MATH 303 Practice Problems Solutions - Week 8

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Problem 1

Let $N(t)$ be a rate λ Poisson process, and let $S_n = \min\{t : N(t) \geq n\}$ be the time of the nth event. Compute the following:

a. If $X \sim \text{Poisson}(\lambda)$ then $\mathbb{E}(X) = \lambda$ and $\text{Var}(X) = \lambda$. We know from the lectures that $N(3) \sim \text{Poisson}(3\lambda)$ so that
$$\mathbb{E}[N(3)] = 3\lambda.$$

b. Similarly,
$$\text{Var}(N(4)) = 4\lambda.$$

c. S_5 is the instant of the 5-th jump and is thus equal to the sum of 5 i.i.d. exponential distributions of parameter λ. We know that this sum will have the law of $\text{Gamma}(5, \lambda)$ whose density function f is given by

$$f(x) = \frac{\lambda^5 x^4 e^{-\lambda x}}{4!} \mathbf{1}_{x \geq 0}$$

We can now compute the average of this random variable. We obtain

$$\begin{aligned}
\mathbb{E}(S_5) &= \frac{1}{4!} \int_0^\infty x(\lambda x)^4 e^{-\lambda x} \lambda \mathrm{d}x \\
&= \frac{1}{4!\lambda} \int_0^\infty (\lambda x)^5 e^{-\lambda x} \lambda \mathrm{d}x.
\end{aligned}$$

Using the change of variable $\lambda x = u$ gives

$$\begin{aligned}
\mathbb{E}(S_5) &= \frac{1}{4!\lambda} \int_0^\infty u^5 e^{-u} \mathrm{d}u \\
&= \frac{5!}{4!\lambda} = \frac{5}{\lambda}.
\end{aligned}$$

d. The event $\{S_3 > 4\}$ means that the 3-rd jump happens after time 4 which is equivalent to saying that $\{N(4) < 3\}$. This, in turn, is equal to the following disjoint union of events $\{N(4) = 0\} \cup \{N(4) = 1\} \cup \{N(4) = 2\}$. With this expression we are able to compute that

$$\begin{aligned}
\mathbb{P}(S_3 > 4) &= \mathbb{P}(N(4) = 0) + \mathbb{P}(N(4) = 1) + \mathbb{P}(N(4) = 2) \\
&= e^{-4\lambda} \frac{(4\lambda)^0}{0!} + e^{-4\lambda} \frac{(4\lambda)^1}{1!} + e^{-4\lambda} \frac{(4\lambda)^2}{2!} \\
&= e^{-4\lambda} \left(1 + 4\lambda + 8\lambda^2\right).
\end{aligned}$$

e. We argue similarly as in question (d.) to see that $\{S_2 > 4\} = \{N(4) < 2\}$. So

$$\begin{aligned}
\mathbb{P}(S_2 > 4 | N(1) = 3) &= \mathbb{P}(N(4) < 2 | N(1) = 3) \\
&= 0.
\end{aligned}$$

In words, the probability that the second jump happens after time 4 given that at time 1 one already has had exactly 3 jumps is zero since this is impossible. A drawing may be useful.

Problem 2

Arrivals of the number 14 bus form a Poisson process with rate 4 per hour, while arrivals of the number 99 bus form an (independent) Poisson process with rate 10 per hour.

a. The number of 14 busses arriving by time t is modelized by a Poisson process N_{14} of rate $\lambda = 4\,h^{-1}$. The same holds for the busses 99 with rate $\mu = 10\,h^{-1}$ for the process that we denote N_{99}. An we assume that N_{14} and N_{99} are independent. h^{-1} is the appropriate physical unit which stands for $\frac{1}{\text{hours}}$. So our unit of time here is chosen to be the hour and not the second. We let $M(t) = N_{14}(t) + N_{99}(t)$ be the sum of the two Poisson processes, then M is again a Poisson process with rate $\lambda + \mu = 14\,h^{-1}$. To see this, note that M satisfies the axioms of a Poisson process with rate $\lambda + \mu$:

- By independence of N_{14} and N_{99} and since both are Poisson processes we have

$$M(t+s) - M(t) = N_{14}(t+s) - N_{14}(t) + N_{99}(t+s) - N_{99}(t).$$

Since $N_{14}(t+s) - N_{14}(t)$ is independent of $N_{14}(s)$ and $N_{99}(t+s) - N_{99}(t)$ is independent of $N_{99}(s)$ we get that $M(t+s) - M(t)$ is independent of $N_{14}(s) + N_{99}(s) = M(s)$.

