strong form

equilibrium equation $\boxed{\mathrm{div}_{cyl}\, \underline{\underline{\sigma}} = \underline{0}}$ in cylindrical coordinates to weak form

$$\to \int_{B_t} \underline{\eta} \cdot \mathrm{div}_{cyl}\, \underline{\underline{\sigma}}^{3D}\, dv = 0 \qquad \text{with test function } \underline{\eta} = \begin{bmatrix} \eta_r \\ \eta_z \\ \eta_\theta \end{bmatrix}$$

$$\text{with } \mathrm{div}_{cyl}(\underline{\eta} \cdot \underline{\underline{\sigma}}) \overset{?}{\rightarrow} \mathrm{div}_{cyl}(\underline{\underline{\sigma}} \cdot \underline{\eta}) = (\mathrm{div}_{cyl}\, \underline{\underline{\sigma}}) \cdot \underline{\eta} + \underline{\underline{\sigma}} : \nabla_{cyl}\, \underline{\eta}$$

$$\Leftrightarrow \underline{\eta} \cdot \mathrm{div}_{cyl}\, \underline{\underline{\sigma}}^{3D} = \mathrm{div}_{cyl}(\underline{\underline{\sigma}} \cdot \underline{\eta}) - \underline{\underline{\sigma}} : \nabla_{cyl}\, \underline{\eta}$$

with $\underline{\underline{\sigma}} = \underline{\underline{\sigma}}^T$

$$= \mathrm{div}_{cyl}(\underline{\eta} \cdot \underline{\underline{\sigma}}) - \nabla_{cyl}\, \underline{\eta} : \underline{\underline{\sigma}}$$

Gauss integral theorem: $\int \mathrm{div}(\cdot)\, dv = \int (\cdot)\, \underline{n}\, da$ (also in cyl. coords.)

$$\int \mathrm{div}_{cyl}(\underline{\eta} \cdot \underline{\underline{\sigma}})\, dv = \int_{\partial\Omega} \underline{\eta} \cdot \underline{\underline{\sigma}} \cdot \underline{n}\, da$$

boundary conditions: $\underline{\eta} = \underline{0}$ on $\partial\Omega_D$, $\underline{t} = \underline{\underline{\sigma}} \cdot \underline{n} = \bar{\underline{t}}$ on $\partial\Omega_N$

$$\int_{\partial\Omega} \underline{\eta} \cdot \underline{\underline{\sigma}} \cdot \underline{n}\, da = \overset{0}{\cancel{\int_{\partial\Omega_D} \underline{\eta} \cdot \underline{\underline{\sigma}}\, \underline{n}\, da}} + \int_{\partial\Omega_N} \underline{\eta} \cdot \underline{\underline{\sigma}}\, \underline{n}\, da = \int_{\partial\Omega_N} \underline{\eta} \cdot \bar{\underline{t}}\, dA \quad \text{with } \bar{\underline{t}} = \underline{0} \text{ on } \partial\Omega \notin \partial\Omega_N$$

$$\overset{!}{=} \int_{\partial\Omega} \underline{\eta} \cdot \bar{\underline{t}}\, da$$

$\to$ weak form: $\boxed{\int_{B_t} \nabla_{cyl}\, \underline{\eta} : \underline{\underline{\sigma}}\, dv - \int_{\partial\Omega} \underline{\eta} \cdot \bar{\underline{t}}\, da = 0}$

$$\text{with } \nabla_{cyl}\, \underline{\eta} \equiv \begin{bmatrix} \partial\eta_r/\partial r & \partial\eta_r/\partial z & \frac{1}{r}\left[\frac{\partial\eta_r}{\partial\theta} - \eta_\theta\right] \\ \partial\eta_z/\partial r & \partial\eta_z/\partial z & \frac{1}{r}\, \partial\eta_z/\partial\theta \\ \partial\eta_\theta/\partial r & \partial\eta_\theta/\partial z & \frac{1}{r}\left[\frac{\partial\eta_\theta}{\partial\theta} + \eta_r\right] \end{bmatrix}$$

simplification for axialsymmetry ($\partial/\partial\theta = 0$, no change in θ-direction, no theta shear stress)

cyl. coord.: $\int_{B_t} \nabla_{cyl}\,\underline{\eta} : \underline{\underline{\sigma}}\, dv - \int_{\partial\Omega} \underline{\eta}\cdot\underline{\bar{t}}\, da = 0$

$\downarrow$

axial symmetry: $\nabla_{ax}\,\underline{\eta} = \begin{bmatrix} \partial\eta_r/\partial r & \partial\eta_r/\partial z & -\frac{1}{r}\eta_\theta \\ \partial\eta_z/\partial r & \partial\eta_z/\partial z & 0 \\ \partial\eta_\theta/\partial r & \partial\eta_\theta/\partial z & \frac{1}{r}\eta_r \end{bmatrix}$; $(\partial/\partial\theta = 0)$

$$\underline{\underline{\sigma}}^{ax} = \begin{bmatrix} \sigma_{rr} & \sigma_{rz} & 0 \\ \sigma_{zr} & \sigma_{zz} & 0 \\ 0 & 0 & \sigma_{\theta\theta} \end{bmatrix}, \quad \sigma_{r\theta} = \sigma_{z\theta} = \sigma_{\theta r} = \sigma_{\theta z} = 0$$

$$\rightarrow \nabla_{ax}\,\underline{\eta} : \underline{\underline{\sigma}}^{ax} = \begin{bmatrix} \partial\eta_r/\partial r & \partial\eta_r/\partial z \\ \partial\eta_z/\partial r & \partial\eta_z/\partial z \end{bmatrix} : \begin{bmatrix} \sigma_{rr} & \sigma_{rz} \\ \sigma_{zr} & \sigma_{zz} \end{bmatrix} + \frac{1}{r}\eta_r\sigma_{\theta\theta}$$

$$= \nabla_{2D}\,\underline{\eta}^{2D} : \underline{\underline{\sigma}}^{2D} + \frac{1}{r} N_r \sigma_{\theta\theta}$$

