MATH 303 Midterm Exam B Solution

Problem 1

Consider the transition matrix P obtained from running the notebook with your student ID.

The notebook will give the following possible outputs

- Case (a): $P = \begin{pmatrix} 0.0 & 1.0 & 0.0 & 0.0 & 0.0 \\ 0.4 & 0.6 & 0.0 & 0.0 & 0.0 \\ 0.5 & 0.0 & 0.0 & 0.5 & 0.0 \\ 0.0 & 0.0 & 0.5 & 0.0 & 0.5 \\ 0.0 & 0.0 & 0.0 & 0.0 & 1.0 \end{pmatrix}$, $i = 1$, $j = 3$

- Case (b):: $P = \begin{pmatrix} 0.5 & 0.5 & 0.0 & 0.0 & 0.0 \\ 1.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.4 & 0.6 \\ 0.0 & 0.6 & 0.4 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 1.0 \end{pmatrix}$, $i = 2$, $j = 3$

- Case (c):: $P = \begin{pmatrix} 0.6 & 0.4 & 0.0 & 0.0 & 0.0 \\ 1.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.5 & 0.0 & 0.5 & 0.0 \\ 0.0 & 0.0 & 0.5 & 0.0 & 0.5 \\ 0.0 & 0.0 & 0.0 & 0.0 & 1.0 \end{pmatrix}$, $i = 2$, $j = 4$

- Case (d):: $P = \begin{pmatrix} 0.0 & 1.0 & 0.0 & 0.0 & 0.0 \\ 0.5 & 0.5 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.4 & 0.6 \\ 0.0 & 0.6 & 0.4 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 1.0 \end{pmatrix}$, $i = 1$, $j = 4$

a. Draw the transition diagram associated with P with the states corresponding to their row index in the matrix (i.e. first row correspond to state 1, second to state 2 etc.).

Solution: See Figure 1

b. Determine all the communication states (no need to justify).

Solution: Overall, There are 3 classes.

- Case (a): $\{\{1,2\},\{3,4\},\{5\}\}$
- Case (b): $\{\{1,2\},\{3,4\},\{5\}\}$
- Case (c): $\{\{1,2\},\{3,4\},\{5\}\}$
- Case (d): $\{\{1,2\},\{3,4\},\{5\}\}$

c. Determine which states are recurrent and which are transient (briefly justify).

Solution: Recurrence and transience are class properties, so we can look at each class:

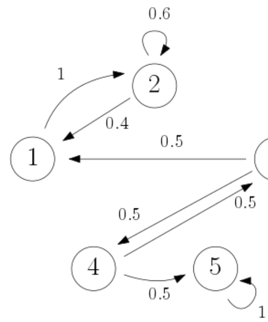


Figure 1: Transition diagram for the case (a). Other cases are similar (permute the states and change transion probabilites accordingly)

- Case (a) :
  - $\{1, 2\}$ is recurrent since it is closed and finite.
  - $\{3, 4\}$ is transient since it is not closed.
  - $\{5\}$ is recurrent since it is closed and finite.
- Case (b): (similar justification as in (a))
  $\{1, 2\}$ is recurrent, $\{3, 4\}$ is transient, $\{5\}$ is recurrent.
- Case (c) : (similar justification as in (a))
  $\{1, 2\}$ is recurrent, $\{3, 4\}$ is transient, $\{5\}$ is recurrent.
- Case (d) : (similar justification as in (a))
  - $\{1, 2\}$ is recurrent, $\{3, 4\}$ is transient, $\{5\}$ is recurrent.

d. Determine the period of each state (briefly justify).

