3. Find curl $F$ and div $F$, where

$$F = \left\langle \frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2}, 0 \right\rangle .$$

$$\text{curl } F = \det \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \frac{x}{x^2+y^2} & -\frac{y}{x^2+y^2} & 0 \end{vmatrix} = \left(0 - \frac{\partial}{\partial z}\left(-\frac{y}{x^2+y^2}\right)\right)\hat{\imath}$$

$$- \left(\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z}\left(\frac{x}{x^2+y^2}\right)\right)\hat{\jmath} + \left(\frac{\partial}{\partial x}\left(-\frac{y}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right)\right)\hat{k}$$

$$\nabla \times F = 0\hat{\imath} - 0\hat{\jmath} + \left(\frac{2xy}{(x^2+y^2)^2} + \frac{2xy}{(x^2+y^2)^2}\right)\hat{k}$$

$$\boxed{\nabla \times F = \frac{4xy}{(x^2+y^2)^2}\hat{k} .}$$

$$\text{div } F = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(-\frac{y}{x^2+y^2}\right) + \frac{\partial}{\partial z}(0)$$

$$\boxed{\nabla \cdot F = \frac{2(y^2-x^2)}{(x^2+y^2)^2} .}$$

5. Show that for a vector field $F(x,y,z)$ and a function $f(x,y,z)$,

$$\nabla \cdot (fF) = f(\nabla \cdot F) + F \cdot \nabla f .$$

<u>Proof</u>: Let $F = \langle P, Q, R \rangle$. Then

$$\nabla \cdot (fF) = \frac{\partial}{\partial x}(fP) + \frac{\partial}{\partial y}(fQ) + \frac{\partial}{\partial z}(fR)$$

(product rule) $$= f\frac{\partial P}{\partial x} + P\frac{\partial f}{\partial x} + f\frac{\partial Q}{\partial y} + Q\frac{\partial f}{\partial y} + f\frac{\partial R}{\partial z} + R\frac{\partial f}{\partial z}$$

(re-grouping) $$= f\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) + P\frac{\partial f}{\partial x} + Q\frac{\partial f}{\partial y} + R\frac{\partial f}{\partial z}$$

(definition of div.) $$= f(\nabla \cdot F) + \langle P, Q, R \rangle \cdot \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

(definition of grad) $$= f(\nabla \cdot F) + F \cdot \nabla f .$$

8. Determine if there exists a vector field $G$ such that

$$\operatorname{curl}(G) = x\sin y\,\hat{\imath} + \cos y\,\hat{\jmath} + (z - xy)\,\hat{k}.$$

Claim: No such $G$ exists!

Proof. Suppose such a $G$ did exist. Then $\operatorname{div}(\operatorname{curl}(G)) = 0$, since $\operatorname{div}(\operatorname{curl}(F)) = 0$ for any vector field $F$. But

$$\begin{aligned}\operatorname{div}(\operatorname{curl}(G)) &= \operatorname{div}\left(x\sin y\,\hat{\imath} + \cos y\,\hat{\jmath} + (z - xy)\,\hat{k}\right)\\ &= (\sin y) + (-\sin y) + (1)\\ &= 1.\end{aligned}$$

This is a contradiction! So no such $G$ can exist.