Math 302, Summer 2023

Midterm exam

Wednesday, July 26, 2023

Problem 1	12	
Problem 2	12	
Problem 3	16	
Problem 4	10	
Total	50	

Name: ____

- There are 4 questions on this exam.
- You have 60 minutes to complete the exam.
- **You must show your work on all problems.** The correct answer with no supporting work may result in no credit. Put a box around your FINAL ANSWER for each problem and cross out any work that you don't want to be graded.
- Give exact answers. You may leave your answer as a sum or product of fractions or binomial coefficients.
- Any student found engaging in academic misconduct will receive a score of 0 on this exam.

GOOD LUCK!

1. (12 points; 4 each) Suppose X is a random variable with PDF

$$f(x) = \begin{cases} \frac{1}{8}, & -3 < x < -1 \\ cx + 1, & 0 < x < \frac{1}{4} \\ 0, & \text{otherwise} \end{cases}$$

(a) Find the value of c.

(b) Let F denote the CDF of X. Find $F\left(\frac{1}{8}\right)$.

(c) Find $\mathbb{E}[X]$.

2. (12 points; 3 each) For each statement below, say whether it is true or false. No justification is required. No partial credit will be awarded on this problem.

   (a) Let B be a discrete random variable with PMF $\mathbb{P}(B = 0) = 1/4$ and $\mathbb{P}(B = 1) = 3/4$. B has the same PMF as B^3.

   (b) If $Z \sim \text{Poisson}(3)$, then $\mathbb{P}(Z = 5) = e^{-3} \cdot \frac{5^3}{5!}$.

   (c) For any probability measure $\mathbb{P}$ on the set $\Omega = \{1, 2, 3, 4\}$, there exists an event E with $\mathbb{P}(E) = 1/4$.

   (d) Let Y be a continuous random variable with PDF $f(y)$. Suppose that $\mathbb{P}(Y > 3) = 1/4$. Then

$$\int_{-\infty}^{3} f(y)\, dy = 3/4.$$

3. (16 points; 4 each) Roll a fair die 100 times.

(a) Let A denote the event that you get exactly 10 1s in the first 30 rolls, B the event that you get exactly 20 2s in rolls 31-60, and C the event that you get all even numbers in rolls $61 - 100$. Find $\mathbb{P}(A), \mathbb{P}(B), \mathbb{P}(C)$, and $\mathbb{P}(A \cap B \cap C)$.

(b) Let D denote the event that you get exactly 10 1s in total, and E the event that you get exactly 20 2s in total. Find $\mathbb{P}(D \cap E)$.

(c) Use the inclusion-exclusion principle and your answer from part (b) to find $\mathbb{P}(D \cup E)$. You can still get credit on this part even if you didn't do part (b).

(d) Let X be the number of times in the 100 rolls that you rolled a 1, and then the next nine rolls were all 6s. Use linearity of expectation to find $\mathbb{E}[X]$.

4. (10 points) Jacob has infinitely many bags, labeled $1, 2, \ldots$. For every n, the bag labeled n has $n-1$ green balls and 1 red ball. Jacob generates a random variable N with Geometric(1/2) distribution, then draws a ball uniformly at random from the bag labeled N. Let $A_n = \{N = n\}$, and let R be the event that Jacob draws a red ball.

   (a) (8 points) Use the law of total probability to write $\mathbb{P}(R)$ as an infinite sum. The expression in the sum should be explicit (evaluate any probabilities).

   (b) (2 points) Use the fact that $-\log(1-x) = \sum_{n=1}^{\infty} \frac{1}{n} x^n$ for $x \in (0, 1)$ to evaluate your expression from part (a).