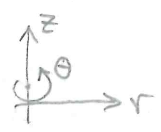
$\rightarrow$ axisymmetric weak form

$$\boxed{\int_{B_t} \nabla_{2D}\,\underline{\eta}^{2D} : \underline{\underline{\sigma}}^{2D} + \underline{\frac{1}{r}\eta_r\sigma_{\theta\theta}}\, dv - \int_{\partial\Omega} \underline{\eta}\cdot\underline{\bar{t}}\, da = 0}$$

AXISYM. RESIDUAL

spatial

$$\boxed{\int_{B_t} \underbrace{\nabla_x^{sym} \underline{N}^i_{2D} : \underline{\underline{\sigma}}^{2D}} + \underbrace{N_r^i \cdot \frac{1}{r}\, \sigma_{\theta\theta}}\, dv}$$



material

$$\boxed{R^i = \int_{B_0} \underbrace{\left[\underline{\underline{F}}_{2D}^T \cdot \nabla_x \underline{N}^i\right]^{sym} : \underline{\underline{S}}^{2D}}_{\text{standard (plane strain)}} + \underbrace{N_r^i \cdot \frac{r}{R^2}\, S_{\theta\theta}}_{\text{axisym.}}\, dV}$$

with $\underline{\underline{\sigma}}^{2D} = \begin{bmatrix} \sigma_{rr} & \sigma_{rz} \\ \sigma_{zr} & \sigma_{zz} \end{bmatrix}$

$\underline{\underline{F}}_{2D} = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} = \begin{bmatrix} F_{rr} & F_{rz} \\ F_{zr} & F_{zz} \end{bmatrix}$

$\underline{\underline{\sigma}}^{3D} = \begin{bmatrix} \sigma_{rr} & \sigma_{rz} & 0 \\ \sigma_{zr} & \sigma_{zz} & 0 \\ 0 & 0 & \sigma_{\theta\theta} \end{bmatrix}$: 3D Cauchy stress tensor

$\underline{\underline{F}}_{3D} = \begin{bmatrix} F_{11} & F_{12} & 0 \\ F_{21} & F_{22} & 0 \\ 0 & 0 & F_{\theta\theta} \end{bmatrix}$: 3D deformation gradient

$\underline{N}^i_{2D} = \begin{bmatrix} N_r^i \\ N_z^i \end{bmatrix}$: 2D (in-plane) shape function vector

$F_{\theta\theta} = \frac{r}{R} = 1 + \frac{u_r}{R}$: hoop "strain"

$r = R + u_r$: spatial radial coordinate

$\underline{\underline{C}}_{3D} = \underline{\underline{F}}_{3D}^T \cdot \underline{\underline{F}}_{3D}$: 3D Right Cauchy-Green tensor (RCG)

R: undeformed/initial radial coordinate

$C_{\theta\theta}$: (θ,θ) component of $\underline{\underline{C}}_{3D}$

$\underline{u} = \begin{bmatrix} u_r \\ u_z \end{bmatrix}$: 2D displacement vector

$\underline{\underline{S}}$: Piola-Kirchhoff stress (PK2)

∇_x^{sym} : symmetric spatial gradient

$\frac{\partial \underline{\underline{S}}}{\partial \underline{\underline{C}}}$: derivative of PK2 with respect to RCG

dv: spatial volume element, $dv = 2\pi r\, dr\, dz = J\, dV$

$\frac{\partial \underline{\underline{S}}^{2D}}{\partial C_{\theta\theta}}$: the (i,j,θ,θ) components of $\left[\frac{\partial S}{\partial C}\right]^{3D}$

dV: initial volume element

$J = \det(\underline{\underline{F}}^{3D})$

$\Delta \underline{\underline{C}}^j_{2D}$: linearisation of RCG in 2D, $\Delta \underline{\underline{C}}^j_{2D} = 2\left[\nabla_x^T \underline{N}^j \cdot \underline{\underline{F}}^{2D}\right]^{sym}$

$\nabla_x^{sym} \underline{N}^i : \sigma^{2D}\, dv = \left[\underline{\underline{F}}_{2D}^T \cdot \nabla_x \underline{N}^i\right]^{sym} : \underline{\underline{S}}^{2D}\, dV$

$\underline{\underline{\varepsilon}}^{3D} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & 0 \\ \varepsilon_{21} & \varepsilon_{22} & 0 \\ 0 & 0 & \varepsilon_{\theta\theta} \end{bmatrix}$: geom. linear strain

$\underline{\underline{\sigma}}^{3D} = \frac{1}{J}\, \underline{\underline{F}}_{3D} \cdot \underline{\underline{S}}^{3D} \cdot \underline{\underline{F}}_{3D}^T$

