RANDOM WALKS ON REGULAR TREES CAN NOT BE SLOWED DOWN

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ABSTRACT. A random walk on a regular tree has constant speed. We ask whether the walk can be slowed down by carefully chosen time dependent permutations of the vertices. We prove that it can not be slowed down.

1. INTRODUCTION

A particle performing a simple random walk on a d-regular trees escapes from the origin with positive speed if $d > 2$. *Can we slow the particle down?* If we observe the particle, then we can constantly push it back to the origin so that it never moves. Our main finding is that the particle can not be slowed down without observing it. In a nutshell, the reason is that the distribution of a random walk on trees is inherently aligned with sets with minimum vertex boundary.

We start by considering lazy random walks, because the result in this case is cleaner. Fix $d \geq 2$, and let $T = T_d$ denote the rooted infinite d-regular tree. The vertex set is denoted by $V = V(T_d)$. The root is denoted by v_0. The depth $|v|$ of a vertex $v \in V$ is its distance from the root. The neighborhood $N(v)$ of v the set of vertices u that are of distance at most one from v. The size of $N(v)$ is $d + 1$.

Let $(X_t)_{t=0}^{\infty}$ be a lazy random walk on T started at the root. That is, $X_0 = v_0$ and X_{t+1} is a uniformly random element of $N(X_t)$. The speed of (X_t) is defined to be the process $(|X_t|/t)$. The strong law of large numbers says that the speed a.s. converges to $\frac{d-2}{d+1}$. For more on random walks on trees, see [1, 3] and references therein.

Here is the model we suggest for studying the slowing down of particles. Before the particle starts to move, we can choose a sequence $\pi = (\pi_1, \pi_2, \ldots)$ of permutations of V. The permutation π_t is applied on the random walk at time t with the goal of slowing it down, pushing it backwards towards the origin. The permuted random walk (Y_t) starts at the root, and its position at time $t + 1$ is defined as $Y_{t+1} = \pi_{t+1}(Y'_{t+1})$, where Y'_{t+1} is a uniformly random vertex in $N(Y_t)$. The speed of (Y_t) is not always defined. The lower speed of the permuted random walk is defined to be the liminf of $\frac{|Y_t|}{t}$.

Our main result is that (Y_t) is not slower than (X_t).

Theorem 1. *For every $d \geq 2$, for every sequence of permutations π of $V(T_d)$, and every time $t \geq 1$, the distance of permuted random walk $|Y_t|$ stochastically dominates the distance of lazy random walk $|X_t|$. That is, for all $t, n \geq 0$,*

$$\Pr[|Y_t| \geq n] \geq \Pr[|X_t| \geq n].$$

In particular, $\mathbb{E}\,|Y_t| \geq \mathbb{E}\,|X_t|$ for all $t \geq 0$, and $\liminf_{t \to \infty} \frac{|Y_t|}{t} \geq \frac{d-2}{d+1}$ almost surely.

The theorem is informative even for $d = 2$, where the speed is zero. For a non-lazy walk, the theorem is false; even for the 2-regular tree without self-loops, we can have $\mathbb{E}\,|Y_2| < \mathbb{E}\,|X_2|$.

The distributions of (X_t) and (Y_t) correspond to a lazy random walk that stays put with probability $\frac{1}{d+1}$. Theorem 1 holds as long as the probability to stay put is at least $\frac{1}{d+1}$. In particular, it holds when the chance to stay put is one half. For more details, see the remark after the proof of Theorem 2.

The theorem is a special case of a more general phenomenon, which we describe in the next two theorems. For a distribution p on V, define $p^* \colon \mathbb{N} \to [0, 1]$ by letting $p^*(s)$ be the mass of the s largest atoms in p, i.e., $p^*(s) = \max_{J \subset V, |J| = s} p(J)$. The distribution p is *more concentrated* than another distribution q if $p^*(s) \geq q^*(s)$ for all s.

Denote by p_t the distribution of X_t and by q_t the distribution of Y_t. The stochastic domination asserted in Theorem 1 is implied by the following more technical statement upon noting that p_t is spherically symmetric and non-increasing in distance (in the sense that $p_t(v) \geq p_t(u)$ whenever $|v| \leq |u|$) for every $t \geq 0$.

