1. (12 points; 4 each) Suppose X is a random variable with PDF

$$f(x) = \begin{cases} \frac{1}{8}, & -3 < x < -1 \\ cx + 1, & 0 < x < \frac{1}{4} \\ 0, & \text{otherwise} \end{cases}$$

(a) Find the value of c.

$\int_{-\infty}^{\infty} f(x) = 1$, so

$$1 = \int_{-3}^{-1} \frac{1}{8}\,dx + \int_0^{1/4} (cx+1)\,dx$$

$$= \frac{1}{8}\cdot 2 + \frac{1}{2}cx^2\Big|_0^{1/4} + \frac{1}{4}$$

$$= \frac{1}{2} + \frac{1}{2}c\cdot\frac{1}{16}$$

$$\Rightarrow \boxed{c = 16}$$

(b) Let F denote the CDF of X. Find $F\left(\frac{1}{8}\right)$.

$$F\left(\tfrac{1}{8}\right) = 1 - \int_{1/8}^{1/4} (16x+1) \qquad \left(\text{since } F\left(\tfrac{1}{8}\right) = 1 - P\left(X > \tfrac{1}{8}\right)\right)$$

$$= 1 - 8x^2\Big|_{1/8}^{1/4} + \frac{1}{8}$$

$$= 1 - 8\cdot\left(\frac{1}{16} - \frac{1}{64}\right) + \frac{1}{8}$$

$$= 1 - \frac{1}{2} + \frac{1}{8} + \frac{1}{8}$$

$$= 1 - \frac{1}{2} + \frac{1}{4}$$

$$= \boxed{\frac{3}{4}}$$

(c) Find $\mathbb{E}[X]$.

$$\mathbb{E}X = \int_{-\infty}^{\infty} x f(x)\,dx = \int_{-3}^{-1} \frac{1}{8}x\,dx + \int_0^{1/4} (16x^2 + x)\,dx$$

$$= \frac{1}{16}x^2\Big|_{-3}^{-1} + \frac{16}{3}x^3\Big|_0^{1/4} + \frac{1}{2}x^2\Big|_0^{1/4}$$

$$= \frac{1}{16}(1-9) + \frac{1}{12} + \frac{1}{32}$$

$$= -\frac{1}{2} + \cancel{\frac{1}{8}} + \frac{1}{32} = \cancel{\boxed{-\frac{11}{32}}} \quad \boxed{-\frac{37}{96}}$$

$\frac{1}{12}$

2. (12 points; 3 each) For each statement below, say whether it is true or false. No justification is required. No partial credit will be awarded on this problem.

(a) Let B be a discrete random variable with PMF $\mathbb{P}(B=0)=1/4$ and $\mathbb{P}(B=1)=3/4$. B has the same PMF as B^3.

True: $0^3 = 0$ and $1^3 = 1$,

so $P(B^3 = b) = P(B = b)$ for $b \in \{0,1\}$

(b) If $Z \sim \text{Poisson}(3)$, then $\mathbb{P}(Z=5) = e^{-3} \cdot \frac{5^3}{5!}$.

False: $P(Z=5) = e^{-3} \cdot \frac{3^5}{5!}$

(c) For any probability measure $\mathbb{P}$ on the set $\Omega = \{1,2,3,4\}$, there exists an event E with $\mathbb{P}(E) = 1/4$.

False: e.g. $\mathbb{P}(1) = P(2) = P(3) = \frac{1}{6}$, $P(4) = \frac{1}{2}$.

(d) Let Y be a continuous random variable with PDF $f(y)$. Suppose that $\mathbb{P}(Y > 3) = 1/4$. Then

$$\int_{-\infty}^{3} f(y)\, dy = 3/4.$$

True: $\int_{-\infty}^{3} f(y) dy = P(Y < 3) = 1 - P(Y > 3) = 1 - 1/4 = 3/4$.

since Y cts, $P(Y=3) = 0$.

3. (16 points; 4 each) Roll a fair die 100 times.