$\varepsilon_{\theta\theta} = u_r / r$

$$= \frac{1}{J} \begin{bmatrix} F_{11} & F_{12} & 0 \\ F_{21} & F_{22} & 0 \\ 0 & 0 & F_{33} \end{bmatrix} \cdot \begin{bmatrix} S_{rr} & S_{rz} & 0 \\ S_{zr} & S_{zz} & 0 \\ 0 & 0 & S_{\theta\theta} \end{bmatrix} \cdot \begin{bmatrix} F_{11} & F_{21} & 0 \\ F_{12} & F_{22} & 0 \\ 0 & 0 & F_{33} \end{bmatrix}$$

$\frac{\partial \varepsilon_{\theta\theta}}{\partial u_r} = \frac{1}{r}$

$\rightarrow \sigma_{\theta\theta} = \frac{1}{J} F_{33}^2\, S_{\theta\theta} = \frac{1}{J} \frac{r^2}{R^2}\, S_{\theta\theta}$

AXISYM. LINEARISATION

$$\boxed{\Delta R^{ij} = \Delta R^{ij}_{standard} + \Delta R^{ij}_{ax1} + \Delta R^{ij}_{ax2}}$$

Geometrically nonlinear

ΔR^{ij}_{ax1} : $\underline{\underline{S}}^{2D} = f(\underline{\underline{C}}^{3D}) = f(\underbrace{\underline{\underline{C}}^{2D}}_{\text{linearised in } \Delta R^{ij}_{standard}}, \underbrace{C_{\theta\theta}(r)}_{\to \Delta R^{ij}_{ax1}})$

$$\rightarrow \Delta R^{ij}_{ax1} = \left[F^T \cdot \nabla_X \underline{N}^i\right]^{sym} : \frac{\partial S^{2D}}{\partial C_{\theta\theta}} \cdot \frac{\partial C_{\theta\theta}}{\partial u_r} \Delta u_r$$

with $C_{\theta\theta} = F_{\theta\theta}^2 = \frac{r^2}{R^2} = \frac{1}{R^2}\left[R + u_r\right]^2$

$$\frac{\partial C_{\theta\theta}}{\partial u_r} = \frac{2}{R^2}\left[R + u_r\right]$$

$$\boxed{\Delta R^{ij}_{ax1} \doteq \left[F^T \cdot \nabla_X \underline{N}^i\right]^{sym} : \frac{\partial S^{2D}}{\partial C_{\theta\theta}} \cdot 2 \, \frac{r}{R^2} N_r^j}$$

ΔR^{ij}_{ax2} : $R^i_{ax2} = N_r^i \cdot \frac{r}{R^2} S_{\theta\theta}$

$$\rightarrow \boxed{\Delta R^{ij}_{ax2} = N_r^i / R^2 \cdot \left[N_r^j S_{\theta\theta} + r \cdot \left[\frac{\partial S_{\theta\theta}}{\partial \underline{\underline{C}}^{2D}} : \Delta \underline{\underline{C}}^j_{2D} + \frac{\partial S_{\theta\theta}}{\partial C_{\theta\theta}} \cdot 2 \, \frac{r}{R^2} N_r^j \right]\right]}$$

Geometrically linear ("r=R")

ΔR^{ij}_{ax1} :

$$\rightarrow \boxed{\Delta R^{ij}_{ax1} = \nabla_X^{sym} \underline{N}^i : \frac{\partial \sigma}{\partial \varepsilon_{\theta\theta}} \cdot \frac{1}{r} N_r^j}$$

ΔR^{ij}_{ax2} :

$$\rightarrow \boxed{\Delta R^{ij}_{ax2} = N_r^i / r \left[\frac{\partial \sigma_{\theta\theta}}{\partial \underline{\underline{\varepsilon}}^{2D}} : \nabla_X^{sym} \underline{N}^j + \frac{\partial \sigma_{\theta\theta}}{\partial \varepsilon_{\theta\theta}} \cdot \frac{1}{r} N_r^j \right]}$$

["Mechanics of deformable solids" by Doghri, 2000]

A. Cylindrical coordinates

Many problems are such that it is advantageous to use cylindrical (r, θ, z) instead of Cartesian (x, y, z) coordinates. Cylindrical basis vectors (e_r, e_θ, e_z) are expressed in the Cartesian basis (e_x, e_y, e_z) as follows:

$$e_r = \cos(\theta)e_x + \sin(\theta)e_y, \quad e_\theta = -\sin(\theta)e_x + \cos(\theta)e_y \tag{A.1}$$

See Fig. 8.1 for an illustration. Both cylindrical and Cartesian bases are orthonormal. The position vector of a point $M(r, \theta, z)$ w.r.t. the frame (O, e_x, e_y, e_z) is:

$$x \equiv \overrightarrow{OM} = re_r + ze_z \tag{A.2}$$

Differentiation of (A.1) gives:

$$de_r = (d\theta)e_\theta, de_\theta = -(d\theta)e_r \tag{A.3}$$

Let v be a *vector field* defined in the cylindrical basis as follows:

$$\boxed{v(r, \theta, z) = F(r, \theta, z)e_r + G(r, \theta, z)e_\theta + H(r, \theta, z)e_z} \tag{A.4}$$

The *gradient* (∇v) of v is defined by:

$$dv = (\nabla v) \cdot (dx) \tag{A.5}$$

Differentiating (A.4) and using (A.3), we obtain:

$$\begin{aligned} dv &= \left(\frac{\partial F}{\partial r}dr + \frac{\partial F}{\partial \theta}d\theta + \frac{\partial F}{\partial z}dz\right) e_r + F(d\theta)e_\theta \\ &+ \left(\frac{\partial G}{\partial r}dr + \frac{\partial G}{\partial \theta}d\theta + \frac{\partial G}{\partial z}dz\right) e_\theta - G(d\theta)e_r \\ &+ \left(\frac{\partial H}{\partial r}dr + \frac{\partial H}{\partial \theta}d\theta + \frac{\partial H}{\partial z}dz\right) e_z \end{aligned} \tag{A.6}$$