Theorem 2. *For every $t \geq 0$, the distribution p_t is more concentrated than q_t.*

The theorem follows from a more general statement which we now describe. Fix an enumeration $\{v_0, v_1, v_2, \dots\}$ of V so that for every $i < j$ and $v \in V$, if v_i and v_j are both descendants of v of the same depth, then so are $v_{i+1}, \dots, v_{j-1}$. Say that a distribution p is *greedily arranged* if $p(v_i) \geq p(v_{i+1})$ for every $i \geq 0$.

Note that p_0 is clearly greedily arranged. Note also that if a distribution p is more concentrated than another distribution q, then it is also more concentrated than any rearrangement of q (i.e. any measure $q \circ \pi$ for a permutation π on V). The following theorem therefore easily implies Theorem 2.

Theorem 3. *Let p and q be distributions on V. Let p' and q' be the corresponding distributions after a single step of lazy random walk started at p and q, respectively. If p is greedily arranged and is more concentrated than q, then p' is greedily arranged and is more concentrated than q'.*

The theorem implies that for every $t \geq 0$ and finite $J \subset V$,

$$(1) \qquad\qquad p_t(\{v_0, v_1, \dots, v_{|J|-1}\}) \geq q_t(J).$$

This is a strong statement on the nature of lazy random walks on regular trees. For non-lazy walks, this is too strong to be true: the distribution of a simple non-lazy random walk on a regular tree is not greedily arranged because the tree is bipartite and thus (1) fails already for $t = 1$. On the other hand, as we will see, the statement of Theorem 2 does hold for non-lazy walks.

We now move to the results for simple (non-lazy) walks. The speed of a simple random walk is $\frac{d-2}{d}$ a.s. For such random walks, the same stochastic domination as in Theorem 1 does not hold. Nonetheless, we prove that it almost holds (at least for $d > 2$, when the tree is not the line).

Denote by $N'(v)$ the d neighbors of v not including v. Let (S_t) be a simple random walk so that $S_0 = v_0$ and S_{t+1} is uniform in $N'(S_t)$. Let (Z_t) be a permuted simple random walk so that $Z_0 = v_0$ and Z_{t+1} is $\pi_{t+1}(Z'_{t+1})$ where Z'_{t+1} is uniform in $N'(Z_t)$.

Theorem 4. *For every $d > 2$, for every sequence of permutations π of $V(T_d)$, and for every time $t \geq 1$, we have that $|Z_t| + 2$ stochastically dominates $|S_t|$. In particular, $\mathbb{E}\,|Z_t| \geq \mathbb{E}\,|S_t| - 2$ for all $t \geq 0$, and $\liminf_{t \to \infty} \frac{|Z_t|}{t} \geq \frac{d-2}{d}$ almost surely.*

For $d = 2$, the bound on the lower speed of (Z_t) trivially holds, but the theorem is false. For example, we can always wait until a large even time T, and then map via π_{T+1} all even integers in the range $[-T, T]$ to all integers in half of that range so that $\mathbb{E}\,|Z_{T+1}| = \frac{1}{2}\,\mathbb{E}\,|S_{T+1}|$.

We shall deduce Theorem 4 from the following modification of Theorem 3 which takes into account the periodicity of the non-lazy walk. Say that a distribution p is *half-greedily arranged* if it is supported on either the even vertices (those of even depth) or on the odd vertices (those of odd depth), and $p(v_i) \geq p(v_j)$ for every $i > j$ for which v_i and v_j have the same parity.

Theorem 5. *Let p and q be distributions on V. Let p' and q' be the corresponding distributions after a single step of non-lazy random walk started at p and q, respectively. If p is half-greedily arranged and is more concentrated than q, then p' is half-greedily arranged and is more concentrated than q'.*

Note that the theorem implies that the distribution of S_t is more concentrated than that of Z_t for every $t \geq 0$.

One natural approach for proving the results above is using spectral methods (see [2] and references within), but the spectral gap of the tree (analyzed by Kesten) does not seem to be informative enough to yield the accurate bounds we seek. The proof, not surprisingly, uses an isoperimetric inequality for the tree; see Propositions 7 and 12. Proposition 7 is a non-standard isoperimetric inequality, which takes into account the amount of "isolated" points in the set of interest. The proof also uses the fact that the structure of the tree is simple enough so that we exactly know the sizes of balls, spheres, etc.; see the proof of Lemma 9 and Proposition 13.

