#4. Find the gradient of $f(x,y) = 17y - 2xy^2$, and compute $D_u f(8,1)$, where $u = \frac{1}{\sqrt{17}} \langle -4, 1 \rangle$.

Solution: $\nabla f(x,y) = \frac{\partial f}{\partial x}(x,y)\,\hat{\imath} + \frac{\partial f}{\partial y}(x,y)\,\hat{\jmath}$

$$= -2y^2\,\hat{\imath} + (17 - 4xy)\,\hat{\jmath}$$

$$\frac{\partial f}{\partial x} = 0 - 2(1)y^2 = -2y^2$$

$$\frac{\partial f}{\partial y} = 17(1) - 2(2xy) = 17 - 4xy$$

Thus, using the formula $D_u f(x,y) = \nabla f(x,y) \cdot u$, (since u is a unit vector),

$$D_u f(8,1) = (\nabla f(8,1)) \cdot \frac{1}{\sqrt{17}}(-4\hat{\imath} + \hat{\jmath})$$

$$= (-2\hat{\imath} + (17-32)\hat{\jmath}) \cdot \frac{1}{\sqrt{17}}(-4\hat{\imath} + \hat{\jmath})$$

$$= \frac{1}{\sqrt{17}}(-2\hat{\imath} - 15\hat{\jmath}) \cdot (-4\hat{\imath} + \hat{\jmath})$$

$$= \frac{1}{\sqrt{17}}(8 - 15) = \boxed{\frac{7}{\sqrt{17}}}.$$

5. Find (the) two points on $\frac{x^2}{4} + \frac{y^2}{9} + z^2 = 1$ where the tangent plane is normal to $v = \langle 1, 1, -2 \rangle$.

<u>Solution</u>: Recall that if $F(x,y,z) = \frac{x^2}{4} + \frac{y^2}{9} + z^2$, then $\nabla F(x,y,z)$ is normal to the surface at (x,y,z) for any point (x,y,z) on the ellipsoid.

Thus, we want to find (x,y,z) satisfying

$x^2/4 + y^2/9 + z^2 = 1$, and also $\nabla F(x,y,z)$ is parallel to v. Well,

$$\nabla F(x,y,z) = \frac{x}{2}\hat{\imath} + \frac{2y}{9}\hat{\jmath} + 2z\hat{k} \quad \text{and} \quad \nabla F(x,y,z) \text{ parallel to } v$$

means $\nabla F(x,y,z) = \frac{x}{2}\hat{\imath} + \frac{2y}{9}\hat{\jmath} + 2z\hat{k} = \alpha v = \alpha\hat{\imath} + \alpha\hat{\jmath} - 2\alpha\hat{k}$

for some $\alpha > 0$. Thus $x = 2\alpha$, $y = \frac{9}{2}\alpha$, $z = -\alpha$. Plugging into the ellipsoid equation,

$$\frac{(2\alpha)^2}{4} + \frac{(9\alpha/2)^2}{9} + (-\alpha)^2 = 1, \quad \text{or} \quad \alpha^2 + \frac{9}{4}\alpha^2 + \alpha^2 = 1$$

$\Longrightarrow \alpha = \pm\frac{2}{\sqrt{17}}$. Thus $(x,y,z) = \boxed{\left(\pm\frac{4}{\sqrt{17}}, \pm\frac{9}{\sqrt{17}}, \mp\frac{2}{\sqrt{17}}\right)}$ are

#8. $f(x,y,z) = x + yz$, C = line segment from $(0,0,0) \longrightarrow (6,2,2)$.

Give a linear parameterization of C that traverses the segment in one time unit, and evaluate $\int_C f \cdot ds$.

Do the same integral with the parameterization $s(t) = (6t^2, 2t^2, 2t^2)$, $0 \le t \le 1$.

Solution: A linear parameterization of C is $r(t) = (6t, 2t, 2t)$ $= (x(t), y(t), z(t))$ for $0 \le t \le 1$. Then

$x'(t) = 6$
$y'(t) = 2$
$z'(t) = 2$

$$\int_C f \cdot ds = \int_0^1 f(6t, 2t, 2t)\sqrt{(6)^2 + (2)^2 + (2)^2}\, dt$$

$$= \int_0^1 (6t + (2t)(2t))\sqrt{44}\, dt$$

$$= \sqrt{44}\int_0^1 (6t + 4t^2)\, dt = \sqrt{44}\left(3 + \frac{4}{3}\right) = \frac{26\sqrt{11}}{3}.$$

$s'(t) = (12t, 4t, 4t)$

Also, with the second parameterization,

$$\int_C f \cdot ds = \int_0^1 (6t^2 + (2t^2)(2t^2))\sqrt{(12t)^2 + (4t)^2 + (4t)^2}\, dt$$

$$= \int_0^1 (6t^2 + 4t^4)\sqrt{176\, t^2}\, dt$$

$$= 4\sqrt{11}\int_0^1 (6t^3 + 4t^5)\, dt = 4\sqrt{11}\left(\frac{6}{4} + \frac{4}{6}\right) = \frac{26\sqrt{11}}{3}.$$

The two integrals are equal, because line integrals are independent of parameterization!