

Math 408A

Line Search Methods

The Backtracking Line Search

One Dimensional Optimization and Line Search Methods

Line Search Methods

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How should the search direction and stepsize be chosen.

The Basic Backtracking Algorithm

Assume that $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is differentiable and $d \in \mathbb{R}^n$ is a direction of strict descent at x_c , i.e., $f'(x_c; d) < 0$.

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STEP 1: Compute the backtracking stepsize

$$\begin{aligned} t^* &:= \max \gamma^\nu \\ \text{s.t. } &\nu \in \{0, 1, 2, \dots\} \text{ and} \\ &f(x_c + \gamma^\nu d) \leq f(x_c) + c\gamma^\nu f'(x_c; d). \end{aligned}$$

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STEP 2: Set $x_+ = x_c + t^* d$.

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Therefore, there is a $\bar{t} > 0$ such that

$$\frac{f(x_c + td) - f(x_c)}{t} < cf'(x_c; d) \quad \forall t \in (0, \bar{t}),$$

that is

$$f(x_c + td) < f(x_c) + ctf'(x_c; d) \quad \forall t \in (0, \bar{t}).$$

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Since $0 < \gamma < 1$, $\gamma^\nu \downarrow 0$ as $\nu \uparrow \infty$, there is a ν_0 such that $\gamma^\nu < \bar{t}$ for all $\nu \geq \nu_0$.

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Consequently,

$$f(x_c + \gamma^\nu d) \leq f(x_c) + c\gamma^\nu f'(x_c; d) \quad \forall \nu \geq \nu_0,$$

that is, the backtracking line search is finitely terminating.

Programming the Backtracking Algorithm

Pseudo-Matlab code:

```
[  
     $f_c = f(x_c)$   
     $\Delta f = cf'(x_c; d)$   
    newf =  $f(x_c + d)$   
     $t = 1$   
    while newf >  $f_c + t\Delta f$   
         $t = \gamma t$   
        newf =  $f(x_c + td)$   
    endwhile
```

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3. Newton-Like Direction:

$$d = -H \nabla f(x_c),$$

where $H \in \mathbb{R}^{n \times n}$ is symmetric and constructed so that

$$H \approx \nabla^2 f(x_c)^{-1} .$$

Descent Condition

For all of these directions we have

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In all other cases, $H \approx \nabla^2 f(x_c)^{-1}$. The condition that H be pd is related to the second-order sufficient condition for optimality, a local condition.