MATH 303 Practice Problems Solutions - Week 4

February 10, 2021

Problem 1

1. $\begin{pmatrix} \frac{1}{6} & \frac{2}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & \frac{2}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{2}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{2}{6} \\ \frac{2}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \end{pmatrix}$

2.
1 gives access to all states and is accessible from all states, so there is only one communicating class. The chain is irreducible and the state space is finite, so it is positive recurrent. It is also aperiodic (for example, $p_{11} > 0$ and periodicity is a class property). So the chain is ergodic.

3.
The chain is ergodic, so the limiting probabilities exist and noticing that the transition matrix is doubly stochastic, we know from class that these limiting probabilities are uniform (equal to $\frac{1}{5}$ here). From the course, we know that the mean number of turns to re-visit a given state i is $\frac{1}{\pi_i} = 5$.

Problem 2

1. The transition matrix is doubly stochastic, so the uniform distribution is stationary.

2. We denote by $(Y_n)_{0 \leq n \leq N}$ the reverse chain, i.e. $Y_n = X_{N-n}$. By the result in the course, Y is a Markov chain with transitions $q_{ij} = \frac{\pi_j}{\pi_i} p_{ji} = p_{ji}$, so the transition matrix of Y is

$$Q = \begin{pmatrix} 0 & 1/4 & 1/4 & 1/2 \\ 1/2 & 0 & 1/4 & 1/4 \\ 1/4 & 1/2 & 0 & 1/4 \\ 1/4 & 1/4 & 1/2 & 0 \end{pmatrix}.$$

In particular, we have

$$\mathbb{P}\left(X_{N-1} = 1 | X_N = 2\right) = \mathbb{P}\left(Y_1 = 1 | Y_0 = 2\right) = q_{21} = \frac{1}{2}.$$

3. We apply the Chapman–Kolmogorov to the chain Y:

$$\mathbb{P}\left(X_{N-2} = 3 | X_N = 4\right) = \mathbb{P}\left(Y_2 = 3 | Y_0 = 4\right) = q_{43}^2 = \sum_{i=1}^{4} q_{4i} q_{i3} = \frac{1}{4} \times \frac{1}{4} + \frac{1}{4} \times \frac{1}{4} = \frac{1}{8}.$$

Problem 3

1. The state space is $\{ABC, ACB, BAC, BCA, CAB, CBA\}$. The transition matrix is:

$$P = \begin{pmatrix} p_A & 0 & p_B & 0 & p_C & 0 \\ 0 & p_A & p_B & 0 & p_C & 0 \\ p_A & 0 & p_B & 0 & 0 & p_C \\ p_A & 0 & 0 & p_B & 0 & p_C \\ 0 & p_A & 0 & p_B & p_C & 0 \\ 0 & p_A & 0 & p_B & 0 & p_C \end{pmatrix}.$$

2. If Alice reads book B and book A on the next day, the order of the pile will be ABC whatever it was in the beginning, so ABC is accessible from all other states. The same reasoning applies to any ordering of the books, so the chain is irreducible.
The chain has finite state space, so it has at least one positive recurrent state. Since it has only one communicating class, all states are positive recurrent. Moreover, we have $p_{ii} > 0$ for every state i, so the chain is also aperiodic, so it is ergodic.
3. Since the chain is irreducible and ergodic, there is a unique stationary distribution π. If we write the equation $\pi = \pi P$, we obtain one equation for each of the 6 possible orderings. In particular, we have

$$\pi_{BAC} = p_B \pi_{BAC} + p_B \pi_{ABC} + p_B \pi_{ACB},$$

$$\pi_{BCA} = p_B \pi_{BCA} + p_B \pi_{CBA} + p_B \pi_{CAB}.$$

Summing up these two equations, we obtain

$$\pi_{BAC} + \pi_{BCA} = p_b \left(\pi_{BAC} + \pi_{ABC} + \pi_{ACB} + \pi_{BCA} + \pi_{CBA} + \pi_{CAB} \right) = p_B.$$

We also have

$$\pi_{ABC} = p_A \pi_{ABC} + p_A \pi_{BAC} + p_A \pi_{BCA} = p_A \pi_{ABC} + p_A p_B,$$

so $(1 - p_A)\pi_{ABC} = p_A p_B$, which gives $\pi_{ABC} = \frac{p_A p_B}{1-p_A}$. By the theorem from the course we conclude

$$\lim_{n \to +\infty} \mathbb{P}\left(X_n = ABC\right) = \pi_{ABC} = \frac{p_A p_B}{1 - p_A}.$$

Finally, we have $p_{ABC,CAB} = p_C > 0$ but $p_{CAB,ABC} = 0$ so the chain is not reversible.

Problem 4

1. The transitions are $p_{i,i-1} = \frac{2}{3}$ for all $i \geq 1$, $p_{i,i+1} = \frac{1}{3}$ for all $i \geq 0$, $p_{0,0} = \frac{2}{3}$ and $p_{ij} = 0$ everywhere else.
2. The chain (X_n) is irreducible. Moreover $p_{00} > 0$, so the state 0 has period 1, so all the states are aperiodic. However, since the state space is infinite, it is not immediate that the chain is positive recurrent. To prove it, we look for a stationary distribution π, and try to do so using reversibility. We need, for every $i \geq 1$:

$$\pi_i \times \frac{1}{3} = \pi_{i+1} \times \frac{2}{3},$$

so $\pi_{i+1} = \frac{1}{2}\pi_i$ for all i, so $\pi_i = \frac{1}{2^i}\pi_0$. Since the sum of π_i must be 1, we obtain $\pi_0 = \frac{1}{2}$ and $\pi_i = \frac{1}{2^{i+1}}$.
In particular (X_n) has a stationary distribution, so it is positive recurrent, and the limiting probabilities are given by $\pi_i = \frac{1}{2^{i+1}}$.