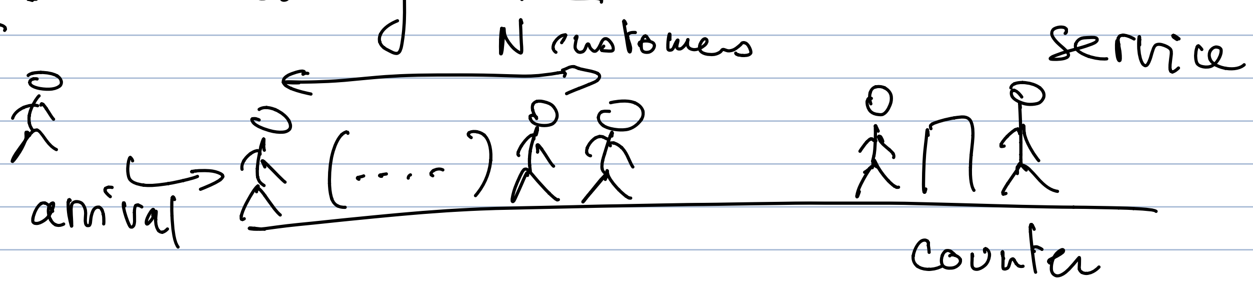
Chap 2: The Exponential Distribution and the Poisson Process

March 2 ①

Introduction: After studying processes in discrete time, we focus on how processes in continuous time can be modeled and studied

Ex: → Waiting line



→ Customers arrive at random times

Q: How to model the length of the line at time t? What is the waiting time of a customer?

→ This is an example of counting process (counting # customers at time t)

→ This type of processes has many applications in many domains (e.g. communication, physics, neuroscience ....)

Before studying the process, we will first review the exponential distribution, which is closely connected with the process of this chapter

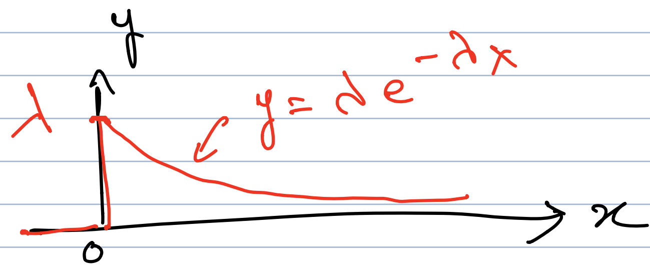
I. The Exponential Distribution

1) Def. and first properties

Def: A continuous variable follows an exponential distribution with parameter $\lambda > 0$ ($Exp(\lambda)$) if its p.d.f. is given by

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

Graph:



Properties: If $X \sim Exp(\lambda)$

- $E(X) = \boxed{\dfrac{1}{\lambda}}$
- $E(X^2) = \dfrac{2}{\lambda^2}$

- $Var(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2$

$$= \boxed{\frac{1}{\lambda^2}}$$

- The cumulative distribution function (cdf) of X is

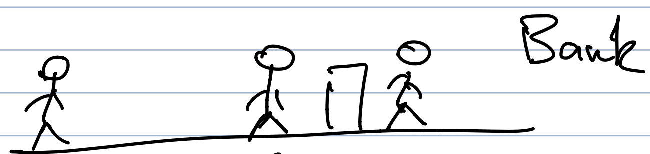
$$F(x) = P(X \le x) = \int_0^x \lambda e^{-\lambda t}\, dt$$

$$= \boxed{1 - e^{-\lambda x}}$$

(in particular, $P(\underset{(\geq)}{X > x}) = e^{-\lambda x}$)

2) The memoryless property

One key property of the Exponential r.v. is that is memoryless. We illustrate this concept with the following example



customer 2 customer 1, service starts at time $t=0$

Consider that the service time X at a bank for a single customer follows $Exp(\lambda)$