(a) Let A denote the event that you get exactly 10 1s in the first 30 rolls, B the event that you get exactly 20 2s in rolls 31-60, and C the event that you get all even numbers in rolls $61 - 100$. Find $\mathbb{P}(A), \mathbb{P}(B), \mathbb{P}(C)$, and $\mathbb{P}(A \cap B \cap C)$.

$P(A) = \binom{30}{10}\left(\frac{1}{6}\right)^{10}\left(\frac{5}{6}\right)^{20}$

$P(B) = \binom{30}{20}\left(\frac{1}{6}\right)^{20}\left(\frac{5}{6}\right)^{10}$

$P(C) = \left(\frac{1}{2}\right)^{40}$

A, B, C all indep

$\Rightarrow P(A \cap B \cap C) = P(A)P(B)P(C)$,

↑ product

(b) Let D denote the event that you get exactly 10 1s in total, and E the event that you get exactly 20 2s in total. Find $\mathbb{P}(D \cap E)$.

$P(D \cap E) = \binom{100}{10}\binom{90}{20} \cdot \left(\frac{1}{6}\right)^{30} \cdot \left(\frac{4}{6}\right)^{70}$

or $\binom{100}{20}\binom{80}{10} \cdot \left(\frac{1}{6}\right)^{30} \cdot \left(\frac{4}{6}\right)^{70}$

"choose which $\overset{10}{\wedge}$ of the 100 are 1's,

then " " 20" " 90 remaining are 2s,

then prob $\left(\frac{1}{6}\right)^{30} \cdot \left(\frac{\cancel{\;}}{\cancel{\;}}\right)^{70}$ the prob to get those #s.

↑ 4/6

(c) Use the inclusion-exclusion principle and your answer from part (b) to find $\mathbb{P}(D \cup E)$. You can still get credit on this part even if you didn't do part (b).

$$P(D) = \binom{100}{10}\left(\frac{1}{6}\right)^{10}\left(\frac{5}{6}\right)^{90}$$

$$P(E) = \binom{100}{20}\left(\frac{1}{6}\right)^{20}\left(\frac{5}{6}\right)^{80}$$

$$P(D \cup E) = P(D) + P(E) - P(D \cap E)$$

(d) Let X be the number of times in the 100 rolls that you rolled a 1, and then the next nine rolls were all 6s. Use linearity of expectation to find $\mathbb{E}[X]$.

$P($Roll i was 1, then rolls $i+1, i+2, \ldots, i+9$ were 6$)$

$= \left(\frac{1}{6}\right)^{10}$ ↑ event A_i

$X = \sum_{i=1}^{90} \mathbb{1}_{A_i}$, and $\mathbb{E}\,\mathbb{1}_{A_i} = P(A_i) = \left(\frac{1}{6}\right)^{10}$

So $\mathbb{E}X = 90 \cdot \left(\frac{1}{6}\right)^{10}$

4. (10 points) Jacob has infinitely many bags, labeled $1, 2, \ldots$. For every n, the bag labeled n has $n-1$ green balls and 1 red ball. Jacob generates a random variable N with Geometric(1/2) distribution, then draws a ball uniformly at random from the bag labeled N. Let $A_n = \{N = n\}$, and let R be the event that Jacob draws a red ball.

(a) (8 points) Use the law of total probability to write $\mathbb{P}(R)$ as an infinite sum. The expression in the sum should be explicit (evaluate any probabilities).

The events $A_1, A_2, \ldots$ partition Ω. So by LOTP,

$$P(R) = \sum_{n \geq 1} P(R \mid A_n) P(A_n)$$

$$= \sum_{n \geq 1} \frac{1}{n} \cdot 2^{-n}.$$

$$P(A_n) = P(\mathrm{Geo}(1/2) = n) = 2^{-n}$$

$P(R \mid A_n) = \frac{1}{n}$, since we pick unif. from n balls, 1 is Red.

(b) (2 points) Use the fact that $-\log(1-x) = \sum_{n=1}^{\infty} \frac{1}{n} x^n$ for $x \in (0,1)$ to evaluate your expression from part (a).

$$\sum_{n \geq 1} \frac{1}{n} 2^{-n} = -\log\left(1 - \frac{1}{2}\right) = \boxed{\log 2}$$

$x = \frac{1}{2}$