Reminder: Register to the midterm session of your choice (before Friday 11:59 PM)

Poisson process properties:
- stationarity
- independent increment
- superposition
- Thinning

Thinning: $N(t)$ a P.P. of rate λ,
we label each event independently
of $\begin{cases} \text{type } 1 \text{ w.p. } p & \to N_1(t) = \# \text{ events of type } 1 \text{ by } t \\ \text{type } 2 \text{ w.p. } 1-p & \to \quad (2) \qquad\qquad (2) \end{cases}$

Then $N_1(t)$ and $N_2(t)$ are P.P. of intensity λp & $\lambda(1-p)$.

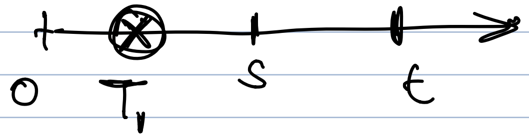
Rmk: This thinning property generalizes to $k \geq 2$ types of events, with probabilities $p_1, p_2, \dots, p_k$

- Conditional distribution of arrival times

Let $N(t)$ be a P.P. of rate λ

Q: Suppose we know that $N(t) = 1$. When did

The event in $[0, t]$ occur?



$$P(T_1 \leq s \mid N(t) = 1) = ?$$

$\underline{A}$: $P(T_1 < s \mid N(t) = 1) = \dfrac{P(T_1 < s,\ N(t) = 1)}{P(N(t) = 1)}$

$$= \frac{P(N(s) = 1,\ N(t) - N(s) = 0)}{P(N(t) = 1)}$$

$$= \frac{P(N(s) = 1) \cdot P(N(t) - N(s) = 0)}{P(N(t) = 1)}$$

independent increments

stationarity

$$= \frac{\cancel{\lambda} s\, \cancel{e^{-\lambda s}} \cdot \cancel{e^{-\lambda(t-s)}}}{\cancel{\lambda} t\, \cancel{e^{-\lambda t}}}$$

$$= \boxed{\frac{s}{t}} \qquad (0 \leq s \leq t)$$

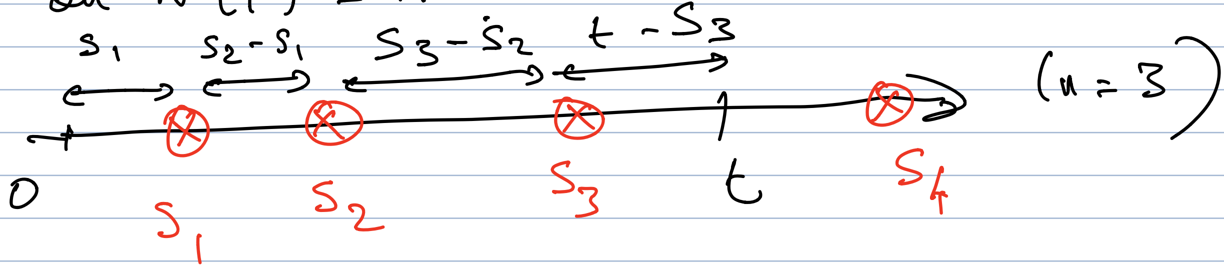
i.e. The conditional pdf of T_1

is $\dfrac{d}{ds}\left(\dfrac{s}{t}\right) = \boxed{\dfrac{1}{t}}$

i.e. $\boxed{T_1 \mid N(t) = 1 \ \sim\ \text{Uniform}\,[0; t]}$

More generally:

Q: Let $S_1, S_2, \ldots, S_n$ be the 1st n arrival times of $N(t)$. What is their joint pdf, conditioned on $N(t) = n$?



A: Conditional joint pdf, for $0 < s_1 < s_2 < \ldots < s_n < t$

$$f(s_1, \ldots, s_n \mid \underbrace{N(t) = n}_{\text{same as } S_n \le t \le S_{n+1}})$$

$(S_{k+1} - S_k \sim Exp(\lambda))$

$$= \frac{\int_t^\infty ds_{n+1} \cdot \lambda e^{-\lambda s_1} \cdot \lambda e^{-\lambda(s_2 - s_1)} \cdots\cdots \lambda e^{-\lambda(s_{n+1} - s_n)}}{\frac{(\lambda t)^n e^{-\lambda t}}{n!}}$$

integrating over all possible values of S_{n+1}

$$= \frac{\cancel{\lambda^n} e^{-\lambda(\cancel{s_1} - \cancel{s_2} + \cancel{s_1} - \cancel{s_3} + \cancel{s_2} \cdots)}}{\frac{\cancel{\lambda^n} t^n \cancel{e^{-\lambda t}}}{n!}} \underbrace{\int_t^\infty ds_{n+1} \lambda e^{-\lambda s_{n+1}}}_{\frac{\cancel{\lambda} \cancel{e^{-\lambda t}}}{\cancel{\lambda}}}$$

$$= \boxed{\frac{n!}{t^n}} \quad \text{for } 0 < s_1 < \ldots\ldots < s_n < t$$

Rmk: We can interpret this result as the fact that the n events (unordered) are conditionally independent and uniform on $[0, t]$