Differentiation of (A.2) gives:

$$dx = (dr)e_r + (rd\theta)e_\theta + (dz)e_z \tag{A.7}$$

Using (A.6) and (A.7), Eq. (A.5) can be written in the following matrix form:

$$\begin{bmatrix} \frac{\partial F}{\partial r}dr + \frac{1}{r}(\frac{\partial F}{\partial \theta} - G)rd\theta + \frac{\partial F}{\partial z}dz \\ \frac{\partial G}{\partial r}dr + \frac{1}{r}(\frac{\partial G}{\partial \theta} + F)rd\theta + \frac{\partial G}{\partial z}dz \\ \frac{\partial H}{\partial r}dr + \frac{1}{r}\frac{\partial H}{\partial \theta}(rd\theta) + \frac{\partial H}{\partial z}dz \end{bmatrix} = [\nabla \boldsymbol{v}] \underbrace{\begin{bmatrix} dr \\ rd\theta \\ dz \end{bmatrix}}_{d\boldsymbol{x}}, \tag{A.8}$$

where the 3×3 matrix $(\nabla \boldsymbol{v})$ is given in the cylindrical basis as follows:

$$\boxed{\nabla \boldsymbol{v} = \begin{bmatrix} \frac{\partial F}{\partial r} & \frac{1}{r}(\frac{\partial F}{\partial \theta} - G) & \frac{\partial F}{\partial z} \\ \frac{\partial G}{\partial r} & \frac{1}{r}(\frac{\partial G}{\partial \theta} + F) & \frac{\partial G}{\partial z} \\ \frac{\partial H}{\partial r} & \frac{1}{r}\frac{\partial H}{\partial \theta} & \frac{\partial H}{\partial z} \end{bmatrix}} \tag{A.9}$$

The *divergence* of $\boldsymbol{v}$ is given by:

$$\text{div } \boldsymbol{v} \equiv \text{ tr}(\nabla \boldsymbol{v}) = \frac{\partial F}{\partial r} + \frac{1}{r}(\frac{\partial G}{\partial \theta} + F) + \frac{\partial H}{\partial z}, \tag{A.10}$$

using (A.9). Now let $g(r, \theta, z)$ be a *scalar field*. The *gradient* (∇g) of g is defined by:

$$dg = (\nabla g) \cdot (d\boldsymbol{x}), \tag{A.11}$$

where the 3×1 array (∇g) is given in the cylindrical basis as follows:

$$\nabla g = \begin{bmatrix} \frac{\partial g}{\partial r} \\ \frac{1}{r}\frac{\partial g}{\partial \theta} \\ \frac{\partial g}{\partial z} \end{bmatrix}, \tag{A.12}$$

using (A.7). The gradient $(\nabla \nabla g)$ of (∇g) is found from (A.9) and (A.12) as follows:

$$\nabla \nabla g = \begin{bmatrix} \frac{\partial^2 g}{\partial r^2} & \frac{1}{r}(\frac{\partial^2 g}{\partial r \partial \theta} - \frac{1}{r}\frac{\partial g}{\partial \theta}) & \frac{\partial^2 g}{\partial r \partial z} \\ \frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial g}{\partial \theta}) & \frac{1}{r}(\frac{1}{r}\frac{\partial^2 g}{\partial \theta^2} + \frac{\partial g}{\partial r}) & \frac{1}{r}\frac{\partial^2 g}{\partial \theta \partial z} \\ \frac{\partial^2 g}{\partial r \partial z} & \frac{1}{r}\frac{\partial^2 g}{\partial \theta \partial z} & \frac{\partial^2 g}{\partial z^2} \end{bmatrix} \tag{A.13}$$

The *Laplacian* (Δg) of g is defined by:

$$\Delta g \equiv \text{tr } (\nabla \nabla g) = \frac{\partial^2 g}{\partial r^2} + \frac{1}{r}(\frac{1}{r}\frac{\partial^2 g}{\partial \theta^2} + \frac{\partial g}{\partial r}) + \frac{\partial^2 g}{\partial z^2} \tag{A.14}$$

This can be rewritten as follows:

$$\Delta g = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial g}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 g}{\partial \theta^2} + \frac{\partial^2 g}{\partial z^2} \tag{A.15}$$

Consequently, the following result follows:

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$$\Delta\Delta g = \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial(\Delta g)}{\partial r}\right] + \frac{1}{r^2}\frac{\partial^2(\Delta g)}{\partial \theta^2} + \frac{\partial^2(\Delta g)}{\partial z^2} \tag{A.16}$$

Consider a second-order symmetric tensor $\boldsymbol{a}$ (e.g., stress $\boldsymbol{\sigma}$ or strain $\boldsymbol{\epsilon}$) and a vector $\boldsymbol{u}$. In Cartesian coordinates, the following result is easily established:

$$\frac{\partial}{\partial x_i}(a_{ij}u_j) = (\frac{\partial a_{ij}}{\partial x_j})u_i + a_{ji}\frac{\partial u_i}{\partial x_j} \tag{A.17}$$

