\# 1. Evaluate $\iiint_E x\,dV$, where E is enclosed by $z=0$, $z=x+y+5$, and $x^2+y^2=4$, $x^2+y^2=9$.

<u>Solution:</u> In cylindrical coordinates,

$$E = \left\{ (r,\theta,z) : \begin{array}{l} 2 \le r \le 3 \\ 0 \le \theta \le 2\pi \\ 0 \le z \le r\cos\theta + r\sin\theta + 5 \end{array} \right\}.$$

This works since $x+y+5 > 0$ on the region $2 \le r \le 3$. [$x+y$ is smallest when $x=y=-\sqrt{9/2}$, or $x+y=-\sqrt{18}$, which is greater than -5. So the plane $x+y+5=z$ doesn't intersect $z=0$ for $x^2+y^2 \le 9$.]

Now parameterize: $$\iiint_E x\,dV = \int_0^{2\pi}\int_2^3\int_0^{r\cos\theta + r\sin\theta + 5} r^2\cos\theta \, dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_2^3 \Big(r^3\cos^2\theta + \underbrace{r^3\sin\theta\cos\theta + 5r^2\cos\theta}_{\text{both integrate to } 0 \text{ over } 0 \le \theta \le 2\pi}\Big)\,dr\,d\theta$$

$$= \int_0^{2\pi} \tfrac{1}{4}(3^4-2^4)\cos^2\theta\,d\theta = \boxed{\frac{65\pi}{4}}$$

\# 4. Express the integral

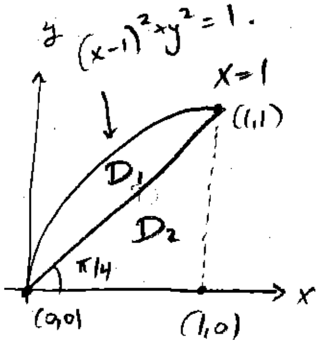
$$\int_0^1 \int_0^{\sqrt{2x-x^2}} \int_0^{\sqrt{x^2+y^2}} f(x,y,z)\,dz\,dy\,dx$$ in cylindrical coordinates.

Solution: We want to express the region

$$E = \{(x,y,z) : 0 \le x \le 1,\ 0 \le y \le \sqrt{2x-x^2},\ 0 \le z \le \sqrt{x^2+z^2}\}.$$

Note that $0 \le y \le \sqrt{2x-x^2}$ is equivalent

to $x^2+y^2 \le 2x$, or $(x-1)^2+y^2 \le 1$ (and $y \ge 0$).



This is the inside of the circle of radius 1 centered at (1,0), for $y \ge 0$ and $x \le 1$.

In polar, we can write the region as a union of two regions D_1 and D_2.

D_1 is bounded by the polar curve

$$x^2+y^2 \le 2x \longrightarrow r^2 \le 2r\cos\theta, \quad \text{or}$$

$$r \le 2\cos\theta, \quad \text{for } \tfrac{\pi}{4} \le \theta \le \tfrac{\pi}{2}.$$

D_2 is bounded by the polar curve

$$x \le 1 \longrightarrow r\cos\theta \le 1, \quad \text{or}$$

$$r \le \sec\theta, \quad \text{for } 0 \le \theta \le \pi/4.$$

Thus

$$\iiint_E f(x,y,z)\,dV = \int_0^{\pi/4}\int_0^{\sec\theta}\int_0^{r} f(r,\theta,z)\cdot r\,dz\,dr\,d\theta$$

$$+ \int_{\pi/4}^{\pi/2}\int_0^{2\cos\theta}\int_0^{r} f(r,\theta,z)\cdot r\,dz\,dr\,d\theta.$$

