MATH 303 Practice Problems Solutions - Week 3

Jamuary 29, 2021

Problem 1

1. Consider the stochastic matrices

$$M_1 = \begin{pmatrix} * & * & 0 & 0 \\ * & * & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix}, \qquad M_2 = \begin{pmatrix} * & * & 0 & * \\ * & * & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix}, \qquad M_3 = \begin{pmatrix} * & * & 0 & * \\ * & * & 0 & 0 \\ 0 & 0 & * & * \\ * & 0 & * & * \end{pmatrix},$$

where all entries marked with $*$ are greater than 0. For each of them, find the number of communicating classes, and decide if they are closed (briefly justify).
2. Show that if a state j is accessible from i but i is not accessible from j, then i is transient. [Hint: remember how transience of some states was proved in the Gambler's ruin problem.] Note that as a corollary, if i is recurrent and j is accessible from i, then i is accessible from j.
3. Show that if i is recurrent, then it belongs to a closed class. Conclude that if the state space is finite, then there is at least one closed communicating class.
4. If the state space is finite, show that all the states of a closed communicating class are recurrent.
5. Is this still true in general (i.e. for arbitrary state spaces)? Justify.

Problem 2

Consider the asymmetric random walk in $\mathbb{Z}$, where one moves one step to the right with probability p, and to the left with probability $1-p$, where $p \neq \frac{1}{2}$. Show that the walk is transient. [Hint: apply the method used in class for 1-D symmetric (simple) random walk.]

Problem 3

Consider the following Markov Chain with state space $\{0, 1, 2, 3, 4\}$: if current state is 0 then the next state is 4 w.p. 1; if the state is $i \neq 0$ then the next state is equally likely to be any of the states $\{0, \ldots, i-1\}$.
1. Determine the transition matrix for this Markov chain.
2. Calculate the stationary distribution of this Markov chain.
3. Suppose the Markov chain has been running for a long time. Approximately what fraction of time has it spent in state 0?

Problem 4

We consider a Markov chain $(X_n)_{n\geq 0}$ on states 1, 2, 3 and 4, with transition matrix

$$\begin{pmatrix} 1/5 & 2/5 & 1/5 & 1/5 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ * & * & * & * \\ * & * & * & * \end{pmatrix}$$

Assuming $X_0 = 1$,
1. what is the probability to enter state 3 before state 4?

2. what is the mean number of transitions until either state 3 or 4 is entered? (hint: for both questions, do a one-step analysis, i.e. condition on the outcome of X_1)
3. Solve **2.** using the method introduced last week to find the mean time in transient states (see week 2 notebook).

Problem 5 (Jupyter Notebooks)

1. Modify the 2D simulation of the random walk in the Jupyter notebook slightly, so that instead of moving North/South/East/West with equal probability, the random walker goes North or East with probability .3, and South or West with probability .2. What do you think the path of the random walk will like? Do you think the new Markov chain will be recurrent? Check by simulating a few instances.
2. The Jupyter notebook has a simulation of 3D simple random walk, that counts the total number of visits to 0. Simulate the walk 10^3 times, each time running it for 10^3 steps, and record how many times the walk visited 0 in each case. Plot the result in a histogram. According to the computation from lecture, the distribution should be exponential – check that the histogram decays exponentially, and use the data to estimate the return probability.
3. Modify the Markov chain in the Jupyter notebook by adding .1 to an entry in the first row, and subtracting .1 from another entry in the first row. How do you think this will affect the limiting probabilities for being in all three states? (Which ones will increase, which will decrease, and which will stay the same?) Check by running the code to count the total number of visits to each state.