Independent Study – Derivative of Laplacian formula

Peter Yun-shao Sung*

1. Laplacian formula. Give the definition of normalized Laplacian formula as followed:

$$L := I - D^{\frac{1}{2}} W D^{\frac{1}{2}} \tag{1.1}$$

D is degree matrix defined as the diagnal matrix with degrees $d_1, d_2, \ldots, d_n$, which d_i is defined as followed:

$$d_i = \sum_{i \neq j}^{n} w_{ij} \tag{1.2}$$

Althought the condition of $i \neq j$ is not emphasized in [1], but this is the condition when I checked scipy.sparse.csgraph.laplacian.
Here is the setps to calculate equation 1.1

$$D^{\frac{1}{2}} W D^{\frac{1}{2}} = \begin{pmatrix} \frac{1}{\sqrt{d_1}} & 0 & \cdots & 0 \\ 0 & \frac{1}{\sqrt{d_2}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{\sqrt{d_n}} \end{pmatrix} \begin{pmatrix} w_{11} & w_{12} & \cdots & w_{1n} \\ w_{21} & w_{22} & \cdots & w_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ w_{n1} & w_{n2} & \cdots & w_{nn} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{d_1}} & 0 & \cdots & 0 \\ 0 & \frac{1}{\sqrt{d_2}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{\sqrt{d_n}} \end{pmatrix} \tag{1.3}$$

After multiplication and the result of equation 1.1 can be rewrite as:

$$L := I - D^{\frac{1}{2}} W D^{\frac{1}{2}} = \begin{pmatrix} 1 - \frac{w_{11}}{d_1} & \frac{-w_{12}}{\sqrt{d_1 d_2}} & \cdots & \frac{-w_{1n}}{\sqrt{d_1 d_n}} \\ \frac{-w_{21}}{\sqrt{d_2 d_1}} & 1 - \frac{w_{22}}{d_2} & \cdots & \frac{-w_{2n}}{\sqrt{d_2 d_n}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{-w_{n1}}{\sqrt{d_n d_1}} & \frac{-w_{n2}}{\sqrt{d_n d_2}} & \cdots & 1 - \frac{w_{nn}}{d_n} \end{pmatrix} \tag{1.4}$$

and can be generalized as followed:

$$L_{i,j} = \begin{cases} \frac{-w_{i,j}}{\sqrt{d_i, d_j}} & \text{if } i \neq j \\ 1 - \frac{w_{i,j}}{d_i} & \text{if } i = j \end{cases} \tag{1.5}$$

REFERENCES

[1] A Tutorial on Spectral Clustering

*yss265@nyu.edu