6 Evaluate $\iiint_E \sqrt{x^2+y^2+z^2}\, dV$, where E is the region $x^2+y^2+z^2 \le 2z$.

Solution: E is the sphere centered at $(0,0,1)$ of radius 1 (can see this by completing the square: $x^2+y^2+z^2-2z+1 = 1$). In spherical coordinates, $x^2+y^2+z^2 \le 2z \longrightarrow \rho^2 \le 2\rho\cos\phi$, or

$$\rho \le 2\cos\phi.$$

Thus, since E lies above the plane $z=0$, (and contains the origin),

$$E = \left\{(\rho,\theta,\phi): \begin{array}{c} 0 \le \theta \le 2\pi \\ 0 \le \phi \le \pi/2 \\ 0 \le \rho \le 2\cos\phi \end{array}\right\}, \quad \text{so}$$

$$\iiint_E \sqrt{x^2+y^2+z^2}\, dV = \int_0^{2\pi}\int_0^{\pi/2}\int_0^{2\cos\phi} \rho \cdot \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

$$= 2\pi \int_0^{\pi/2} \frac{1}{4}\sin\phi \,(2\cos\phi)^4\, d\phi = \boxed{\frac{\pi}{5}}$$

#8. Evaluate $\iiint_E x e^{x^2+y^2+z^2}\, dV$, where

E is the part of $x^2+y^2+z^2 \leq 1$ where $x \leq 0$,

$z \leq 0$, and $y \geq 0$.

<u>Solution</u>: In spherical coordinates, E is

parameterized as $E = \left\{ (\rho, \theta, \phi) : \begin{array}{l} 0 \leq \rho \leq 1 \\ \pi/2 \leq \theta \leq \pi \end{array} \right.$, and $\frac{\pi}{2} \leq \phi \leq \pi$

Thus

$$\iiint_E x e^{x^2+y^2+z^2}\, dV = \int_{\pi/2}^{\pi}\int_{\pi/2}^{\pi}\int_0^1 (\rho \cos\theta \sin\phi)\, e^{\rho^2} \cdot \rho^2 \sin\phi \; d\rho\, d\theta\, d\phi.$$

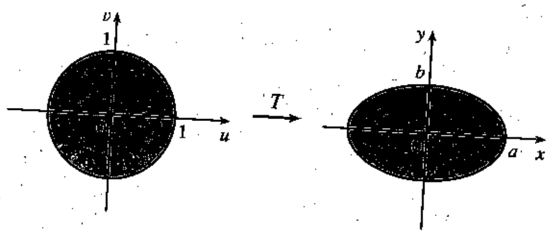
$$= \int_{\pi/2}^{\pi} \cos\theta \, d\theta \cdot \int_{\pi/2}^{\pi} \sin^2\phi \, d\phi \cdot \int_0^1 \rho^3 e^{\rho^2} d\rho \qquad \left(\begin{array}{l}\text{let } \rho^2 = u \\ 2\rho\, d\rho = du\end{array}\right)$$

$$= (-1) \cdot \left(\frac{\pi}{4}\right) \cdot \int_0^1 \frac{1}{2} u e^u \, du \qquad (\text{now integrate by parts})$$

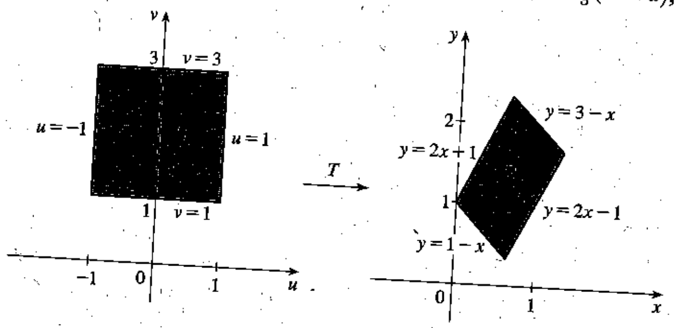
$$= -\frac{\pi}{8}\left[u e^u \Big|_0^1 - \int_0^1 e^u \, du \right]$$

