VI - Branching processes

We present here an important example of stochastic process, which originated with Galton & Watson (1874). They studied the dynamics and possible extinction of family names in the British nobility.

- We can represent this as a population that evolves in generations
- Each individual has a random number of offsprings (with same distribution) and independtly of other individuals

$i=0$ $\quad Z_0 = 1$

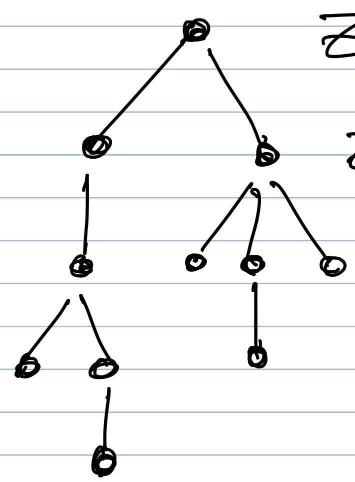
$i=1$ $\quad Z_1 = 2$

$i=2$ $\quad Z_2 = 4$

$i=3$ $\quad Z_3 = 3$

$i=4$ $\quad Z_4 = 1$

$i=5$ $\quad Z_5 = 0$



- We model the number of individuals at generation i

Z_i

②

Def (Branching process): We call branching process (or Galton Watson process) the sequence of random variables $(Z_n)_{n\geq 0}$ defined by

$$\begin{cases} Z_0 = 1 \\ Z_{n+1} = \begin{cases} \sum_{i=1}^{Z_n} Y_{n,i} & \text{if } Z_n > 0 \\ 0 & \text{if } Z_n = 0 \end{cases} \end{cases} \quad \text{for } n \geq 0,$$

where the $Y_{n,i}$ are iid r.v.'s on $\mathbb{N}$. The law that the $Y_{n,i}$'s follow is called the reproduction law. Z_n thus describes the number of individuals at the n-th generation, and $Y_{n,i}$ the number of offsprings of the i-th individual at the n-th generation

Rmk: • Z_n defines a discrete time MC on $\mathbb{N}$

• $\{0\}$ defines a recurrent (absorbing) class

• Assuming $P(Y_{n,i} = 0) > 0$, 0 is accessible from all the other states $i > 0 \Rightarrow$ All the other states $i > 0$ are transient.

• Since any finite set of transient states can only be visited finitely often, either

Z_n is eventually 0, or $Z_n \to +\infty$

Q: The original question asked by Galton & Watson:
What is the probability of extinction of the population?

Rmk: This process has many important applications in Genetics, ecology, epidemiology etc.

• To study the probability of extinction (or equivalently the survival probability), we use generating functions

Def (generating function). Let X be a r.v. on $\mathbb{N}$.
We call the generating function of X the power series

$$G_X(s) = \mathbb{E}(s^X) = \sum_{k=0}^{+\infty} P(X=k)\, s^k$$

Rmk: • The radius of convergence of G_X (largest value of $s > 0$ s.t. $G_X(s) < +\infty$) is greater than or equal to 1, since

$$G_X(1) = \sum_{k=0}^{+\infty} P(X=k)\, \cancel{1^k} = 1$$

$\Rightarrow G_X(s)$ is well defined for all $s \in [-1,1]$

- $\boxed{G_X(0) = P(X=0)}$
- For other properties $\to$ see hw problem.

Thm: If X and Y are 2 independent r.v.'s on $\mathbb{N}$

$\forall t \in [-1,1] \quad G_{X+Y}(t) = G_X(t)\, G_Y(t)$

Proof: $G_{X+Y}(t) = \mathbb{E}\left(t^{X+Y}\right) = \mathbb{E}\left(t^X \cdot t^Y\right)$

$$= \mathbb{E}(t^X)\, \mathbb{E}(t^Y) = G_X(t) \cdot G_Y(t)$$

$\uparrow$ by independence of X and Y $\boxtimes$

Prop: Let $X_1, X_2, \ldots$ be i.i.d r.v.'s with generating function G_X (same generating function for all X_i's).

Let N be a r.v. independent of the X_i's, with generating function G_N.

Let $T = X_1 + X_2 + \cdots + X_N$

Then $G_T(s) = G_N(G_X(s))$

Proof: $G_T(s) = \mathbb{E}(s^T)$

$$\underset{(\text{condition on } N)}{=} \mathbb{E}_{(N)}\left(\mathbb{E}(s^T \mid N)\right)$$

$$= \sum_{n=0}^{+\infty} P(N=n)\, \mathbb{E}(s^T \mid N=n)$$

$$= \sum_{n=0}^{+\infty} P(N=n)\, \underbrace{\mathbb{E}\left(s^{X_1+\cdots+X_N} \mid N=n\right)}_{\mathbb{E}\left(s^{X_1+\cdots+X_n}\right)}$$

$$\underbrace{\mathbb{E}\left(s^{X_1+\cdots+X_n}\right)}_{\shortparallel} = G_{X_1+\cdots+X_n}(s) \underset{\text{Thm}}{=} (G_X(s))^n$$

$$\Rightarrow \quad G_T(s) = \sum_{n=0}^{+\infty} P(N=n)\,(G_X(s))^n$$

$$= \mathbb{E}\left((G_X(s))^N\right) = G_N(G_X(s)) \quad \boxtimes$$

Application to B.P

$$Z_{n+1} = \sum_{i=1}^{Z_n} Y_{n,i} = Y_{n,1} + Y_{n,2} + \cdots + Y_{n,Z_n}$$

$$\Rightarrow G_{Z_{n+1}}(s) = G_{Z_n}(G_Y(s)) = G_{Z_{n-1}}(G_Y \circ G_Y(s))$$

$$(\ldots) = \underbrace{G_Y \circ \cdots\cdots \circ G_Y}_{n+1 \text{ times}}(s)$$

$$\Rightarrow \boxed{G_n(s) = G_1^n(s)}$$

generating function of Z_n

n-th iteration of the generating function of Z_1!