Math 324 D - Winter 2018
Final exam
Wednesday, March 14th, 2018

Name: ______________________________

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- There are 6 problems (7 pages) in this exam. Make sure you have them all.
- You must show your work on all problems. The correct answer with no supporting work may result in no credit. **Put a box around your FINAL ANSWER for each problem and cross out any work that you don't want to be graded.**
- Give exact answers, and simplify as much as possible. For example, $\frac{\pi}{\sqrt{2}}$ is acceptable, but $\frac{1}{2}+\frac{3}{4}$ should be simplified to $\frac{5}{4}$.
- If you need more room, use the backs of the pages and indicate to the grader that you have done so.
- Raise your hand if you have a question.
- Any student found engaging in academic misconduct will receive a score of 0 on this exam.
- You have 2 hours to complete the exam. Budget your time wisely!

GOOD LUCK!

1. (15 pts) Let C be the cylinder $x^2 + z^2 = 4$ for $1 \leq y \leq 4$, oriented inward (i.e. normal points toward the y-axis). Use Stokes' theorem to evaluate

$$\iint_C \nabla \times \langle yz, 0, 0 \rangle \cdot dS.$$

Make sure to indicate how you are orienting the boundary.

2. (15 pts) Evaluate the surface integral

$$\iint_S \langle x+y, z, z-x\rangle \cdot d\mathbf{S},$$

where S is the boundary of the region between the paraboloid $z = 9 - x^2 - y^2$ and the xy-plane, oriented inward.

3. (19 pts) Consider a uniform magnetic field B with constant strength $b > 0$ in the z-direction, i.e. B is the vector field $B = b\hat{k}$.

(a) (5 pts) Let r be the vector field $r = x\hat{\imath} + y\hat{\jmath}$. Verify that $A = \frac{1}{2}B \times r$ is a 'vector potential' for B, i.e. $\nabla \times A = B$.

(b) (6 pts) Calculate the flux of B through the disk $x^2 + y^2 \leq 1$ in the plane $z = 1$, oriented upward.

(c) (8 pts) Use the result from part (a) and Stokes' theorem to calculate the flux of B through the disk bounded by the curve $s(t) = \cos t\hat{\imath} + \frac{\sqrt{2}}{2}\sin t\hat{\jmath} - \frac{\sqrt{2}}{2}\sin t\hat{k}$ for $0 \leq t \leq 2\pi$.

4. (16 pts; 4 pts each) For each of the statements below, circle **True** if you think it is true and **False** if you think it is false.

   (a) **True** **False** If S is any closed surface enclosing the origin, oriented away from the origin, then

$$\iint_S \frac{r}{|r|^3} \cdot dS = 4\pi,$$

   where $r = \langle x, y, z \rangle$.

   (b) **True** **False** Suppose F is a vector field, and g is a function with $\nabla \cdot F = g$. Then there exists a vector field H which is different from F and satisfies $\nabla \cdot H = g$.

   (c) **True** **False** If G is a vector field and $\nabla \cdot G = 0$, then

$$\iint_S G \cdot dS = 0$$

   for any oriented surface S.

   (d) **True** **False** For any conservative vector field F,

$$\text{grad}(\text{div}(F)) = 0.$$

5. (20 pts) Let D be the part of the ellipse $x^2 + \frac{y^2}{8} = 1$ in the $x - y$ plane with $y \geq 0$.

(a) Evaluate

$$\iint_D e^{8x^2+y^2}\, dA.$$

(b) Find all points on the boundary curve $C = \partial D$ where the tangent vector to C is parallel to the vector $\sqrt{2}\hat{\imath} + \hat{\jmath}$.

6. (15 pts) Let $F = \langle xy^2, x+y \rangle$ be a vector field in the $xy-$plane, and let C denote the upper half of the unit circle $x^2+y^2 = 1, y \geq 0$ oriented counter-clockwise. Also, let D be the upper half of the unit disk $x^2+y^2 \leq 1, y \geq 0$.

(a) (6 pts) Draw a picture of D and C, and evaluate $\int_C F \cdot d\mathbf{r}$.

(b) (4 pts) Henry, Jacob's evil twin brother, claims he is able to use Green's theorem to solve part a. He reasons as follows: "The curve C bounds the region D with the positive orientation, so by Green's theorem,

$$\int_C F \cdot d\mathbf{r} = \iint_D \left(\frac{d}{dx}(x+y) - \frac{d}{dy}(xy^2) \right) dA = \int_0^\pi \int_0^1 (1 - 2\sin\theta\cos\theta) r\,dr\,d\theta = \frac{\pi}{2}."$$

What is wrong with Henry's argument?

(c) (5 pts) Use Green's theorem correctly to relate a double integral over D to a line integral. Explain why Henry got the right answer, even though his reasoning is flawed.