Rmk: last lecture (typo)

$$Q = \begin{pmatrix} -v_1 & q_{\cancel{21}\,12} & q_{\cancel{31}\,13} & \cdots \\ q_{\cancel{12}\,21} & -v_2 & & \\ \vdots & & \ddots & \end{pmatrix}$$

Recall: We defined the intensity matrix

$$\boxed{(Q)_{ij} = \begin{cases} q_{ij} & \text{if } i \neq j \\ -v_i & \text{if } i = j \end{cases}} \qquad (q_{ij} = v_i p_{ij})$$

$$\boxed{\textbf{Thm}: \ \pi \text{ is stationary} \iff \pi Q = 0}$$

ex: • Poisson process of intensity λ $\quad 0 \to 1 \to 2 \cdots$

$$Q = \begin{array}{c} \\ \\ \end{array} \begin{array}{cc} & \begin{array}{ccccc} 0 & 1 & 2 & \cdots & \end{array} \\ & \begin{pmatrix} -\lambda & \lambda & & (0) & \\ 0 & -\lambda & \lambda & & \\ & & \ddots & \ddots & \\ & (0) & & -\lambda & \ddots \\ & & & & \ddots \end{pmatrix} \begin{array}{c} 0 \\ 1 \\ 2 \\ \vdots \\ \vdots \end{array} \end{array}$$

(For a P.P. of intensity λ, $v_0 = v_1 = \cdots = \lambda$)

$$P_{01} = 1$$

$$P_{10} = 0$$

$$\to q_{01} = v_0 \cdot P_{01} = \lambda \cdot 1 = \lambda$$

$$\to q_{10} = v_1 \cdot P_{10} = \lambda \cdot 0 = 0$$

Does the P.P. admit a stationary distribution?

→ If it admits a stationary distribution π,
then $\pi Q = 0$

$$(\pi_0, \pi_1, \dots) \begin{pmatrix} -\lambda & \lambda & & (0) \\ 0 & -\lambda & \lambda & \\ & & \ddots & \ddots \\ & (0) & & \ddots \end{pmatrix} = (0, 0, \dots\dots)$$

$$\Longleftrightarrow \begin{cases} -\lambda \pi_0 + 0.\pi_1 + 0.\pi_2 \dots = 0 \\ \lambda \pi_0 - \lambda \pi_1 + 0.\pi_2 \dots = 0 \\ \vdots \end{cases}$$

$$\Longleftrightarrow \begin{cases} \pi_0 = 0 \\ \pi_1 = 0 \\ \vdots \\ \pi_i = 0 \qquad i \geq 0 \end{cases}$$

So π cannot be a distribution, so there is no stationary distribution

- 2-state CTMC: $\quad 0 \underset{\mu}{\overset{\lambda}{\rightleftarrows}} 1 \qquad q_{12} = v_1 = \lambda$
$\qquad q_{21} = v_2 = \mu$

$$Q = \begin{pmatrix} -\lambda & \lambda \\ \mu & -\mu \end{pmatrix}$$

$$\pi Q = (\pi_0, \pi_1)\begin{pmatrix} -\lambda & \lambda \\ \mu & -\mu \end{pmatrix}$$

$$= \left(-\lambda\pi_0 + \mu\pi_1 \, ; \, \lambda\pi_0 - \mu\pi_1\right)$$

$$\pi Q = (0,0) \iff \lambda\pi_0 = \mu\pi_1 \iff \pi_0 = \frac{\mu}{\lambda}\pi_1$$

To fully solve (π_0, π_1), we use $\pi_0 + \pi_1 = 1$

$$\Longrightarrow \boxed{(\pi_0, \pi_1) = \left(\frac{\lambda}{\lambda+\mu}, \frac{\mu}{\lambda+\mu}\right)}$$