2. Lazy random walks

2.1. **An isoperimetric inequality.** Recall that $N(v)$ is the neighborhood of v, including v. For $J \subset V$, the neighborhood of J is

$$N(J) = \bigcup_{v \in J} N(v).$$

Let $\kappa_1(J)$ denote the number of connected components induced by J. Let $\kappa_2(J)$ denote the number of connected components induced by J in the graph in which edges are added between all pairs of vertices that are at distance 2 from each other in the tree.

Proposition 6. *For every finite $J \subset V$,*

$$|N(J)| = (d - 1)|J| + \kappa_1(J) + \kappa_2(J).$$

Proof. We prove the claim by induction on $|J|$. The base case when $|J| = 0$ is trivial. Let J be non-empty. Let $v \in J$ be such that $J \setminus \{v\}$ is contained in a subtree not containing v. Denote $N_1 = N(v) \cap J \setminus \{v\}$ and $N_2 = N(N(v)) \cap J \setminus \{v\}$. Note that

$$\kappa_1(J \setminus \{v\}) = \kappa_1(J) - \mathbf{1}_{\{N_1 = \emptyset\}}, \qquad \kappa_2(J \setminus \{v\}) = \kappa_2(J) - \mathbf{1}_{\{N_2 = \emptyset\}},$$

Therefore, by the induction hypothesis,

$$|N(J \setminus \{v\})| = (d-1)|J| + \kappa_1(J) + \kappa_2(J) + d - 1 + \mathbf{1}_{\{N_1 = \emptyset\}} + \mathbf{1}_{\{N_2 = \emptyset\}}.$$

Thus, we need to show that

$$|N(J)| - |N(J \setminus \{v\})| = d - 1 + \mathbf{1}_{\{N_1 = \emptyset\}} + \mathbf{1}_{\{N_2 = \emptyset\}}.$$

The left-hand side equals $|N(v) \setminus N(J \setminus \{v\})| = d + 1 - |N(v) \cap N(J \setminus \{v\})|$. So we need to show that

$$|N(v) \cap N(J \setminus \{v\})| = \mathbf{1}_{\{N_1 \neq \emptyset\}} + \mathbf{1}_{\{N_2 \neq \emptyset\}}.$$

This is straightforward to check by separating into cases according to the three possible values of $(\mathbf{1}_{\{N_1 \neq \emptyset\}}, \mathbf{1}_{\{N_2 \neq \emptyset\}})$. $\qquad\square$

The set of isolated points in J is defined to be

$$\mathsf{iso}(J) = \{v \in J : N(v) \cap J = \{v\}\}.$$

The set of connected points is defined to be $\mathsf{con}(J) = J \setminus \mathsf{iso}(J)$.

Proposition 7. *For every non-empty $J \subset V$, if $\mathsf{con}(J) = \emptyset$ then*

$$|N(J)| \geq 1 + d|J|,$$

and if $\mathsf{con}(J) \neq \emptyset$ then

$$|N(J)| \geq 2 + d|\mathsf{iso}(J)| + (d-1)|\mathsf{con}(J)|,$$

Proof. This follows immediately from Proposition 6 upon noting that

$$|J| = |\mathsf{iso}(J)| + |\mathsf{con}(J)|, \qquad \kappa_1(J) \geq |\mathsf{iso}(J)| + \mathbf{1}_{\{\mathsf{con}(J) \neq \emptyset\}}, \qquad \kappa_2(J) \geq 1. \qquad\square$$

2.2. **Minimizing the mass loss.** This section presents the key link between the isoperimetric inequality and the behavior of the random walks.

Denote by B_n the ball of radius $n \geq 0$ in the tree (i.e., the set of vertices of depth at most n) and by $\partial B_n = B_n \setminus B_{n-1}$ the sphere of radius n (i.e., the set of vertices of depth exactly n). Recall the fixed enumeration $\{v_0, v_1, \dots\}$ of V from before Theorem 3. A set of the form $\{v_0, v_1, \dots, v_i\}$ for some $i \geq 0$ is called a *quasi-ball*.