- We now check the infinitesimal relations to check that S has the correct rate.

$$\begin{aligned}
\mathbb{P}(M(t+h) - M(t) = 1) &= \mathbb{P}(N_{14}(t+h) - N_{14}(t) = 1, N_{99}(t+h) - N_{99}(t) = 0) \\
&\quad + \mathbb{P}(N_{14}(t+h) - N_{14}(t) = 0, N_{99}(t+h) - N_{99}(t) = 1) \\
&= (\lambda h + o(h))(1 - O(h)) + (\mu h + o(h))(1 - O(h)) \\
&= (\lambda + \mu)h + o(h).
\end{aligned}$$

- The fact that

$$\mathbb{P}(M(t+h) - M(t) > 1) = o(h)$$

is obtained similarly.

We conclude that M is a Poisson process with rate $\lambda + \mu$. And we have re-proved the superposition property which states that if you sum two independent Poisson processes then you obtain another Poisson process with a new rate equal to the sum of the rates of the two summed processes. So it is now easy to compute that at least 8 buses total arrive in one hour. This corresponds to the probability of the event $\{M(1) \geq 8\}$. By the standard properties of the Poisson process we have

$$\begin{aligned}
\mathbb{P}(M(1) \geq 8) &= \sum_{i=8}^{\infty} e^{-(\lambda+\mu)} \frac{(\lambda+\mu)^i}{i!}, \\
&= \sum_{i=8}^{\infty} e^{-14} \frac{14^i}{i!}.
\end{aligned}$$

b. Suppose you arrive at the bus stop, and will only take a number 99 bus. What is the probability that exactly two number 14 buses go by while you wait?

This question reformulates as: What is the probability that the first jump of N_{99} happens exactly between the second and third jumps of N_{14}?

$$\{N_{99}(S_{14}(2)) = 0\} \cap \{N_{99}(S_{14}(3)) \geq 1\}.$$

We can condition on $S_{14}(2) \sim \text{Gamma}(2, \lambda)$. Also, since $S_{14}(3) = S_{14}(2) + T$ where $T \sim \text{Exp}(\lambda)$, we obtain, by independence of N_{14} and N_{99}, that

$$\begin{aligned}
\mathbb{P}(\{N_{99}(S_{14}(2)) = 0\} \cap \{N_{99}(S_{14}(3)) \geq 1\}) &= \int_0^\infty \mathbb{P}(\{N_{99}(t) = 0\} \cap \{N_{99}(T + t) \geq 1\})\lambda^2 t e^{-\lambda t} \mathrm{d}t \\
&= \int_0^\infty \mathbb{P}(N_{99}(T + t) \geq 1 \mid N_{99}(t) = 0)\mathbb{P}(N_{99}(t) = 0)\lambda^2 t e^{-\lambda t} \mathrm{d}t
\end{aligned}$$

We use the memoryless property of N_{99} to see that

$$\begin{aligned}
\mathbb{P}(N_{99}(T + t) \geq 1 \mid N_{99}(t) = 0) &= \mathbb{P}(N_{99}(T) \geq 1) \\
&= \int_0^\infty \mathbb{P}(N_{99}(u) \geq 1)\lambda e^{-\lambda u} \mathrm{d}u \\
&= \int_0^\infty (1 - \mathbb{P}(N_{99}(u) = 0))\lambda e^{-\lambda u} \mathrm{d}u \\
&= \int_0^\infty (1 - e^{-\mu})\lambda e^{-\lambda u} \mathrm{d}u \\
&= 1 - \frac{\lambda}{\lambda + \mu} = \frac{\mu}{\lambda + \mu}.
\end{aligned}$$

