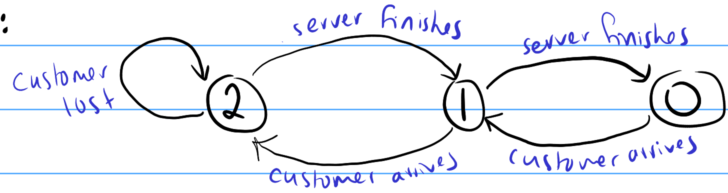
Example ② (5.47 in Ross) (Queueing Problem)

- Two server queueing system where customers arrive according to a Poisson process with rate λ, and service times are exponentially distributed with rate μ.
- Assume that arrivals finding both servers busy leave immediately without receiving any service (i.e. customer is lost), whereas finding at least one server free they immediately enter service & depart when their service is completed.

We can sketch a diagram to visualize the transitions between how many servers are busy:



(a) If both servers are presently busy, find the expected time until the next customer enters the system.

(b) Starting empty, find the expected time until both servers are busy

(c) Find the expected time between two successive lost customers

Solution:

(a) Let T_0 = time until one of the two servers is free

T_1 = time after T_0 until the next customer arrives

$$\mathbb{E}[\text{time until next customer enters system} \mid \text{both servers busy}]$$
$$= \mathbb{E}[T_0 + T_1 \mid \text{both servers busy}]$$
$$= \underbrace{\mathbb{E}[T_0 \mid \text{both servers busy}]}_{\substack{\min(\text{time } S_1 \text{ done},\ \text{time } S_2 \text{ done}) \\ \sim \text{Exp}(\mu+\mu) = \text{Exp}(2\mu)}} + \underbrace{\mathbb{E}[T_1 \mid \text{both servers busy}]}_{\substack{\text{Time until next customer arrives if process} \\ \text{started at time } T_0 \\ \sim \text{Exp}(\lambda) \\ \text{by memoryless}}}$$

$$= \boxed{\frac{1}{2\mu} + \frac{1}{\lambda}}$$

(b) Let T_i = time until both servers are busy given we start with i busy servers

Then we want to find:

$$\mathbb{E}[T_0] = \mathbb{E}[\text{ time until one customer arrives + time remaining until both are busy }]$$
$$= \frac{1}{\lambda} + \mathbb{E}[T_1]$$

Let:

S_i = time of the 1st event (arrival or departure) given we start w/ i busy servers

$$X_i = \begin{cases} 1 & \text{if the 1st event is an arrival given we start with } i \text{ busy servers} \\ 0 & \text{if " " " a departure " " " " " " "} \end{cases}$$

Y_i = additional time after the first event until both servers are busy given we start with i busy servers

Then we have:

$$\mathbb{E}[T_1] = \mathbb{E}[S_1] + \mathbb{E}[Y_1]$$

Note: in this step we have conditioned on which of the two events occurs (arrival or departure) in an analogous way to how we addressed some discrete time Markov chain problems previously

where:

$$\mathbb{E}[S_1] = \frac{1}{\lambda+\mu} \quad (\text{min of two Exp. r.v.})$$

$$\mathbb{E}[Y_1] = \mathbb{E}[Y_1 \mid X_1 = 0]\, P(X_1 = 0) + \cancel{\mathbb{E}[Y_1 \mid X_1 = 1]\, P(X_1 = 1)}$$
$$= \mathbb{E}[Y_1 \mid X_1 = 0] \frac{\mu}{\lambda+\mu}$$

memoryless

$$= \frac{\mu}{\lambda+\mu} \mathbb{E}[T_0]$$

$$\Rightarrow \mathbb{E}[T_1] = \frac{1}{\lambda+\mu} + \frac{\mu}{\lambda+\mu} \mathbb{E}[T_0]$$

and,

$$\Rightarrow \mathbb{E}[T_0] = \frac{1}{\lambda} + \frac{1}{\lambda+\mu} + \frac{\mu}{\lambda+\mu} \mathbb{E}[T_0]$$

From here we can solve to find $\boxed{\mathbb{E}[T_0] = \frac{2\lambda+\mu}{\lambda^2}}$

(c) Imagine a customer was just lost, then we must have that both servers are busy. Now let:

L_i = time until a customer is lost when you start with i busy servers

Want to find: $\mathbb{E}[L_2] = \mathbb{E}[S_2 + Y_2]$

$$= \mathbb{E}[S_2] + \mathbb{E}[Y_2]$$

where

$$\mathbb{E}[S_2] = \frac{1}{2\mu+\lambda} \quad \text{(min of 3 Exp. r.v.)}$$

$$\mathbb{E}[Y_2] = \mathbb{E}[Y_2 \mid X_2 = 1]\, P(X_2 = 1) + \mathbb{E}[Y_2 \mid X_2 = 0]\, P(X_2 = 0)$$

$$= \mathbb{E}[L_1] \frac{2\mu}{2\mu+\lambda}$$

where $\mathbb{E}[L_1] = \mathbb{E}[T_1 + L_2]$

$$= \underbrace{\mathbb{E}[T_1]}_{\text{can find from (b) to be } \frac{\lambda+\mu}{\lambda^2}} + \mathbb{E}[L_2]$$

Putting it all together:

$$\mathbb{E}[L_2] = \frac{1}{2\mu+\lambda} + \frac{2\mu}{2\mu+\lambda}\left(\frac{\lambda+\mu}{\lambda^2} + \mathbb{E}[L_2]\right)$$

Solve for $\mathbb{E}[L_2]$:

$$\boxed{\mathbb{E}[L_2] = \frac{1}{\lambda} + \frac{2\mu(\lambda+\mu)}{\lambda^3}}$$