For $J \subset V$ and $i \in [d+1]$, define

$$K_i(J) = \{v \in V : |N(v) \cap J| \geq i\}, \qquad k_i(J) = |K_i(J)|.$$

Fix J and let B be the quasi-ball of the same size. Denote $k_i = k_i(J)$ and $m_i = k_i(B)$.

Proposition 8. *For any distribution q and any set J,*

$$\sum_i q^*(k_i) \leq \sum_i q^*(m_i).$$

The following is the main step toward establishing the proposition.

Lemma 9. *For any $r \in [d+1]$,*

$$\sum_{i \leq r} k_i \geq \sum_{i \leq r} m_i.$$

Proof. For ease of notation, write

$$|J| = |B| = |B_n| + a(d-1) + c,$$

where $B_n \subset B \subsetneq B_{n+1}$, $a \in \{0, 1, \ldots, |\partial B_n| - 1\}$ and $c \in \{0, 1, \ldots, d-2\}$.

Suppose $|J| > 1$ (the result is trivial if $|J| = 1$). Observe that

- $m_1 = (d-1)|J| + 2$
- $m_2 = |J|$
- $m_3 = \cdots = m_{c+2} = |B_{n-1}| + a + 1 = \lceil \frac{(d+1)|J| - m_1 - m_2}{d-1} \rceil = \lceil \frac{|J|-2}{d-1} \rceil$
- $m_{c+3} = \cdots = m_{d+1} = |B_{n-1}| + a = \lfloor \frac{(d+1)|J| - m_1 - m_2}{d-1} \rfloor = \lfloor \frac{|J|-2}{d-1} \rfloor$

These values are easy to check.

Let us now prove the inequality for the partial sums. Consider first the case $r = 1$, which is simply the inequality $k_1 \geq m_1$. By Proposition 6,

$$k_1 = |N(J)| \geq (d-1)|J| + 2 = m_1.$$

Consider next the case $r = 2$. By Proposition 7, and using that $\mathsf{con}(J) \subset K_2(J)$,

$$k_1 + k_2 \geq k_1 + |\mathsf{con}(J)| \geq d|J| + 2 = m_1 + m_2.$$

For $r \geq 3$, we proceed by induction. Denote $s = (d+1)|J| = \sum_i k_i = \sum_i m_i$. Since the k_i are decreasing,

$$k_r \geq \frac{s - \sum_{i \leq r-1} k_i}{d - r + 2}.$$

Thus,

$$\begin{aligned}
\sum_{i \leq r} k_i &\geq \sum_{i \leq r-1} k_i + \frac{s - \sum_{i \leq r-1} k_i}{d - r + 2} \\
&= \frac{d - r + 1}{d - r + 2} \sum_{i \leq r-1} k_i + \frac{s}{d - r + 2} \\
&\geq \frac{d - r + 1}{d - r + 2} \sum_{i \leq r-1} m_i + \frac{s}{d - r + 2} \\
&= \sum_{i \leq r-1} m_i + g_r,
\end{aligned}$$

where $g_r = \frac{s - \sum_{i \leq r-1} m_i}{d - r + 2}$. Since $\lceil g_r \rceil = m_r$, and since the k_i and m_i are integers, the desired inequality follows. $\qquad\square$

Proposition 8 immediately follows from the previous lemma and the following basic fact (taking $\mu_i = k_i$ and $\lambda_i = m_i$):

Lemma 10. *If $(\mu_i)_{i=1}^t$, $(\lambda_i)_{i=1}^t$ are nonincreasing sequences satisfying*

$$\sum_{i \leq r} \mu_i \geq \sum_{i \leq r} \lambda_i \text{ for every } r \in [t] \text{ and } \sum_i \mu_i = \sum_i \lambda_i,$$

then $\sum_i f(\mu_i) \leq \sum_i f(\lambda_i)$ for any increasing, concave function f.

