap. 2)

$$\Rightarrow \boxed{T_i \sim Exp(v_i)} \text{ for some } v_i > 0$$

- In addition, T_i must be independent of the next state that the chain jumps to (otherwise, the Markov property would be violated, as the waiting time in a state would affect the next jump outcome)

Conclusion: We can fully describe a CTMC by

- $T_i \sim Exp(v_i)$, $v_i > 0$ (characterizes the sojourn time in a state)

- P_{ij} = transition probability from i to j which satisfy $P_{ii} = 0$ & $\sum_j P_{ij} = 1$

Rmk: This characterization provides a way to simulate the MC ($\to$ next ~~wk~~ week's notebook)

Example (6.11 txt book): 2-state CTMC

state = 0 working machine
1 broken —

$$0 \rightleftarrows 1$$

Assume: $\begin{cases} \text{time to breakdown} \sim Exp(\lambda) \\ \text{— repair} \sim Exp(\mu) \end{cases}$

$$\to \begin{cases} v_0 = \lambda & P_{01} = 1 \\ v_1 = \mu & P_{10} = 1 \end{cases}$$

- Birth and Death Processes

We consider $X(t)$ = Population size at time t, where
- birth (arrival) occurs at a rate λ_n when pop. size is n
- death (departure) — μ_n —

Given $\lambda_0, \lambda_1, \ldots ; \mu_0, \mu_1, \ldots$, with $\mu_0 = 0$, this defines a CTMC $0 \rightleftarrows 1 \rightleftarrows 2 \cdots$,
where

$$\begin{cases} \nu_0 = \lambda_0 \\ \nu_n = \lambda_n + \mu_n \\ (n > 0) \end{cases}$$

← We need to consider for a given population size the time that it takes for either a birth or a death event to occur.

$$T_n = \min\left(\mathrm{Exp}(\lambda_n), \mathrm{Exp}(\mu_n)\right)$$

(birth: $\mathrm{Exp}(\lambda_n)$; death: $\mathrm{Exp}(\mu_n)$)

$$\sim \mathrm{Exp}(\lambda_n + \mu_n)$$

chap. 2

$$\begin{cases} P_{01} = 1 \\ P_{n,n+1} = P(\text{birth time} < \text{death time}) = \dfrac{\lambda_n}{\lambda_n + \mu_n} \\ P_{n,n-1} = P(\text{birth time} > \text{death time}) = \dfrac{\mu_n}{\lambda_n + \mu_n} \\ n > 0 \end{cases}$$

chap. 2

ex: (cf. Ross)

6.2: Poisson process: $\mu_n = 0$, $\lambda_n = \lambda$ $\forall n \geq 0$

6.3: Yule process: $\mu_n = 0$, $\lambda_n = n\lambda$ $\forall n \geq 0$

6.4: Linear growth with immigration: $\mu_n = n\mu$, $\lambda_n = n\lambda + \theta$ $\forall n \geq 0$

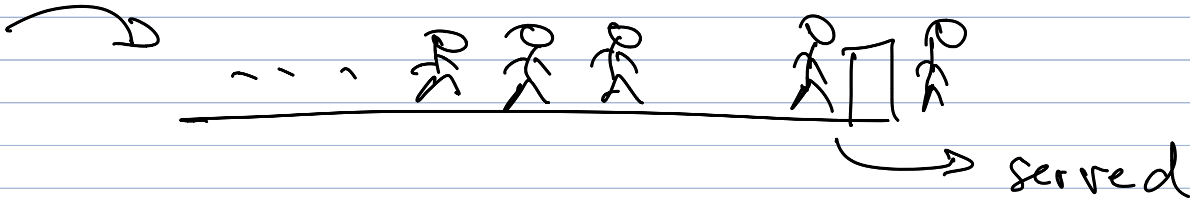
(θ: immigration rate)

6.5: M/M/1 queue (queuing theory)

(M: Markov arrivals; M: Markov arrivals; 1: # server)

→ A single server receives customers as a λ-Poisson process, and service times are independent $Exp(\mu)$ r.v.'s.

arrival



$X(t)$ = # customers in the system (waiting to be served) at time t

$\downarrow$

it is a B-D process

$$0 \rightleftarrows 1 \rightleftarrows 2 \cdots\cdots$$

$\rightarrow \lambda_n = \lambda \qquad \mu_n = \mu$

6.6. M|M|s queue (s servers)

$$\lambda_n = \lambda, \quad \mu_n = \begin{cases} n\mu & 1 \le n \le s \\ s\mu & \text{else} \end{cases}$$

II. Transition probabilities & Kolmogorov eqs.

Goal: Study

$$\boxed{\begin{aligned} P_{ij}(t) &= P(X(s+t)=j \mid X(s)=i) \\ &= P(X(t)=j \mid X(0)=i) \end{aligned}}$$

(stationarity)

Rmk: $P_{ij}(t)$ is the continuous-time analog of $P_{ij}^{(n)}$ studied in discrete time.

Similarly, it satisfies the Chapman-Kolmogorov eq. (7)

Prop (C-K-eq.) $\forall s,t \geq 0$

$$P_{ij}(s+t) = \sum_{k} P_{ik}(s) P_{kj}(t)$$