This can be written in the following *intrinsic* form which is valid in cylindrical coordinates for instance

$$\operatorname{div}(\boldsymbol{a}\cdot\boldsymbol{u}) = (\operatorname{div}\,\boldsymbol{a})\cdot\boldsymbol{u} + \boldsymbol{a} : (\nabla\boldsymbol{u}), \tag{A.18}$$

where *div* designates the *divergence* operator. We now apply the result to the cylindrical basis by taking $\boldsymbol{u}$ to be equal to $\boldsymbol{e}_r$, $\boldsymbol{e}_\theta$ and $\boldsymbol{e}_z$, successively,

eq.(A.9)

$$\begin{aligned}
(\operatorname{div}\,\boldsymbol{a})\cdot\boldsymbol{e}_r &= \operatorname{div}(\boldsymbol{a}\cdot\boldsymbol{e}_r) - \boldsymbol{a} : (\nabla\boldsymbol{e}_r) \\
&= \frac{\partial a_{rr}}{\partial r} + \frac{1}{r}(\frac{\partial a_{\theta r}}{\partial \theta} + a_{rr}) + \frac{\partial a_{zr}}{\partial z} - \frac{1}{r}a_{\theta\theta}, \\
(\operatorname{div}\,\boldsymbol{a})\cdot\boldsymbol{e}_\theta &= \operatorname{div}(\boldsymbol{a}\cdot\boldsymbol{e}_\theta) - \boldsymbol{a} : (\nabla\boldsymbol{e}_\theta) \\
&= \frac{\partial a_{r\theta}}{\partial r} + \frac{1}{r}(\frac{\partial a_{\theta\theta}}{\partial \theta} + a_{r\theta}) + \frac{\partial a_{z\theta}}{\partial z} + \frac{1}{r}a_{r\theta}, \\
(\operatorname{div}\,\boldsymbol{a})\cdot\boldsymbol{e}_z &= \operatorname{div}(\boldsymbol{a}\cdot\boldsymbol{e}_z) - \boldsymbol{a} : (\nabla\boldsymbol{e}_z) \\
&= \frac{\partial a_{rz}}{\partial r} + \frac{1}{r}(\frac{\partial a_{\theta z}}{\partial \theta} + a_{rz}) + \frac{\partial a_{zz}}{\partial z},
\end{aligned} \tag{A.19}$$

For the projection along $\boldsymbol{e}_r$, we used (A.9) with $\boldsymbol{v} = \boldsymbol{e}_r$, i.e. $F = 1$ and $G = H = 0$, and Eq. (A.10) with

$$\boldsymbol{v} = \boldsymbol{a}\cdot\boldsymbol{e}_r = a_{rr}\boldsymbol{e}_r + a_{\theta r}\boldsymbol{e}_\theta + a_{zr}\boldsymbol{e}_z,$$

i.e. $F = a_{rr}$, $G = a_{\theta r}$ and $H = a_{zr}$. The projections along $\boldsymbol{e}_\theta$ and $\boldsymbol{e}_z$ are obtained in a similar fashion. As an application of results (A.19), *equilibrium equations* (div $\boldsymbol{\sigma} + \boldsymbol{f} = \boldsymbol{0}$) are obtained in cylindrical coordinates by setting $\boldsymbol{a} = \boldsymbol{\sigma}$, i.e.

$$\begin{aligned}
&\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\partial \sigma_{zr}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + f_r = 0; \\
&\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{z\theta}}{\partial z} + \frac{2}{r}\sigma_{r\theta} + f_\theta = 0 \\
&\frac{\partial \sigma_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta z}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\sigma_{rz}}{r} + f_z = 0
\end{aligned} \tag{A.20}$$

The infinitesimal *strain* tensor is defined by: $\boldsymbol{\epsilon} = (\nabla\boldsymbol{u} + \nabla^T\boldsymbol{u})/2$, where $\boldsymbol{u}$ is the displacement. Using (A.9), the components in the cylindrical basis are given as follows:

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High-order curvilinear finite elements for axisymmetric Lagrangian hydrodynamics

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ABSTRACT

In this paper we present an extension of our general high-order curvilinear finite element approach for solving the Euler equations in a Lagrangian frame [1] to the case of axisymmetric problems. The numerical approximation of these equations is important in a number of applications of compressible shock hydrodynamics and the reduction of 3D problems with axial symmetry to 2D computations provides a significant computational advantage. Unlike traditional staggered-grid hydrodynamics (SGH) methods, which use the so-called "area-weighting" scheme, we formulate our semi-discrete axisymmetric conservation laws directly in 3D and reduce them to a 2D variational form in a meridian cut of the original domain. This approach is a natural extension of the high-order curvilinear finite element framework we have developed for 2D and 3D problems in Cartesian geometry, leading to a rescaled momentum conservation equation which includes new radial terms in the pressure gradient and artificial viscosity forces. We show that this approach exactly conserves energy and we demonstrate via computational examples that it also excels at preserving symmetry in problems with symmetric initial conditions. The results also illustrate that our computational method does not produce spurious symmetry breaking near the axis of rotation, as is the case with many area-weighted approaches.