Plugging that in the previous expression we get

$$\begin{aligned}
\mathbb{P}(\{N_{99}(S_{14}(2)) = 0\} \cap \{N_{99}(S_{14}(3)) \geq 1\}) &= \frac{\lambda^2 \mu}{\lambda + \mu} \int_0^\infty \mathbb{P}(N_{99}(t) = 0) t e^{-\lambda t} \mathrm{d}t \\
&= \frac{\lambda^2 \mu}{\lambda + \mu} \int_0^\infty t e^{-(\lambda+\mu)t} \mathrm{d}t \\
&= \frac{\lambda^2 \mu}{(\lambda + \mu)^3} \\
&\simeq 0.058.
\end{aligned}$$

This was for the formal proof. An other a bit less formal argument is that you first need to observe a 14 bus before a 99 bus which happens with probability $\frac{\lambda}{\lambda+\mu}$, then observe again a 14 bus before a 99 bus which by memoryless-ness has probability $\frac{\lambda}{\lambda+\mu}$ and lastly you need to see a 99 bus before a 14 bus which happens with probability $\frac{\mu}{\lambda+\mu}$. By multiplying these three probabilities, we get the result again.

c. Using the process M with halved rates introduced in (a.) and the memoryless property of M, we see that the probability that no bus arrives between 8:30 and 8:45 AM is equal to the probability that no bus arrives in a time window that starts at $t = 0$ of 15 min $= \frac{1}{4} h$ which means that we are looking for

$$\begin{aligned}
\mathbb{P}(M(1/4) = 0) &= e^{-(\lambda/2+\mu/2)1/4} \\
&= e^{-7/4}.
\end{aligned}$$

Problem 3

a. Let τ be the time that we wait before getting bored and $\lambda = 1$. We condition on S_1 which is the time at which the first firework explodes, i.e. the first jump in the Poisson process, we obtain that

$$\mathbb{E}[\tau] = \int_0^{\infty} \mathbb{E}(\tau \mid S_1 = t)\lambda e^{-\lambda t} \mathrm{d}t$$

In the integral if $t \geq 4$ then $\mathbb{E}[\tau \mid S_1 = t] = 4$ by definition of τ. If $t \leq 4$ then, by memorylessness, given that $\{S_1 = t\}$ has occured, we will get bored exactly as if the firework had started all over again but we have already waited t. This gives

$$\mathbb{E}(\tau \mid S_1 = t) = t + \mathbb{E}(\tau).$$

Putting this into the previous expression leads us to

$$\mathbb{E}[\tau] = \int_0^4 (t + \mathbb{E}[\tau])\lambda e^{-\lambda t}\, dt + 4 \int_4^{\infty} \lambda e^{-\lambda t}\, dt.$$

This an equation with a single unknown and thus solving this equation with unknown $\mathbb{E}(\tau)$ and with $\lambda = 1$ gives

$$\begin{aligned}
\mathbb{E}(\tau) &= \int_0^4 te^{-t}\mathrm{d}t + \mathbb{E}(\tau)\int_0^4 e^{-t}\mathrm{d}t + 4\int_4^{\infty} e^{-t}\mathrm{d}t \\
&= \left[-te^{-t}\right]_0^4 + \int_0^4 e^{-t}\mathrm{d}t + \mathbb{E}(\tau) \times \left[-e^{-t}\right]_0^4 + 4\left[-e^{-t}\right]_4^{\infty} \\
&= -4e^{-4} + 1 - e^{-4} + (1 - e^{-4})\mathbb{E}(\tau) + 4e^{-4} \\
&= 1 - e^{-4} + (1 - e^{-4})\mathbb{E}(\tau),
\end{aligned}$$

and thus

$$\mathbb{E}(\tau) = e^4 - 1.$$

b. If $\{T > 4k\}$ occurs then this implies that on the time intervals of length 4

$$(0, 4], (4, 2 \times 4], (2 \times 4, 3 \times 4], \cdots (4(k-1), 4k],$$

there is a firework going off. This latter event has probability

$$\mathbb{P}(N(4) - N(0) \geq 1, N(8) - N(4) \geq 1, \cdots, N(4k) - N(4(k-1)) \geq 1).$$

Since $N(t+s) - N(t)$ is independent of the $(N(u))_{u \leq t}$, i.e. of all the Poisson process up to time t we obtain that all the random variables $N(4i) - N(4(i-1))$ are mutually independent for any $i \geq 1$ which implies by also using stationnarity of the increments and $N(0) = 0$ that

$$\mathbb{P}(N(4) - N(0) \geq 1, N(8) - N(4) \geq 1, \cdots, N(4k) - N(4(k-1)) \geq 1) \leq \mathbb{P}(N(4) \geq 1)^k.$$

The power is k because there are k intervals of length 4 in total. Finally since $\mathbb{P}(N(4) \geq 1) = 1 - e^{-4}$ we obtain the claimed upper bound.

