Math 302, PSET 3

(1) (a) Define the function
$$f(x) = \begin{cases} 3x - b & x \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$
Show that there is no value of b for which this is the p.d.f. of some RV X.

(b) Let
$$f(x) = \begin{cases} \frac{1}{2}\cos x & x \in [-b,b] \\ 0 & \text{otherwise} \end{cases}$$
Show that there is exactly one value of b for which this could be the p.d.f. of some RV X.

(2) Let $c > 0$ and $X \sim \text{Unif}[0,c]$. Show that the RV $Y = c - X$ has the same c.d.f. and therefore also the same p.d.f. as X.

(3) Let X be a random variable with p.d.f.
$$f(x) = \begin{cases} cx^{-3} & x > 2 \\ 0 & \text{otherwise} \end{cases}$$
(a) Find c so that f is a p.d.f.
(b) Compute the c.d.f. of X.
(c) Find $\mathbb{P}(X > 3 | X < 5)$.
(d) Find the median of X, i.e. the value m such that $\mathbb{P}(X > m) = \mathbb{P}(X \le m)$.
(e) Calculate $\mathbb{E}\sqrt{X}$.

(4) Let X be an Exp(2) random variable. Find a number a such that $\{X \in [0,1]\}$ is independent of $\{X \in [a,2]\}$.

(5) Let X be a standard normal random variable. Compute $\mathbb{E}\, X^n$ for all $n \in \mathbb{N}$.

(6) You have two dice, one with three sides labeled $0,1,2$ and one with 4 sides, labeled $0,1,2,3$. Let X_1 be the outcome of rolling the first die, and X_2 the outcome of rolling the second. The rolls are independent.
(a) What is the joint p.m.f. of (X_1, X_2)?
(b) Let $Y_1 = X_1 \cdot X_2$ and $Y_2 = \max\{X_1, X_2\}$. Make a table for the joint p.m.f. of (Y_1, Y_2).
(c) Are Y_1, Y_2 independent? Compute $\text{Cov}(Y_1, Y_2)$.

(7) A fair die is rolled three times with outcomes X_1, X_2, X_3. Let Y_3 be the maximum of the values obtained.
(a) Show that
$$\mathbb{P}(Y_3 \le j) = \mathbb{P}(X_1 \le j)^3$$
for any $j = 1, 2, \ldots, 6$. Use this to find the distribution of Y_3.

(b) Suppose instead we sample n independent random variables $U_1, U_2, \ldots, U_n$ with Unif$(0,1)$ distribution, and let M_n be their maximum. Find the PDF of M_n.

(c) Show that, for any $x \in \mathbb{R}$, $\mathbb{P}(n \cdot (1 - M_n) \leq x) \to 1 - e^{-x}$ as $n \to \infty$.

(8) Compute the moment generating functions of $X \sim \text{Geom}(p)$, $Y \sim \text{Exp}(\lambda)$ and of $Z \sim \text{Poisson}(\mu)$.

(9) Proof of the 'law of the unconscious statistician'

(a) Let X be a continuous random variable with p.d.f. $f(x)$ and $g : \mathbb{R} \to \mathbb{R}$ be a strictly increasing function for which the set $A := \{x : g'(x) = 0\}$ is finite. Show that the following is a p.d.f. of $g(X)$:

$$f_{g(X)}(y) = \begin{cases} \frac{f(g^{-1}(y))}{|g'(g^{-1}(y))|} & \text{there exists some } x \in \mathbb{R} \setminus A \text{ s.t. } g(x) = y \\ 0 & \text{otherwise} \end{cases}.$$

(Note that the set $g(A) := \{y \in \mathbb{R} : \text{ s.t. } g(x) = y \text{ for some } x \in A\}$ is finite. The values of a function on a finite set does not affect its integral on any interval. Thus one need not worry about the value of $f_{g(X)}$ on A.)

(b)Let X be a continuous random variable with density function f_X. Let g be a differentiable, strictly increasing or strictly decreasing function for which $\mathbb{E}[g(X)]$ and $|A|$ are finite (where A is as above)). Prove that

$$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x) dx.$$

Hint: Use part (a). When solving part (b) use the definition for the expectation for the continuous random variable $Y := g(X)$, which is $\mathbb{E}[Y] := \int_{-\infty}^{\infty} y f_Y(y) dy$, where f_Y is the p.d.f. of Y. (Note that we have been using the formula $\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x) dx$ without a proof. Part (a) allows us to prove this formula in the special case considered at part (b).)