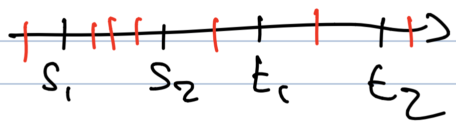
<u>Recall</u>: • We defined the Poisson Process (with rate or intensity λ)
• We established several properties
→ Stationarity
→ Independent increments
$N(t_2 - t_1)$ and $N(s_2 - s_1)$ are independent



if $[s_1; s_2]$ and $[t_1; t_2]$ are disjoint intervals.

• $N(t)$ can be also defined "axiomatically"

<u>Thm</u>: The Poisson process is the unique increasing jump process $X(t): \mathbb{R}^+ \to \{0, 1, \dots\}$ satisfying:

1) $X(0) = 0$
2) independent increments
3) $P(X(t+\Delta t) - X(t) = 1) = \lambda \Delta t + o(\Delta t)$
4) $P(X(t+\Delta t) - X(t) \geq 2) = o(\Delta t)$

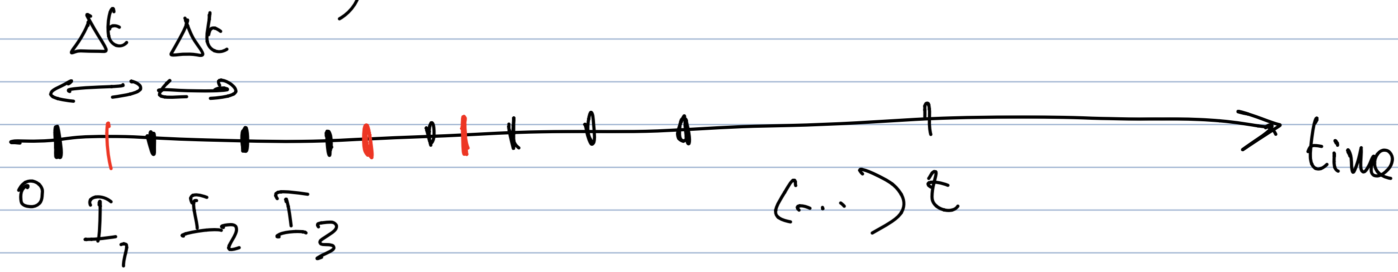
<u>Rmk</u>: • A function $f(x)$ is $o(x)$ if $\lim_{x \to 0} \frac{f(x)}{x} = 0$

→ it means that $o(\Delta t)$ is something that can be neglected compared with Δt as $\Delta t \to 0$.

(ex: $f(x) = x$ is not $o(x)$; $f(x) = x^2$ is $o(x)$)

- Key idea: (2) "independent increments" $\longleftrightarrow$ memorylessness
  (3) & (4) $\longleftrightarrow$ events happen at a rate λ.

This second definition allows to have an alternative construction of the Poisson Process (another way to simulate it)



With $\Delta t \ll 1$, we can perform the following simulation:

- In each interval I_k, sample a Bernoulli$(\lambda \Delta t)$ r.v., all i.i.d. Y_i

$\rightarrow$ Then $N(t) \simeq \sum_{i=1}^{t/\Delta t} Y_i$ $\rightarrow$ # of 1's by time t.

In other words this process converges to the Poisson process as $\Delta t \to 0$.

To prove it exactly, one notably uses that

$$\text{Binomial}\left(n, \frac{\mu}{n}\right) \longrightarrow \text{Poisson}(\mu) \quad \text{as } n \to \infty$$

- For a comparison using stochastic simulations
  $\longrightarrow$ see this week Jupyter Notebook

- More properties (superposition | thinning)

Let $N(t)$ and $M(t)$ be independent PP's of intensity λ_1 and λ_2, respectively.

Q: What is the distribution of $N(t) + M(t)$?

Prop (superposition): $N(t)+M(t)$ is a P.P. of intensity $\lambda_1+\lambda_2$

Proof (sketch): In the "Δt interval" description, adding $N(t)$ to $M(t)$ is like sampling $Y_i \sim Ber(\lambda_1 \Delta t)$ and $Y_i' \sim Ber(\lambda_2 \Delta t)$ in each Δt interval.

So the probability of a success in any Δt interval is

$$1 - P(Y_i = Y_i' = 0) = 1 - (1-\lambda_1 \Delta t)(1-\lambda_2 \Delta t)$$

$$\left(P(Y_i = 1) = \lambda_1 \Delta t \right)$$

$$= (\lambda_1 + \lambda_2)\Delta t + \underbrace{\lambda_1 \lambda_2 (\Delta t)^2}_{= o(\Delta t)}$$

so $1 - P(Y_i = Y_i' = 0) = (\lambda_1 + \lambda_2)\Delta t + o(\Delta t)$

(Similarly, the probability of both sucesses in one interval $Y_i = Y_i' = 1$ is $\lambda_1 \Delta t . \lambda_2 \Delta t = o(\Delta t)$
so it's negligible)

Thus, $N(t) + M(t)$ is like a P.P. of rate $\lambda_1 + \lambda_2$ $\boxtimes$

Rmk : this result is close to what we saw for the minimum of two exponentials.

→ The time for an event to happen is the minimum of the times for either one event from N or M to happen, and we know that this follows $Exp(\lambda_1 + \lambda_2)$

ex : . Cars traveling on a highway

$N(t)$ = # cars seen going east up to t

$M(t)$ = ——————— west ———

Suppose that $N(t)$ and $M(t)$ are P.P.'s with rates λ and μ

From the superposition property, we know that $(N+M)(t)$ is a Poisson process of intensity $\lambda + \mu$.

Q : (i) Find P(first two cars are going East)

(ii) What is the distribution of the # of cars seen going East by time t, given that there were C cars in total seen by time t?

A: (i) p = Prob. of first car going East $= P(Exp(\lambda) < Exp(\mu))$

(Time of observing 1st car going East $\sim Exp(\lambda)$
West $\sim Exp(\mu)$)

so $p = \frac{\lambda}{\lambda+\mu}$.

$\Rightarrow$ By memorylessness of the process,
P(first 2 cars going East) $= \boxed{\left(\frac{\lambda}{\lambda+\mu}\right)^2}$

(ii) The answer is Binomial$\left(C, \frac{\lambda}{\lambda+\mu}\right)$

($\rightarrow$ see Next week)