Rmk: Last week Notes → for the Poisson Process

$$P_{ii}'(t) = \lambda\left(\cancel{P_{i-1,i}(t)} - P_{ii}(t)\right)$$

This week: • We conclude Chap 3

→ Limiting probabilities of CTMC

→ Applications

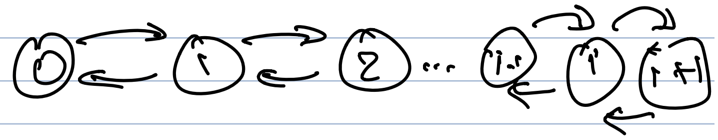
• HW solutions for Pbm 1-3 to be posted today

Next week (last class): Notebook session + review for the final exam.

Recall: We saw that a CTMC satisfies the Kolmogorov backwards and forward equations

$$\boxed{\begin{aligned} K\text{-}b\text{-}E:&\quad P_{ij}'(t) = \sum_{k\neq i} q_{ik} P_{kj}(t) - v_i P_{ij}(t) \\ K\text{-}f\text{-}E:&\quad P_{ij}'(t) = \sum_{k\neq j} P_{ik}\, q_{kj} - v_j P_{ij}(t) \end{aligned}}$$

(where $\underline{q_{ij} = v_i p_{ij}}$)



Ex: For a Birth-death process, find the K-b-E and K-f-E satisfied by $P_{i\,i+1}(t)$

Recall: For a B-D process with birth rates and death rates $(\lambda_i)_{i\geq 0}$ and $(\mu_i)_{i\geq 0}$ respectively

$$p_{i,i+1} = \frac{\lambda_i}{\lambda_i+\mu_i} ; \quad p_{i,i-1} = \frac{\mu_i}{\lambda_i+\mu_i}, \quad p_{ij} = 0 \text{ else}$$

$v_i = \lambda_i + \mu_i$ (rmk: when $i=0$, $\mu_i = \mu_0 = 0$)

$$\Rightarrow \quad q_{i\,i+1} = \lambda_i, \quad q_{i\,i-1} = \mu_i, \quad q_{ij} = 0 \text{ else.}$$

- **Backwards eq.**:

$$P_{i\,i+1}'(t) = \sum_{k\neq i} q_{ik} P_{k,i+1}(t) - v_i P_{i,i+1}(t)$$

only 2 terms: $q_{i\,i+1}, q_{i\,i-1}$ (all others are 0)

$$= q_{i\,i-1} P_{i-1,i+1}(t) + q_{i,i+1} P_{i+1,i+1}(t) - v_i P_{i\,i+1}(t)$$

$$\boxed{P_{i,i+1}'(t) = \mu_i P_{i-1,i+1}(t) + \lambda_i P_{i+1,i+1}(t) - (\lambda_i+\mu_i) P_{i,i+1}(t)}$$

• <u>Forward eq.</u>

③

$$P_{i\,i+1}'(t) = \sum_{k \neq i+1} P_{ik}(t)\, q_{k\,i+1} - v_{i+1} P_{i\,i+1}(t)$$

$$= P_{i\,i+2}(t)\, q_{i+2,\,i+1} + P_{ii}(t)\, q_{i\,i+1} - v_{i+1} P_{i\,i+1}(t)$$

$$\boxed{P_{i\,i+1}'(t) = P_{i\,i+2}(t)\, \mu_{i+2} + P_{ii}(t)\, \lambda_i - (\lambda_{i+1} + \mu_{i+1}) P_{i\,i+1}(t)}$$

<u>exercise</u>: $i = 0$, $P_{i,i-1}'(t)$, $P_{ii}'(t)$ etc...

<u>Rmk</u>: As a set (or a system) of linear ordinary differential equations, Kolmogorov eq's can be solved by considering the associated matrix exponential (if you're familiar with the theory of linear diff. eq.). This can only be done explicitly in simple cases (for example, check HW problem 1.4), so we'll now focus on limiting behaviour.

III. Limiting probabilities

- Like in discrete time, we're now interested in the long term behaviour of the MC. In particular, finding, if it exists

$$P_j = \lim_{t\to+\infty} P_{ij}(t) \quad , \text{ independent of } i.$$

- Rmk : if $\lim\limits_{t\to+\infty} P_{ij}(t)$ exists, then $\lim\limits_{t\to+\infty} P'_{ij}(t) = 0$

Let's plug this in the Kolmogorov backwards eq.

$$\underset{\substack{\uparrow\, t\to+\infty \\ P'_{ij}(t)}}{0} = \sum_{k\neq i} q_{ik} \underset{\substack{\uparrow\, t\to+\infty \\ P_{kj}(t)}}{P_j} - v_i \underset{\substack{\uparrow\, t\to+\infty \\ P_{ij}(t)}}{P_j}$$

$$\Longleftrightarrow \left(\sum_{k\neq i} q_{ik}\right) \cancel{P_j} = v_i \cancel{P_j}$$

$$\Longleftrightarrow \sum_{k\neq i} q_{ik} = v_i$$

, which is something we already know (cf. last week)

So we haven't gained any new info. on the limiting behaviour.

→ For the forward eq:

$$\underset{\substack{\uparrow \\ P_{ij}'(t)}}{0}{}^{t\to+\infty} = \sum_{k\neq j} \underset{\substack{\uparrow\, t\to+\infty \\ P_{ik}(t)}}{P_k} q_{kj} - \nu_j \underset{\substack{\uparrow\, t\to+\infty \\ P_{ij}(t)}}{P_j}$$

so

and besides

$$\boxed{\begin{cases} \nu_j P_j = \sum_{k\neq j} q_{kj} P_k \\ \sum_j P_j = 1 \end{cases}} \leftarrow (*)$$

Interpretation of $(*)$

LHS : $\boxed{\nu_j P_j}$ → overall flux at which the CTMC leaves j

ν_j: rate to leave j when the process is at j

P_j: proba to be in j (long-term)

RHS : $\boxed{\sum_{k\neq j} q_{kj} P_k}$ → overall flux at which the CTMC enters j

q_{kj}: rate at which the CTMC enters j starting from k

Cc : At equilibrium ($P_{ij}' = 0$, $\forall i,j$), the overall flux of entrance into a state equals the flux of exit

↪ (*) describes "balance equations"

(rmk: this is not "detailed" balance yet....)

(*) can be rewritten as $\begin{cases} \pi Q = 0 \\ \sum \pi_j = 1 \end{cases}$,

where $\pi = (P_1, P_2, \ldots\ldots)$ is a probability distribution and

$$Q = \begin{pmatrix} -v_1 & q_{21} & q_{31} & \cdots\cdots \\ q_{12} & -v_2 & q_{32} & \cdots\cdots \\ & \ddots & \ddots & \\ & & & \ddots \end{pmatrix}$$ is the "intensity matrix" of the CTMC

$$\left(\sum_{k \neq j} q_{kj} P_k - v_j P_j = 0 \right)$$

Thm: The distribution π is stationary

(i.e. $P(X(t) = y \mid X(0) \sim \pi) = \pi_y$)

$$\Updownarrow$$

$$\pi Q = 0$$

ex: stationary distribution for 2-state CTMC or B-D process (if it exists) (next time)

. Find the intensity matrix for the Poisson process