Solution: States are in the same communicating class have the same period, so we only need to find the period for one state in each class:

- Case (a):
  - For $\{1, 2\}$:
    $d(1) = d(2) = 1$ since $P_{22}^{(1)} > 0$.
  - For $\{3, 4\}$:
    $d(3) = d(4) = 2$ since $P_{33}^{(1)} = 0$, $P_{33}^{(2)} > 0$, $P_{33}^{(2k-1)} = 0$, and $P_{33}^{(2k)} > 0$ for all $k \in \mathbb{N}_{>0}$.
  - For $\{5\}$:
    $d(5) = 1$ since $P_{55}^{(1)} > 0$.
- Case (b) (similar justification as in (a)): $\{1, 2\}$: aperiodic. $\{3, 4\}$: period 2. $\{5\}$: aperiodic.
- Case (c) (similar justification as in (a)): $\{1, 2\}$: aperiodic. $\{3, 4\}$: period 2. $\{5\}$: aperiodic.
- Case (d) (similar justification as in (a)): $\{1, 2\}$: aperiodic. $\{3, 4\}$: period 2. $\{5\}$: aperiodic.

e. Consider the state j obtained from running the notebook and assuming $X_0 = j$ (cf. previous question), what is the probability that the chain gets absorbed to the class of i? (justify your calculation with key steps; answers directly written won't be accepted).

Solution:

- Case (a) From the transition diagram, we see that starting from 3, the chain gets either absorbed to 1 or moves to 4. From 4, the chain gets either absorbed into 5, or moves to 3. Let $p(k)$ be the probability to be absorbed into the class of 1 starting from state k, where $k \in \{3, 4\}$. By doing a one-step analysis, we obtain

$$\begin{cases} p(3) = 0.5 + 0.5p(4) \\ p(4) = 0.5p(3) \end{cases} \Leftrightarrow \begin{cases} 0.75p(3) = 0.5 \\ p(4) = 0.5p(3) \end{cases} \Leftrightarrow \begin{cases} p(3) = \frac{2}{3} \\ p(4) = \frac{1}{3} \end{cases}$$

  The probability to find is thus $p(3) = \dfrac{2}{3}$.

- Case (b): Similarly as in (a)

$$\begin{cases} p(3) = 0.4p(4) \\ p(4) = 0.6 + 0.4p(3) \end{cases} \Leftrightarrow \begin{cases} p(3) = 0.4p(4) \\ 0.84p(4) = 0.6 \end{cases} \Leftrightarrow \begin{cases} p(3) = \frac{2}{7} \\ p(4) = \frac{0.6}{0.84} = \frac{5}{7} \end{cases}$$

  The probability to find is thus $p(3) = \frac{2}{7} = 0.2857$.

- Case (c): Similarly as in (a)

$$\begin{cases} p(4) = 0.5p(3) \\ p(3) = 0.5 + 0.5p(4) \end{cases} \Leftrightarrow \begin{cases} p(3) = \frac{2}{3} \\ p(4) = \frac{1}{3} \end{cases}$$

  The probability to find is thus $p(4) = \dfrac{1}{3}$.

- Case (d): Similarly as in (a)

$$\begin{cases} p(4) = 0.6 + 0.4p(3) \\ p(3) = 0.4p(4) \end{cases} \Leftrightarrow \begin{cases} p(3) = \frac{2}{7} \\ p(4) = \frac{0.6}{0.84} = \frac{5}{7} \end{cases}$$

  The probability to find is thus $p(4) = \frac{5}{7} = 0.7143$.

Problem 2

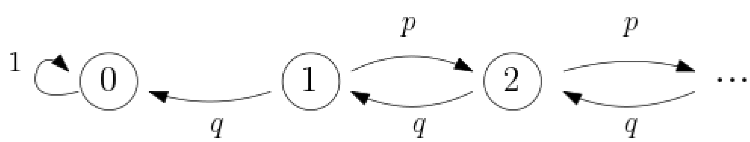
Consider the Markov Chain $(X_n)_{n\geq 0}$ defined on $\mathbb{N} = \{0, 1, \ldots\}$ with transition probabilities obtained from running the notebook.

The notebook gives $\forall i > 0$ $p_{i,i+1} = p$; $p_{i,i-1} = q$ and $p_{0,0} = 1$. Values of p, q are the following

- Case (a): $(p, q) = (0.8, 0.2)$
- Case (b): $(p, q) = (0.6, 0.4)$
- Case (c): $(p, q) = (0.75, 0.25)$
- Case (d): $(p, q) = (0.9, 0.1)$

a. Draw the transition diagram.