1. Introduction and motivation

The Euler equations of compressible hydrodynamics describe complex, multi-material, high speed flow and shock wave propagation over general 2D and 3D computational domains. We are interested in Lagrangian numerical methods for these problems, where the equations are discretized and solved on a generally unstructured computational mesh that moves with the fluid velocity. Specifically, the goal of Lagrangian hydrodynamics is to solve the following system of conservation laws:

$$\text{Momentum Conservation}: \quad \rho\frac{dv}{dt} = \nabla\cdot\sigma, \tag{1}$$

$$\text{Mass Conservation}: \quad \frac{1}{\rho}\frac{d\rho}{dt} = -\nabla\cdot v, \tag{2}$$

$$\text{Energy Conservation}: \quad \rho\frac{de}{dt} = \sigma : \nabla v, \tag{3}$$

$$\text{Equation of Motion}: \quad \frac{dx}{dt} = v, \tag{4}$$

$$\text{Equation of State}: \quad \sigma = -EOS(\rho, e)I, \tag{5}$$

which involves the material derivative $\frac{d}{dt}$, the kinematic variables for the fluid velocity v and position x, and the thermodynamic variables for the density ρ, pressure $p = EOS(\rho, e)$ and internal energy e of the fluid [2,3]. The equation of state, EOS, is a constitutive relation which in the simplest case of a polytropic ideal gas with a constant adiabatic index $\gamma > 1$ has the form $p = (\gamma - 1)\rho e$. Our formulation uses a general stress tensor σ in order to accommodate the inclusion of anisotropic tensor artificial viscosity stresses (see Section 3.2) as well as more complex material constitutive relations. We focus on purely Lagrangian methods, and do not consider the other components of a full Arbitrary Lagrangian–Eulerian (ALE) framework in this paper.

Three dimensional simulations of Lagrangian shock hydrodynamics are of great practical importance [3–5], but are also substantially more expensive than 2D calculations. Therefore, for problems with axial symmetry, the reduction of (1)–(5) to computations in a 2D meridian cut provides a significant computational advantage. In previous articles [6,1], we developed a general framework for high-order Lagrangian discretization of the Euler equations using curvilinear finite elements. In this paper, we present the extension of this framework to axisymmetric problems and demonstrate its ability to both conserve energy exactly and maintain symmetry. The realization of both these goals concurrently has proven challenging for many axisymmetric discretization schemes.

Traditional staggered-grid hydrodynamics (SGH) Lagrangian methods for axisymmetric problems have used the "area-weighted"

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method where the momentum equation is solved in 2D planar coordinates using the "area masses" at nodes while the internal energy equation is solved over the real volumes [7,8]. Generally, this approach does not conserve total energy exactly (unless the compatible approach of [9] is used) and can often lead to incorrect shock speeds, or cause spurious symmetry breaking in the internal energy field near the axis of rotation, leading to non-physical results as illustrated in Figs. 1 and 2. Preservation of physical symmetries is critical for inertial confinement fusion (ICF) simulations as uncertainties in whether non-symmetric results are due to numerical errors or physical processes can limit predictive capability. New compatible approaches have been proposed to address this deficiency [9–14], which have led to significant improvements in energy and symmetry preservation. Other successful methods in this area include the special finite elements proposed in [15,16] and the recent cell-centered hydro approach of [17,18].

In contrast to the above schemes, our finite element numerical method is derived by a faithful reduction of the 3D axisymmetric problem to a 2D variational form in a meridian cut of the domain. This approach conserves total energy exactly by construction. Unlike the area-weighted scheme, it leads to a rescaled momentum conservation equation, which also includes new terms in the pressure gradient and artificial viscosity forces. As in Cartesian coordinates, the high-order finite element approach uses high-order basis function expansions obtained via a high-order mapping from a standard reference element. This enables the use of curvilinear zone geometry and higher order approximations for the fields within a zone.

The remainder of the paper is organized as follows. In Section 2 we introduce notation and recall some basic facts about axisymmetric scalar, vector and tensor fields. These are used in Section 3, where we describe the derivation of our axisymmetric semi-discrete finite element method, followed by discussion of the artificial viscosity, the fully-discrete algorithm and the relation to some classical SGH methods. In Section 4, we present an extensive set of numerical results that demonstrate the robustness of our algorithm with respect to symmetry and energy conservation on a range of challenging axisymmetric problems. Finally, we summarize our experience and draw some conclusions in Section 5.

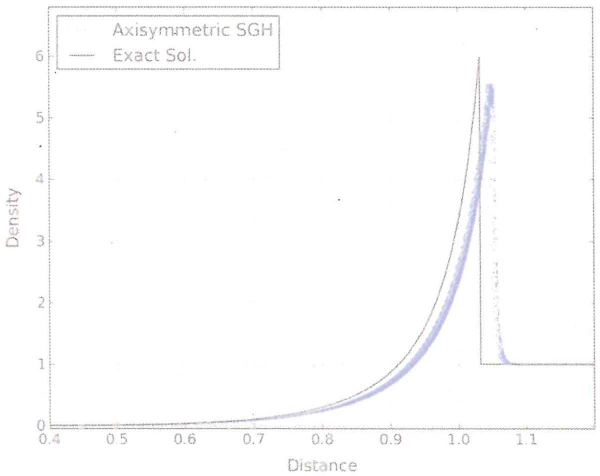

Fig. 1. Scatter plot of the density from a traditional SGH calculation of the spherical Sedov blast wave in axisymmetric mode [19]. The exact solution corresponds to the black line. While the "area-weighted" approach preserves the symmetry of accelerations, the corresponding energy update is not conservative. In this calculation this results in a 6% spurious gain in energy leading to incorrect shock speed and location. These do not improve under mesh refinement.