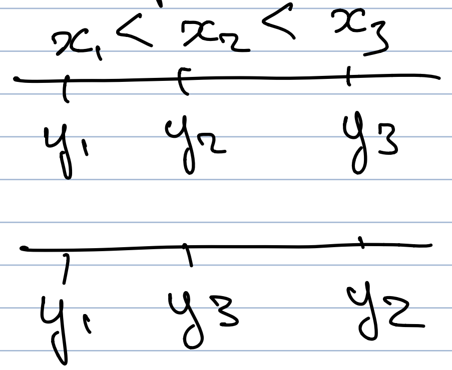
→ If $Y_1 = 5$, $Y_2 = -3$, $Y_3 = 1$

Let's define the ordered sequence

$$(Y_{(1)}, Y_{(2)}, Y_{(3)}) = (-3, 1, 5)$$

If Y_i's are continuous i.i.d. r.v's with pdf f, then the joint pdf of $(Y_{(1)}, Y_{(2)}, \dots Y_{(n)})$ is obtained by summing over all configurations $(Y_{\sigma(1)}, Y_{\sigma(2)}, \dots Y_{\sigma(n)})$, where σ is a permutation of $\{1, \dots, n\}$

ex: for $n = 3$



→ the joint pdf of $Y_{(1)}, \dots Y_{(n)}$ is

$$f(x_1, \dots, x_n) = \#\text{ permutations} \cdot f(x_1) \cdot f(x_2) \cdot \dots \cdot f(x_n)$$

$$= n!\, f(x_1) \dots f(x_n)$$ 5

$y_2 \quad y_1 \quad y_3$

$y_2 \quad y_3 \quad y_1$

$y_3 \quad y_1 \quad y_2$

$y_3 \quad y_2 \quad y_1$

$\rightarrow 6 = 3!$ configurations of (y_1, y_2, y_3) that generate $(Y_{(1)}, Y_{(2)}, Y_{(3)})$

$$= (x_1, x_2, x_3)$$

$\rightarrow$ In particular, for $Y_i \sim \text{Uniform}[0,t]$ we obtain

$$\boxed{f(x_1, \dots, x_n) = \frac{n!}{t^n}}$$

Thm: Consider the "point process" $Y = \{Y_1, \dots, Y_n\}$ of n independent uniform r.v.'s on $[0,t]$. Then Y has the same distribution as the set of arrival times $\{S_1, \dots, S_n\}$ in $[0,t]$ of a Poisson process $N(t)$, conditioned on $N(t) = n$

Rmk: $P(X_1 < s \mid N(t) = 1) \neq P(X_1 < s \mid X_1 < t)$
(exercise: show it by calculation)

$$\hookrightarrow P(X_1<s \mid X_1<t) = \frac{1-e^{-\lambda s}}{1-e^{-\lambda t}}$$

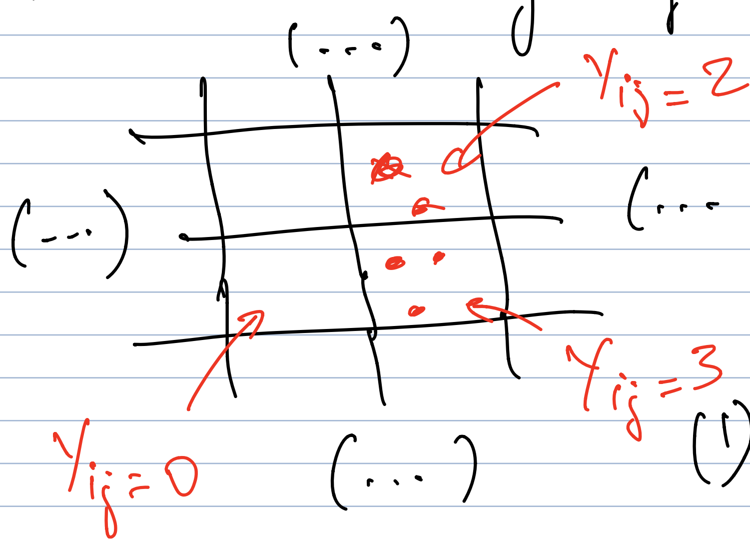
$$\left(\neq \frac{s}{t}\right)$$

This concludes our chapter on Poisson processes, which have many applications. To go further one can consider extensions and generalizations of the homogeneous 1-D Poisson process:

- → Non homogeneous processes in time, or $\neq$ distribution of arrival times (→ Notebooks, " Ross Chap. 5)
- → Poisson process in dimensions ≥ 2. (→ Notebook)

(Extra topic): Poisson process in dim. 2

Let's consider a grid of squares in $\mathbb{R}^2$



- Let's generate random points in $\mathbb{R}^2$: S_{ij}

(1) In each square, sample $Y_{ij} \sim \text{Poisson}(\lambda)$

(2) Then generate Y_{ij} uniform r.v.'s on the square S_{ij}

(This generalizes what we saw in dim. 1)

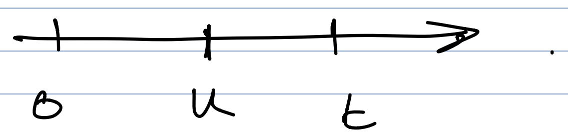
HW problem:

Let $N(t)$ be a λ-rate P.P.
Given that $N(t) = n$, what is the conditional probability of $N(s) = k$, for $0 \le s < t$ and $k = 0, 1, \dots, n$?

A: . Given that $N(t) = n$, what is the probability for one single event to occur in $[0, u]$?

- from what we saw, we know that the n events $E_1, \dots E_n$ are independent and uniformly distributed on $[0, t]$

$\rightarrow P(T_{E_1} \le u) = \dfrac{u}{t}$



- $\boxed{N(u) \sim \text{Binomial}(n, u/t)}$: Calling the event $T_{E_i} \le u$ a "success", $N(u)$ is the number of successes in n independent Bernoulli trials with probability of success $\frac{u}{t}$