MATH 303 Practice Problems - Week 2

Jamuary 22, 2021

Problem 1

1. Complete the following transition matrix, associated with a Markov chain defined on $\{1, \dots 6\}$, such that transition probabilities from a given state are uniform:

$$\begin{pmatrix} \cdot & \frac{1}{2} & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \cdot & 0 & 0 & 0 & 0 \\ 0 & 0 & \cdot & 0 & \cdot & 0 \\ \frac{1}{4} & \frac{1}{4} & 0 & \cdot & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & \cdot & 0 & \cdot & 0 \\ 0 & \cdot & 0 & \cdot & \cdot & 0 \end{pmatrix}$$

2. Draw the transition diagram and find the communicating classes (briefly justify your answers).
3. Find the period of the states, and if they are transient or recurrent (briefly justify).

Problem 2

We consider the random walk on the vertices of a polygon, represented by $S = \{1, \dots N\}$, where the transition probabilities from a given vertex are given by p, q and $r > 0$ according to Figure 1.

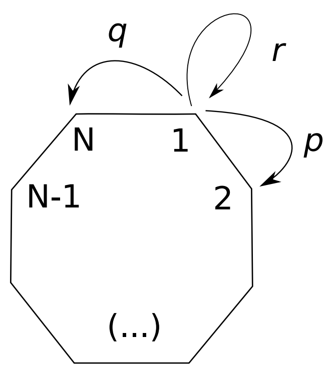

Figure 1: Random walk on the vertices of a polygon with N vertices. We only represent the transition from state 1 (same rule for the other states).

1. Write the transition matrix for $N = 6$.
2. We suppose $p, q > 0$. Show that the Markov chain is irreducible.
3. We suppose $p, q > 0$ and $r > 0$. Show that the chain is aperiodic.
4. We suppose $p, q > 0$ and $r = 0$. Show that if N is even, then the chain has period 2, and if N is odd, then the chain is aperiodic.
5. Suppose $p, q > 0$ and $r = 0$. What is the probability to visit all the other states before returning to its initial position (hint: think of the gambler's ruin problem)?

Problem 3

Let $(X_n)_{n \geq 0}$ be a Markov chain on $\{0, 1, 2\}$ with transition matrix

$$M = \frac{1}{2}\begin{array}{c} \begin{matrix} 0 & \ 1 & \ 2 \end{matrix} \\ \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \end{array}$$

1. Find the 2-step transition matrix of X_n.
2. We assume for the rest of the problem that $X_0 = 0$. Find $\mathbb{E}(X_2)$.
3. What is the mean number of steps until X_n visits 2?

Problem 4

We study here the Gambler's Ruin Problem (seen in class, or in Ross' 4.5.1), with same hypothesis and notations (p being the probability to win a single game). As shown in class, we know that the wealth of the gambler follows a Markov chain X_k, that either reaches 0 or N with probability 1, so we can define the random time $T = \min\{k \geq 0 \text{ s.t. } X_k = 0 \text{ or } N\}$. We want to find the mean duration of the whole game, given initial wealth k ($0 \leq k \leq N$), i.e. $\mathbb{E}_k(T) = \mathbb{E}(T|X_0 = k)$.
1. What are $\mathbb{E}_0(T)$ and $\mathbb{E}_N(T)$? (boundary conditions)
2. We write $x_k = \mathbb{E}_k(T)$. Show that for $1 \leq k \leq N-1$, x_k satisfies

$$p x_{k+1} - x_k + (1-p) x_{k-1} = 1 \qquad (*)$$

3. Solutions of $(*)$ can be written as $x_k = y_k + f(k)$, where y_k is solution of the homogeneous equation $p y_{k+1} - y_k + (1-p) y_{k-1} = 0$, and $f(k)$ is a particular solution of $(*)$.
3.a Solve the homogeneous equation (cf. the Probability of ruin seen in class).
3.b We assume that $p \neq \frac{1}{2}$. Find a particular solution of the form $f(k) = Ck$, where C is a constant.
3.c Using the boundary conditions, conclude that

$$\mathbb{E}_k(T) = \frac{1}{1-2p}\left(k - N\frac{1-(\alpha-1)^k}{1-(\alpha-1)^N}\right), \text{ where } \alpha = \frac{1}{p}.$$

Problem 5 (Jupyter Notebooks)

1a. Use the scripts available from the jupyter notebooks to estimate the probabilities of return for the transient states in Problem 1. As we studied in class the distribution of the time spent in a given transient state s, get a histogram of this time (do it for one of the transient states found in Problem 1).
1b. Use the results and script on the mean time in transient states introduced in the Jupyter notebook, to find the probability of return for s. Use this to plot the theoretical distribution of time spent in s (see lecture notes), and compare it with the empirical histogram found in 1a.
2. Do the same for the transient states in the Gambler ruin's problem, for a goal of $N = 50$, and a probability of winning a bet of $p = 0.4$.