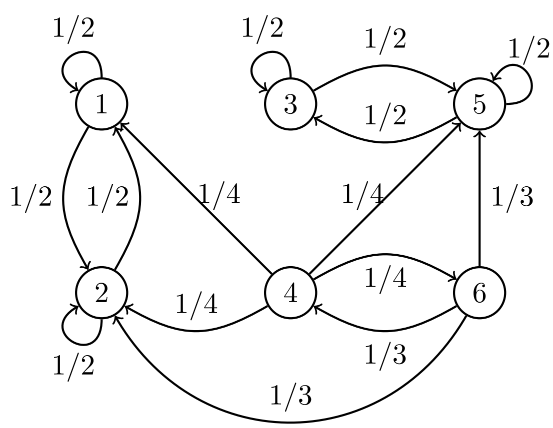
MATH 303 Practice Problems Solutions - Week 2 *January 22, 2021*

Problem 1

1.

$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{4} & 0 & 0 & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} & 0 \end{pmatrix}$$

2. The transition diagram may be represented as:



There are three communicating classes:

- $C_1 = \{1,2\}$ ($P_{12} > 0$ and $P_{21} > 0$ so $1 \leftrightarrow 2$, and 1 and 2 do not give access to other states).
- $C_2 = \{3,5\}$ (by the same argument as above).
- $C_3 = \{4,6\}$ ($4 \leftrightarrow 6$ and they do not communicate with any other classes).

3. Since periodicity, transience and recurrence are class properties, we can consider each class to study the properties of all the states they contain.

States in:

- C_1 are aperiodic (since $p_{11}^{(1)} > 0$ for example).
- C_2 are aperiodic (same as above).
- C_3 are of period 2 (starting from any state it takes an even number of steps to revisit the state)

States in:

- C_1 and C_2 are recurrent (since the class is closed and finite)
- C_3 are transient (for example $4 \to 2$ and $2 \not\to 4$).

Problem 2

1.

$$\begin{pmatrix} r & p & 0 & 0 & 0 & q \\ q & r & p & 0 & 0 & 0 \\ 0 & q & r & p & 0 & 0 \\ 0 & 0 & q & r & p & 0 \\ 0 & 0 & 0 & q & r & p \\ p & 0 & 0 & 0 & q & r \end{pmatrix}$$

2. For all states i, j, $i \leftrightarrow j$ since one can move from any vertex to any other by successively moving in one direction. So the chain has only one communicating class and it is irreducible.

3. If $r > 0$, then $p_{11} = r > 0$, so the period divides 1, so it is equal to 1 and the chain is aperiodic.

4. If N is even, then there exists a proper 2-coloring of the N-gon: the states can be colored red or blue such that $P_{ij} > 0$ if and only if i and j have different colors. Suppose state 1 is colored blue, and consider any path $1 \to 1$ of length n, say $(X_0 = 1, X_1 = x_1, X_2 = x_2, \ldots, X_{n-1} = x_{n-1}, X_n = 1)$, that has positive probability to occur. Since x_j and x_{j-1} have different colors for $j = 1, 2, \ldots, n$, and $X_n = 1$ is blue, there are exactly $\frac{n}{2}$ blue states in the sequence $(x_0, x_1, \ldots, x_{n-1})$, namely $x_0, x_2, \ldots, x_{n-2}$. Thus $\frac{n}{2}$ is an integer, so n must be even, and $P_{11}^n = 0$ if n is odd. Finally, $p_{11}^2 > p_{12}^1 p_{21}^1$, so $d(1) = 2$.

   If N is odd then,
   $$p_{11}^{(N)} \geq p_{12}p_{23}p_{34}\ldots p_{N1} = q^N > 0$$
   so $d(1)|N$. As we observed previously, $d(1)|2$ (this did not depend on the parity of N). So $d(1) = 1$ and the chain is aperiodic.

5. Let $(X_n)_{n\geq 0}$ describe the position of the random walk at time n, and suppose that we start at $X_0 = 1$. If $X_1 = 2$, the the probability to visit all states before returning to 1 is the same as the probability of hitting $N - 1$ before getting ruined in the gambler's ruin problem, with a probability of winning a single game p and starting from 1. From class we know that this probability is
   $$P_1 = \begin{cases} \frac{a-1}{a^{N-1}-1}, \text{ where } a = \frac{1}{p} - 1, & \text{if } p \neq \frac{1}{2} \\ \frac{1}{N-1} & \text{if } p = \frac{1}{2}. \end{cases}$$
   Similarly, if $X_1 = N$, we have that the probability of visiting all states before returning to 1 is
   $$P_2 = \begin{cases} \frac{b-1}{b^{N-1}-1}, \text{ where } b = \frac{1}{q} - 1, & \text{if } q \neq \frac{1}{2} \\ \frac{1}{N-1} & \text{if } q = \frac{1}{2}. \end{cases}$$
   The reasoning is the same for any initial position X_0, so we conclude that
   $$P(\text{visiting all states before returning to initial position}) = pP_1 + qP_2$$
   $$= \begin{cases} p\left(\frac{a-1}{a^{N-1}-1}\right) + q\left(\frac{b-1}{b^{N-1}-1}\right), \text{ where } a = \frac{1}{p} - 1,\ b = \frac{1}{q} - 1, & \text{if } p \neq \frac{1}{2} \\ \frac{1}{N-1} & \text{if } p = \frac{1}{2}. \end{cases}$$