2.3. **Putting it together.** Before proving Theorem 3, we note the following simple observation:

Observation 11. *Let B be a quasi-ball. Each $K_i(B)$ is also a quasi-ball. In particular, if p is a greedily arranged distribution, then $p(K_i(B)) = p^*(|K_i(B)|)$.*

Proof of Theorem 3. Let $J \subset V$ and let B be a quasi-ball of the same size. Denote $k_i = k_i(J)$ and $m_i = k_i(B)$. We have

$$q'(J) = \frac{1}{d+1} \sum_{i \leq d+1} q(K_i(J))$$

$$(2) \qquad\qquad \leq \frac{1}{d+1} \sum_{i \leq d+1} q^*(k_i)$$

$$(3) \qquad\qquad \leq \frac{1}{d+1} \sum_{i \leq d+1} q^*(m_i)$$

$$(4) \qquad\qquad \leq \frac{1}{d+1} \sum_{i \leq d+1} p^*(m_i)$$

$$(5) \qquad\qquad = \frac{1}{d+1} \sum_{i \leq d+1} p(K_i(B))$$

$$= p'(B),$$

where the first and last equalities follow from the definition of the lazy random walk, (2) follows from the definition of q^*, (3) follows from Proposition 8, (4) follows from the assumption that p is more concentrated than q, and (5) follows from the above observation and the assumption that p is greedily arranged. Applying this inequality with $q = p$ yields that p' is greedily arranged. Finally, for any $s \in \mathbb{N}$, choose any J such that $(q')^*(s) = q'(J)$ and apply the above to obtain

$$(q')^*(s) \leq p'(B) = (p')^*(s). \qquad \square$$

Remark. *Theorem 3 (and thus also Theorem 1 and Theorem 2) extends to the lazy random walk in which the probability to stay put is $\gamma \geq \frac{1}{d+1}$. Indeed, in this case,*

$$q'(J) = \left(\gamma - \frac{1}{d+1}\right)q(J) + \left(1 - \gamma + \frac{1}{d+1}\right)\frac{1}{d+1}\sum_i q(K_i(J))$$

$$\leq \left(\gamma - \frac{1}{d+1}\right)p(B) + \left(1 - \gamma + \frac{1}{d+1}\right)\frac{1}{d+1}\sum_i p(K_i(B)) = p'(B).$$

Proof of Theorem 1. Recall that Theorem 3 implies (1). In particular, $p_t(B_n) \geq q_t(B_n)$ for all $n, t \geq 0$. In other words, $|Y_t|$ stochastically dominates $|X_t|$ for every $t \geq 0$. In particular, $\mathbb{E}\,|Y_t| \geq \mathbb{E}\,|X_t|$.

It remains to show that $\liminf_{t \to \infty} \frac{Y_t}{t} \geq \frac{d-2}{d+1}$ almost surely. For every $\epsilon > 0$, standard concentration bounds imply that

$$\sum_t \Pr[\tfrac{|X_t|}{t} < \tfrac{d-2}{d+1} - \epsilon] < \infty.$$

Since $|Y_t|$ stochastically dominates $|X_t|$ for every $t \geq 0$, the same holds with Y_t instead of X_t, and the Borel–Cantelli lemma gives the result. $\qquad\square$

3. Simple random walks

In this section, we consider simple (non-lazy) walks. The argument is similar to the lazy case, so we move quicker below. The main difficulty is handling the fact that the tree is bipartite. This is done via the "half" balls:

$$B'_n = \{v \in B_n : |v| = n \bmod 2\}.$$

A *half-quasi-ball* is the restriction of a quasi-ball to either the even vertices or the odd vertices.

For $J \subset V$, let

$$N'(J) = \bigcup_{v \in J} N'(v).$$

Proposition 12. *For every non-empty $J \subset V$,*

$$|N'(J)| \geq 1 + (d-1)|J|.$$

Proof. First assume that J is contained in either $V_0 = \{v \in V : |v| = 0 \bmod 2\}$ or $V_1 = V \setminus V_0$. In this case, $\kappa_1(J) = |J|$ so that Proposition 6 implies that

$$|N'(J)| = |N(J)| - |J| = (d-1)|J| + \kappa_2(J) \geq (d-1)|J| + 1.$$

For arbitrary J, note that $N'(v) \cap N'(w) = \emptyset$ if $|v| \neq |w| \bmod 2$, so the result follows by applying the above to $J \cap V_0$ and $J \cap V_1$ separately. $\qquad\square$

For $J \subset V$ and $i \in [d]$, define

$$K_i(J) = \{v \in V : |N'(v) \cap J| \geq i\}, \qquad k_i(J) = |K_i(J)|.$$

Fix J and let B be a half-quasi-ball of the same size. Denote $k_i = k_i(J)$ and $m_i = k_i(B)$.

