Kirsch equations

Analytical displacements and stresses for plane strain/stress infinite plate with circular hole under uniaxial tension

Table 1: Nomenclature

Symbol	Description	Unit
w	half plate width	mm
a	hole radius	mm
σ	uniaxial tension	MPa
E	Young's modulus	MPa
ν	Poisson's ratio	–
$\boldsymbol{T} = \boldsymbol{\sigma} \cdot \boldsymbol{N}$	traction	MPa
$\boldsymbol{N}$	unit normal vector	–
$\boxed{\varepsilon}$	plane strain state	–
$\boxed{\sigma}$	plane stress state	–
$R \equiv \frac{a^2}{r^2}$	quadratic ratio of hole radius a to r-coordinate	–

Polar and Cartesian coordinates: (only valid for quarter plate in first quadrant: $x \geq 0, y \geq 0$)

$$r = \sqrt{x^2 + y^2} \qquad \theta = \arctan \frac{y}{x} \tag{1}$$

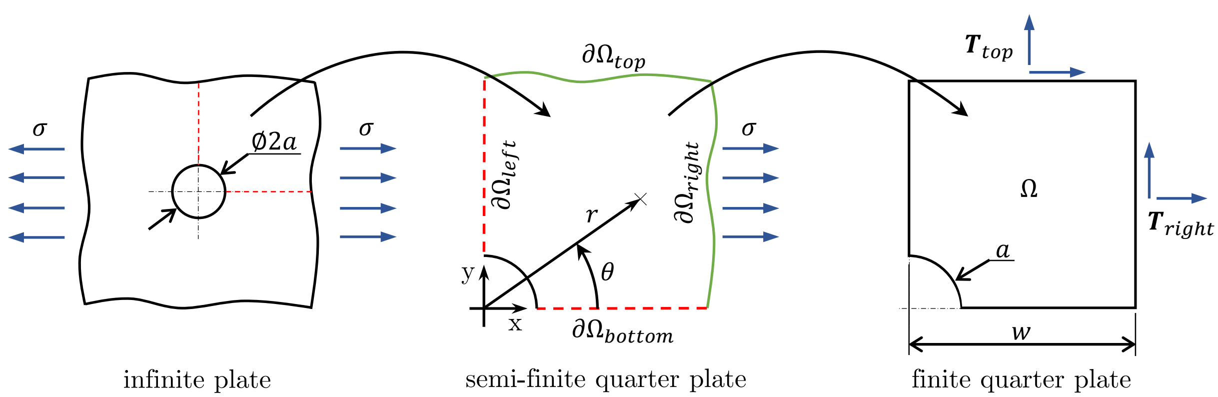

Figure 1: Benchmark problem with load (left) simplified to semi-finite quarter plate with additional boundary conditions (middle) and finite quarter plate with coupling tractions (right).

Stresses

Kirsch equations (polar stress components):

$$\sigma_{rr}^{\boxed{\varepsilon}} = \sigma_{rr}^{\boxed{\sigma}} = \frac{\sigma}{2} \left[1 - R + \left[1 + 3R^2 - 4R\right] \cdot \cos(2\theta)\right] \tag{2}$$

$$\sigma_{\theta\theta}^{\boxed{\varepsilon}} = \sigma_{\theta\theta}^{\boxed{\sigma}} = \frac{\sigma}{2} \left[1 + R - \left[1 + 3R^2\right] \cdot \cos(2\theta)\right] \tag{3}$$

$$\tau_{r\theta}^{\boxed{\varepsilon}} = \tau_{r\theta}^{\boxed{\sigma}} = -\frac{\sigma}{2} \left[1 - 3R^2 + 2R\right] \cdot \sin(2\theta) \tag{4}$$

Kirsch equations (Cartesian stress components):

$$\sigma_{xx}^{\boxed{\varepsilon}} = \sigma_{xx}^{\boxed{\sigma}} = \frac{\sigma}{2}\left[R^2\left[-3+6\,[\cos(2\,\theta)]^2\right] - R\,[3+4\,\cos(2\,\theta)]\cos(2\,\theta)+2\,R+2\right] \tag{5}$$

$$\sigma_{yy}^{\boxed{\varepsilon}} = \sigma_{yy}^{\boxed{\sigma}} = -\frac{\sigma}{2}R\,\left[[\cos(2\,\theta)]^2\,[6\,R-4]+\cos(2\,\theta)-3\,R+2\right] \tag{6}$$