- Birth-Death Processes $\left(\textcircled{0} \rightleftarrows \textcircled{1} \rightleftarrows \cdots\cdots \right)$

with b-rate $(\lambda_i)_{i\geq 0}$

d-rate $(\mu_i)_{i\geq 0}$

$(\mu_0 = 0)$

$$\nu_i = \lambda_i + \mu_i$$

$$q_{i\,i+1} = \lambda_i \, , \quad q_{i\,i-1} = \mu_i$$

$$Q = \begin{pmatrix} -\lambda_0 & \lambda_0 & & & & (0) & \\ \mu_1 & -\lambda_1-\mu_1 & \lambda_1 & & & & \\ & \mu_2 & -\lambda_2-\mu_2 & \ddots & & & \\ & (0) & \ddots & \ddots & \lambda_{i-1} & & \\ & & & \mu_i & -\lambda_i-\mu_i & \lambda_i & \\ & & & & \mu_{i+1} & \ddots & \ddots \end{pmatrix}$$

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$$\pi Q = 0 \iff \begin{cases} -\lambda_0 \pi_0 + \mu_1 \pi_1 = 0 \\ \lambda_0 \pi_0 + (-\lambda_1 - \mu_1)\pi_1 + \mu_2 \pi_2 = 0 \\ \vdots \\ \lambda_{i-1} \pi_{i-1} + (-\lambda_i - \mu_i)\pi_i + \mu_{i+1} \pi_{i+1} = 0 \\ \vdots \end{cases}$$

We try to express everything in terms of π_0:

$$\rightarrow \pi_1 = \frac{\lambda_0}{\mu_1} \pi_0$$

$$\rightarrow \pi_2 = \frac{1}{\mu_2} \left(\lambda_1 \pi_1 + \mu_1 \pi_1 - \lambda_0 \pi_0 \right)$$

$$\Rightarrow \pi_2 = \frac{1}{\mu_2} \left(\frac{\lambda_1 \lambda_0}{\mu_1} \pi_0 + \cancel{\frac{\mu_1}{\mu_1} \lambda_0 \pi_0} - \cancel{\lambda_0 \pi_0} \right)$$

$$\Rightarrow \pi_2 = \frac{\lambda_1 \lambda_0}{\mu_2 \mu_1} \pi_0$$

In general, (we can show it by induction)

$$\boxed{\pi_i = r_i \pi_0, \quad \text{where } r_i = \frac{\lambda_0 \lambda_1 \cdots \lambda_{i-1}}{\mu_1 \mu_2 \cdots \mu_i} \quad (i > 0)}$$

Finally, we also must have $\boxed{1 = \sum_{i \geq 0} \pi_i = \left(1 + \sum_{i > 0} r_i\right) \pi_0}$

$\Rightarrow$ • If $\sum r_i = +\infty$, then there is no stationary distribution

• If $\sum r_i$ is finite, then we have found the stationary distribution

$$\boxed{\pi_0 = \frac{1}{1+\sum_i r_i} \quad \text{and} \quad \pi_n = \frac{r_n}{1+\sum_i r_i} \quad \text{for } n \geq 1}$$

Application (exercise): study the existence of the stationary distribution for particular cases of the B-D process seen in class.

IV. Applications & Reversibility

• As in Chap. 1, the stationary distribution is key to study limiting behaviour properties (also, the study simplifies when the chain is time reversible)

1) Interpretation of limiting probabilities

Def: • For a CTMC $\{X(t), t \geq 0\}$, 2 states $x, y \in S$ are said to communicate whenever $P_{xy}(s) > 0$ and $P_{yx}(t) > 0$, for some $s, t \geq 0$

(→ see chap. 1)

• As communication defines an equivalence relation (see chap. 1 again), we say that $X(t)$ is irreducible if it has 1 communicating class

Thm: Let $\{X(t), t \geq 0\}$ be an irreducible CTMC and let's consider

$$L_y = \lim_{t \to +\infty} \frac{1}{t} \int_0^t \mathbb{1}_{X_s = y}(s)\, ds$$

= Average fraction of time spent in state y in the long run (as $t \to +\infty$)

Then we have the following alternative

1) There is no stationary distribution and $L_y = 0$

2) There is a unique stationary distribution and $L_y = \pi_y$

Like in chap. 1, we can also define analogs of transience, null and positive recurrence and from the thm. below, the alternative yields:

Prop: 1) All states are transient or all states are null-recurrent

2) All states are positive recurrent with mean recurrence time $= \dfrac{1}{\nu_y \pi_y}$

→ Next week:

- We will conclude chap. 3 (time reversibility $\hookrightarrow \pi_i q_{ij} = \pi_j q_{ji}$)
- review some HW problems
- Jupyter notebook session on CTMC