For the lower bound, we reason similarly. We need to find a sufficient event for $\{T > 4k\}$ to hold. Note that for $\{T > 4k\}$ to hold, it is enough that on the time intervals

$$(0, 2], (2, 4], (4, 6], \cdots, (4k - 2, 4k],$$

there is a firework going off since then you can not find any time interval of length 4 with no firework going off. Exactly as before, this event has probability $\mathbb{P}(N(2) \geq 1)^{2k}$ but with power $2k$ now because there are $2k$ disjoint intervals of length 2 of the form $(2i, 2(i+1)]$ inside $[0, 4k]$. Since $\mathbb{P}(N(2) \geq 1) = 1 - e^{-2}$ we get the lower bound. Again, it is very useful to do a drawing to understand the proof.

Problem 4

Jacob and Khanh are on a skiing trip at Whistler mountain. Tired from shredding the halfpipe, they take a breather by the gondola, and watch skiers and snowboarders go by. Assume that people arrive at the times of a Poisson process with rate ten per minute, and that $\frac{2}{3}$ of riders are skiers, and $\frac{1}{3}$ are snowboarders.

a. Every time that a person passes by, there is probability 2/3 that it is a skier and probability 1/3 that it is a snowboarder. So we see that if 4 people passes by,

- $\mathbb{P}(0 \text{ of them were skiers}) = \binom{4}{0}(2/3)^0(1/3)^4$;
- $\mathbb{P}(1 \text{ of them were skiers}) = \binom{4}{1}(2/3)^1(1/3)^3$;
- $\mathbb{P}(2 \text{ of them were skiers}) = \binom{4}{2}(2/3)^2(1/3)^2$;
- $\mathbb{P}(3 \text{ of them were skiers}) = \binom{4}{3}(2/3)^3(1/3)^1$;
- $\mathbb{P}(4 \text{ of them were skiers}) = \binom{4}{4}(2/3)^4(1/3)^0$.

b. We can answer this answer from different perspectives. Here is one : Assume that, when someone arrive, the probability that it is a skier is p and the probability that it is a snowboarder is $1-p$. Now we see skiers arriving as a Poisson process with rate $\lambda = p \times 10$ per minute and snowboarders as a Poisson process with rate $\mu = (1-p) \times 10$ per minute. Then the probability that **exactly** k skiers arrive before a snowboarder arrive is denoted P_k and satisfies

$$P_k = \left(\frac{\lambda}{\lambda+\mu}\right) P_{k-1}.$$

Indeed the probability that $\mathbb{P}(\text{Exp}(\lambda) < \text{Exp}(\mu)) = \frac{\lambda}{\lambda+\mu}$ and after a skier arrives we use the memoryless property to see that we still need to observe exactly $k-1$ skiers and then a snowboarder, and it is as if we started from $t=0$ again. Thus, iteratively

$$P_k = \left(\frac{\lambda}{\lambda+\mu}\right)^{k-1} P_1.$$

Then P_1 satisfies

$$P_1 = \frac{\lambda}{\lambda+\mu}\frac{\mu}{\lambda+\mu}$$

since we need to observe a skier before a snowboarder and then, by the memoryless property, we need to observe a snowboarder before a skier, starting from $t=0$ again. We are looking for the probability that **at least** 2 skiers arrive which is then equal to

$$\begin{aligned}
\sum_{k\geq 2} P_k &= \frac{\mu}{\lambda+\mu}\sum_{k\geq 2}\left(\frac{\lambda}{\lambda+\mu}\right)^k \\
&= \frac{\mu}{\lambda+\mu}\left(\frac{\lambda}{\lambda+\mu}\right)^2 \frac{1}{1-\frac{\lambda}{\lambda+\mu}} \\
&= \left(\frac{\lambda}{\lambda+\mu}\right)^2
\end{aligned}$$