$$\tau_{xy}^{\boxed{\varepsilon}} = \tau_{xy}^{\boxed{\sigma}} = \frac{\sigma}{2}R\,[6\,\cos(2\,\theta)\,R-4\,\cos(2\,\theta)-1]\sin(2\,\theta) \tag{7}$$

$$\sigma_{zz}^{\boxed{\sigma}} = 0 \tag{8}$$

$$\sigma_{zz}^{\boxed{\varepsilon}} = \nu\left[\sigma_{xx}^{\boxed{\varepsilon}} + \sigma_{yy}^{\boxed{\varepsilon}}\right] \tag{9}$$

von Mises stress (identically with polar stress components):

$$\sigma_{vM}^{\boxed{\varepsilon}} = \sqrt{[\nu^2-\nu+1]\,[\sigma_{xx}{}^2+\sigma_{yy}{}^2]+\sigma_{xx}\sigma_{yy}\,[2\,\nu^2-2\,\nu-1]+3\,\tau_{xy}{}^2} \tag{10}$$

$$\sigma_{vM}^{\boxed{\sigma}} = \sqrt{\sigma_{xx}{}^2+\sigma_{yy}{}^2-\sigma_{xx}\sigma_{yy}+3\,\tau_{xy}{}^2} \tag{11}$$

Displacements

Plane strain polar displacement components:

$$u_r^{\boxed{\varepsilon}} = -\frac{\sigma r}{2E}\left[\left[-1+R^2+[4\,\nu-4]\,R\right]\cos(2\,\theta)-R+2\nu-1\right][1+\nu] \tag{12}$$

$$u_\theta^{\boxed{\varepsilon}} = -\frac{\sigma r}{2E}\sin(2\,\theta)\,[1+\nu]\left[R^2+[-4\nu+2]\,R+1\right] \leq 0 \tag{13}$$

Plane stress polar displacement components:

$$u_r^{\boxed{\sigma}} = -\frac{\sigma r}{2E}\left[\left[[\nu+1]\,R^2-4R-\nu-1\right]\cos(2\theta)-[\nu+1]\,R+\nu-1\right] \tag{14}$$

$$u_\theta^{\boxed{\sigma}} = -\frac{\sigma r}{2E}\left[[\nu+1]\,R^2+2\,[-\nu+1]\,R+\nu+1\right]\sin(2\theta) \leq 0 \tag{15}$$

Transformation from polar displacements to Cartesian components (identically for plane strain/stress):

$$u_x = \sqrt{u_r{}^2+u_\theta{}^2}\cdot\cos(\theta+\phi) \qquad u_y = \sqrt{u_r{}^2+u_\theta{}^2}\cdot\sin(\theta+\phi) \tag{16}$$

$$\text{with } \phi = \begin{cases} \arctan\left(\frac{u_\theta}{u_r}\right) & \text{if } u_r > 0 \\ \arctan\left(\frac{u_\theta}{u_r}\right)+\pi & \text{if } u_r < 0 \\ \frac{3\pi}{2} & \text{if } u_r = 0 \end{cases} \tag{17}$$

Coupling tractions (on green lines, identically for plane strain/stress)

$$\left[(T_{right})_i\right] = \sigma_{ij}\cdot(N_1)_j = \begin{bmatrix} \sigma_{xx} & \tau_{xy} \\ \tau_{xy} & \sigma_{yy} \end{bmatrix}\cdot\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \sigma_{xx} \\ \tau_{xy} \end{bmatrix} \quad \text{on} \quad \partial\Omega_{right} \tag{18}$$

$$\left[(T_{top})_i\right] = \sigma_{ij}\cdot(N_2)_j = \begin{bmatrix} \tau_{xy} \\ \sigma_{yy} \end{bmatrix} \quad \text{on} \quad \partial\Omega_{top} \tag{19}$$

Symmetry constraints (on red dashed lines)

$$\text{Polar constraints: } u_\theta = 0 \quad \text{on} \quad \partial\Omega_{left} \Leftrightarrow (\theta=\pi/2) \qquad u_\theta = 0 \quad \text{on} \quad \partial\Omega_{bottom} \Leftrightarrow (\theta = 0) \tag{20}$$

$$\text{Cartesian constraints: } u_x = 0 \quad \text{on} \quad \partial\Omega_{left} \Leftrightarrow (x=0) \qquad u_y = 0 \quad \text{on} \quad \partial\Omega_{bottom} \Leftrightarrow (y = 0) \tag{21}$$