Math 324 E - Fall 2017
Midterm exam 1
Wednesday, October 18, 2017

Name: Solutions

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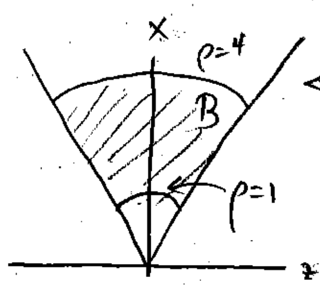
Problem 2	12	
Problem 3	12	
Problem 4	12	
Total	50	

- There are 4 questions on this exam. Make sure you have all four.
- **You must show your work on all problems.** The correct answer with no supporting work may result in no credit. Put a box around your FINAL ANSWER for each problem and cross out any work that you don't want to be graded.
- Give exact answers, and simplify as much as possible. For example, $\frac{\pi}{\sqrt{2}}$ is acceptable, but $\frac{1}{2}+\frac{3}{4}$ should be simplified to $\frac{5}{4}$.
- If you need more room, use the backs of the pages and indicate to the grader that you have done so.
- Raise your hand if you have a question.
- Any student found engaging in academic misconduct will receive a score of 0 on this exam.
- You have 50 minutes to complete the exam. Budget your time wisely!

GOOD LUCK!

1. (a) (7 pts) Let $B \subset \mathbb{R}^3$ be the region inside the sphere $x^2+y^2+z^2=16$, inside the half space $x \geq 0$, inside the cone $x^2 = 3y^2+3z^2$, and outside the sphere $x^2+y^2+z^2=1$. Set up an integral to find the volume of B. **You do not need to evaluate it.** (Hint: Use a rotated version of spherical coordinates.)

Use $x = \rho\cos\phi, \quad y = \rho\sin\theta\sin\phi, \quad z = \rho\cos\theta\cos\phi$



$\leftarrow \quad x^2 = 3y^2+3z^2$

in spherical, $\rho^2\cos^2\phi = 3\rho^2\sin^2\phi$

$\iff \tan\phi = \frac{1}{\sqrt{3}} \iff \phi = \pi/6.$

So $\text{vol}(B) = \iiint_B 1\, dV = \int_0^{\pi/6}\int_0^{2\pi}\int_1^{4} \rho^2\sin\phi \, d\rho\, d\theta\, d\phi.$

(b) (7 pts) Let S denote the sphere of radius 2 centered at $(0,0,0)$, and suppose S is filled with a fluid with density function $f(x,y,z) = z^3 - z + 8$. Find the total mass of fluid inside S by integrating the function f over S. (Hint: use symmetry.)

Note that $g(x,y,z) = z^3 - z$ has $g(x,y,-z) = -g(x,y,z)$,

so $\iiint_{S^-} g\, dV = -\iiint_{S^+} g\, dV$, where $S^+ = S \cap \{z \geq 0\}$, $S^- = S \cap \{z \leq 0\}$

Thus $\iiint_S g\, dV = 0$, so $\iiint_S f\, dV = \iiint_S g\, dV + \iiint_S 8\, dV$

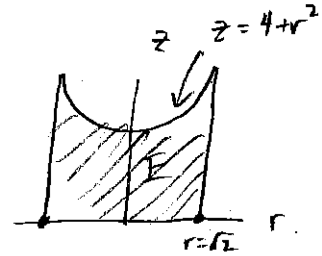
$= 8 \cdot \text{vol}(S)$

$= 8 \cdot \frac{4}{3}\pi (2)^3 = \boxed{\frac{256\pi}{3}}$

2. (12 pts) Use cylindrical coordinates to evaluate

$$\iiint_E e^z \, dV,$$

where E is the region bounded by the parabaloid $z = 4 + x^2 + y^2$, the cylinder $x^2 + y^2 = 2$, and the plane $z = 0$.



