Math 302
Final exam
Friday, June 25, 7pm

Instructions

- There are 7 questions on this exam.
- You have 120 minutes to complete the exam, then an additional 20 minutes to upload pictures/scans of your solutions to Canvas.
- Write your name on the top of each page of work that you submit.
- **You must show your work on all problems.** The correct answer with no supporting work may result in no credit. Put a box around your FINAL ANSWER for each problem and cross out any work that you don't want to be graded.
- Give exact answers.
- Any student found engaging in academic misconduct will receive a score of 0 on this exam.

GOOD LUCK!

1. (10 points) There are twelve pitch classes in the chromatic musical scale. Suppose my cat hops onto my piano and steps on six random keys, playing a sequence of six pitches. Assume the pitches are independent, and that each pitch is selected uniformly from all twelve possibilities.

   (a) (3 pts) What is the probability that my cat doesn't play any pitch twice?

   Solution: We can think of this one combinatorially. The number of ways to play six different pitches is $(12)_6$, 12 falling factorial 6, and the total number of possible sequences is 12^6. So the probability is $\frac{12 \cdot 11 \cdots 8 \cdot 7}{12^6}$.

   (b) (3 pts) What is the probability that the cat plays either the first six notes of Beethoven's Fur Elise or the first six notes of Hot Cross Buns? (Note: those two sequences of six notes are different.)

   Solution: The probability of playing any given sequence is 12^{-6}, so since the two sequences are different, the probability of matching one of them is $2/12^6$.

   (c) (4 pts) I will give my cat one pet for each F sharp she plays, two pets for each B, and six pets each time she plays a D, then a G immediately after, then a C immediately after that. What is the expected number of pets my cat gets?

   Solution: we get $(1 + 2)/12$ pets per note on average from the first three conditions, and $6/12^3$ pets for each sequence of three adjacent notes on average. Since there are six notes and four sequences of three adjacent notes, the average number of pets (by linearity of expectation) is $3 \cdot 1/2 + 4 \cdot (6/12^3) = 3/2 + 1/72$.

2. (10 points) An ice cream shop has three kinds of cones: waffle, sugar, and cake. I buy eight ice cream cones, and choose randomly between the three cone options for each order. Suppose I choose each of the eight cones independently, with probability $1/2$ for a waffle cone, probability $1/3$ for a sugar cone, and probability $1/6$ for a cake cone. Let W, S, C denote the total numbers of waffle/sugar/cake cones I get, respectively.

   (a) (2 pts) Compute the conditional expectation $\mathbb{E}[W|S+C]$.

   Solution: Note that $W+S+C=8$, so $W=8-(S+C)$. Thus, given $S+C$, we can completely determine W. So $\mathbb{E}[W|S+C]=W$. Alternatively, by linearity,

   $$\mathbb{E}[W|S+C]=\mathbb{E}[8-S-C|S+C]=8-\mathbb{E}[S+C|S+C]=8-(S+C)=W.$$

   (b) (5 pts) Compute the conditional expectation $\mathbb{E}[W|S]$.

   Solution: Note that conditionally on the event $S=s$, W is distributed as $\text{Binomial}(8-s, \frac{1/2}{1/2+1/6}) = \text{Binomial}(8-s, 3/4)$. To see this, just compute the conditional distribution:

   $$\mathbb{P}(W=w|S=s)=\frac{\binom{8}{s}\binom{8-s}{w}2^{-w}3^{-s}6^{-(8-s-w)}}{\binom{8}{s}3^{-s}(1/2+1/6)^{8-s}}$$
   $$=\binom{8-s}{w}3^{w}4^{s-8}=\mathbb{P}(\text{Bin}(8-s,3/4)=w).$$

   The numerator comes from first choosing which cones were sugar, then which ones were waffle – the remaining cones must be cake. Each such sequence has the same probability. The denominator is also a binomial probability: the marginal distribution of S is $\text{Binomial}(8,1/3)$. Since the expected value of $\text{Binomial}(n,p)$ is np, we get

   $$\mathbb{E}[W|S]=(8-S)\cdot 3/4=6-\frac{3S}{4}.$$

   (c) (3 pts) Compute the conditional probability $\mathbb{P}(W=4|S=2)$.

   Solution: Using the calculation from the last part, it's $\binom{6}{4}3^4 4^{-6}\approx .3$

3. (11 points) Let (X, Y) be a random point in the square

$$R = [1, 2] \times [0, 1] = \{(x, y) : 1 \leq x \leq 2 \text{ and } 0 \leq y \leq 1\}$$

with joint density function

$$f_{X,Y}(x, y) = C \cdot \frac{xy^3}{1 + y^4}$$

where $C = \frac{8}{3 \log 2}$.

