MATH 303 Practice Problems - Weeks 11-12

April 2, 2021

Problem 1

A frog is in a pond with 5 water lilies numbered from 1 to 5. With exponential rate 1, the frog leaves its current water lily, chooses a new one uniformly among the four others and jumps to it. We assume the frog starts from lily 1. Let $X(t)$ be the number of the lily where the frog is at time t.

1. Admitting that $X(t)$ is a continuous-time Markov chain, give its parameters (i.e. the v_i and p_{ij} of the course). No proof is required.

2. Let $p_{1j}(t) = \mathbb{P}\left(X(t) = j | X(0) = 1\right)$. Explain why $p_{12}(t) = p_{13}(t) = p_{14}(t) = p_{15}(t)$ (no computations required).

3. Write the backward Chapman–Kolmogorov equation, and prove that

$$p'_{11}(t) = \frac{1}{4} - \frac{5}{4} p_{11}(t).$$

4. (**optional**) Solve this equation to compute $p_{11}(t)$.

Problem 2

A one-cell organism can be in two possible states called A and B. An organism in state A switches to state B with exponential rate α. An organism in state B divides into two organism in state A with exponential rate β.

Define an appropriate continuous-time Markov chain for a population of such organisms, and give the parameters of the model. No proof is required.

Hint: The state space is formed of pairs of integers such as $\{2, 1\}$. For example, the probability to pass from 2 organisms of type A and 1 of type B to 4 of type A and none of type B can be denoted by $p_{\{2,1\},\{4,0\}}$.

Problem 3

We consider a group of 4 students among which a rumour is spreading. At time 0, only one student is aware of the rumour. Whenever two students meet, if one is aware of the rumour and not the other, the second one becomes aware of it. We assume that for any pair of students, the times at which they meet forms a Poisson process with rate 1. We also assume that, for the 6 possible pairs of students, we get 6 independent Poisson processes.

1. Let $A(t)$ be the number of students aware of the rumour at time t. We admit this is a continuous-time Markov chain. Give its parameters. No proof is required.

2. Let T be the first time at which all students become aware of the rumour. Compute $\mathbb{E}[T]$.

Problem 4

We consider a machine that can work for an exponential time duration of parameter λ before failing. It can have two types of failures: when the machine fails, it is a failure of type 1 with probability p, and of type 2 with probability $1 - p$. For a type 1 failure, the time to repair the machine is exponential with parameter μ_1. For a type 2 failure, it is exponential with parameter μ_2.

1. Describe this situation using a continuous-time Markov chain with 3 states and give the parameters of the model.

2. In the long run, what is the proportion of time where the machine is working? Where it is down because of a type 1 failure? Because of a type 2 failure?

Problem 5

In a factory, there are 4 machines and 2 repairmen. The duration of a machine before breaking is exponential with rate $\frac{1}{20}$. Once broken, the amount of time it takes to a repairman to repair it is $Exp(1/5)$. We assume the the two repairmen cannot repair the same machine at the same time.

1. In the long run, what is the proportion of time where both repairmen are busy?
2. In the long run, what is the average number of broken machines?

Problem 6

We consider a counter with two servers, where customers arrive at exponential rate λ and join a queue. When a server completes a service, the first customer in the queue joins this server. If a customer finds both servers free, he joins server 1 with probability $\frac{1}{2}$ and server 2 with probability $\frac{1}{2}$. The service time is $Exp(\mu_1)$ for server 1 and $Exp(\mu_2)$ for server 2. We assume $\mu_1 + \mu_2 > \lambda$.

1. What is the set of all possible states of the system?
Hint: The number of customers in the system is not always sufficient to describe the state.
2. Describe this situation using a continuous-time Markov chain and give the parameters of the model.
3. Find the limiting probabilities of this chain.
4. We now assume that server 1 is more efficient than server 2, i.e. $\mu_1 > \mu_2$, but when a customer finds both servers free, he always joins server 1. According to the value of λ, μ_1 and μ_2, which server will be the busiest? Prove that if $\mu_2 < \mu_1 < 2\mu_2$, then server 1 is always busier, but if $\mu_1 > \mu_2$, then it depends on λ.

Problem 7 (From Final exam 2020)

In a factory, there are 3 machines and one repairman. We assume that each machine has a failure with rate 1, in which case it breaks. Moreover, the machines are linked in such a way that, every time a machine has a failure, there is a probability $\frac{1}{2}$ that all the other machines break at the same time. On the other hand, the repairman can only repair one machine at the same time, and does so whenever there is a broken machine. The repair times are exponential variables with rate 4.

1. Let $W(t)$ be the number of working machines at time t. Admitting that $W(t)$ is a continuous-time Markov chain, give its parameters and justify your answers (we only ask for the transition rates q_{ij}, as defined in class).
2. Is $W(t)$ reversible? Why?
3. Compute the limiting probabilities for $W(t)$.
4. The repairman would like to have at least one working machine as often as possible. For this, he tries another strategy: he repairs a machine only when all machines are broken. Which is the better strategy?

Problem 8 (From Final exam 2020)

In a factory, there are two machines, each of which can be in three states: working, half-broken or fully broken. Each working machine becomes half-broken with rate 1, and each half-broken machine becomes fully broken with rate 2. There is one repairman, who can only repair one machine at a time. The time to turn a fully broken machine to half-broken is an exponential variable with parameter 2. The time to turn a half-broken machine into working is also exponential with parameter 2. We assume that the repairman prioritizes having as many working machines as possible (i.e. when there is a choice, he fixes a half-broken machine first if possible).

1. What are the possible states of this system? *Hint:* You should get 6 states in total. If there are more, you can do something simpler!
2. Describe the factory as a continuous-time Markov chain, and give all the parameters of the model (we only ask for the transition rates q_{ij}, as defined in class). Justify your answers.
3. Find the limiting probabilities of this chain.
4. In the long run, what is the average number of working machines?

Jupyter notebook problems

Check the week 12 notebook

Recommended Problems

These provide additional practice. Textbook Chapter 6 (12th ed.): Exercises 3, 4, 9, 16, 24, 29.