$$= \boxed{-\frac{\pi}{8}}.$$

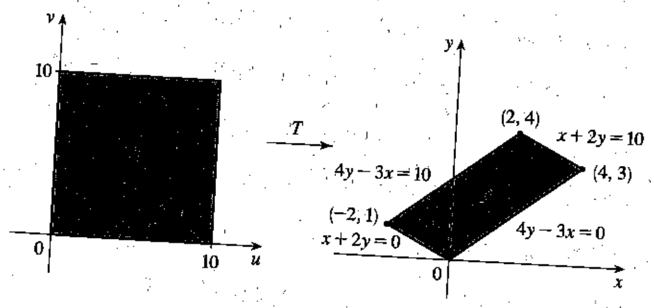
10. Substituting $u = \dfrac{x}{a}$, $v = \dfrac{y}{b}$ into $u^2 + v^2 \le 1$ gives $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} \le 1$, so the image of $u^2 + v^2 \le 1$ is the elliptical region $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} \le 1$.



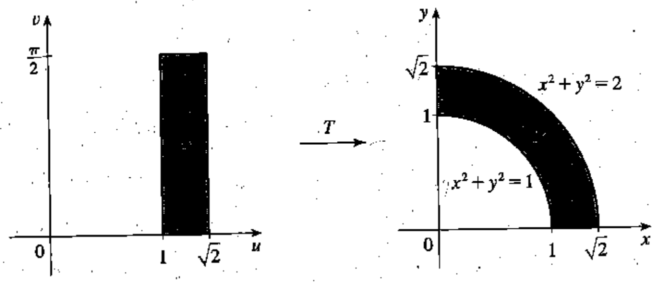
11. R is a parallelogram enclosed by the parallel lines $y = 2x - 1$, $y = 2x + 1$ and the parallel lines $y = 1 - x$, $y = 3 - x$. The first pair of equations can be written as $y - 2x = -1$, $y - 2x = 1$. If we let $u = y - 2x$ then these lines are mapped to the vertical lines $u = -1$, $u = 1$ in the uv-plane. Similarly, the second pair of equations can be written as $x + y = 1$, $x + y = 3$, and setting $v = x + y$ maps these lines to the horizontal lines $v = 1$, $v = 3$ in the uv-plane. Boundary curves are mapped to boundary curves under a transformation, so here the equations $u = y - 2x$, $v = x + y$ define a transformation T^{-1} that maps R in the xy-plane to the square S enclosed by the lines $u = -1$, $u = 1$, $v = 1$, $v = 3$ in the uv-plane. To find the transformation T that maps S to R we solve $u = y - 2x$, $v = x + y$ for x, y: Subtracting the first equation from the second gives $v - u = 3x \;\Rightarrow\; x = \frac{1}{3}(v - u)$ and adding twice the second equation to the first gives $u + 2v = 3y \;\Rightarrow\; y = \frac{1}{3}(u + 2v)$. Thus one possible transformation T (there are many) is given by $x = \frac{1}{3}(v - u)$, $y = \frac{1}{3}(u + 2v)$.



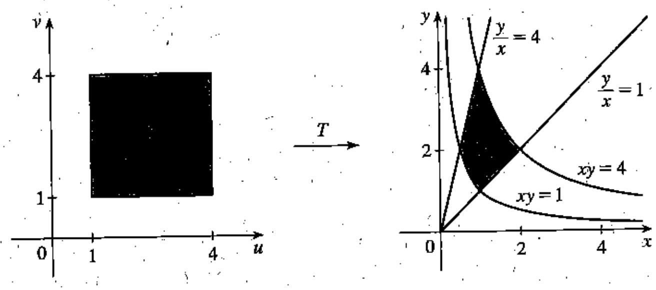
12. The boundaries of the parallelogram R are the lines $y = \frac{3}{4}x$ or $4y - 3x = 0$, $y = \frac{3}{4}x + \frac{5}{2}$ or $4y - 3x = 10$, $y = -\frac{1}{2}x$ or $x + 2y = 0$, $y = -\frac{1}{2}x + 5$ or $x + 2y = 10$. Setting $u = 4y - 3x$ and $v = x + 2y$ defines a transformation T^{-1} that maps R in the xy-plane to the square S enclosed by the lines $u = 0$, $u = 10$, $v = 0$, $v = 10$ in the uv-plane. Solving $u = 4y - 3x$, $v = x + 2y$ for x and y gives $2v - u = 5x \;\Rightarrow\; x = \frac{1}{5}(2v - u)$, $u + 3v = 10y \;\Rightarrow\; y = \frac{1}{10}(u + 3v)$. Thus one possible transformation T is given by $x = \frac{1}{5}(2v - u)$, $y = \frac{1}{10}(u + 3v)$.