(a) (2 pts) Find the marginal PDF of Y.

Solution: $f_Y(y) = \int_1^2 f(x, y)\, dx = \frac{Cy^3}{1+y^4} \int_1^2 x\, dx = \frac{3C}{2} \frac{y^3}{1+y^4}$.

(b) (3 pts) Compute $\mathbb{P}(Y \leq \frac{1}{2})$. [*Hint: Use the substitution* $u = y^4$.]

Solution: $\mathbb{P}(Y \leq 1/2) = \frac{3C}{2} \int_0^{1/2} \frac{y^3}{1+y^4}\, dy = \frac{3C}{8} \log(17/16)$.

(c) (3 pts) Compute the conditional probability $\mathbb{P}(X \leq \frac{3}{2} | Y \leq \frac{1}{2})$.

Solution: Note that $\mathbb{P}(X \leq 3/2, Y \leq 1/2) = \int_0^1 \int_1^{3/2} C \frac{xy^3}{1+y^4}\, dx\, dy = \frac{5C}{32} \log(17/16)$.

(d) (3 pts) Are X, Y independent?

Solution: The marginal of X is $f_X(x) = Cx \int_0^1 \frac{y^3}{1+y^4}\, dy = C\frac{\log 2}{4} x$. The condition $f_X f_Y = f_{X,Y}$ is equivalent to

$$C \frac{\log 2}{4} \cdot \frac{3C}{2} = C,$$

which is true. So X and Y are independent.

4. (11 points) Let $X_1, X_2, \ldots$ be an iid sequence with common distribution $\mathrm{Exp}(\log(2))$, i.e. common PDF

$$f_X(x) = (\log 2) \cdot 2^{-x} \text{ for } x > 0,$$

and for each $n \geq 1$, let Y_n be the random variable

$$Y_n = \begin{cases} 0, & \text{if } X_n \in [0,1] \\ 1, & \text{if } X_n \in [1,2] \\ 2, & \text{if } X_n > 2 \end{cases}.$$

Also, for each $n \geq 1$, let $S_n = \sum_{i=1}^n Y_i Y_{i+1}$.

(a) (2 pts) Find the CDF of X_1 and the PMF of Y_1.

Solution: The CDF of X is $F_X(x) = \int_0^x (\log 2) 2^{-t}\, dt = 1 - 2^{-x}$ for $x > 0$. Thus, the PDF of Y is $\mathbb{P}(Y = 0) = F_X(1) - F_X(0) = 1/2 - 0 = 1/2$, $\mathbb{P}(Y = 1) = 1/4$, and $\mathbb{P}(Y = 2) = 1 - \mathbb{P}(Y = 1) - \mathbb{P}(Y = 0) = 1/4$.

(b) (3 pts) Compute $\mathbb{E}[Y_1]$ and $\mathbb{E}[S_n]$ for each $n \geq 1$.

Solution: $\mathbb{E}Y_1 = 1 \cdot 1/4 + 2 \cdot 1/4 = 3/4$. Since the Y_i are independent, $\mathbb{E}[Y_i Y_{i+1}] = \mathbb{E}[Y]^2 = 9/16$. So by linearity, $\mathbb{E}[S_n] = 9n/16$.

(c) (3 pts) Compute $\mathrm{Cov}(Y_1Y_2, Y_2Y_3)$ for each $n \geq 1$.

Solution: Note $\mathbb{E}[Y_1Y_2Y_2Y_3] = \mathbb{E}[Y_1Y_2^2Y_3] = \mathbb{E}[Y^2]\mathbb{E}[Y]^2$. The former term is $\mathbb{E}[Y^2] = 1 \cdot 1/4 + 4 \cdot 1/4 = 5/4$, so we get

$$\mathrm{Cov}(Y_1Y_2, Y_2Y_3) = (5/4) * (9/16) - (9/16)^2 = 99/256.$$

(d) (3 pts) Compute $\mathrm{Var}(Y_1)$ and $\mathrm{Var}(S_n)$ for each $n \geq 1$.

Solution: Combining the above, $\mathrm{Var}(Y_iY_{i+1}) = (5/4)^2 - (9/16)^2 = 319/256$. By the formula for the variance of a sum, and since Y_iY_{i+1} and Y_jY_{j+1} are independent (and thus covariance 0) if and only if $i \notin \{j-1, j+1\}$,

$$\mathrm{Var}(S_n) = \sum_{i=1}^n \mathrm{Var}(Y_iY_{i+1}) + \sum_{i \neq j} \mathrm{Cov}(Y_iY_{i+1}, Y_jY_{j+1})$$
$$= \frac{319n}{256} + (2n-2)(99/256) = \frac{517n - 198}{256}$$

(There are $2 * (n-2) + 1 * 2 = 2n - 2$ many nonzero terms in the second sum.) Also, $\mathrm{Var}(Y) = 5/4 - 9/16 = 11/16$.