Proposition 13. *For any distribution q and any set J,*

$$\sum_i q^*(k_i) \leq \sum_i q^*(m_i).$$

Proof. As before, the proposition will follow Lemma 10 once we show that for any $r \in [d]$,

$$\sum_{i \leq r} k_i \geq \sum_{i \leq r} m_i.$$

For ease of notation, write

$$|J| = |B| = |B'_n| + a(d-1) + c,$$

where $B'_n \subset B \subsetneq B'_{n+2}$, $a \in \{0, 1, \ldots, |\partial B_{n+1}| - 1\}$ and $c \in \{0, 1, \ldots, d-2\}$.

We first observe that

- $m_1 = (d-1)|J| + 1$
- $m_2 = \cdots = m_{c+1} = |B'_{n-2}| + a + 1 = \lceil \frac{d|J|-m_1}{d-1} \rceil = \lceil \frac{|J|-1}{d-1} \rceil$
- $m_{c+2} = \cdots = m_d = |B'_{n-2}| + a = \lfloor \frac{d|J|-m_1}{d-1} \rfloor = \lfloor \frac{|J|-1}{d-1} \rfloor$

These values are easy to check.

Let us now prove the inequality for the partial sums. Consider first the case $r = 1$, which is simply the inequality $k_1 \geq m_1$, which follows from Proposition 12. For $r \geq 2$, one proceeds by induction in a similar manner as in the proof of Lemma 9. $\qquad\square$

Since a non-empty quasi-ball is contained in either the even or odd vertices, it has a parity. A half-greedily arranged distribution also has a parity according to whether its support is contained in the even or odd vertices.

Observation 14. *Let B be a half-quasi-ball. Each $K_i(B)$ is a half-quasi-ball of opposite parity from B. In particular, if p is a half-greedily arranged distribution of opposite parity from B, then $p(K_i(B)) = p^*(|K_i(B)|)$.*

Proof of Theorem 5. Let $J \subset V$ and let B be a half-quasi-ball of the same size, whose parity is opposite from p. Denote $k_i = k_i(J)$ and $m_i = k_i(B)$. We have

$$q'(J) = \frac{1}{d} \sum_{i \leq d+1} q(K_i(J))$$

$$(6) \qquad\qquad \leq \frac{1}{d} \sum_{i \leq d+1} q^*(k_i)$$

$$(7) \qquad\qquad \leq \frac{1}{d} \sum_{i \leq d+1} q^*(m_i)$$

$$(8) \qquad\qquad \leq \frac{1}{d} \sum_{i \leq d+1} p^*(m_i)$$

$$(9) \qquad\qquad = \frac{1}{d} \sum_{i \leq d+1} p(K_i(B))$$

$$= p'(B),$$

where the first and last equalities follow from the definition of the non-lazy random walk, (6) follows from the definition of q^*, (7) follows from Proposition 13, (8) follows from the assumption that p is more concentrated than q, and (9) follows from the above observation and the assumption that p is half-greedily arranged. The result follows the same way as in the proof of Theorem 3. $\qquad\square$

Proof of Theorem 4. Denote by p_t the distribution of S_t, and denote by q_t the distribution of Z_t. We need to show that for every $n, t \geq 0$,

$$q_t(B_n) \leq p_t(B_{n+2}).$$

Since $d > 2$,

$$|B_n| = 1 + d + d(d-1) + \ldots + d(d-1)^{n-1}$$
$$\leq 1 + d(d-1)^n$$
$$\leq |B'_{n+1}|$$
$$\leq |B'_{n+1+\mathbf{1}_{\{n=t \bmod 2\}}}|.$$

So using Theorem 5,

$$
\begin{aligned}
q_t(B_n) &\leq q_t^*(|B_n|) \\
&\leq p_t^*(|B_n|) \\
&\leq p_t^*(|B'_{n+1+\mathbf{1}_{\{n=t \bmod 2\}}}|) \\
&= p_t(B'_{n+1+\mathbf{1}_{\{n=t \bmod 2\}}}) \\
&= p_t(B_{n+2}).
\end{aligned}
$$

This completes the proof that $|Z_t| + 2$ stochastically dominates $|S_t|$. The lower bound on the liminf of $\frac{|Z_t|}{t}$ now follows in a similar manner as in the proof of Theorem 1. □

References

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