Feb 2 (1)

last week (recall)

<u>Thm</u>: For an irreducible ergodic MC,
$\pi_j = \lim_{j \to \infty} P_{ij}^n$ exists for all j and is independent of i, and:

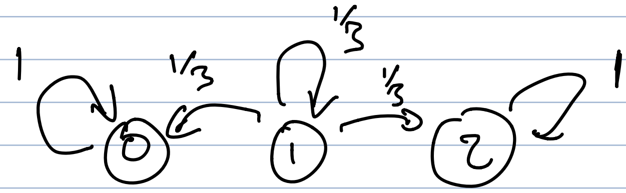
(i) $\underline{\pi}$ is the unique solution of $\begin{cases} \underline{\pi} = \underline{\pi}\,\underline{P} \\ \sum_j \pi_j = 1 \end{cases}$

(ii) $\pi_j = \dfrac{1}{m_j}$, where m_j = mean time to return to j

(iii) $\pi_j = \lim_{n \to \infty} \dfrac{\#\text{ visits to } j \text{ by time } n}{n}$

<u>Rmk</u>: • Without irreducibility, we lose uniqueness of the stationary distribution.

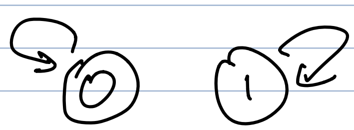
$$\underline{P} = \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 1 \end{pmatrix} \end{array}$$



There are 3 communicating classes

$$\pi_1 = \left(\frac{3}{4}, 0, \frac{1}{4}\right) \text{ and } \pi_2 = \left(\frac{1}{3}, 0, \frac{2}{3}\right)$$

are both stationary distributions

(rmk: for  all distributions are stationary)

- Without aperiodicity, the eq. in (i) still has a unique solution, and (ii) and (iii) also hold, but we do not have $\pi_j = \lim_{n \to +\infty} P_{ij}^n$ (limit does not exist)

ex : $\textcircled{0} \rightleftarrows \textcircled{1}$ $\qquad P_{01}^n = \begin{cases} 0 & \text{if } n \text{ is even} \\ > 0 & \text{if } n \text{ is odd} \end{cases}$

$\Rightarrow$ the limit does not exist

Prop : For an irreducible MC

(i) if there is no solution of $\pi P = \pi$, then the MC is transient or null-recurrent and $\pi_j = 0$

(ii) if there is a solution, then the MC is positive recurrent

In practice, if you have an irreducible MC, then you can try to solve $\begin{cases} \Sigma \pi P = \pi \\ \Sigma \pi_k = 1 \end{cases}$

if you solve it (or guess a solution that works), then π is the stationary distribution and the chain is positive recurrent

Example: MC with transition matrix

$$\underline{P} = \begin{array}{c} \begin{pmatrix} .5 & .4 & .1 \\ .3 & .4 & .3 \\ .2 & .3 & .5 \end{pmatrix} \end{array} \begin{array}{c} 0 \\ 1 \\ 2 \end{array}$$

Q: find the long run proportion of time spent in each state.

Solution: Since the chain is ergodic and irreducible we can solve

$$\begin{cases} \underline{\Pi}\,\underline{P} = \underline{\Pi} \\ \Pi_0 + \Pi_1 + \Pi_2 = 1 \end{cases}$$

to get exercise $\underline{\Pi} = \left(\frac{21}{62}, \frac{23}{62}, \frac{18}{62}\right)$

so the proportion of time spent in state

$$\begin{cases} 0 \text{ is } \frac{21}{62} \\ 1 \text{ is } 23/62 \\ 2 \text{ is } \frac{18}{62} \end{cases}$$

Example: A specific case where the stationary distribution is straightforward is if we have doubly stochastic matrices.

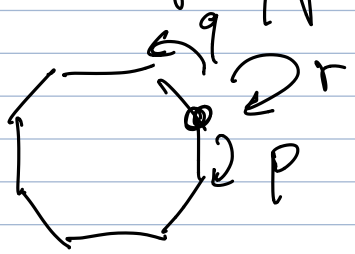
Def: Let M be a stochastic matrix ($m_{ij} \geq 0$ $\forall i,j$ and $\sum_{j=1}^{n} m_{ij} = 1$ for all i). Then M is **doubly stochastic** ${}^tM$ is stochastic

i.e. $\sum_{i=1}^{n} m_{ij} = 1$ for all j

(rmk: $({}^tM)_{ij} = (M)_{ji}$)

rmk: In other words, all columns and rows sum to 1

ex: RW on a polygon (HW week 2)



$p+q+r=1$

$$\underline{P} = \begin{pmatrix} r & p & & (0) & q \\ q & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & (0) & \ddots & \ddots & p \\ p & & & q & r \end{pmatrix}$$

$\underline{P}$ is doubly stochastic

Prop: The uniform distribution ($\underline{\pi} = [\frac{1}{n}, \dots, \frac{1}{n}]$, where n is the number of states) is stationary for a doubly stochastic matrix