Solution: (Replace p, q by their numerical values)



b. For all $i > 0$, let p_i be the probability that given $X_0 = i$ the chain ever returns at state 0 (i.e. $P_i = P(X_n = 0 \text{ for some } n > 0 \; | X_0 = i))$. By conditioning on the outcome of the first step, find a relation between p_1 and p_2 (justify your answer).

Solution: From state 1, the chain can only move to 0 or 2, so by conditioning on the first step, we obtain

$$\begin{aligned} p_1 &= P(X_n = 0 \text{ for some } n > 0 | X_0 = 1) \\ &= P(X_n = 0 \text{ for some } n > 0, X_1 = 0 | X_0 = 1) + P(X_n = 0 \text{ for some } n > 0, X_1 = 2 | X_0 = 1) \\ &= P(X_1 = 0 | X_0 = 1) + P(X_n = 0 \text{ for some } n > 0 | X_1 = 2) P(X_1 = 2 | X_0 = 1) \\ &= q + p_2 p \end{aligned}$$

(remark: we used the fact that the MC is homogeneous so $P(X_n = 0 \text{ for some } n > 0 | X_1 = 2)) = p_2$

c. We now also assume that $p_2 = p_1^2$ and $p_1 < 1$. Under these assumptions, use **b.** to find p_1.

Solution: Replacing p_2 by p_1^2 in the relation found in **b.** yields $p_1 = q + p p_1^2$. Solving this equation for p_1, we obtain two roots $\frac{q}{p}$ and 1. Since we assume that $p_1 < 1$, we conclude that $p_1 = \frac{q}{p}$.

d. For $i > 1$, show that $p_k = p_{k-1} p_1$ (*hint: If the chain starts at $i > 1$, the chain has to visit state* 1 *before visiting 0*). In particular, this shows the previous assumption $p_2 = p_1^2$. Deduce the general formula for p_k as a function of k.

Solution: If the chain starts at $i > 1$, the chain has to visit state 1 before visiting 0, so we can write

$$\begin{aligned}
p_k &= P(X_n = 0 \text{ for some } n > 0 | X_0 = k) \\
&= P(X_m = 1 \text{ for some } m > 0, X_n = 0 \text{ for some } n > m | X_0 = k) \\
&= \sum_{m>0} P(X_n = 0 \text{ for some } n > m | X_m = 1) P(X_m = 1 | X_0 = k) \\
&\quad \text{(we used the Markov property)} \\
&= \sum_{m>0} p_1 P(X_m = 1 | X_0 = k) \qquad \text{(since the MC is homogeneous)} \\
&= p_1 \sum_{m>0} P(X_m = 1 | X_0 = k) \\
&= p_1 p_{k-1}.
\end{aligned}$$

Using this result, we obtain $p_k = p_1^k = \left(\frac{q}{p}\right)^k$.

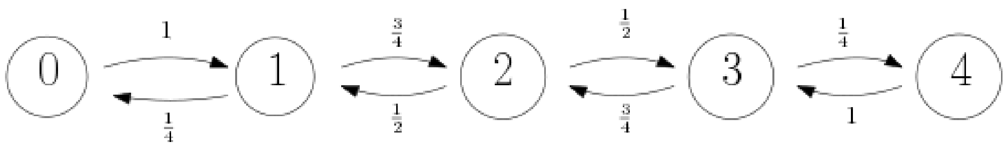
Problem 3

Your refrigerator has four cans of soda in it. You only buy two flavors of soda: purple berry and orange blast. At each step, you put your hand in the fridge, remove a soda (selecting one uniformly at random from those in the fridge), drink it, then replace it with a new can of the *opposite* flavor. Let X_n be the number of cans of purple berry in the fridge after you have drunk n cans.

The notebook gives four possible values of i to be used in question b.

a. Draw the transition diagram for X.

Solution:



b. Consider the state i obtained from running the notebook. Find $\mathbb{P}(X_2 = i | X_0 = i)$. (justify your calculation; answers directly written won't be accepted).

