MATH 303 Practice Problems Solutions - Week 11-12 *April, 2021*

Problem 1

1. We have $v_i = 1$ for all i, and $p_{ij} = \frac{1}{4}$ for all i and j from 1 to 5 such that $i \neq j$.

2. If we start from lily number 1, then the lilies 2, 3, 4 and 5 play similar roles in the problem, so the chances to be on each of them should be the same.

3. The forwards Chapman–Kolmogorov equation can be written:

$$p'_{11}(t) = \sum_{k=2}^{5} p_{1k}(t) \times \frac{1}{4} - p_{11}(t) \times 1.$$

Since the $p_{1k}(t)$ for $k = 2, 3, 4, 5$ are all equal and $\sum_{k=1}^{5} p_{1k}(t) = 1$, we have

$$p_{1k}(t) = \frac{1}{4}(1 - p_{11}(t))$$

for $k = 2, 3, 4, 5$, so the forwards Chapman–Kolmogorov equation becomes

$$p'_{11}(t) = 4 \times \frac{1}{4}(1 - p_{11}(t)) \times \frac{1}{4} - p_{11}(t) = \frac{1}{4} - \frac{5}{4} p_{11}(t).$$

4. We have a linear differential equation of the form $p'(t) = ap(t) + \theta$, with $a = -\frac{5}{4}$ and $\theta = \frac{1}{4}$. The solutions are of the form

$$p_{11}(t) = Ce^{-5t/4} + \frac{1}{5},$$

where C is a constant (see MATH 215/255, or look for how to solve first order differential equations with constant coefficients). To find C, we use $p_{11}(0) = 1$, so $C + \frac{1}{5} = 11$, so $C = \frac{4}{5}$ so

$$p_{11}(t) = \frac{4}{5} e^{-5t/4} + \frac{1}{5}.$$

Problem 2

We will denote by $\{i, j\}$ the state where there are i cells of type A and j of type B.
If we have i individuals of type A and j individuals of type B, the rate at which there is an A cell switching to B is $i\alpha$, and the rate at which there is a B cell that splits is $j\beta$, so we have

$$v_{\{i,j\}} = i\alpha + j\beta.$$

Moreover, when an event occurs, the probability that is is an A cell switching to B is $\frac{i\alpha}{i\alpha + j\beta}$, and the probability that it is a B cell splitting is $\frac{j\beta}{i\alpha + j\beta}$. In the first case, the new state of the population is $\{i-1, j+1\}$. In the second, the new state is $\{i+2, j-1\}$. Therefore, we have

$$p_{\{i,j\},\{i-1,j+1\}} = \frac{i\alpha}{i\alpha + j\beta}, \quad p_{\{i,j\},\{i+2,j-1\}} = \frac{j\beta}{i\alpha + j\beta}$$

and $p_{\{i,j\},\{k,\ell\}} = 0$ in all other cases.

Problem 3

1. New students can only become aware of the rumour one at a time, so the process has to start from 1 and then jump to 2, then 3 ad finally 4. In other words, we have $p_{12} = p_{23} = p_{34} = 1$. Therefore, the only thing we need to compute are the v_i.
When $A(t) = 1$, only one student (let us call her Alice) is aware of the rumour, so three possible students meetings can make $A(t)$ jump to 2 (the meetings where Alice meets one of the three other students). Each meeting occurs at rate 1, so such meetings occur at rate 3, so $v_1 = 3$.
When $A(t) = 2$, four possible meetings can make $A(t)$ jump to 3 (one of the two aware students meets one of the two unaware ones), so $v_2 = 4$.
When $A(t) = 3$, three possible meetings can make $A(t)$ jump to 4 (one of the three aware students meets the last unaware one), so $v_3 = 3$.

2. The time it takes to jump from 1 to 2 is $Exp(3)$, so it has expectation $\frac{1}{3}$. Similarly, the time it takes to jump from 2 to 3 has expectation $\frac{1}{4}$ and the time it takes to jump from 3 to 4 has expectation $\frac{1}{3}$. Since T is the sum of these three durations, it has expectation

$$\mathbb{E}[T] = \frac{1}{3} + \frac{1}{4} + \frac{1}{3} = \frac{11}{12}.$$

Problem 4

1. We denote by 0 the state where the machine is working, by 1 the state where it has a failure of type 1 and by 2 the state where it has a failure of type 2. The statement of the problem can be translated as:

$$v_0 = \lambda, \quad v_1 = \mu_1 \quad \text{and} \quad v_2 = \mu_2,$$

with

$$p_{0,1} = p, \quad p_{0,2} = 1 - p, \quad p_{1,0} = p_{2,0} = 1 \quad \text{and} \quad p_{1,2} = p_{2,1} = 0.$$

2. This is an irreducible CTMC with finite state space, so it has limiting probabilities (π_i) satisfying the equations:

$$\lambda\pi_0 = \mu_1\pi_1 + \mu_2\pi_2,$$
$$\mu_1\pi_1 = p\lambda\pi_0,$$
$$\mu_2\pi_2 = (1-p)\lambda\pi_0,$$
$$\pi_0 + \pi_1 + \pi_2 = 1.$$

In particular, the second and third equations give $\pi_1 = \frac{p\lambda}{\mu_1}\pi_0$ and $\pi_2 = \frac{(1-p)\lambda}{\mu_2}\pi_0$, so the fourth equations gives

$$\left(1 + \frac{p\lambda}{\mu_1} + \frac{(1-p)\lambda}{\mu_2}\right)\pi_0 = 1,$$

so

$$\pi_0 = \frac{1}{1 + \frac{p\lambda}{\mu_1} + \frac{(1-p)\lambda}{\mu_2}}, \quad \pi_1 = \frac{p\lambda/\mu_1}{1 + \frac{p\lambda}{\mu_1} + \frac{(1-p)\lambda}{\mu_2}} \quad \text{and} \quad \pi_2 = \frac{(1-p)\lambda/\mu_2}{1 + \frac{p\lambda}{\mu_1} + \frac{(1-p)\lambda}{\mu_2}}.$$

Note: It is also possible to find π by using reversibility (to be covered at the end of the chapter).