Vector fields and gradients (MATH 324, 5/17/19)

1. Consider the vector field

$$F(x,y) = \langle P(x,y), Q(x,y)\rangle = \frac{\langle -y, x\rangle}{x^2+y^2}.$$

Note that F is defined everywhere except at $(0,0)$.

a. Suppose D is any simply connected region that does not contain $(0,0)$. Show that F is conservative as a vector field on D by checking that $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$.

b. Let C be the unit circle, oriented counter-clockwise. Compute $\int_C F \cdot dr$, and use this to show that F is not conservative on $\mathbb{R}^2 \setminus \{(0,0)\}$.

c. Compute $F(x,y)$ at the points $(1,0), (0,1), (-1,0), (0,-1), (1,1)$, and $(-1,1)$, and draw those vectors in $\mathbb{R}^2$. How would you describe F in words?

d. Suppose F describes the flow of water at each point in a pond. If a boat was dropped in the pond with no initial velocity, and was taken by the current of F, what path would the boat take?

2. In this problem, you will investigate the gradient using polar coordinates. So far, we've been working with the formula for the gradient $\nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle$ in Cartesian coordinates. Using the chain rule, we can write the gradient in polar coordinates. To start, for a function $f(r, \theta)$,

$$\frac{\partial f}{\partial x} = \frac{\partial f}{dr}\frac{\partial r}{\partial x} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial x}.$$

We know $r^2 = x^2 + y^2$ and $\tan\theta = \frac{y}{x}$, so differentiating implicitly gives

$$2r\frac{\partial r}{\partial x} = 2x, \text{ or } \frac{\partial r}{dx} = \frac{x}{r} = \cos\theta,$$

and

$$\sec^2\theta \frac{\partial \theta}{\partial x} = -\frac{y}{x^2}, \text{ or } \frac{\partial \theta}{\partial x} = -\frac{r\sin\theta\cos^2\theta}{r^2\cos^2\theta} = -\frac{1}{r}\sin\theta.$$

In other words,

$$\frac{\partial f}{\partial x} = \cos\theta\frac{\partial f}{\partial r} - \frac{1}{r}\sin\theta\frac{\partial f}{\partial \theta}.$$

a. Use the chain rule the same way to derive a similar equation for $\frac{\partial f}{\partial y}$ in terms of r's and θ's.

b. Give a formula for $\nabla f(r, \theta) = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle$ in polar coordinates using your result from part a.

c. Use the vectors $\hat{r} = \langle \cos\theta, \sin\theta \rangle$ and $\hat{\theta} = \langle -\sin\theta, \cos\theta \rangle$ to write your formula from b for ∇f as

$$\nabla f = \hat{r}\frac{\partial f}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial f}{\partial \theta}.$$

The unit vectors $\hat{r}$ and $\hat{\theta}$ represent the radial and angular directions on $\mathbb{R}^2$: $\hat{r}$ points radially outward, and $\hat{\theta}$ is normal to $\hat{r}$, pointing in the counter-clockwise direction. Think of them as the unit vectors corresponding to $\hat{\imath}$ and $\hat{\jmath}$, but in polar-land. (The vectors $\hat{r}$ and $\hat{\theta}$ are an orthonormal basis for $\mathbb{R}^2$ at each point.)

d. Let $f(r, \theta) = r^2$, and $g(r, \theta) = r\sin\theta$: find ∇f and ∇g in polar coordinates using the formula above. Find $f(x, y)$ and $g(x, y)$, and check that the usual gradient formula agrees with your result.

e. Recall the vector field from problem 1, i.e.

$$F(x, y) = \frac{\langle -y, x \rangle}{x^2 + y^2}.$$

Write F in polar coordinates. That is, find functions $u(r, \theta)$ and $v(r, \theta)$ such that

$$F(r\cos\theta, r\sin\theta) = u(r, \theta)\hat{r} + v(r, \theta)\hat{\theta}.$$

f. Find a potential function for F in polar coordinates. That is, find a function $h = h(r, \theta)$ such that $\nabla h = F$. Use this to explain why F is not conservative.