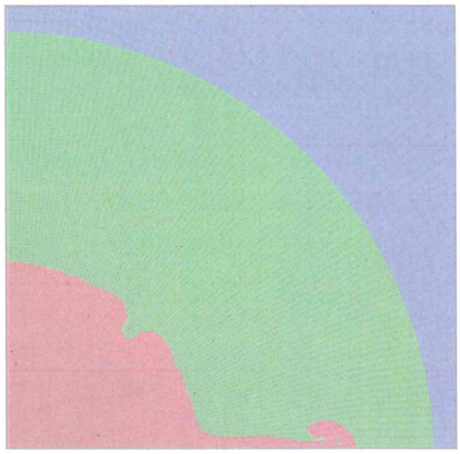

Fig. 2. Example of numerical symmetry breaking in an axisymmetric multi-material inertial confinement fusion (ICF) simulation. This is an ALE calculation where different colors are used to identify the different materials. The jet at the axis of rotation is spurious and does not disappear under mesh refinement.

2. Axisymmetric scalar, vector and tensor fields

In this section we recall some basic facts about axisymmetric fields that will be used in the development of our finite element discretization method in the following section.

We assume that at any given time, the domain Ω occupied by the fluid is a body of revolution, as illustrated in Fig. 3. In cylindrical coordinates (r,θ,z), Ω can be obtained from a "meridian cut" Γ in the r–z plane by a rotation around the axis $r=0$:

$$\Omega = \{(r,\theta,z) : (r,z) \in \Gamma\}.$$

A scalar function f, defined on the axisymmetric domain Ω, is itself called axisymmetric if it is independent of θ, i.e. $f(r,\theta,z) = f(r,z)$, so f is uniquely determined by its values in Γ. If f is given in Cartesian coordinates, it is axisymmetric if and only if

$$\frac{\partial}{\partial\theta} f(r\cos\theta, r\sin\theta, z) = 0,$$

i.e. if f is only spatially varying in the r–z plane

A key property of axisymmetric functions is that their integrals over Ω can be reduced to integrals over Γ:

$$\int_\Omega f(r,\theta,z) = 2\pi \int_\Gamma r f(r,z). \tag{6}$$

The local cylindrical coordinate system vectors at a point (r,θ,z) are given by

$$\vec{e}_r = (\cos\theta, \sin\theta, 0), \quad \vec{e}_\theta = (-\sin\theta, \cos\theta, 0), \quad \vec{e}_z = (0,0,1).$$

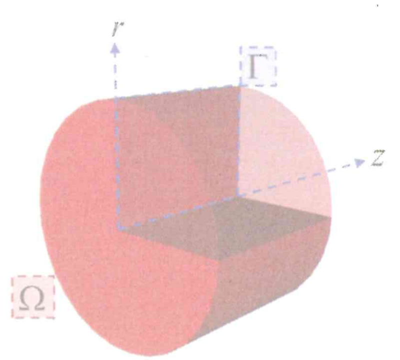

Fig. 3. Schematic depiction of the reduction of a 3D axisymmetric problem to a 2D "meridian cut" in the r–z plane.

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Let x and $\xi = (r, \theta, z)$ be the Cartesian and cylindrical coordinates of a point, respectively, so that

$$x = x(\xi) = (r\cos\theta, r\sin\theta, z).$$

The material derivative of ξ can be expressed as

$$\frac{d\xi}{dt} = \frac{\partial \xi}{\partial x}\frac{dx}{dt},$$

which after some simple manipulations can be written as

$$\frac{d}{dt}(r, \theta, z) = \left(v_r, \frac{v_\theta}{r}, v_z\right), \tag{7}$$

where (v_r, v_θ, v_z) are the cylindrical components of the velocity:

$$\frac{dx}{dt} = v = v_r\vec{e}_r + v_\theta\vec{e}_\theta + v_z\vec{e}_z.$$

Identity (7) represents the cylindrical version of the equation of motion (4). Furthermore, using (7) together with the definitions of $\vec{e}_r, \vec{e}_\theta$, and $\vec{e}_z$, the acceleration vector can be expressed as

$$\frac{dv}{dt} = \left(\frac{dv_r}{dt} - \frac{v_\theta^2}{r}\right)\vec{e}_r + \left(\frac{dv_\theta}{dt} + \frac{v_r v_\theta}{r}\right)\vec{e}_\theta + \frac{dv_z}{dt}\vec{e}_z. \tag{8}$$

The material derivative of a scalar field f is given by

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial \xi}\frac{d\xi}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial r}v_r + \frac{\partial f}{\partial \theta}\frac{v_\theta}{r} + \frac{\partial f}{\partial z}v_z,$$

hence if f, v_r, and v_z are axisymmetric then so is $\frac{df}{dt}$.

A vector field v, defined on the axisymmetric domain Ω, is called axisymmetric if

$$v = v_r(r,z)\vec{e}_r + v_\theta(r,z)\vec{e}_\theta + v_z(r,z)\vec{e}_z,$$

i.e. if v remains invariant under arbitrary rotation around the axis $r = 0$. This is the most general axial symmetry assumption for a vector field which requires the use of the additional (compared to a 2D method) velocity component v_θ and is generally more complex to handle. Therefore, here we consider the more standard additional assumption that $v_\theta \equiv 0$, i.e. axisymmetric vector fields without components in the normal direction of the meridian cut. With this assumption the equation of motion (7) and the left hand side of the momentum conservation equation (1), as well as (8), simplify and become identical to the 2D case in the meridian cut Γ.

The gradient operator in cylindrical coordinates is given by

$$\nabla_{rz} f = \frac{\partial f}{\partial r}\vec{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\vec{e}_\theta + \frac{\partial f}{\partial z}\vec{e}_z. \tag{9}$$

Therefore, $\nabla_{rz} f$ is axisymmetric if and only if f is. In this case, the formula simplifies to

$$\nabla_{rz} f = \frac{\partial f}{\partial r}\vec{e}_r + \frac{\partial f}{\partial z}\vec{e}_z,$$

which is just the regular 2D gradient in Γ. Note that this property is one of the motivating factors for using area-weighting schemes, as it implies that gradient operators are unchanged in axisymmetric coordinates.

