# Independent Study – Learning Music Structure by Laplacian Formula

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**Abstract.** There are many approaches to analyzing music structure by features extracted from dimension of time series. With contruction of similarity matrix, repeated pattern can be captured which is the building block for large-scale structure. This is the work based on the Laplacian Matrix, which is essential start point of spectral clustering. We introduce variables that are trainable to reduce the cost of Laplacian Matrix from true lable, and run this method on wide variable of music recordings. Finally, we demonstrate using these trained variable for performing proper music segmentation.

**1. Laplacian formula.** Normalized Laplacian matrix is the essential start point for identify music segmentation, and the correct boundary detection is done in my baseline approach (Ref2). For proper boundary detection, we woule like to train the initial laplacian matrix ($L$) close to true laplacian ($L^*$) from true inerval annotation from SALAMI dataset. Therefore, to train and update the model, this section is for deriving the $L$ and $\frac{\partial L}{\partial w_{i,j}}$
Given the definition of normalized laplacian matrix:

$$L := I - D^{\frac{1}{2}} W D^{\frac{1}{2}} \tag{1.1}$$

D is degree matrix defined as the diagnal matrix with degrees $d_1, d_2, \ldots, d_n$, which $d_i$ is accumulation of similarity coefficient $w_{i,j}$ betoween time point $i$ and $j$ which defined as followed:

$$d_i = \sum_{j \neq i}^{n} w_{ij} \tag{1.2}$$

After multiplication and the result of equation 1.1 can be rewrite as:

$$L := I - D^{\frac{1}{2}} W D^{\frac{1}{2}} = \begin{pmatrix} 1 & \frac{-w_{12}}{\sqrt{d_1 d_2}} & \cdots & \frac{-w_{1n}}{\sqrt{d_1 d_n}} \\ \frac{-w_{21}}{\sqrt{d_2 d_1}} & 1 & \cdots & \frac{-w_{2n}}{\sqrt{d_2 d_n}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{-w_{n1}}{\sqrt{d_n d_1}} & \frac{-w_{n2}}{\sqrt{d_n d_2}} & \cdots & 1 \end{pmatrix} = \begin{cases} \frac{-w_{i,j}}{\sqrt{d_i d_j}} & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases} \tag{1.3}$$

To take the derivative of $L$ w.r.t $w_{i,j}$. Results is as follow and detail derivation is in appendix 2.1:

$$\frac{\partial L}{\partial w_{i,j}} = \begin{cases} 0 & \text{, if } i = j \\ \frac{-1}{\sqrt{d_i d_j}} + \frac{w_{i,j}(d_i + d_j)}{2(d_i d_i)^{\frac{3}{2}}} & \text{, for position } (i,j), (j,i) \\ \frac{w_{l,k}}{2\sqrt{d_k} d_l^{\frac{3}{2}}} & \text{, for all position } (l,k), (k,l) \text{, where } k \neq i \& j \text{ and } l = i \| j \\ 0 & \text{for any other position} \end{cases} \tag{1.4}$$

The key idea of this derivation is that, when taking derivative of $w_{i,j}$ the i-th and j-th row and column will be altered because of the degree changes in equation 1.2.

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## REFERENCES

[1] A Tutorial on Spectral Clustering
[2] "https://github.com/jfriend08/IS/blob/master/reports/midway/report.pdf"


## 2. Appendix.

### 2.1. Differential of Laplacian Matrix.

If we would like to take derivative of Laplacian w.r.t variable $w_{i,j}$ in the symmetric matrix $W$. Basically, except for $L_{i,j}$ the components of $L_{i,k}$, $L_{k,i}$, $L_{k,j}$, $L_{j,k}$ will need to consider.
For position $L_{i,j}$:

$$\begin{aligned} L_{i,j} &= L_{j,i} = \frac{-w_{i,j}}{\sqrt{d_i d_j}} \\ \frac{\partial L_{i,j}}{\partial w_{i,j}} &= \frac{\partial L_{j,i}}{\partial w_{i,j}} = \frac{-1}{\sqrt{d_i d_j}} + \frac{w_{i,j}}{2(d_i d_i)^{\frac{3}{2}}}\left(\frac{\partial d_i}{\partial w_{i,j}} d_j + d_i \frac{\partial d_j}{\partial w_{i,j}}\right) \\ &= \frac{-1}{\sqrt{d_i d_j}} + \frac{w_{i,j}(d_i + d_j)}{2(d_i d_i)^{\frac{3}{2}}} \end{aligned} \tag{2.1}$$

For position $L_{l,k}$, $L_{k,l}$, where $k \neq i \& j$ and $l = i \| j$:

$$\begin{aligned} L_{k,l} &= L_{l,k} = \frac{-w_{k,l}}{\sqrt{d_k d_l}} \\ \frac{\partial L_{k,l}}{\partial w_{i,j}} &= \frac{\partial L_{k,l}}{\partial w_{i,j}} = \frac{w_{k,l}}{2\sqrt{d_k} d_l^{\frac{3}{2}}} \end{aligned} \tag{2.2}$$