(5)

Proof: For π uniform and for all j

$$(\underline{\pi}\,\underline{P})_j = \sum_i \underbrace{\pi_i}_{=\,1/n} P_{ij} = \frac{1}{n}\overbrace{\sum_i P_{ij}}^{=\,1 \text{ because } P \text{ is doubly stochastic}} = \frac{1}{n}\cdot 1 = \pi_j \qquad \boxtimes$$

$\underline{P}$: doubly stochastic

$(\underline{\pi}\,\underline{P})_j$: we want to show that this is equal to π_j

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Summary of the results: (How to study a MC)

Discrete-time MC
- transition probabilities
  - transition matrix
    - n-step transition probabilities (C-K eq.)
      - $\sum_n P^n_{xy} = \text{ or } \neq \infty \Rightarrow$ recurrence or transience
    - stationary distribution (when it exists): $\pi P = \pi$
  - transition diagram
    - communication class
      - infinite state space $\rightarrow$ recurrence or transience
      - finite state space
        - closed class $\rightarrow$ positive recurrent: $E(\text{time in } i) = \frac{1}{\pi_i}$
        - non-closed class $\rightarrow$ transience (no stationary distribution)
    - period

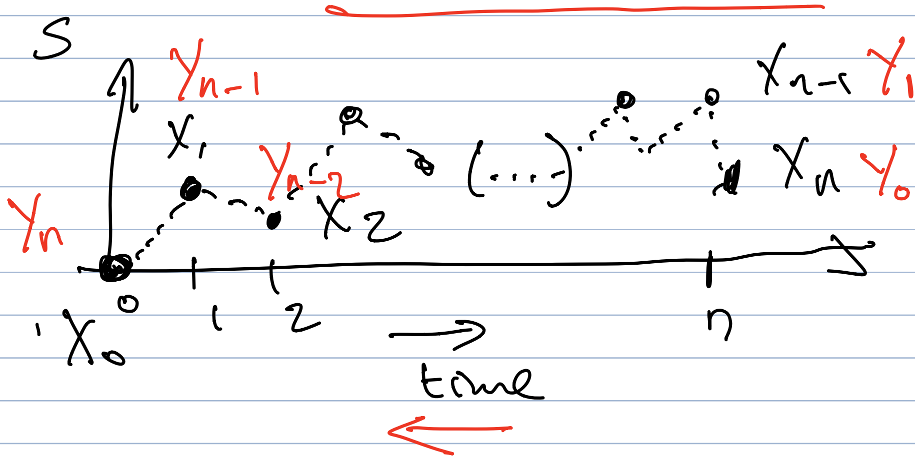
Aperiodicity ($d=1$) + unique stationary distribution $\longleftrightarrow$ positive recurrent

$\rightarrow$ ergodicity $\longrightarrow$ (then) $P^n_{xy} \xrightarrow[n\to\infty]{} \pi(y)$

V. Time reversibility

- We see one last concept useful to find limiting probabilities.

- Given a MC $X_0, X_1, \dots, X_n$, we can observe it backwards in time



- The Markov property (past and future are independent given the present) is symmetric under this reversal, so this allows us to see Y_k also as a MC.

Question: • What is the relation between the original and the reversed process?

- Can we say something about the limiting probabilities?

<u>Thm</u>: Given a MC $(X_n)_{0 \le n \le N}$ with stationary distribution π (we assume that it exists) and with $P(X_0 = j) = \pi_j$ (we initially sample the MC from π), let $Y_n = X_{N-n}$.

Then $(Y_n)_{0 \le n \le N}$ is a MC with stationary distribution π and with transition probabilities

$$Q_{ij} = P_{ji} \frac{\pi_j}{\pi_i}$$

<u>Proof</u>: i) Markov property for Y_n (to be shown):

$$\underset{LHS}{P(Y_n = j \mid Y_{n-1} = i, Y_{n-2} = \dots)} = \underset{RHS}{P(Y_n = j \mid Y_{n-1} = i)}.$$

$$LHS = \frac{P(X_{N-n} = j, X_{N-n+1} = i, \overbrace{X_{N-n+2} = \dots, \dots}^{E})}{P(X_{N-n+1} = i, E)}$$

Markov property

$$= \frac{P(E \mid \cancel{X_{N-n} = j}, X_{N-n+1} = i) \cdot P(X_{N-n} = j, X_{N-n+1} = i)}{P(X_{N-n+1} = i) \cdot P(E \mid X_{N-n+1} = i)}$$

$$= \frac{P(\overset{Y_n}{X_{N-n}} = j, \overset{Y_{n-1}}{X_{N-n+1}} = i)}{P(X_{N-n+1} = i)} = P(Y_n = j \mid Y_{n-1} = i) \qquad \boxtimes$$

Y_{n-1}''

RHS

<u>exercise</u> : Take a 2-state MC and try to find $P(Y_0 = 1 \mid Y_1 = 0)$