

## Independent Study – Derivative of Laplacian formula

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**1. Laplacian formula.** Give the definition of normalized Laplacian formula as followed:

$$L := I - D^{\frac{1}{2}} W D^{\frac{1}{2}} \quad (1.1)$$

D is degree matrix defined as the diagonal matrix with degrees  $d_1, d_2, \dots, d_n$ , which  $d_i$  is defined as followed:

$$d_i = \sum_{i \neq j}^n w_{ij} \quad (1.2)$$

Althought the condition of  $i \neq j$  is not emphasized in [1], but this is the condition when I checked `scipy.sparse.csgraph.laplacian`.

Here is the steps to calculate equation 1.1

$$D^{\frac{1}{2}} W D^{\frac{1}{2}} = \begin{pmatrix} \frac{1}{\sqrt{d_1}} & 0 & \cdots & 0 \\ 0 & \frac{1}{\sqrt{d_2}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{\sqrt{d_n}} \end{pmatrix} \begin{pmatrix} w_{11} & w_{12} & \cdots & w_{1n} \\ w_{21} & w_{22} & \cdots & w_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ w_{n1} & w_{n2} & \cdots & w_{nn} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{d_1}} & 0 & \cdots & 0 \\ 0 & \frac{1}{\sqrt{d_2}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{\sqrt{d_n}} \end{pmatrix} \quad (1.3)$$

After multiplication and the result of equation 1.1 can be rewrite as:

$$L := I - D^{\frac{1}{2}} W D^{\frac{1}{2}} = \begin{pmatrix} 1 - \frac{w_{11}}{d_1} & \frac{-w_{12}}{\sqrt{d_1 d_2}} & \cdots & \frac{-w_{1n}}{\sqrt{d_1 d_n}} \\ \frac{-w_{21}}{\sqrt{d_2 d_1}} & 1 - \frac{w_{22}}{d_2} & \cdots & \frac{-w_{2n}}{\sqrt{d_2 d_n}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{-w_{n1}}{\sqrt{d_n d_1}} & \frac{-w_{n2}}{\sqrt{d_n d_2}} & \cdots & 1 - \frac{w_{nn}}{d_n} \end{pmatrix} \quad (1.4)$$

and can be generalized as followed:

$$L_{i,j} = \begin{cases} \frac{-w_{i,j}}{\sqrt{d_i d_j}} & \text{if } i \neq j \\ 1 - \frac{w_{i,i}}{d_i} & \text{if } i = j, \text{ not-scipy} \\ 1 & \text{if } i = j, \text{ scipy} \end{cases} \quad (1.5)$$

**2. Derivative of Laplacian formula.** Now if we would like to take derivative of Laplacian w.r.t variable  $w_{i,j}$  in the symmetric matrix  $W$ . Basically the components  $L_{i,k}$ ,  $L_{k,i}$ ,  $L_{k,j}$ ,  $L_{j,k}$ ,  $L_{i,j}$ , and  $L_{j,i}$ , where  $k \neq i, k \neq j$  will need to consider.

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For position  $L_{i,j}$ , where  $i \neq j$ :

$$\begin{aligned} L_{i,j} &= L_{j,i} = \frac{-w_{i,j}}{\sqrt{d_i d_j}} \\ \frac{\partial L_{i,j}}{\partial w_{i,j}} &= \frac{\partial L_{j,i}}{\partial w_{i,j}} = \frac{-1}{\sqrt{d_i d_j}} + \frac{w_{i,j}}{2(d_i d_i)^{\frac{3}{2}}} \left( \frac{\partial d_i}{\partial w_{i,j}} d_j + d_i \frac{\partial d_j}{\partial w_{i,j}} \right) \\ &= \frac{-1}{\sqrt{d_i d_j}} + \frac{w_{i,j}(d_i + d_j)}{2(d_i d_i)^{\frac{3}{2}}} \end{aligned} \quad (2.1)$$

For position  $L_{i,j}$ , where  $i = j$ :

$$\begin{aligned} L_{i,j} &= 1 - \frac{w_{i,j}}{d_i} \\ \frac{\partial L_{i,j}}{\partial w_{i,j}} &= -(d_i)^{-1} + w_{i,j}(d_i)^{-2} \frac{\partial d_i}{\partial w_{i,j}} \\ &= \frac{w_{i,j} - d_i}{d_i^2} \end{aligned} \quad (2.2)$$

For position  $L_{i,k}$ , where  $k \neq i$  and  $k \neq j$ :

$$\begin{aligned} L_{i,k} &= L_{k,i} = \frac{-w_{i,k}}{\sqrt{d_i d_k}} \\ \frac{\partial L_{i,k}}{\partial w_{i,j}} &= \frac{\partial L_{k,i}}{\partial w_{i,j}} = \frac{w_{i,k}}{2\sqrt{d_k d_i^{\frac{3}{2}}}} \end{aligned} \quad (2.3)$$

For position  $L_{k,j}$ , where  $k \neq i$  and  $k \neq j$ :

$$\begin{aligned} L_{k,j} &= L_{j,k} = \frac{-w_{j,k}}{\sqrt{d_j d_k}} \\ \frac{\partial L_{k,j}}{\partial w_{i,j}} &= \frac{\partial L_{j,k}}{\partial w_{i,j}} = \frac{w_{j,k}}{2\sqrt{d_k d_j^{\frac{3}{2}}}} \end{aligned} \quad (2.4)$$

Therefore the generalized results is as followed:

$$\frac{\partial L}{\partial w_{i,j}} = \begin{cases} \frac{w_{i,j} - d_i}{d_i^2} & \text{if } i = j, \text{ for position (i,j), not-scipy} \\ 0 & \text{if } i = j, \text{ for position (i,j), scipy} \\ \frac{-1}{\sqrt{d_i d_j}} + \frac{w_{i,j}(d_i + d_j)}{2(d_i d_i)^{\frac{3}{2}}} & \text{if } i \neq j, \text{ for position (i,j) and (j,i)} \\ \frac{w_{i,k}}{2\sqrt{d_k d_i^{\frac{3}{2}}}} & \text{if } k \neq i \& k \neq j, \text{ for position (i,k) and (k,i)} \\ \frac{w_{j,k}}{2\sqrt{d_k d_j^{\frac{3}{2}}}} & \text{if } k \neq i \& k \neq j, \text{ for position (k,j) and (j,k)} \\ \frac{w_{i,k} - d_i}{d_i^2} & \text{if } k = i, \text{ for position (i,i), not-scipy} \\ \frac{w_{j,k} - d_j}{d_j^2} & \text{if } k = j, \text{ for position (j,j), not-scipy} \\ 0 & \text{for any other position} \end{cases} \quad (2.5)$$

## REFERENCES

- [1] A TUTORIAL ON SPECTRAL CLUSTERING