The divergence in cylindrical coordinates is more complicated:

$$\nabla_{rz} \cdot v = \frac{\partial v_r}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} + \frac{v_r}{r}, \tag{10}$$

but $\nabla_{rz} \cdot v$ is still axisymmetric, provided that v is. In this case, the formula simplifies to

$$\nabla_{rz} \cdot v = \frac{\partial v_r}{\partial r} + \frac{\partial v_z}{\partial z} + \frac{v_r}{r},$$

which has an extra term compared to the regular 2D divergence in Γ.

The gradient of the axisymmetric vector function $v = v_r(r,z)\vec{e}_r + v_z(r,z)\vec{e}_z$ can be written as

$$\begin{aligned}\nabla_{rz} v &= \vec{e}_r \otimes \frac{\partial v}{\partial r} + \frac{1}{r}\vec{e}_\theta \otimes \frac{\partial v}{\partial \theta} + \vec{e}_z \otimes \frac{\partial v}{\partial z} \\ &= \frac{\partial v_r}{\partial r}\vec{e}_r \otimes \vec{e}_r + \frac{\partial v_z}{\partial r}\vec{e}_r \otimes \vec{e}_z + \frac{v_r}{r}\vec{e}_\theta \otimes \vec{e}_\theta \\ &\quad + \frac{\partial v_r}{\partial z}\vec{e}_z \otimes \vec{e}_r + \frac{\partial v_z}{\partial z}\vec{e}_z \otimes \vec{e}_z,\end{aligned}$$

so the matrix form of the gradient in the $z - r - \theta$ ordering is

$$\nabla_{rz} v = \begin{pmatrix} \frac{\partial v_z}{\partial z} & \frac{\partial v_r}{\partial z} & 0 \\ \frac{\partial v_z}{\partial r} & \frac{\partial v_r}{\partial r} & 0 \\ 0 & 0 & \frac{v_r}{r} \end{pmatrix} = \begin{pmatrix} \nabla_{2d} v & 0 \\ 0 & \frac{v_r}{r} \end{pmatrix}. \tag{11}$$

A tensor field σ is axisymmetric if its components in the local cylindrical basis are independent of θ and have the form

$$\sigma = \begin{pmatrix} \sigma_{zz} & \sigma_{zr} & 0 \\ \sigma_{rz} & \sigma_{rr} & 0 \\ 0 & 0 & \sigma_{\theta\theta} \end{pmatrix} = \begin{pmatrix} \sigma_{2d} & 0 \\ 0 & \sigma_{\theta\theta} \end{pmatrix}.$$

Since the local cylindrical basis is orthonormal, the contraction (or double dot product) of σ and $\nabla_{rz} v$ is given by

$$\boxed{\sigma : \nabla_{rz} v = \sigma_{2d} : \nabla_{2d} v + \sigma_{\theta\theta}\frac{v_r}{r},}$$

which is a scalar axisymmetric field, i.e. it is independent of θ.

3. Finite element discretization

In this section we derive and discuss a finite element-based numerical approximation scheme for the Euler equations (1)–(5) in axisymmetric form. The presentation follows the finite element form of the general semi-discrete Lagrangian discretization method from [1], to which we refer for additional details.

3.1. Semi-discrete formulation

We first discuss the semi-discrete axisymmetric method, which is concerned only with the spatial approximation of the continuum equations. The fully-discrete methods that incorporate time discretization will be presented in Section 3.3.

Let $\Omega(t)$ be the continuous 3D axisymmetric medium (fluid or elastic body) which is deforming in time according to (1)–(5) starting from an initial configuration at time $t = t_0$. Let $\Gamma(t)$ be the corresponding meridian cut, as discussed in Section 2. Following [1], we introduce a 2D finite element mesh on $\tilde{\Gamma} \equiv \Gamma(t_0)$ with zones (or elements) $\{\Gamma_z(t_0)\}$. This also induces a decomposition of $\tilde{\Omega} \equiv \Omega(t_0)$ into toroidal zones $\{\Omega_z(t_0)\}$ obtained by revolution of the 2D mesh elements around the axis $r = 0$ in cylindrical coordinates:

$$\Omega_z(t_0) = \{(r, \theta, z) : (r, z) \in \Gamma_z(t_0)\}. \tag{12}$$

A main feature of our approach is that the finite element mesh is described through the locations of high-order particles (or control points) that are tracked by the semi-discrete algorithm. This results in curvilinear zones that can better represent the naturally developing curvature in the flow. Specifically, the current position at time t, $x = (r, z) \in \Gamma(t)$, corresponding to a particle at an initial position $\tilde{x} = (\tilde{r}, \tilde{z}) \in \tilde{\Gamma}$ is discretized using the expansion

$$x(\tilde{x}, t) = \sum_{i=1}^{N_V} \mathbf{x}_i(t) w_i(\tilde{x}) = \mathbf{x}(t)^{\mathrm{T}} \mathbf{w}(\tilde{x}), \tag{13}$$

where $\mathbf{x}(t)$ is an unknown time-dependent vector of coefficients in the kinematic basis $\{w_i\}_{i=1}^{N_V}$, and $\mathbf{w}$ is a column vector of all the basis

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