Finally, we see that here

$$\left(\frac{\lambda}{\lambda+\mu}\right)^2 = \left(\frac{10p}{10(1-p)+10p}\right)^2$$
$$= p^2.$$

If $p = 2/3$, then $p^2 = 4/9 < 1/2$ and Jacob has more chances to loose his bet. For Jacob's bet to be fair, we would need $p^2 = 1/2$ and therefore $p = 1/\sqrt{2}$.

c. Denote S for skiers, SB for snowboarders and P for people. We could imagine that the skiers and snowboarders process are two independent process in the first place and then the answer would be immediate by independence: we would have

$$\begin{aligned}\mathbb{P}(\text{exactly 6 S in 1 min} \mid \text{exactly 4 SB in 1 min}) &= \mathbb{P}(\text{exactly 6 S in 1 min})\\ &= e^{-2/3\times 10}\frac{(10\times 2/3)^6}{6!}.\end{aligned}$$

But the way the problem is presented here makes it less clear. Indeed, there is one people process for which every time there is a jump we randomly (and independently) decide if that is a skier or a snowboarder. So since the the skiers process and snowboarders process are made both "made" from the people process, it is less obvious that they are independent. It turns out that **they are from the thinning property** seen in the lectures and thus one can conclude directly by the above. Since this solution was written before the thinning property was seen, I give hereafter a direct proof but it is tedious as you can see.

Then we are looking for

$$\begin{aligned}&\mathbb{P}(\text{exactly 6 S in 1 min} \mid \text{exactly 4 SB in 1 min})\\ &\quad= \mathbb{P}(\text{exactly 10 P in 1 min} \mid \text{exactly 4 SB in 1 min})\\ &\quad= \frac{\mathbb{P}(\{\text{exactly 10 P in 1 min}\}, \{\text{exactly 4 SB in 1 min}\})}{\mathbb{P}(\text{exactly 4 SB in 1 min})}\\ &\quad= \mathbb{P}(\text{exactly 4 SB in 1 min} \mid \text{exactly 10 P in 1 min})\frac{\mathbb{P}(\text{exactly 10 P in 1 min})}{\mathbb{P}(\text{exactly 4 SB in 1 min})}\end{aligned}$$

And we see that

$$\mathbb{P}(\text{exactly 4 SB in 1 min} \mid \text{exactly 10 P in 1 min}) = \binom{10}{4}(1/3)^4(2/3)^6,$$

since we only need to choose which among the ten people are snowboarders. Then

$$\mathbb{P}(\text{exactly 10 P in 1 min}) = e^{-(1/3+2/3)\times 10}\frac{((1/3+2/3)\times 10\times 1)^{10}}{10!},$$

since the arrival of people (S + SB) is a Poisson process of rate $(2/3+1/3)\times 10$. Also,

$$\mathbb{P}(\text{exactly 4 SB in 1 min}) = e^{-1/3\times 10\times 1}\frac{(1/3\times 10\times 1)^4}{4!},$$

for a similar reason. Combining the three above expressions gives

$$\begin{aligned}\mathbb{P}(\text{exactly 6 S in 1 min} \mid \text{exactly 4 SB in 1 min}) &= \binom{10}{4}(1/3)^4(2/3)^6 e^{-20/3}\frac{10^{10}3^4 4!}{10^4 10!}\\ &= e^{-20/3}\frac{(20/3)^6}{6!}.\end{aligned}$$

Problem 5

Complete the following exercises from lecture.

a. If $X_1, X_2 \sim \mathrm{Exp}(\lambda)$, then the probability density function of X_1+X_2 is $f(x) = \lambda^2 x \exp(-\lambda x)$, and

Let $X_1, X_2 \sim \mathrm{Exp}(\lambda)$, then

$$\begin{aligned}
\mathbb{P}(X_1 + X_2 > x) &= \int_0^\infty \mathbb{P}(X_1 + y > x)\lambda e^{-\lambda y}\mathrm{d}y \\
&= \int_0^\infty \mathbb{P}(X_1 > x - y)\lambda e^{-\lambda y}\mathrm{d}y \\
&= \int_0^x \mathbb{P}(X_1 > x - y)\lambda e^{-\lambda y}\mathrm{d}y + \int_x^\infty 1 \times \lambda e^{-\lambda y}\mathrm{d}y \\
&= \int_0^x e^{-\lambda(x-y)}\lambda e^{-\lambda y}\mathrm{d}y + e^{-\lambda x} \\
&= \int_0^x \lambda e^{-\lambda x}\mathrm{d}y + e^{-\lambda x} \\
&= \lambda x e^{-\lambda x} + e^{-\lambda x}.
\end{aligned}$$

Differentiating with respect to x gives $-f$ and thus the result.