Problem 3

1. The 2-step transition matrix is $M^2 = \dfrac{1}{4}\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$

2. $(1,0,0)M^2 = \frac{1}{4}(2,1,1)$, so $\mathbb{E}(X_2) = \dfrac{1}{4}(0\times 2 + 1\times 1 + 2\times 1) = \dfrac{3}{4}$.

3. Let $N(i)$ be the mean number of transitions before visiting 2, given that $X_0 = i$, $i = 0, 1$. Then, by conditioning on the outcome of X_1, $N(0) = (1+N(0))p_{00} + (1+N(1))p_{01} + p_{02}$, and $N(1) = (1+N(0))p_{10} + (1+N(1))p_{11} + p_{12}$. Solving for $N(0)$ and $N(1)$, we obtain that the answer is $N(0) = 4$.

Problem 4

1.
$$\mathbb{E}_0(T) = 0, \quad \mathbb{E}_N(T) = 0.$$

2.
$$\begin{aligned}\mathbb{E}_k(T) &= \mathbb{E}_k(T \mid \text{win first game})P(\text{win first game}) + \mathbb{E}_k(T \mid \text{lose first game})P(\text{lose first game}) \\ &= (\mathbb{E}_{k+1}(T)+1)p + (\mathbb{E}_{k-1}(T)+1)(1-p) \\ &\iff x_k = (x_{k+1}+1)p + (x_{k-1}+1)(1-p) \\ &\iff px_{k+1} - x_k + (1-p)x_{k-1} = -1\end{aligned}$$

3. (a) We solve the characteristic equation
$$pX^2 - X + (1-p) = 0$$
with discriminant
$$\Delta = 1 - 4p(1-p) = (2p-1)^2$$
   - If We assume $p \neq \frac{1}{2}$ so $\Delta > 0$ and
   $$y_n = A\left(\frac{1+2p-1}{2p}\right)^n + B\left(\frac{1-2p+1}{2p}\right)^n$$
   so
   $$y_n = A + B\left(\frac{1-p}{p}\right)^n \text{ if } p \neq \frac{1}{2}.$$
   - If $p = \frac{1}{2}$ then $\Delta = 0$ and
   $$y_n = A\left(\frac{1}{2p}\right)^n + Bn\left(\frac{1}{2p}\right)^n$$
   so
   $$y_n = A + Bn \text{ if } p = \frac{1}{2}.$$

(b) Assuming $p \neq \frac{1}{2}$, replacing x_i with Ci in $(*)$ gives $C = -\frac{1}{2p-1}$. (*remark:* when $p = \frac{1}{2}$, replacing x_i with Ci^2 in $(*)$ gives $C = -1$)

(c) Initial conditions impose

$$\begin{cases} A + B = 0, \\ A + B(\alpha-1)^N = -\frac{N}{2p-1}, \text{ where } \alpha = \frac{1}{p} \end{cases} \iff \begin{cases} A = -B, \\ -B(1-(\alpha-1)^N) = \frac{N}{2p-1}, \end{cases} \tag{1}$$

$$\iff \begin{cases} A = -\frac{N}{1-2p}\frac{1}{1-(\alpha-1)^N}, \\ B = \frac{N}{1-2p}\frac{1}{1-(\alpha-1)^N}, \end{cases} \tag{2}$$

So

$$\mathbb{E}_k(T) = x_k = y_k + Ck = \frac{1}{1-2p}\left(\frac{-N}{1-(\alpha-1)^N} + \frac{N(\alpha-1)^k}{1-(\alpha-1)^N} + k\right)$$

for $p \neq \frac{1}{2}$,

$$\mathbb{E}_k(T) = \frac{1}{1-2p}\left(k - N\frac{1-(\alpha-1)^k}{1-(\alpha-1)^N}\right)$$

where $\alpha = \frac{1}{p}$.

Remark: If $p = \frac{1}{2}$, similarly, we find

$$\begin{cases} A = 0 \\ B = N \end{cases}$$

and $\mathbb{E}_k(T) = Nk - k^2$ so for $p = \frac{1}{2}$ we have $\mathbb{E}_k(T) = k(N-k)$.