Jan 19 ①

Recall: • We introduce the n-step transition probabilities

$$P_{ij}^{n} = P(X_n = j \mid X_0 = i)$$

• Chapman - Kolmogorov eq.

$$P_{ij}^{n+m} = \sum_{k \in S} P_{ik}^{n} P_{kj}^{m}$$

Proof:

$$P_{ij}^{n+m} = P(X_{n+m} = j \mid X_0 = i)$$

$$= \sum_{k \in S} P(X_{n+m} = j, X_n = k \mid X_0 = i)$$

$$= \sum_{k \in S} \frac{P(X_{n+m} = j, X_n = k, X_0 = i)}{P(X_0 = i)}$$

$P(A \cap B)$
$\shortparallel$
$P(A \mid B) \cdot P(B)$

$$= \sum_{k \in S} \frac{P(X_{n+m} = j \mid X_n = k, X_0 = i) \times P(X_n = k, X_0 = i)}{P(X_0 = i)}$$

• $$\frac{P(X_n = k, X_0 = i)}{P(X_0 = i)} = P(X_n = k \mid X_0 = i) = P_{ik}^{n}$$

by definition

• $$P(X_{n+m} = j \mid X_n = k, \cancel{X_0 = i}) = P(X_{n+m} = j \mid X_n = k)$$

Markov property

$$= P(X_m = j \mid X_0 = k)$$

by homogeneity of the MC

$$= P^m_{kj}$$

Conclusion : $P^{n+m}_{ij} = \sum_{k \in S} P^n_{ik} \cdot P^m_{kj}$ ⊠

Week 2 content : • Classification of M-C's and states

• State : Accessibility, communication, periodicity

• M-C : Irreducibility, recurrence, transience

• Gambler's ruin problem.

III. Classification of states

Rmk : • We saw last week that we can get access to the states of a M-C at any given time by using the transition matrix. But this requires some algebra, that in practice can be tedious and become intractable as the number of transitions and states increases.

• In this section, we introduce some concepts that are fundamental and useful to classify and study M-C's and their states, without having to use complicated algebra

1) Accessibility and communication

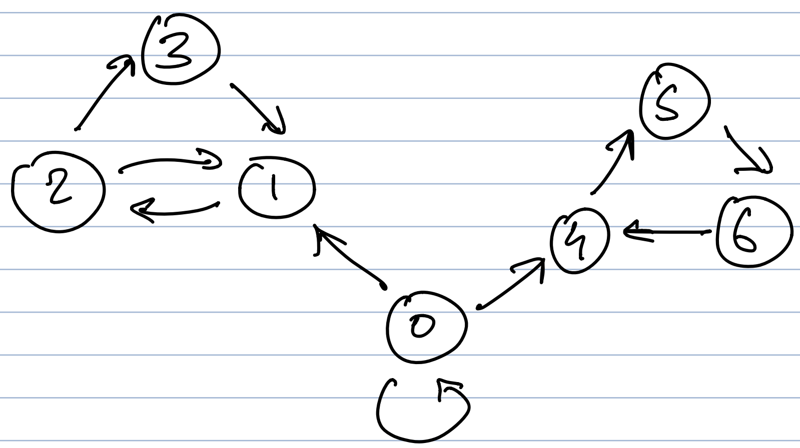
Def: • We say that the state j is accessible from i if $P_{ij}^{(n)} > 0$ for some $n \geq 0$

• We say that the states i and j communicate ($i \leftrightarrow j$) if $\begin{cases} j \text{ is accessible from } i \\ i \text{ '' '' } j \end{cases}$

Interpretation: "There is a path of size n that joins i to j in the transition diagram"

Rmk: The transition diagram is useful to determine accessible and communicating states

Ex:



- All states are accessible from 0 (T or F)
- All states communicate with 0 (T or F)
- 1 & 2 communicate (T or F)
- 1 & 4 communicate (T or F)

Prop: Communication is an equivalence relation

Recall: $\mathcal{R}$ is an equivalence relation if

$$\begin{cases} \text{(i) } x\mathcal{R}x \text{ (reflexivity)} \\ \text{(ii) } x\mathcal{R}y \Rightarrow y\mathcal{R}x \text{ (symmetry)} \\ \text{(iii) } x\mathcal{R}y \text{ and } y\mathcal{R}z \Rightarrow x\mathcal{R}z \\ \text{(transitivity)} \end{cases}$$

Proof: (i) $i \leftrightarrow i$ ($\underset{(=1)}{P_{ii}^{n}} > 0$ for $n = 0$)

(ii) $i \leftrightarrow j \Rightarrow j \leftrightarrow i$ by definition

(iii) Suppose $\exists m \mid P_{ij}^{m} > 0$ and $\exists n \mid P_{jk}^{n} > 0$

We will show is that $P_{ik}^{n+m} > 0$ (so $i \to k$).