Solution: For example, for $i = 2$ we notice that to return in two steps starting from i, there are only two paths $2 \to 3 \to 4$ and $2 \to 1 \to 2$, so

$$P^2_{22} = P_{2,3}P_{3,2} + P_{2,1}P_{1,2} = \frac{3}{4}.$$

Doing something similar for different possible values of i yields

$$P^2_{ii} = \begin{cases} \frac{5}{8}, & i = 1 \\ \frac{3}{4}, & i = 2 \\ \frac{5}{8}, & i = 3 \\ \frac{1}{4}, & i = 4 \end{cases}$$

Remark: One can answer the question by calculating the 2-step transition matrix P^2, but it is faster to directly use the transition diagram.

c. Show that the distribution $\pi = \frac{1}{16}(1, 4, 6, 4, 1)$ is stationary for X_n, and that the process is reversible.

Solution: We just need to check that detailed balance is satisfied, i.e. $\sum_i \pi_i = 1$ (trivial since the question says that this is a distribution), and

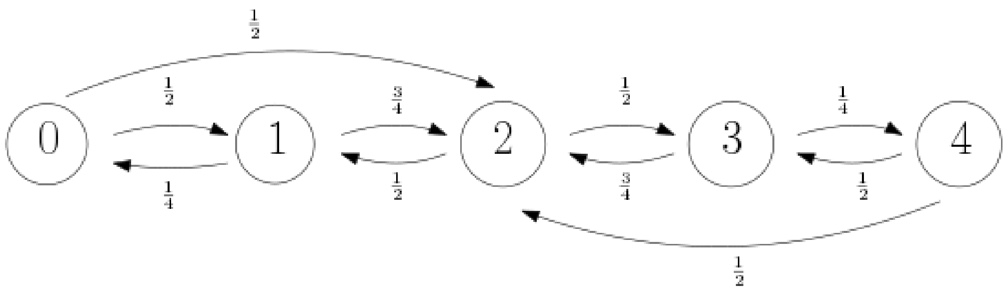
$$\forall i \in \{0, 1, 2, 3\} \quad P_{i,i+1}\pi_i = P_{i+1,i}\pi_{i+1}$$

Remark: There was no need to show the general relation given by a stationary distribution.

Consider a different Markov chain defined by slightly modifying the soda process: when all the soda in the fridge is the same flavor, you flip a fair coin, drink one can if it's heads and two cans if it's tails, and then replace any cans you drank with new cans of the opposite flavor. Let Y_n be the number of purple berry cans in the fridge after n steps in the modified process.

d. We assume that $\sigma = \frac{1}{17}(1,4,7,4,1)$ is stationary for Y. Use a result from class to argue that Y_n converges in distribution to σ as $n \to \infty$. Is Y reversible? Justify.

Solution: Now the transition diagram is



Y_n is aperiodic (since $P_{22}^3 > 0$ and $P_{22}^2 > 0$), and since the chain is irreducible and finite, it is positive recurrent. Thus it is ergodic, so by a result from class, $Y_n \to \sigma$. It is not reversible, since, for example,

$$\sigma_1 Q_{12} = \frac{3}{17} \neq \frac{7}{34} = \sigma_2 Q_{21}.$$

Problem 4

Complete the Problem 4 set in `midtermB.ipynb` (there are two questions).
Submit the completed notebook as `midtermB_complete.ipynb`

The notebook gives 2 values for p, q

Solution to question a: Enter the following matrix

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ (1-p)q & (1-p)(1-q) & p & 0 \\ (1-p)q & (1-p)(1-q) & 0 & p \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Marking: Points are possibly awarded if one enters a 4×4 matrix, the matrix is stochastic, and values are correct.

Solution to question b: Enter the following matrix for P_T to get S:

$$P_T = \begin{pmatrix} (1-p)(1-q) & p \\ (1-p)(1-q) & 0 \end{pmatrix}.$$

Marking: Points are possibly awarded if one enters a square matrix, and the values are correct.

Solution to question c: answer for $Time$ is S_{11}

Marking: Points are possibly awarded if the value is correct.