Rmk: $o(h) \pm o(h) = o(h)$, $\underset{\substack{\uparrow \\ \text{constant}}}{C} \cdot o(h) = o(h)$.

Def (Poisson process): A counting process $\{N(t), t \geq 0\}$ is a Poisson process of rate (or intensity) λ, if it satisfies the following axioms.

(i) $N(0) = 0$

(ii) it has independent increments (cf. previous proposition)

(iii) $P(N(t+h) - N(t) = 1) = \lambda h + o(h)$

(iv) $P(N(t+h) - N(t) \geq 2) = o(h)$

↳ (or equivalently: $P(N(t+h) - N(t) = 0) = 1 - \lambda h + o(h)$)

Rmk: (iv): $P(N(t+h) - N(t) = 0)$

$$= 1 - P(N(t+h) - N(t) = 1) - P(N(t+h) - N(t) \geqslant 2)$$

$$= 1 - \lambda h \underbrace{- o(h) - o(h)}_{= +o(h)} = 1 - \lambda h + o(h)$$

Rmk: A consequence of this 2^{nd} definition is that

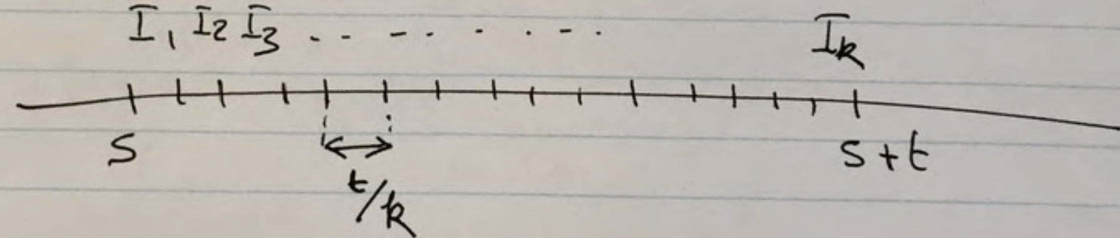
$$N(t+s) - N(s) \sim \text{Poisson}(\lambda t)$$

- (This result will be useful to show that the two definitions of a Poisson Process are equivalent)
- This also shows the property of stationary increments, seen previously

Proof: We use that $\text{Binomial}\left(k, \frac{\lambda}{k}\right) \simeq \text{Poisson}(\lambda)$ for large k.

(this uses Stirling formula)

Let us divide $(s, t+s)$ into k-subintervals



$$P(\text{some } I_i \overset{\text{among } k}{} \text{ has} \geq 2 \text{ events})$$
$$= \sum_{i=1}^{k} P(I_i \text{ has} \geq 2 \text{ events})$$
$$\overset{(iv)}{=} \sum_{i=1}^{k} o\left(\frac{t}{k}\right) = \sum_{i=1}^{k} o\left(\frac{1}{k}\right)$$
$$= k * o\left(\frac{1}{k}\right) = \frac{o(1/k)}{1/k} \xrightarrow[k\to\infty]{} 0$$

(t is constant $o\left(\frac{t}{k}\right) = t * o\left(\frac{1}{k}\right) = o\left(\frac{1}{k}\right)$)

$\Rightarrow$ ~~$N(t)$~~ $N(t+s) - N(s) \sim$ # intervals where 1 event occurs, as $k \to \infty$

$\to P(I_i \text{ has } 1 \text{ event}) \overset{(iii)}{=} \dfrac{\lambda t}{k} + o\left(\dfrac{1}{k}\right)$

$\to \left\{ \begin{array}{l} \cancel{N(t)} = j \\ j \leq k \end{array} \right\} \Leftrightarrow$ independently picking j intervals among k (ii)

$N(t+s) - N(s)$

$$\Rightarrow N(t+s) - N(s) \sim \text{Binomial}\left(k, p = \frac{\lambda t}{k} + o\left(\frac{1}{k}\right)\right)$$
$$\simeq \text{Poisson}\Big(\lambda t + \underbrace{k * o\left(\tfrac{1}{k}\right)}_{\downarrow k\to\infty \; 0}\Big)$$
$$\xrightarrow[k\to\infty]{} \boxed{\text{Poisson}(\lambda t)}$$

Csq: $\boxed{\text{Thm}: \{N(t), t \geq 0\} \text{ is a Poisson process of rate } \lambda \text{ (2nd def.)} \Leftrightarrow N(t) \text{ is a counting process with iid interarrival times} \sim Exp(\lambda) \text{ (1st def.)}}$

Proof: $\boxed{\Leftarrow}$ We already know that (ii) holds (independent increments) and (i) holds by definition.