Parameterize E in cylindrical:

$$E = \left\{ (r,\theta,z) : \begin{array}{l} 0 \le \theta < 2\pi \\ 0 \le r \le \sqrt{2} \\ 0 \le z \le 4+r^2 \end{array} \right\}$$

Thus

$$\iiint_E e^z dV = \int_0^{2\pi} \int_0^{\sqrt{2}} \int_0^{4+r^2} e^z \cdot r \, dz \, dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\sqrt{2}} r\left(e^{4+r^2} - 1\right) dr \, d\theta$$

$$= 2\pi \left(\frac{1}{2} e^{4+r^2} - \frac{1}{2} r^2 \Big|_0^{\sqrt{2}} \right.$$

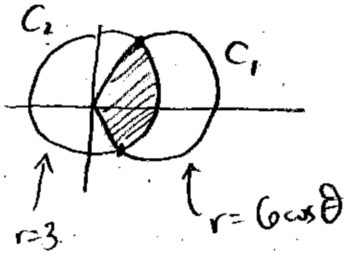
$$= \pi\left(e^6 - e^4 - 2\right)$$

3. (a) (6 pts) Set up a double integral in polar coordinates to find the area of the region inside the circle $(x-3)^2+y^2=9$ and outside the circle $x^2+y^2=9$. **You do not need to evaluate it.**

C_1 C_2

Use polar: C_1 is given by $x^2+y^2=6x$, or $r=6\cos\theta$

C_2 is given by $x^2+y^2=9$, or $r=3$.



area of shaded $= \displaystyle\int_{-\pi/3}^{\pi/3}\int_{6\cos\theta}^{3} r\,dr\,d\theta.$

intersection at $3 = 6\cos\theta \Rightarrow \theta = \pm\frac{\pi}{3}$.

(b) (6 pts) Consider the transformation $T:\mathbb{R}^2\to\mathbb{R}^2$ given by $(u,v)=T(x,y)=(2x+4y,x-3y)$. Solve for the inverse of T in terms of equations $x=x(u,v)$ and $y=y(u,v)$, and find the Jacobian determinant of T.

$\left.\begin{aligned} ①\ u &= 2x+4y \\ ②\ v &= x-3y \end{aligned}\right\}$ solve for x and y.

① $-2$② $\longrightarrow$ $u-2v = 10y \longrightarrow y = \dfrac{u-2v}{10}$

① $+4\cdot$② $\longrightarrow$ $3u+4v = 10x \longrightarrow x = \dfrac{3u+4v}{10}$

So $T^{-1}(u,v) = \left(\dfrac{3u+4v}{10}, \dfrac{u-2v}{10}\right)$.

$\operatorname{Jac}(T) = \begin{pmatrix} 2 & 4 \\ 1 & -3 \end{pmatrix}$, So

$|\det(\operatorname{Jac}(T))| = |-6-4|$

$= 10.$

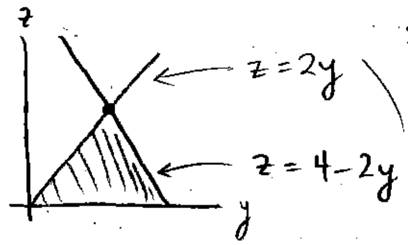
$\left(|\det(\operatorname{Jac}(T^{-1}))| = \dfrac{1}{10}.\right)$

4. (12 pts) Consider the tetrahedron $E \subset \mathbb{R}^3$ bounded by the planes $x = 0, z = 0, z = 2y$ and $2x + 2y + z = 4$. Set up the triple integral

$$\iiint_E xz\, dV$$

with the two given orders of integration. **You do not need to evaluate the integrals.**

(a) $dx\, dy\, dz$.



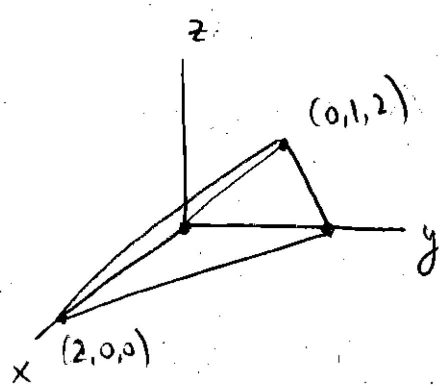
intersect at $4y = 4$, or $y = 1$

$\Rightarrow z = 2.$

$$E = \left\{ (x,y,z) : \begin{array}{l} 0 \le z \le 2 \\ z/2 \le y \le 2 - z/2 \\ 0 \le x \le 2 - y - z/2 \end{array} \right\}$$

So $$\iiint_E xz\, dV = \int_0^2 \int_{z/2}^{2-z/2} \int_0^{2-y-z/2} xz\, dV.$$

(b) $dy\, dz\, dx$.



Intersection of the surfaces $z = 2y$ and $2x + 2y + z = 4$ is

$2x + 2z = 4$, or $z = 2 - x$.

Thus $$\iiint_E xz\, dV = \int_0^2 \int_0^{2-x} \int_{z/2}^{2-x-z/2} xz\, dy\, dz\, dx.$$