MATH 303 Practice Problems Solutions - Week 6

Jamuary 29, 2021

Problem 1

1. We show the result by induction.

   For $i = 1$, differentiating each term of the series $G_X(s) = \sum_{k=0}^{+\infty} P(X = k)s^k$ yields

   $$G'_X(s) = \sum_{k=1}^{+\infty} kP(x = k)s^{k-1} = \sum_{k=1}^{+\infty} \frac{k!}{(k-1)!} P(x = k)s^{k-1}.$$

   For $i \geq 1$, and assuming $G_X^{(i)}(s) = \sum_{k=i}^{+\infty} \frac{k!}{(k-i)!} P(X = k)s^{k-i}$, differentiating $G_X^{(i)}$ yields

   $$G_X^{(i+1)}(s) = \sum_{k=i+1}^{+\infty} \frac{k!}{(k-i)!} P(x = k)(k-i)s^{k-i-1} = \sum_{k=i+1}^{+\infty} \frac{k!}{(k-i-1)!} P(x = k)s^{k-i-1},$$

   which proves the recurrence.

2. Using question 1, we have that for all $i \geq 0$, $G_X^{(i)}(0) = \sum_{k=i}^{+\infty} \frac{k!}{(k-i)!} P(X = k)0^{k-i} = i!P(X = i)$ (all the terms of the series are 0, except for $k = i$), so

   $$P(X = i) = \frac{G_X^{(i)}(0)}{i!}.$$

   Thus, if two random variables X and Y have the same generating function, $P(X = i) = P(Y = i)$ for all $i \in \mathbb{N}$, so they follow the same law.

3. Using question 1, we have $G'_X(1) = \sum_{k=1}^{+\infty} kP(x = k) = \sum_{k=0}^{+\infty} kP(x = k)$, so

   $$\mathbb{E}(X) = G'_X(1).$$

   Similarly, $G''_X(s) = \sum_{k=2}^{+\infty} k(k-1)P(x = k)s^{k-2}$, so $G''_X(1) = \sum_{k=2}^{+\infty} k^2 P(X = k) - kP(X = k) = \sum_{k=1}^{+\infty} k^2 P(X = k) - \sum_{k=1}^{+\infty} kP(X = k) = \mathbb{E}(X^2) - \mathbb{E}(X)$.

   Since $Var(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2$, we obtain

   $$Var(X) = G''_X(1) + G'_X(1) - (G'_X(1))^2.$$

   (or equivalently, $G''_X(1) = Var(X) - \mathbb{E}(X) + (\mathbb{E}(X))^2$)

Problem 2

The probability of extinction P_e is the smallest fixed point of the generating function in (0,1).

1. $G_X(s) = \frac{1}{3} + \frac{2}{3}s^2$; and $G_X(s) = s \iff 1 - 3s + 2s^2 = 0$, with 2 roots $s_1 = \frac{1}{2}$ and $s_2 = 1$, so $P_e = \frac{1}{2}$.

2. $G_X(s) = \left(\frac{3+s}{4}\right)^2$; $\mathbb{E}(X) = \frac{1}{2} < 1$, so $P_e = 1$.

3. $G_X(s) = \sum_{k=0}^{+\infty} \left(\frac{1}{4}\right)\left(1 - \frac{1}{4}\right)^k s^k = \left(\frac{1}{4}\right) \sum_{k=0}^{+\infty} \left(\frac{3s}{4}\right)^k = \frac{1}{4-3s}$.

   $G_X(s) = s \iff 1 - 4s + 3s^2 = 0$, with 2 roots $s_1 = \frac{1}{3}$ and $s_2 = 1$, so $P_e = \frac{1}{3}$.

Problem 3

1. By differentiating $G_{S_N}(s) = G_N(G_X(s))$, we obtain $G'_{S_N}(s) = G'_X(s)G'_N(G_X(s))$. Since $G_X(1) = 1$, we obtain, for $s = 1$, $G'_{S_N}(1) = G'_X(1)G'_N(1)$. From Problem 1, we thus have

$$\mathbb{E}(S_N) = \mathbb{E}(N)\mathbb{E}(X).$$

By differentiating twice, we obtain $G''_{S_N}(s) = G''_X(s)G'_N(G_X(s)) + (G'_X(s))^2G''_N(G_X(s))$, so $G''_{S_N}(1) = G''_X(1)G'_N(1) + (G'_X(1))^2G''_N(1)$. From (1), we thus have $Var(S_N) = G''_{S_N}(1) + G'_{S_N}(1) - (G'_{S_N}(1))^2 = G''_X(1)G'_N(1) + (G'_X(1))^2G''_N(1) + G'_X(1)G'_N(1) - (G'_X(1)G'_N(1))^2$. Using (1) again with the expression for $G''(1)$ and simplifying the equation yields