$\to$ We need to check is (iii) & (iv)

(iii): $P(N(t+h)-N(t)=1) = P(N(h)=1)$ (stationary increments)

$$= \frac{\lambda h e^{-\lambda h}}{1!} = \lambda h e^{-\lambda h}$$

$(N(t) \sim \text{Poisson}(\lambda t))$

$$\left(e^{-\lambda h} = 1 - \lambda h + \frac{(\lambda h)^2}{2!} + \cdots\cdots\right) \qquad = \lambda h (1 + o(1))$$

$$= 1 + o(1) \qquad\qquad = \lambda h + o(h) \checkmark$$

(iv): $P(N(t+h)-N(t) \geq 2)$

$$= 1 - P(N(h)=1) - P(N(h)=0)$$

$$= 1 - (\lambda h + o(h)) - \frac{\lambda^0 h^0 e^{-\lambda h}}{0!}$$

$$= 1 - \lambda h + o(h) - e^{-\lambda h}$$

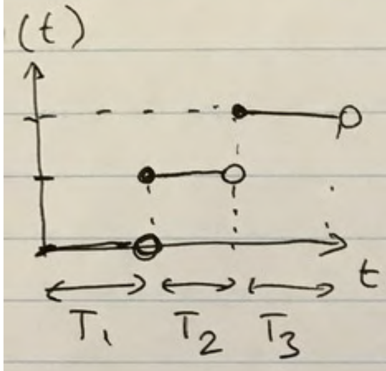
$$= -\lambda h + 1 - \underbrace{e^{-\lambda h}}_{1-\lambda h + o(h)} + o(h)$$

$$= -\cancel{\lambda h} + \cancel{1} - \cancel{1} + \cancel{\lambda h} + \underbrace{o(h) + o(h)}_{=o(h)}$$

$$= o(h) \checkmark$$

$\boxed{\Rightarrow}$ We need to check that interarrival times $T_1, T_2, \ldots$ are iid $\mathrm{Exp}(\lambda)$ r.v.'s.

We'll use the previous result $N(t+s) - N(s) \sim \mathrm{Poisson}(\lambda t)$
so in particular $N(t) \sim \mathrm{Poisson}(\lambda t)$



Distribution of T_1

$$P(T_1 > t) = P(\underbrace{N(t)}_{\sim \mathrm{Poisson}(\lambda t)} = 0)$$
$$= e^{-\lambda t} \longrightarrow \text{cdf of } \mathrm{Exp}(\lambda)$$
$$\Rightarrow T_1 \sim \mathrm{Exp}(\lambda) \checkmark$$

Distribution of T_2

$$P(T_2 > t) \underset{\substack{\uparrow \\ \text{condition on } T_1}}{=} \int_0^{+\infty} P(T_2 > t \mid T_1 = s) \cdot f_{T_1}(s)\, ds.$$

- $P(T_2 > t \mid T_1 = s) = P(N(s+t) - N(s) = 0)$
$$= e^{-\lambda t} \longrightarrow \text{shows that } T_2 \text{ is independent from } T_1$$

$$\Rightarrow P(T_2 > t) = \int_0^{+\infty} e^{-\lambda t} f_{T_1}(s)\, ds = e^{-\lambda t} \cdot 1$$
$$= e^{-\lambda t} \longrightarrow T_2 \sim \mathrm{Exp}(\lambda) \checkmark$$

This argument can be extended to $P(T_{n+1} > t \mid T_1 + \cdots + T_n = s)$ by induction, showing that $T_1, T_2, \ldots$ are iid $\mathrm{Exp}(\lambda)$ ✓

$\boxtimes$