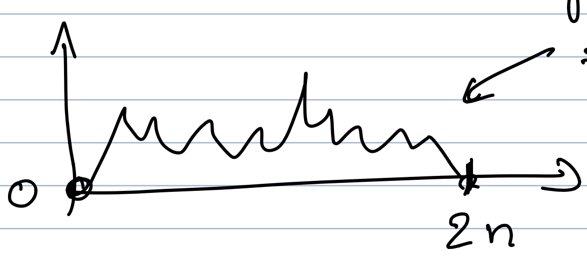
Is the symmetric R.W in $\mathbb{Z}^d$ transient or recurrent?

$\boxed{d=1}$ (Recall: We will use the criterion that $\sum_n P_{ii}^n < \infty$ to show that the M-C is recurrent)

- Starting from 0, one can only come back in an even number of steps



times the MC $\uparrow$ = # times the MC goes $\downarrow$ = n

P_{00}^{2n} = Proba. to move n times backward and n times forward.

1) What is the probability of observing a specific trajectory of 2n steps?
$\rightarrow \dfrac{1}{2^{2n}}$

$(\text{ex}: \underbrace{\uparrow \downarrow \uparrow \uparrow \downarrow \ldots\ldots}_{2n \text{ steps}})$

2) How many trajectories satisfy the condition above.

$\rightarrow \binom{2n}{n}$

$$\Rightarrow P_{00}^{2n} = \frac{1}{2^{2n}} \times \binom{2n}{n}$$

$$\Rightarrow \sum_{n=0}^{+\infty} P_{00}^{2n} = \sum_{n=0}^{+\infty} \frac{\binom{2n}{n}}{2^{2n}} = \sum_{n=0}^{+\infty} \underbrace{\frac{2n!}{(n!)^2\, 2^{2n}}}_{= U_n}$$

Stirling Formula : $n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$

$$\Rightarrow U_n \sim \left(\frac{\cancel{2n}}{\cancel{e}}\right)^{2n} \sqrt{2\pi 2n} \left(\frac{\cancel{e}}{\cancel{n}}\right)^{2n} \times \frac{1}{2\pi n} \times \frac{1}{\cancel{2}^{2n}}$$

$$U_n \sim \frac{1}{\sqrt{\pi n}} = \frac{1}{(\pi n)^{\frac{1}{2}}} \quad \tfrac{1}{2}) < 1$$

$$\Rightarrow \sum_{n=0}^{+\infty} P_{00}^{2n} = \sum_{n=0}^{+\infty} P_{00}^{n} = +\infty$$ so the $\boxed{\text{R-W is recurrent}}$

Rmk : • When the R-W is asymetric $\left(\overset{1-p}{\curvearrowleft}\ \overset{p}{\curvearrowright} \quad p \neq \frac{1}{2} \right)$ we can show that the R-W is transient

• For $d = 2$, one can use a similar approach (and the walk is recurrent), but the probleem gets more and more complex combinatorically as the dimension increases

(For a proof in $d = 3$ - using characteric functions -, see Notes online)

$$\boxed{\text{Result : The walk is } \begin{cases} \text{recurrent if } d = 1, 2 \\ \text{transient if } d \geq 3 \end{cases}}$$

symmetric random

IV - Limiting probabilities

Recall the example of 2 state MC seen in HW 1

$$\begin{tikzcd} 0 \arrow[loop left, "1-p"] \arrow[r, bend left, "p"] & 1 \arrow[l, bend left, "q"] \arrow[loop right, "1-q"] \end{tikzcd}$$

$\rightarrow$ we saw $\lim_{n \to +\infty} P(X_n = i) = \boxed{\begin{cases} \frac{q}{p+q} & i = 0 \\ \frac{p}{p+q} & i = 1 \end{cases}}$

This distribution defines some **limiting probabilities** of the M.C.

Def: Let $(X_n)_{n \geq 0}$ be a M.C with transition matrix $\underline{P}$ and let μ be a probability distribution on its state space. We say that μ is a **stationary distribution** of (X_n) if

$$\mu \underset{\sim}{P} = \mu$$

Prop: If (X_n) converges to a distribution μ as $n \to \infty$ then μ is stationary

Proof: Let μ_k be the distribution of X_k

then $\mu_{k+1} = \mu_k \underset{\sim}{P}$

$$\begin{array}{ccc} \mu_{k+1} & = & \mu_k \underset{\sim}{P} \\ \downarrow k \to +\infty & & \downarrow k \to +\infty \\ \mu & = & \mu \underset{\sim}{P} \end{array}$$

$\boxtimes$

Rmk : The limiting probabilities also account for the proportion of time the MC stays in each state in the long run

- Let m_i = mean time to return starting from i.

Then the proportion of time in state i must be $\frac{1}{m_i}$ in the long run.

Q : Can we guarantee the existence of a stationary distribution?

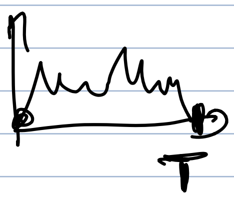
Def : - A recurrent state i is positive-recurrent if $m_i < \infty$ (expected return time to i)

- If i is recurrent but not positive recurrent ($m_i = \infty$) then i is null-recurrent

Ex : Symmetric R.W in $\mathbb{Z}$

Let T = first time $X_n = 0$, starting from 0

so $m_0 = \mathbb{E}(T)$



One can show that :

$$\begin{cases} P(T = 2n+1) = 0 \\ P(T = 2n) = \frac{1}{2n-1}\binom{2n}{n}\frac{1}{2^{2n}} \sim \frac{1}{2\sqrt{\pi}\, n^{3/2}} \end{cases}$$

So $m_0 = \sum_n 2n \underbrace{P(T=2n)}_{\sim \frac{1}{\sqrt{\pi}\sqrt{n}}} = +\infty$

$\Rightarrow$ The chain is null recurrent

Prop (admitted):
- Positive and null recurrence are class properties.
- If the state space is finite, then all recurrent states are positive recurrent

Def: An aperiodic positive recurrent state is called ergodic. An ergodic MC is a MC whose states are all ergodic

Thm (admitted): For an irreducible ergodic MC $\Pi_j = \lim_{n \to +\infty} P_{ij}^n$ exists for all j, and is independent of i. In addition

(i) $\underline{\Pi}$ is the unique solution of $\begin{cases} \underline{\Pi} = \underline{\Pi}\, \underline{P} \\ \sum_j \Pi_j = 1 \end{cases}$

(ii) $\Pi_j = \frac{1}{m_j}$, where m_j = mean return time to j (in particular $\Pi_j > 0$)

(iii) $\Pi_j = \lim_{n \to \infty} \frac{\text{\# visits to } j \text{ by time } n}{n}$

= proportion of time, in the long run, spent in state j

Rmk: In practice, if you have an irreducible MC, then you can try to solve

$$(i) \quad \begin{cases} \pi = \pi P \\ \sum_j \pi_j = 1 \end{cases}$$

and if you solve it (or guess a solution that works), then π is the stationary distribution and the chain is positive recurrent.