Using the C-K. eq.

$$P_{ik}^{m+n} = \sum_{\ell \in S} P_{i\ell}^{m} P_{\ell k}^{n} \geq \overbrace{P_{ij}^{m}}^{>0} \overbrace{P_{jk}^{n}}^{>0} > 0$$

(we only keep $\ell = j$)

so k is accessible from i. Similarly, we can also show that i is accessible from k so $i \leftrightarrow k$ $\boxtimes$

Consequence: We can partition (split) the state space into communicating classes (if $i \leftrightarrow j$, they belong to the same class)

Ex: In the previous example, communication classes are $\{0\}$, $\{1,2,3\}$, $\{4,5,6\}$.

Def: A M.C is irreducible if it has only one communicating class

2) Periodicity

Def: i has period d (or $d(i)$) > 0 if d is the greatest common divisor (gcd) of all n, such that $P_{ii}^n > 0 \Rightarrow d(i) | n$

Rmk: • A state can have no period (there is no $n > 0$ s.t. $P_{ii}^n > 0$ ex: $0 \to 0$)

• Interpretation from the transition diagram

$i \rightleftarrows j$

$P_{ii}^1 = 0$

$P_{ii}^2 > 0$ $(i \to j \to i)$

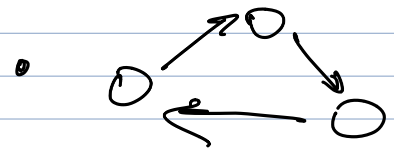
$\boxed{d = 2}$

$P_{ii}^3 = 0$

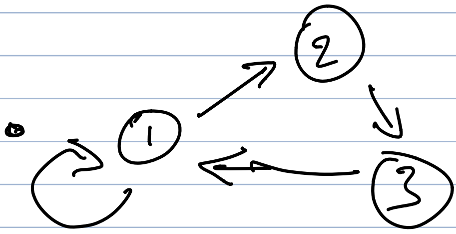
(all the n's s.t. $P_{ii}^n > 0$ are even)

$P_{ii}^4 > 0$ $(i \to j \to i \to j \to i)$

(...)

- 

$\boxed{d = 3}$

- 

we have the paths $(2 \to 3 \to 1)$ and $(2 \to 3 \to 1 \to 1 \to 2)$ so $P^3_{22} > 0$ and $P^4_{22} > 0$

$\Rightarrow d(2) \mid 3 \;\&\; d(2) \mid 4 \Rightarrow \boxed{d(2) = 1}$

Prop: If $i \leftrightarrow j$, then $d(i) = d(j)$

("states in the same communicating class have same period")

Proof: We first prove $d(i) \le d(j)$:

$i \leftrightarrow j$ so $\begin{cases} \exists m \mid P^m_{ij} > 0 \text{ and} \\ \exists n \mid P^n_{ji} > 0 \end{cases}$

$$\Rightarrow P^{n+m}_{ii} \underset{(C\text{-}K \text{ eq.})}{=} \sum_k P^m_{ik} P^n_{ki} \ge \underbrace{P^m_{ij}}_{>0} \underbrace{P^n_{ji}}_{>0} > 0$$

so $\boxed{d(i) \mid m + n}$

For ℓ s.t. $P^\ell_{jj} > 0$ (we know that ℓ exists since $i \leftrightarrow j$)

$$P^{m+n+\ell}_{ii} \ge P^m_{ij} P^\ell_{jj} P^n_{ji} > 0$$

(use the C-K eq. again)

so $\boxed{d(i) \mid m + n + \ell}$

$\Rightarrow d(i) \mid \cancel{m} + \cancel{n} + \ell - (\cancel{m} + \cancel{n})$

$\Rightarrow d(i) \mid \ell$

$\Rightarrow \boxed{d(i) \leq d(j)}$.

(because $d(j)$ is the gcd of all ℓ's s.t. $P^{\ell}_{jj} > 0$)

Similarly, one can also prove that

$\boxed{d(j) \leq d(i)}$

Conclusion: $\boxed{d(i) = d(j)}$ ⊠

<u>Rmk</u>: In practice to study the periods of a M-C, first decompose the M-C into communicating classes, then calculate the period for just one state in each class.

<u>Ex</u>: In the previous example

$\{0\}$: period 1

$\{1,2,3\}$: period 1 $\left(\begin{array}{l} 1 \to 2 \to 1 \quad P^2_{11} > 0 \\ 1 \to 2 \to 3 \to 1 \quad P^3_{11} > 0 \end{array} \right)$

$\{4,5,6\}$: period 3.