5. (10 points) Let $B_1, B_2, \ldots$ be an iid sequence with common distribution Binomial$(6, \frac{1}{2})$, fix a number $\alpha \in \mathbb{R}$, and let

$$S_n = \sum_{i=1}^{n} (B_i - \alpha) .$$

(a) (1 pt) Show that if $\alpha = 3$, then $\frac{1}{\sqrt{n}} S_n$ converges in distribution as $n \to \infty$, and identify the distribution of the limit random variable Z.

Solution: Straightforward application of CLT.

(b) (3 pts) Use your result from part (a) to fill in the blank in the approximation

$$\mathbb{P}(S_n > x\sqrt{n}) \approx \mathbb{P}(Z > x + \underline{\qquad})$$

Your answer should be an expression involving n and α.

Solution: Let Q_n denote the version of S_n with $\alpha = 3$. Then from (a),

$$\begin{aligned} \mathbb{P}(S_n > x\sqrt{n}) &= \mathbb{P}(\frac{1}{\sqrt{n}} S_n > x) \\ &= \mathbb{P}(\frac{1}{\sqrt{n}} Q_n - (\alpha - 3)\sqrt{n} > x) \\ &\approx \mathbb{P}(N(0,1) > x + (\alpha - 3)\sqrt{n}) \end{aligned}$$

(c) (3 pts) Suppose $\alpha > 3$. Use your approximation from part (b) to compute

$$\lim_{n \to \infty} \mathbb{P}(S_n > x\sqrt{n})$$

for $x \in \mathbb{R}$.

Solution: By the prev calculation, the expression in the limit is like integrating the normal density from $x + C\sqrt{n}$ to infinity. Since $n \to \infty$, this integral converges to 0.

(d) (3 pt) Use the result from (c) to show that if $\alpha > 3$, then $\frac{1}{\sqrt{n}} S_n$ does not converge in distribution.

Solution: The calculation in part (c) shows that the CDF of $\frac{1}{\sqrt{n}} S_n$ converges to the function 1, which is not a valid CDF. So it does not converge in distribution.

6. (8 points) Let $W_1, W_2, \ldots$ be an iid sequence with common distribution Poisson(2), and let

$$T_n = \sum_{i=1}^{n} W_i.$$

(a) (2 pts) Does T_n have Poisson(λ) distribution for some value of λ? Justify. (If you answer 'yes', find λ.)

Solution: Yes, Poissons add, so T_n is Poisson with parameter $\lambda = 2n$.

(b) (4 pts) Show that $\frac{1}{n^2}T_n$ converges in probability to 0.

Solution: Simple application of Chebychev.

(c) (2 pts) Show that $\frac{1}{n}T_n$ converges almost surely to a constant random variable, and identify the constant.

Solution: Simple application of the SLLN. The limit constant is $\mathbb{E}W = 2$.

7. (10 points) Let X be a discrete random variable taking values in $\{-1, 0, 1\}$, and let $M_X(t)$ denote its moment generating function. Assume that

$$M_X(\log(3)) = 2.$$

(a) (5 pts) Find $\mathbb{E}[X^2+2X]$. Any solution relying on the information given below will not count. [*Hint: write everything in terms of the PMF values of X.*]

Solution: Let the PMF of X be given by $f(-1) = a, f(0) = b, f(1) = c$. Then we must have $a + b + c = 1$, and $M_X(t) = ae^{-t} + b + ce^t$, so $M_X(\log 3) = 2$ implies $a/3+b+3c = 2$. Subtracting equations, $-2a/3+2c = 1$, or $-a+3c = 3/2$. Also, note that $\mathbb{E}[X^2 + 2X] = (1 - 2)a + (1 + 2)c = -a + 3c$. Thus $\mathbb{E}[X^2 + 2X] = 3/2$.

For parts (b) and (c), assume further that $M_X'(1) = 1/3$, where M_X' is the derivative of M_X with respect to t.

(b) (2 pts) Find the PMF of X.

Solution: since $M_X'(0) = \mathbb{E}[X]$, the condition is equivalent to $-a + c = 1/3$. Solving the system of all three equations gives $a = 1/4, b = 1/3, c = 7/12$.

(c) (3 pts) Let $h(t)$ be the 2021st derivative of $M_X(t)$ with respect to t. Find $h(0)$.

Solution: $h(0) = \mathbb{E}[X^{2021}]$. But since X takes values $\{0, 1, -1\}$, $X^{2021} = X$, so $h = M_X'$, and $h(0) = \mathbb{E}[X] = 1/3$.