Riemann sum method and examples

Given a function $y = f(x)$, we approximate the area under the graph from $x = a$ to $x = b$ as follows:

1. Pick some positive integer n

   $n =$ number of approximating rectangles

   Compute $\Delta x = \frac{b-a}{n} =$ the width

   Label the tick-marks: $x_0 = a, x_1 = a + \Delta x, x_2 = a + 2\Delta x, \ldots, x_n = b$

   General pattern: $x_i = a + i\Delta x$.

   Choose the point x_i^* to determine the height of each rectangle.

   Right-endpoint: $x_i^* = x_i$

   Left-endpoint: $x_i^* = x_{i-1}$

   Midpoint: $x_i^* = \frac{x_i + x_{i-1}}{2}$

2. Area of the ith rectangle $= (\Delta x) f(x_i^*)$

   Add up all the areas for $i = 1, 2, \ldots, n$: in 'sigma' notation,

$$\begin{aligned}\sum_{i=1}^{n} (\Delta x) f(x_i^*) &= (\Delta x)(f(x_1^*) + f(x_2^*) + \cdots + f(x_n^*)) \\ &= \frac{b-a}{n}(f(x_1^*) + f(x_2^*) + \cdots + f(x_n^*))\end{aligned}$$

3. The exact area is

$$\int_a^b f(x)\, dx = \lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^{n} f(x_i^*).$$

Examples

Possible Riemann sums for $f(x) = x^2$, $a = 0, b = 1$

- ($n = 4$, right-endpoints) $\Delta x = \frac{1-0}{4} = \frac{1}{4}$
  $x_i = 0 + i * \frac{1}{4} = \frac{i}{4}$, $i = 1, 2, 3, 4.$
  The Riemann sum is

$$\sum_{i=1}^{4} \frac{1}{4} \cdot \left(\frac{i}{4}\right)^2 = \frac{1}{4}\left(\frac{1^2}{4^2} + \frac{2^2}{4^2} + \frac{3^2}{4^2} + 1\right) = \frac{15}{32} = 0.46875$$

- ($n = 4$, left-endpoints) $\Delta x = \frac{1-0}{4} = \frac{1}{4}$
  $x_i^* = x_{i-1} = 0 + (i-1) * \frac{1}{4} = \frac{i-1}{4}$, $i = 1, 2, 3, 4.$
  The Riemann sum is

$$\sum_{i=1}^{4} \frac{1}{4} \cdot \left(\frac{i-1}{4}\right)^2 = \frac{1}{4}\left(\frac{0^2}{4^2} + \frac{1^2}{4^2} + \frac{2^2}{4^2} + \frac{3^2}{4^2}\right) = \frac{7}{32} = 0.21875$$

- ($n = 40$, right-endpoints) $\Delta x = \frac{1-0}{40} = \frac{1}{40}$
  $x_i = 0 + i * \frac{1}{40} = \frac{i}{40}$, $i = 1, 2, \ldots, 40.$
  The Riemann sum is

$$\sum_{i=1}^{40} \frac{1}{40} \cdot \left(\frac{i}{40}\right)^2 = \frac{1}{40}\left(\frac{1^2}{40^2} + \frac{2^2}{40^2} \cdots + 1\right) \approx 0.345938$$

- For $n = 100, 1000, 10000$ with right-endpoints, I get Riemann sums

$$\sum_{i=1}^{100} \frac{1}{100} \cdot \frac{i^2}{100^2} = 0.33835$$
$$\sum_{i=1}^{1000} \frac{1}{1000} \cdot \frac{i^2}{1000^2} = 0.333834$$
$$\sum_{i=1}^{10000} \frac{1}{10000} \cdot \frac{i^2}{10000^2} = 0.333383$$

These results suggest that

$$\int_0^1 x^2\, dx = \frac{1}{3}.$$

Now you try: use midpoints with $n = 4$ to approximate the area under the curve $f(x) = x^3 - 1$ between $a = 2$ and $b = 4$.

$\Delta x =$ ________________

$x_i =$ ________________

$x_i^* =$ ________________

The Riemann sum is

$$\sum_{i=1}^{4} \_\_\_\_\_\_\_\_\_\_ \times \_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

Exercises

1. You are accelerating a car. You measure the following data for the speed of the car at different times:

time (seconds)	0	.5	1	1.5	2	2.5
speed (meters/second)	0	3.5	6.8	7.4	5.6	9.2

Estimate the total distance traveled by the car.

2. (a) Draw a plot of the function $f(x) = 3\sin x + 4$ for $0 \le x \le 2\pi$.

(b) What are the maximum and minimum values of $f(x)$ for $0 \le x \le 2\pi$?

(c) Draw one rectangle that completely contains the area under $f(x)$ for $0 \le x \le 2\pi$. What can you conclude about the area under the curve?

(d) Draw one rectangle that is completely contained in the area under $f(x)$ for $0 \le x \le 2\pi$. What can you conclude about the area under the curve?

In general if $m \le f(x) \le M$ for $a \le x \le b$, then

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a).$$