If $X_1 \sim \mathrm{Exp}(\lambda_1)$ and $X_2 \sim \mathrm{Exp}(\lambda_2)$, then

$$\begin{aligned}
\mathbb{P}(X_1 < X_2) &= \int_0^\infty \mathbb{P}(x < X_2)e^{-\lambda_1 x}\mathrm{d}x \\
&= \int_0^\infty e^{-\lambda_2 x}\lambda_1 e^{-\lambda_1 x}\mathrm{d}x \\
&= \lambda_1 \int_0^\infty e^{-(\lambda_1+\lambda_2)x}\mathrm{d}x \\
&= \frac{\lambda_1}{\lambda_1 + \lambda_2}.
\end{aligned}$$

b. Suppose that X is memoryless and supported on the positive reals. Write $f(x) = \mathbb{P}(X > x)$, then the memoryless property writes

$$\mathbb{P}(X > x + y \mid X > x) = \mathbb{P}(X > y),$$

which can be rewritten by decomposition conditional probabilities as

$$\frac{f(x+y)}{f(x)} = f(y) \qquad (\forall x, y \geq 0).$$

And thus f satisfies the functional equation

$$f(x+y) = f(x)f(y) \qquad (\forall x, y \geq 0).$$

We don't even need to assume that f is differentiable but eventually need only continuous. The proof goes by proving that $f(n) = f(1)^n$ for any $n \in \mathbb{N}$, then we extend the result to $f(q) = f(1)^q$ for $q \in \mathbb{Q}^+$ and finally we extend this result on $\mathbb{R}^+$ by density of $\mathbb{Q}$ in $\mathbb{R}$ and by continuity of f. Taking $\lambda := \log(f(1))$ gives the result. More details can be found for example on this forum.

c. We use an integration by parts on the expression given for $\mathbb{P}(N(t) \geq k+1)$: we differentiate u^k and integrate $e^{-\lambda t}$. We have

$$\begin{aligned}\mathbb{P}(N(t) \geq k+1) &= \frac{\lambda^{k+1}}{k!} \int_0^t u^k e^{-\lambda u}\,\mathrm{d}u \\ &= \frac{\lambda^{k+1}}{k!} \left[-u^k \frac{1}{\lambda} e^{-\lambda u}\right]_0^t + \frac{\lambda^{k+1}}{k!} \int_0^t \frac{k}{\lambda} e^{-\lambda u} u^{k-1} \mathrm{d}u \\ &= -\frac{(\lambda t)^k}{k!} e^{-\lambda t} + \mathbb{P}(N(t) \geq k),\end{aligned}$$

which, upon reordering, is the result.

d. First note that k is fixed and only $n \to \infty$. We mainly need two ingredients for this question: first

$$(1 - \frac{\mu}{n})^n \to e^{-\mu} \qquad \text{and} \qquad (1 - \frac{\mu}{n})^{-k} \to 1 \qquad \text{as } n \to \infty.$$

Then we need Stirling's asymptotics for $n!$:

$$\begin{gathered} n! \sim \sqrt{2\pi n} n^n e^{-n} \text{ and thus} \\ (n-k)! \sim \sqrt{2\pi(n-k)}(n-k)^{n-k} e^{-(n-k)}. \end{gathered}$$

This implies that

$$\binom{n}{k} \sim \frac{1}{k!} \frac{\sqrt{2\pi n}}{\sqrt{2\pi(n-k)}} e^{-k} \frac{n^n}{(n-k)^{n-k}}$$

And since

$$\begin{aligned}\frac{n^n}{(n-k)^{n-k}} &= (n-k)^k (1 - \frac{k}{n})^{-n} \\ &\sim n^k e^k,\end{aligned}$$

we get that

$$\binom{n}{k} \sim \frac{1}{k!} n^k$$

which implies the result by combining the several asymptotics.