13. R is a portion of an annular region (see the figure) that is easily described in polar coordinates as $R = \{(r, \theta) \mid 1 \le r \le \sqrt{2}, 0 \le \theta \le \pi/2\}$. If we converted a double integral over R to polar coordinates the resulting re of integration is a rectangle (in the $r\theta$-plane), so we can create a transformation T here by letting u play the role of r and role of θ. Thus T is defined by $x = u\cos v$, $y = u \sin v$ and T maps the rectangle $S = \{(u, v) \mid 1 \le u \le \sqrt{2}, 0 \le v \le$ in the uv-plane to R in the xy-plane.



14. The boundaries of the region R are the curves $y = 1/x$ or $xy = 1$, $y = 4/x$ or $xy = 4$, $y = x$ or $y/x = 1$, $y = 4x$ or $y/x = 4$. Setting $u = xy$ and $v = y/x$ defines a transformation T^{-1} that maps R in the xy-plane to the square S enclo the lines $u = 1$, $u = 4$, $v = 1$, $v = 4$ in the uv-plane. Solving $u = xy$, $v = y/x$ for x and y gives $x^2 = u/v \Rightarrow$ $x = \sqrt{u/v}$ [since x, y, u, v are all positive], $y^2 = uv \Rightarrow y = \sqrt{uv}$. Thus one possible transformation T is given by $x = \sqrt{u/v}$, $y = \sqrt{uv}$.



15. $\dfrac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3$ and $x - 3y = (2u + v) - 3(u + 2v) = -u - 5v$. To find the region S in the uv-plane that corresponds to R we first find the corresponding boundary under the given transformation. The line through $(0, 0)$ and ($y = \frac{1}{2}x$ which is the image of $u + 2v = \frac{1}{2}(2u + v) \Rightarrow v = 0$; the line through $(2, 1)$ and $(1, 2)$ is $x + y = 3$ whic image of $(2u + v) + (u + 2v) = 3 \Rightarrow u + v = 1$; the line through $(0, 0)$ and $(1, 2)$ is $y = 2x$ which is the image u $u + 2v = 2(2u + v) \Rightarrow u = 0$. Thus S is the triangle $0 \le v \le 1 - u$, $0 \le u \le 1$ in the uv-plane and

$$\iint_R (x - 3y)\, dA = \int_0^1 \int_0^{1-u} (-u - 5v)\,|3|\, dv\, du = -3 \int_0^1 \left[uv + \tfrac{5}{2}v^2\right]_{v=0}^{v=1-u} du$$
$$= -3 \int_0^1 \left(u - u^2 + \tfrac{5}{2}(1 - u)^2\right) du = -3\left[\tfrac{1}{2}u^2 - \tfrac{1}{3}u^3 - \tfrac{5}{6}(1 - u)^3\right]_0^1 = -3\left(\tfrac{1}{2} - \tfrac{1}{3} + \tfrac{5}{6}\right) =$$

25. Letting $u = y - x$, $v = y + x$, we have $y = \frac{1}{2}(u + v)$, $x = \frac{1}{2}(v - u)$. Then $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{vmatrix} = -\dfrac{1}{2}$ and R is the image of the trapezoidal region with vertices $(-1,1)$, $(-2,2)$, $(2,2)$, and $(1,1)$. Thus

$$\iint_R \cos\frac{y-x}{y+x}\,dA = \int_1^2 \int_{-v}^{v} \cos\frac{u}{v}\left|-\frac{1}{2}\right| du\,dv = \frac{1}{2}\int_1^2 \left[v\sin\frac{u}{v}\right]_{u=-v}^{u=v} dv = \frac{1}{2}\int_1^2 2v\sin(1)\,dv = \tfrac{3}{2}\sin 1$$

