MATH 303 Practice Problems - Week 6

February 26, 2021

Problem 1 (Some more results on generating functions)

We establish here some further properties of the generating function, which has been seen in class to study branching processes. We recall that for a random variable X on $\mathbb{N} = \{0, 1, \ldots\}$, we can associate the probability generating function $G_X(s) = \sum_{k=0}^{+\infty} P(X = k)s^k$, for all $s \in [-1, 1]$.
1. For $i \geq 1$, show that $G_X^{(i)}(s) = \sum_{k=i}^{+\infty} \frac{k!}{(k-i)!} P(X = k)s^{k-i}$, where $G_X^{(i)}$ is the i-th order derivative of G_X.
2. Deduce that two random variables with the same generating function follow the same law (*hint:* Consider the generating function and its derivatives at 0).
3. Use expressions of $G'(s)$ and $G''(s)$ at $s = 1$ to find expressions of $\mathbb{E}(X)$ and $Var(X)$ as a function of G' and G''.

Problem 2 (finding the probability of extinction)

Compute the probability generating function of X, and the probability of extinction of the branching process associated with the following reproduction law X:
1. $P(X = 0) = \frac{1}{3}$ and $P(X = 2) = \frac{2}{3}$.
2. $X \sim$ Binomial $\left(2, \frac{1}{4}\right)$.
3. $X \sim$ Geometric $\left(\frac{1}{4}\right)$. (Remark: $P(X = k) = (1-p)^k p$, where $p = \frac{1}{4}$ and $k \in \{0, 1, 2, \ldots\}$)

Problem 3 (Expectation and variance of the branching process)

1. Let $S_N = \sum_{i=1}^{N} X_i$, where X_i are i.i.d random variables and N is a random variable independent of the X_i's. Show that

$$\mathbb{E}(S_N) = \mathbb{E}(N)\mathbb{E}(X) \text{ , and } Var(S_N) = Var(N)(\mathbb{E}(X))^2 + \mathbb{E}(N)Var(X),$$

where $\mathbb{E}(X)$ and $Var(X)$ respectively are the common expectation and variance of the X_i's. (*hint*: use Problem 1 and the fact -shown in class- that $G_{S_N}(s) = G_N(G_X(s))$ and differentiate $G_{S_N}(s)$, using $(f \circ g(x))' = g'(x)f'(g(x))$).
2. We now consider the branching process Z_n, with μ being the mean of the reproduction law and σ^2 its variance. Show that $\mathbb{E}(Z_n) = \mu\mathbb{E}(Z_{n-1})$ for all $n \geq 1$, and deduce $\mathbb{E}(Z_n)$ as a function of μ and n.
3. Similarly, use question 2 to find $Var(Z_{n+1})$ as a function of $Var(Z_n)$. Deduce that $Var(Z_{n+1}) = \mu^n\sigma^2(1 + \mu + \ldots + \mu^n)$, and

$$Var(Z_n) = \begin{cases} \mu^{n-1}\sigma^2 \frac{1-\mu^n}{1-\mu} & \text{if } \mu \neq 1 \\ n\sigma^2 & \text{if } \mu = 1 \end{cases}$$

4. Application: Compute the expectation and variance of Z_n for the reproduction laws introduced in Problem 1.

Problem 4 (practicing with the exponential distribution)

Remark: We will review the exponential distribution early in week 7.

Let X be an exponential variable with unknown parameter λ. We know that $\mathbb{P}(X \leq 10) = \frac{5}{9}$.

1. Compute $\mathbb{P}(X \geq 15)$.
2. Compute $\mathbb{P}(X \geq 15 | X \geq 10)$ (use the memoryless property).
3. Compute $\mathbb{P}(X \geq 15 | 10 \leq X \leq 20)$.
4. Find the number $y > 0$ for which $\mathbb{P}(y \leq X \leq 2y)$ is the largest.

Problem 5 (Jupyter Notebook)

1. Use the week 6 Jupyter notebook to implement and simulate the branching processes given in Problem 2, and compare the theoretical probability of extinction with the values obtained from simulation. (Remark: Be careful if you use the `numpy.random.geometric` function: Compare the pmf of Problem 3.3 with that in https://numpy.org/doc/stable/reference/random/generated/numpy.random.geometric.html)
2. Extract from simulations the mean and variance of the branching process as a function of n, and compare with results from Problem 3
3. (to go further) We consider a process Z_n, where $Z_0 = 1$, and at each generation $n \geq 1$,

$$Z_{n+1} = I_n + \sum_{i=1}^{Z_n} X_{n,i},$$

where the $X_{n,i}$'s are i.i.d. with common distribution $X \sim Binomial(1, p)$, where $0 < p < 1$, and the I_n's are i.i.d and independent from the $X_{n,i}$'s, with common distribution $I \sim Poisson(\mu)$ where $\mu > 0$ (this can be seen as a branching process where an immigrant population arrives at each generation and independently reproduces at the next one). Implement the process and extract the distribution of Z_n for n large enough. Compare this empirical distribution with the Poisson distribution of parameter $\frac{\mu}{1-p}$, for different values of μ and p. What does it suggest for the limiting probabilities of the process?