$$Var(S_N) = Var(N)(\mathbb{E}(X))^2 + \mathbb{E}(N)Var(X).$$

2. As $Z_{n+1} = \sum_{i=1}^{Z_n} X_{n,i}$, for all $n \geq 1$, where the $X_{n,i}$'s are iid with same law X, we have from question 1, that

$$\mathbb{E}(Z_n) = \mathbb{E}(X)\mathbb{E}(Z_{n-1}) = \mu\mathbb{E}(Z_{n-1}).$$

By induction,

$$\mathbb{E}(Z_n) = \mu^n\mathbb{E}(Z_0) = \mu^n.$$

3. Similarly, considering $Var(Z_{n+1})$ and using question 1 yield

$$Var(Z_{n+1}) = Var(Z_n)(\mathbb{E}(X))^2 + \mathbb{E}(Z_n)Var(X) = Var(Z_n)\mu^2 + \mu^n\sigma^2.$$

By induction, $Var(Z_{n+1}) = \mu^n\sigma^2(1 + \mu + \ldots + \mu^n)$, so

$$Var(Z_n) = \begin{cases} \mu^{n-1}\sigma^2\frac{1-\mu^n}{1-\mu} & \text{if } \mu \neq 1 \\ n\sigma^2 & \text{if } \mu = 1 \end{cases}$$

4. (a) $\mathbb{E}(Z_n) = \left(\frac{4}{3}\right)^n$ and $Var(Z_n) = 2\left(\frac{4}{3}\right)^n\left[\left(\frac{4}{3}\right)^n - 1\right]$.
   (b) $\mathbb{E}(Z_n) = \frac{1}{2^n}$ and $Var(Z_n) = \frac{3}{2^{n+1}}\left(1 - \frac{1}{2^n}\right)$.
   (c) $\mathbb{E}(Z_n) = 3^n$ and $Var(Z_n) = 2(3^n - 1)3^n$.

Problem 4

We first compute λ. We know that $\frac{5}{9} = \mathbb{P}(X \leq 10) = 1 - e^{-\lambda\times 10}$, so $e^{-10\lambda} = \frac{4}{9}$, so $e^{-5\lambda} = \sqrt{\frac{4}{9}} = \frac{2}{3}$ (or if we prefer $\lambda = -\frac{1}{5}\log\frac{2}{3} = \frac{1}{5}\log\frac{3}{2}$).

1. We have

$$\mathbb{P}(X \geq 15) = e^{-15\lambda} = \left(e^{-5\lambda}\right)^3 = \left(\frac{2}{3}\right)^3 = \frac{8}{27}.$$

2. By the memoryless property of X, we have

$$\mathbb{P}(X \geq 15 | X \geq 10) = \mathbb{P}(X \geq 5) = e^{-5\lambda} = \frac{2}{3}.$$

3. Here it is not sufficient to use the memoryless property, so we use the definition of conditional probability:

$$\begin{aligned}
\mathbb{P}\left(X \geq 15 | 10 \leq X \leq 20\right) &= \frac{\mathbb{P}\left(15 \leq X \leq 20\right)}{\mathbb{P}\left(10 \leq X \leq 20\right)} \\
&= \frac{\mathbb{P}\left(X \geq 15\right) - \mathbb{P}\left(X > 20\right)}{\mathbb{P}\left(X \geq 10\right) - \mathbb{P}\left(X > 20\right)} \\
&= \frac{e^{-15\lambda} - e^{-20\lambda}}{e^{-10\lambda} - e^{-20\lambda}} \\
&= \frac{e^{-5\lambda} - e^{-10\lambda}}{1 - e^{-10\lambda}} \\
&= \frac{\frac{2}{3} - \left(\frac{2}{3}\right)^2}{1 - \left(\frac{2}{3}\right)^2} = \frac{2/9}{5/9} = \frac{2}{5}.
\end{aligned}$$

4. For all $y > 0$, we have $\mathbb{P}\left(y \leq X \leq 2y\right) = e^{-\lambda y} - e^{-2\lambda y}$. To find when this is maximal, we compute the derivative:

$$\frac{d}{dy}\mathbb{P}\left(y \leq X \leq 2y\right) = -\lambda e^{-\lambda y} + 2\lambda e^{-2\lambda y} = \lambda e^{-\lambda y}\left(-1 + 2e^{-\lambda y}\right).$$

In particular, this derivative is positive when $e^{-\lambda y} > \frac{1}{2}$, that is $y < \frac{1}{\lambda}\log 2$, and negative when $y > \frac{1}{\lambda}\log 2$. Therefore, the maximum is attained for

$$y = \frac{1}{\lambda}\log 2 = 5\frac{\log 2}{\log(3/2)} \approx 8.55.$$