26. Letting $u = 3x$, $v = 2y$, we have $9x^2 + 4y^2 = u^2 + v^2$, $x = \frac{1}{3}u$, and $y = \frac{1}{2}v$. Then $\dfrac{\partial(x,y)}{\partial(u,v)} = \dfrac{1}{6}$ and R is the image of the quarter-disk D given by $u^2 + v^2 \le 1$, $u \ge 0$, $v \ge 0$. Thus

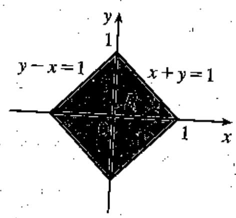
$$\iint_R \sin(9x^2 + 4y^2)\,dA = \iint_D \tfrac{1}{6}\sin(u^2 + v^2)\,du\,dv = \int_0^{\pi/2}\int_0^1 \tfrac{1}{6}\sin(r^2)\,r\,dr\,d\theta = \tfrac{\pi}{12}\left[-\tfrac{1}{2}\cos r^2\right]_0^1 = \tfrac{\pi}{24}(1 - \cos 1)$$

27. Let $u = x + y$ and $v = -x + y$. Then $u + v = 2y \;\Rightarrow\; y = \frac{1}{2}(u + v)$ and $u - v = 2x \;\Rightarrow\; x = \frac{1}{2}(u - v)$.

$\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{vmatrix} = \dfrac{1}{2}$. Now $|u| = |x + y| \le |x| + |y| \le 1 \;\Rightarrow\; -1 \le u \le 1$, and

$|v| = |-x + y| \le |x| + |y| \le 1 \;\Rightarrow\; -1 \le v \le 1$. R is the image of the square region with vertices $(1,1)$, $(1,-1)$, $(-1,-1)$, and $(-1,1)$.

So $\iint_R e^{x+y}\,dA = \frac{1}{2}\int_{-1}^1\int_{-1}^1 e^u\,du\,dv = \frac{1}{2}\left[e^u\right]_{-1}^1\left[v\right]_{-1}^1 = e - e^{-1}$.



28. Let $u = x + y$ and $v = y$, then $x = u - v$, $y = v$, $\dfrac{\partial(x,y)}{\partial(u,v)} = 1$ and R is the image under T of the triangular region with vertices $(0,0)$, $(1,0)$ and $(1,1)$. Thus

$$\iint_R f(x+y)\,dA = \int_0^1\int_0^u (1)\,f(u)\,dv\,du = \int_0^1 f(u)\left[\,v\,\right]_{v=0}^{v=u} du = \int_0^1 u f(u)\,du \quad \text{as desired.}$$

15 Review

CONCEPT CHECK

1. (a) A double Riemann sum of f is $\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\,\Delta A$, where ΔA is the area of each subrectangle and (x_{ij}^*, y_{ij}^*) is a sample point in each subrectangle. If $f(x,y) \ge 0$, this sum represents an approximation to the volume of the solid that lies above the rectangle R and below the graph of f.

(b) $\iint_R f(x,y)\,dA = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\,\Delta A$

(c) If $f(x,y) \ge 0$, $\iint_R f(x,y)\,dA$ represents the volume of the solid that lies above the rectangle R and below the surface $z = f(x,y)$. If f takes on both positive and negative values, $\iint_R f(x,y)\,dA$ is the difference of the volume above R but below the surface $z = f(x,y)$ and the volume below R but above the surface $z = f(x,y)$.

(d) We usually evaluate $\iint_R f(x,y)\,dA$ as an iterated integral according to Fubini's Theorem (see Theorem 15.2.4).