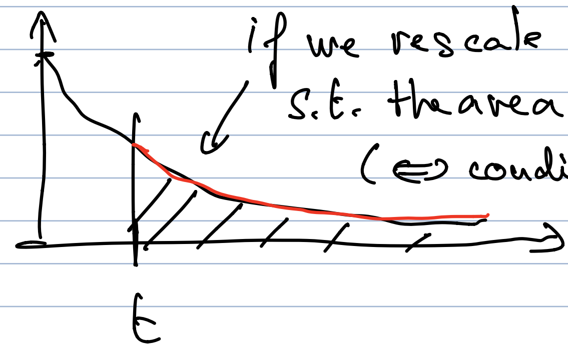
Q: Suppose that customer 2 has been waiting for t minutes. What is the probability to wait at least for another s minutes. (4)

i.e. what is $p = P(X > t+s \mid X > t)$?

$$p = \frac{P(X > t+s)}{P(X > t)} = \frac{e^{-\lambda(t+s)}}{e^{-\lambda t}} = e^{-\lambda s}$$

$$= \boxed{P(X > s)}$$

Graphical interpretation: Graph of $\mathrm{Exp}(\lambda)$ pdf



if we rescale this curve s.t. the area is 1 ($\Leftrightarrow$ conditioning on $X > t$)

we recover the whole curve

Conclusion: $P(X > t+s \mid X > t) = P(X > s)$

Rmk: This means that knowing that the "service" is still going on at time t doesn't give any additional information about the probability that the service will be over in the next units of time.

Ex: $X \sim Exp\left(\frac{1}{2}\right)$ (mean service time is 2 min)

$$P(X > 3 \text{ min}) = e^{-3/2}$$

and $P(X > \underset{= 3+2}{5} \mid X > 2) = P(X > 5-2)$
$= P(X > 3) = e^{-3/2}$

Ex: Bob enters a bank which has 2 tellers. The 2 tellers are currently serving Becca and Khanh. Service times are iid and $Exp(\lambda)$.
What is the proba. that Bob will leave last?

Solution: The solution is rather non-intuitive at first. Suppose that one of the two customers leaves (let's say Becca). From this moment and because of the memoryless property, the remaining time that Khanh will spend with the teller is also $Exp(\lambda)$. Since the service time of Bob exactly follows the same distribution, the probability to finish after Khanh is $\boxed{\frac{1}{2}}$

Def: A r.v. X is said to be without memory, or memoryless if

$$P(X > t+s \mid X > t) = P(X > s) \quad \forall\, t, s \geq 0$$

Rmk: $Exp(\lambda)$ is memoryless and one can actually show that

Thm (admitted): The exponential distribution is the only memoryless real random variable

3) Additional properties

- Minimum of 2 independent Exp r.v.'s

Suppose $X_1 \sim Exp(\lambda_1)$ & $X_2 \sim Exp(\lambda_2)$ are independent and let $X = \min(X_1, X_2)$

(rmk: In the example when 2 customers are served, this is the 1st time that 1 customer leaves).

Q: What is the pdf of X ?

Idea : Look at the event $E = \{X > x\}$ for $x \geq 0$

$X > x \iff X_1 > x$ and $X_2 > x$

so $E = \{X_1 > x, X_2 > x\}$

so $P(E) = P(X_1 > x, X_2 > x)$

$= P(X_1 > x) P(X_2 > x)$

by independence

$= e^{-\lambda_1 x} e^{-\lambda_2 x}$

$= e^{-(\lambda_1 + \lambda_2) x}$

so $F_X(x) = P(X \leq x) = 1 - e^{-(\lambda_1 + \lambda_2) x}$

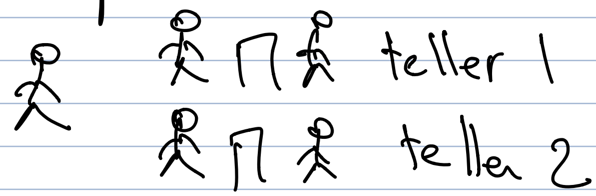
$\Rightarrow$ $X \sim Exp(\lambda_1 + \lambda_2)$ and we can generalize

Prop : Let $X_1, \dots, X_n$ be n independent r.v. s.t.

$X_i \sim Exp(\lambda_i)$, $1 \leq i \leq n$

Then $X = \min(X_1, \dots, X_n) \sim Exp(\lambda_1 + \lambda_2 + \dots + \lambda_n)$

Example: Recall the example with 3 customers at a bank:



Q: What is the expected time until all 3 customers have left?

A: $\boxed{\dfrac{2}{\lambda}}$ (hint: consider the succession of events leading to having all customers being served)